INTERIOR GRADIENT BOUNDS FOR SOLUTIONS TO THE MINIMAL SURFACE SYSTEM

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Abstract. In this article we generalize the classical gradient estimate for the minimal surface equation to higher codimension. We consider a vector-valued function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ that satisfies the minimal surface system, see equation (1.1) in §1. The graph of u is then a minimal submanifold of $\mathbb{R}^{n+m}$. We prove an a priori gradient bound under the assumption that the Jacobian of $du : \mathbb{R}^n \to \mathbb{R}^m$ on any two dimensional subspace of $\mathbb{R}^n$ is less than or equal to one. This assumption is automatically satisfied when du is of rank one and thus the estimate covers the case when $m = 1$, i.e., the original minimal surface equation. This is applied to Bernstein type theorems for minimal submanifolds of higher codimension.

1. Introduction. The interior gradient bound for solutions to the minimal surface equation was discovered, in the case of two variables, by Finn [3] and in the general case by Bombieri, Di Giorgi and Miranda [1]. The a priori bound is a key step in the existence and regularity of minimal surface theory. The estimate has been generalized to other curvature equations and proved by different methods. We refer to the note at the end of chapter 16 of [5] for literature in these directions.

Recall a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a solution to the minimal surface equation if

$$\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} = 0$$

where $g^{ij} = \delta_{ij} - (1 + |du|^2)^{-1} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}$. The graph of u is a minimal hypersurface in $\mathbb{R}^{n+1}$ and the gradient estimate for u says the C^1 norm of u is bounded by an exponential function of the C^0 norm. In this article, we consider the differential systems satisfied by a minimal submanifold of higher codimension.

The following definition is quoted from [16] (see also [11]):

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Definition 1.1. A C^2 vector-valued function $u = (u^{n+1}, \dots, u^{n+m}) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is a solution to the minimal surface system if

$$\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} = 0, \text{ for each } \alpha = n+1, \dots, n+m \tag{1.1}$$

where $g^{ij} = (g_{ij})^{-1}$ and $g_{ij} = \delta_{ij} + \sum_{\beta=n+1}^{n+m} \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}$.

The minimal surface system is indeed equivalent to the Euler-Lagrange equation of the volume functional $\int_\Omega \sqrt{\det g_{ij}}\, dx$, see Theorem 2.2 in [16] for the derivation. It has been considered by Osserman [15], [16] and Lawson-Osserman [11] and it is known that there are nonexistence, nonuniqueness and irregularity examples [11] for the Dirichlet problem. In [22], the author solves the Cauchy-Dirichlet problem of the associated parabolic system, i.e., the mean curvature flow, in arbitrary dimension and codimension assuming the variation of the initial surface is bounded by a constant depending on n. In this article, we prove the following gradient bound for solutions to the minimal surface system.

THEOREM A. *Let Ω be a domain in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}^m$ a C^2 solution to equation (1.1) such that the Jacobian of $du : \mathbb{R}^n \to \mathbb{R}^m$ on any two dimensional subspace of $\mathbb{R}^n$ is less than or equal to one. If each u^α is nonnegative, then for any point $x_0 \in \Omega$, we have estimate*

$$|du(x_0)| \le C_1 \exp\{C_2 |u(x_0)|/d\}$$

where $d = dist(x_0, \partial\Omega)$ and C_1, C_2 are constants depending on n.

The assumption can be described in terms of the singular values of du. du is a linear map and thus $\sqrt{(du)^T du}$ is a nonnegative definite symmetric matrix. If we denote the eigenvalues of $\sqrt{(du)^T du}$ by λ_i (also called the singular values of du), then the condition is equivalent to $|\lambda_i \lambda_j| \le 1$ for $i \neq j$. It was proved in [21] that a Bernstein type theorem for higher codimension minimal submanifolds is true under this assumption. This condition is weaker than the conditions assumed in the Bernstein type theorem established in [8], [4], and [9]. The higher codimension Bernstein type theorem cannot be true without imposing assumptions on u by the examples of non-parametric minimal cones due to Lawson and Osserman [11]. The condition $|\lambda_i \lambda_j| \le 1$ is not achieved by any one of the examples of Lawson and Osserman. However, it seems more counterexamples need to be constructed to make conjectures about the sharp condition. It is not clear whether such condition is necessary for gradient estimates (see the next paragraph for the two-dimensional case).

The underlying principle of the proof is that $|\lambda_i \lambda_j| \le 1$ for $i \neq j$ defines a region on the Grassmannian of n-planes in $\mathbb{R}^{n+m}$ on which $\ln \sqrt{\det(I + (du)^T du)}$ is a convex function. This is in contrast with the codimension one case in which

$\ln\sqrt{1+|du|^2}$ is always a convex function. We remark that when $n = 2$, a gradient bound assuming all but one $|du^\alpha|$ are bounded was obtained by Simon [17]. This condition is similar to our condition $|\lambda_i\lambda_j| \leq 1$ for $i \neq j$. The assumption was removed when $n = 2$ by Gregori in [6].

We provide two proofs for the gradient estimate. The first one follows the approach developed by Michael-Simon [12] and Trudinger [18], [19] as was presented in [5]. The second one follows Korevaar's proof [10] using the maximum principle (see also [23]). As in the codimension one case, the integral method gives the sharper estimate while the maximum principle method is easier. The proofs are contained in §3 and §4. In the last section, we discuss applications to Bernstein type theorems for minimal submanifolds of higher codimension.

Note the summation convention—repeated indices are summed over—is adopted in the rest of this article unless otherwise mentioned.

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2. Preliminaries. In this section, we assume u is a C^2 solution to the minimal surface system (1.1). We remark u is indeed real analytic by a classical theorem of Morrey [13], [14], see also [11]. Denote the graph of u in $\mathbb{R}^{n+m}$ over Ω by $\mathfrak{S}$. By Theorem 2.2 in [16], $\mathfrak{S}$ is a minimal submanifold, i.e., the mean curvature vector vanishes identically. Let $g_{ij} = \delta_{ij} + \sum_\beta \frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}$ be the induced metric on the graph and

$$v = \sqrt{\det g_{ij}} = \sqrt{\det\left(I + (du)^T(du)\right)}$$

be the induced volume element. Let dx denote the volume form $dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^n$ and $dA = v\,dx$ be the volume form on the graph $\mathfrak{S}$.

Since Ω and $\mathfrak{S}$ are canonically identified, a function defined on Ω will be considered as defined on $\mathfrak{S}$ as well. All the coordinates system $\{x^i\}$ we use are coordinates on Ω lifted to $\mathfrak{S}$. An important simplification in our calculation is at any $p \in \Omega$, we can choose a coordinate system $\{x^1, \ldots, x^n\}$ on Ω and $\{y^{n+1}, \ldots, y^{n+m}\}$ on $\mathbb{R}^m$ so that u_i^α is "diagonalized" at p . That is, $\frac{\partial u^\alpha}{\partial x^i} = u_i^\alpha = \lambda_i\delta_{\alpha,n+i}$ where $\lambda_i \geq 0$ are the singular values of du, or eigenvalues of $\sqrt{(du)^T du}$.

Then we have at p,

$$g_{ij} = (1+\lambda_i^2)\delta_{ij}, \text{ and } g^{ij} = \frac{1}{1+\lambda_i^2}\delta_{ij} \text{ for any fixed } i,j$$

and

$$v = \sqrt{\prod_{i=1}^{n}(1+\lambda_i^2)}.$$

The following lemma is a direct consequence of the minimality.

LEMMA 2.1. *For any vector-valued function with compact support $\phi \in C_0^1(\Omega)$,*

$$\int_\Omega v g^{ij} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial \phi^\alpha}{\partial x^j} dx = 0$$

Proof. We compute the variation of volume by ϕ, so $\delta u^\alpha = \phi^\alpha$. Thus

$$0 = \delta\left(\int \sqrt{\det\left(\delta_{ij} + \frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j}\right)}dx\right) = \int \frac{1}{2} v g^{ij} \delta\left(\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j}\right) dx$$

where $\delta(\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j}) = 2\frac{\partial(\delta u^\alpha)}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j}$. □

The next lemma is a key inequality in both methods. It is equivalent to saying $v^{-\frac{1}{n}}$ is a superharmonic function on $\mathfrak{S}$ under the assumption.

LEMMA 2.2. *Let $w = \ln v$. If $\lambda_i\lambda_j \leq 1$ for $i \neq j$ then*

$$\Delta w \geq \frac{1}{n}|\nabla w|^2$$

where ∇ and Δ denote the gradient and Laplacian of the induced metric g_{ij}.

Proof. We derive the equation in the following. First of all we claim the Laplace operator on $\mathfrak{S}$ is given by

$$\Delta = g^{kl}\frac{\partial^2}{\partial x^k \partial x^l}$$

where x^i is any Euclidean coordinate system on $\Omega \subset \mathbb{R}^n$.

The general formula for the Laplacian on any Riemannian manifold with respect to any coordinate system is

$$\Delta f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}g^{kl}\frac{\partial f}{\partial x^l}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}g^{kl})\frac{\partial f}{\partial x^l} + g^{kl}\frac{\partial^2 f}{\partial x^k \partial x^l}$$

where $\sqrt{g} = \sqrt{\det(g_{ij})}$.

Now $\frac{\partial}{\partial x^k}(\sqrt{g}g^{kl}) = 0$ for each l by Theorem 2.2 in [16] and the claim is verified.

We calculate

$$\frac{\partial}{\partial x^l} \ln \sqrt{\det(\delta_{ij} + u_i^\alpha u_j^\alpha)} = u_{il}^\alpha u_j^\alpha g^{ij} \tag{2.1}$$

where we use $(\det A)' = A'_{ij}A^{ji} \det A$.

Therefore

$$\Delta \ln v = g^{kl}\frac{\partial}{\partial x^k}(u_{il}^\alpha u_j^\alpha g^{ij}) = g^{kl}(u_{ilk}^\alpha u_j^\alpha g^{ij}) + g^{kl}(u_{il}^\alpha u_{jk}^\alpha g^{ij}) + g^{kl}u_{il}^\alpha u_j^\alpha (g^{ij})_k.$$

By the equation $g^{kl}u_{kl}^\alpha = 0$, we have $g^{kl}u_{ilk}^\alpha = -(g^{kl})_i u_{lk}^\alpha$. Thus

$$\Delta \ln v = g^{ij}g^{kl}u_{il}^\alpha u_{jk}^\alpha - (g^{kl})_i u_{lk}^\alpha u_j^\alpha g^{ij} + g^{kl}u_{il}^\alpha u_j^\alpha (g^{ij})_k$$

On the other hand, $(g^{kl})_i = -g^{kp}(g_{pq})_i g^{ql}$, we obtain

$$\Delta \ln v = g^{ij}g^{kl}u_{il}^\alpha u_{jk}^\alpha + g^{kp}g^{ql}g^{ij}(g_{pq})_i u_{lk}^\alpha u_j^\alpha - g^{kl}g^{ip}g^{qj}(g_{pq})_k u_{il}^\alpha u_j^\alpha.$$

Plug in $(g_{pq})_i = u_{pi}^\beta u_q^\beta + u_p^\beta u_{qi}^\beta$ and use the "diagonalization" mentioned in the paragraph before Lemma 2.1 so that $g^{ij} = \frac{1}{1+\lambda_i^2}\delta_{ij}$ and $u_i^\beta = \lambda_i \delta_{\beta,n+i}$, we derive

$$\begin{aligned}\Delta \ln v = {} & \frac{1}{(1+\lambda_k^2)(1+\lambda_i^2)}(u_{ik}^\alpha)^2 + \frac{1}{(1+\lambda_k^2)(1+\lambda_l^2)(1+\lambda_i^2)}(u_{ki}^{n+l}\lambda_l + u_{li}^{n+k}\lambda_k)u_{kl}^{n+i}\lambda_i \\ & - \frac{1}{(1+\lambda_k^2)(1+\lambda_i^2)(1+\lambda_j^2)}(u_{ik}^{n+j}\lambda_j + u_{jk}^{n+i}\lambda_i)u_{ik}^{n+j}\lambda_j.\end{aligned}$$

This is simplified to

$$\begin{aligned}\Delta \ln v = {} & \frac{1}{(1+\lambda_k^2)(1+\lambda_i^2)}(u_{ik}^\alpha)^2 - \frac{\lambda_j^2}{(1+\lambda_k^2)(1+\lambda_j^2)(1+\lambda_i^2)}(u_{ik}^{n+j})^2 \\ & + \frac{\lambda_i\lambda_j}{(1+\lambda_k^2)(1+\lambda_j^2)(1+\lambda_i^2)}(u_{ik}^{n+j}u_{jk}^{n+i}).\end{aligned}$$

Notice that $\lambda_i = 0$ if $i > \min\{n, m\}$, so the last equation can be rearranged as

$$\begin{aligned}\Delta \ln v = {} & \sum_{\alpha > n+\min\{n,m\}} \frac{1}{(1+\lambda_k^2)(1+\lambda_i^2)}(u_{ik}^\alpha)^2 \\ & + \sum_{j=1,\ldots,\min\{n,m\}} \frac{1}{(1+\lambda_k^2)(1+\lambda_j^2)(1+\lambda_i^2)}(u_{ik}^{n+j})^2\end{aligned}$$

$$+\sum_{i,j,k}\frac{\lambda_i\lambda_j}{(1+\lambda_k^2)(1+\lambda_j^2)(1+\lambda_i^2)}(u_{ik}^{n+j}u_{jk}^{n+i}).$$

Denote $h_{\alpha ik} = \frac{1}{\sqrt{(1+\lambda_k^2)(1+\lambda_i^2)}}u_{ik}^{\alpha}$ if $\alpha > n+\min\{n,m\}$ and $h_{n+i,jk} = \frac{1}{\sqrt{(1+\lambda_k^2)(1+\lambda_j^2)(1+\lambda_i^2)}}u_{jk}^{n+i}$ if $i = 1\cdots\min\{n,m\}$ and we remark that they indeed represent the second fundamental form of $\mathfrak{S}$.

Split the last term on the right-hand side into two parts that correspond to $i=j$ and $i\neq j$ then we recover a formula in [21] (note $v=\frac{1}{*\Omega}$)

$$\Delta(\ln v)=\sum_{\alpha,l,k}h_{\alpha lk}^2+\sum_{k,i}\lambda_i^2h_{n+i,ik}^2+2\sum_{k,i<j}\lambda_i\lambda_jh_{n+i,jk}h_{n+j,ik}. \tag{2.2}$$

The ranges of the indices are $\alpha=n+1,\ldots,n+m$ and $i,j,k=1,\ldots,n$ and we recall that $\lambda_i=0$ if $i>\min\{n,m\}$. The parabolic version of this equation was first derived in [20].

If $\lambda_i\lambda_j\leq 1$, we can complete the square in equation (2.2) to obtain

$$\Delta(\ln v)\geq\sum_{k,i}\lambda_i^2h_{n+i,ik}^2. \tag{2.3}$$

By equation (2.1), we have

$$(\ln v)_k=\sum_i\frac{\lambda_i}{1+\lambda_i^2}u_{ik}^{n+i}$$

and

$$|\nabla\ln v|^2=g^{kl}(\ln v)_k(\ln v)_l=\frac{\lambda_i\lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)(1+\lambda_k^2)}u_{ik}^{n+i}u_{jk}^{n+j}.$$

Therefore

$$|\nabla w|^2=\sum_k\left(\sum_i\lambda_ih_{n+i,ik}\right)^2\leq n\sum_{k,i}\lambda_i^2h_{n+i,ik}^2$$

The factor $\frac{1}{n}$ can be replaced by $\frac{1}{r}$ where r is the rank of du. Combine this with equation (2.3) and the lemma is proved. □

3. Integral method. We give a proof to Theorem A using integral method adapted from chapter 16 of Gilbarg-Trudinger [5]. All constants, C, C_1, C_2 are generic constants depending only on n.

First we prove a lemma that uses the assumption crucially.

LEMMA 3.1. *If $\lambda_i\lambda_j \le 1$ for $i \ne j$, let $a = a_j^\alpha$*

$$v\nabla u \cdot a = vg^{ij}\frac{\partial u^\alpha}{\partial x^i}a_j^\alpha \le C|a|$$

where C is a constant depending only on n.

Proof. At each point, we apply the singular value decomposition. Thus

$$vg^{ij}\frac{\partial u^\alpha}{\partial x^i}a_j^\alpha \le \sqrt{\prod_i (1+\lambda_i^2)}\left(\sum_i \frac{\lambda_i}{1+\lambda_i^2}\right)|a|$$

Now

$$\sqrt{\prod_i (1+\lambda_i^2)}\left(\sum_{i=1}^n \frac{\lambda_i}{1+\lambda_i^2}\right) = \frac{\sum_i \lambda_i(1+\lambda_1^2)\cdots\widehat{(1+\lambda_i^2)}\cdots(1+\lambda_n^2)}{\sqrt{\prod_i (1+\lambda_i^2)}}$$

Applying the condition $\lambda_i\lambda_j \le 1$ for $i \ne j$, we see for each i,

$$\lambda_i(1+\lambda_1^2)\cdots\widehat{(1+\lambda_i^2)}\cdots(1+\lambda_n^2) \le \lambda_i\left(C_1 + C_2\sum_{j\ne i}\lambda_j^2\right) \le C_3\sum_i \lambda_i$$

where C_1, C_2 and C_3 are constants depending only on n.

Therefore

$$\sum_i \lambda_i(1+\lambda_1^2)\cdots\widehat{(1+\lambda_i^2)}\cdots(1+\lambda_n^2) \le C\sum_i \lambda_i$$

where C depends only on n. □

We assume $x_0 = 0 \in \mathbb{R}^n$, $u(x_0) = 0 \in \mathbb{R}^m$, and $3R < dist(0, \partial\Omega)$. For $\rho \le R$ denote $\mathfrak{S}_\rho = \mathfrak{S} \cap \{(x,y) : |x|^2 + |y|^2 \le \rho^2\}$ and the area of $\mathfrak{S}_\rho$ by $A(\mathfrak{S}_\rho)$.

PROPOSITION 3.1. *If $\lambda_i\lambda_j \le 1$ for $i \ne j$, then*

$$A(\mathfrak{S}_\rho) \le C\rho^n.$$

Proof. Define u_ρ by $u_\rho^\alpha = \min\{u^\alpha, \rho\}$ and substitute the test function $\phi = \eta u_\rho$ into the formula in Lemma 2.1, where η satisfies $\eta \equiv 1$ for $|x| < \rho$, $\eta = 0$ for $|x| > 2\rho$ and $|D\eta| \le 1/\rho$. We derive

$$0 = \int vg^{ij}\frac{\partial u^\alpha}{\partial x^i}\left(\frac{\partial \eta}{\partial x^j}u_\rho^\alpha + \eta\frac{\partial u_\rho^\alpha}{\partial x^j}\right)dx.$$

Now apply Lemma (3.1) to get

$$\int_{|x|<\rho,|u|<\rho} |\nabla u|^2 v\,dx \leq C\rho \int_{|x|<2\rho} |D\eta|\,dx.$$

We notice that at any point $v|\nabla u|^2 = \sqrt{\prod_i (1+\lambda_i^2)} \sum \frac{\lambda_i^2}{1+\lambda_i^2}$ and thus

$$v \leq v|\nabla u|^2 + 1.$$

Therefore

$$\int_{|x|<\rho,|u|<\rho} v dx \leq \int_{|x|<\rho,|u|<\rho} |\nabla u|^2 v\,dx + \int_{|x|<\rho,|u|<\rho} dx$$

and $A(\mathfrak{S}_\rho)$ is bounded by a multiple of $\int_{|x|<\rho,|u|<\rho} v dx$. □

By Lemma 2.2 and the Sobolev inequality for minimal submanifolds (see for example [12]), we have the following sub-mean-value inequality for w:

$$w(0) \leq \frac{1}{\omega_n R^n} \int_{\mathfrak{S}_R} w\,dA. \tag{3.1}$$

Now substitute the test function $\phi = \eta w u_\rho$ into Lemma 2.1, where η satisfies $\eta \equiv 1$ for $|x| < \rho$, $\eta = 0$ for $|x| > 2\rho$ and $|D\eta| \leq 1/\rho$. We derive

$$0 = \int v g^{ij} \frac{\partial u^\alpha}{\partial x^i} \left(\frac{\partial \eta}{\partial x^j} w u_\rho^\alpha + \eta \frac{\partial w}{\partial x^j} u_\rho^\alpha + \eta w \frac{\partial u_\rho^\alpha}{\partial x^j} \right) dx$$

Now apply Lemma 3.1 to get

$$\begin{aligned} &\int_{|x|<\rho,|u|<\rho} w|\nabla u|^2 v\,dx \\ &\quad \leq C\rho \int_{|x|<2\rho} (w|D\eta| + \eta|Dw|)\,dx \\ &\quad \leq C\left(\int_{|x|<2\rho} w\,dx + \rho \int_{|x|<2\rho} \eta|Dw|\,dx \right). \end{aligned} \tag{3.2}$$

To estimate the term $\int_{|x|<2\rho} \eta|Dw|\,dx$, recall Lemma 2.2 and multiply both sides by η^2, and we obtain

$$\int_{\mathfrak{S}} \eta^2 \Delta w \geq \int_{\mathfrak{S}} \frac{1}{n} \eta^2 |\nabla w|^2.$$

Integration by parts on $\mathfrak{S}$ and we derive

$$\frac{1}{n}\int_{\mathfrak{S}}\eta^2|\nabla w|^2 \leq 2\int_{\mathfrak{S}}\eta|\nabla w||\nabla\eta| \leq \frac{1}{2n}\int_{\mathfrak{S}}\eta^2|\nabla w|^2 + 2n\int_{\mathfrak{S}}|\nabla\eta|^2.$$

Therefore

$$\int_{\mathfrak{S}}\eta^2|\nabla w|^2 \leq 4n^2\rho^{-2}Area(\mathfrak{S}\cap supp\,\eta). \tag{3.3}$$

It is not hard to see

$$|Dw| \leq v|\nabla w|.$$

Therefore

$$\int_{|x|<2\rho}\eta|Dw|\,dx \leq \int_{\mathfrak{S}}\eta|\nabla w|dA \leq \left(\int_{\mathfrak{S}}\eta^2|\nabla w|^2\right)^{1/2}(Area(\mathfrak{S}\cap supp\,\eta))^{1/2}.$$

By equation (3.3), we obtain

$$\int_{|x|<2\rho}\eta|Dw|\,dx \leq \frac{C}{\rho}\int_{|x|<2\rho}v\,dx.$$

Since $w \leq v$, we also have

$$\int_{|x|<2\rho}w\,dx \leq \int_{|x|<2\rho}v\,dx.$$

In view of equation (3.2), it only remains to estimate $\int_{|x|<2\rho}v\,dx$. Choose $\phi = \eta u$ where $\eta \equiv 1$ for $|x| < 2\rho$, $\eta \equiv 0$ for $|x| > 3\rho$ and $|D\eta| \leq 1/\rho$. We obtain

$$0 = \int vg^{ij}\frac{\partial u^\alpha}{\partial x^i}\left(\frac{\partial\eta}{\partial x^j}u^\alpha + \eta\frac{\partial u^\alpha}{\partial x^j}\right)dx.$$

Since

$$\frac{1}{\rho}\int_{|x|<3\rho}|u| \leq \frac{1}{\rho}C(3\rho)^n \sup_{|x|<3\rho}|u|.$$

Thus

$$\int_{|x|<2\rho}v|\nabla u|^2 \leq C\rho^{n-1}\sup_{|x|<3\rho}|u|.$$

Use $v \leq v|\nabla u|^2 + 1$ again, we derive

$$\int_{|x|<2\rho} v\,dx \leq C_1\rho^{n-1} \sup_{|x|<3\rho} |u| + C_2\rho^n.$$

Recall equation (3.2),

$$\int_{|x|<\rho,|u|<\rho} w|\nabla u|^2 v\,dx \leq C_1\rho^{n-1} \sup_{|x|<3\rho} |u| + C_2\rho^n. \tag{3.4}$$

This implies

$$\int_{|x|<\rho,|u|<\rho} wv\,dx \leq C_1\rho^{n-1} \sup_{|x|<3\rho} |u| + C_2\rho^n. \tag{3.5}$$

Combining with the sub-mean-value inequality, we arrive at the desired estimate by exponentiating the following inequality:

$$\ln v(0) \leq C_1\rho^{-1} \sup_{|x|<3\rho} |u| + C_2.$$

Notice that $|du| < v$ always holds.

4. Maximum principle. In the section, we shall adapt the proof of Korevaar [10] to the higher codimension case. We remark the method has be applied to obtain gradient estimates for mean curvature flows of hypersurfaces by Ecker-Huisken [2].

We assume $u : B_1 \subset \mathbb{R}^n \to \mathbb{R}^m$ satisfies $u^\alpha \leq 0$ for each α. Let $\tilde{\eta} = \tilde{\eta}(x,y)$ be a cut-off function that is a non-negative and continuous on $B_1 \times (-\infty, 0]^m$ and is zero on $\{(x,y) : |x| = 1, y^\alpha < 0\}$. Let $\eta(x) = \tilde{\eta}(x, u(x))$ be the restriction of $\tilde{\eta}$ to the graph of u. We now consider the function $\eta v^{1/n}$ which achieves a positive maximum in the interior of B_1. At a maximum point p,

$$\begin{aligned} \nabla(\eta v^{1/n}) &= 0 \\ \Delta(\eta v^{1/n}) &\leq 0. \end{aligned}$$

The first equation implies

$$\nabla\eta = -v^{-1/n}\eta\nabla(v^{1/n}). \tag{4.1}$$

By the second equation, we obtain

$$(\Delta\eta)v^{1/n} + 2\nabla\eta \cdot \nabla(v^{1/n}) + \eta\Delta(v^{1/n}) \leq 0.$$

Plug in equation (4.1), we derive

$$(\Delta\eta)\, v^{1/n} + \eta(\Delta(v^{1/n}) - 2v^{-1/n}|\nabla v^{1/n}|^2) \le 0.$$

However, equation (2.2) is

$$\Delta \ln v \ge \frac{1}{n}|\nabla \ln v|^2.$$

It is not hard to check this is equivalent to

$$\Delta(v^{1/n}) - 2v^{-1/n}|\nabla v^{1/n}|^2 \ge 0.$$

Therefore, we have at the point p,

$$\Delta\eta \le 0.$$

Let now $\eta = -1 + \exp \mu\phi$ where the function ϕ will be chosen later and where $\mu > 0$. We infer

$$\Delta\phi \le -\mu|\nabla\phi|^2.$$

By choosing coordinates, we may assume at p, $\max_i \lambda_i = \lambda_1 = u_1^{n+1}$. Set

$$\phi = \left(\frac{1}{2b}y^{n+1} + 1 - |x|^2\right)^+,$$

where $b > 0$ is to be determined. Thus $\phi(x) = (\frac{1}{2b}u^{n+1}(x) + 1 - |x|^2)^+$.

On the set where ϕ is positive, we compute

$$\begin{aligned}
\nabla\phi &= \frac{1}{2b}\nabla u^{n+1} - \nabla|x|^2 \\
|\nabla\phi|^2 &= \frac{1}{4b^2}|\nabla u^{n+1}|^2 + |\nabla|x|^2|^2 - \frac{1}{b}\nabla u^{n+1}\nabla|x|^2 \\
&\ge \frac{1}{4b^2}g^{ij}u_i^{n+1}u_j^{n+1} - \frac{1}{b}g^{ij}u_i^{n+1}(|x|^2)_j \\
&\ge \frac{1}{4b^2}\frac{\lambda_1^2}{1+\lambda_1^2} - \frac{2}{b}\frac{\lambda_1}{1+\lambda_1^2}
\end{aligned}$$

where we use $g^{ij} = \frac{1}{1+\lambda_i^2}\delta_{ij}$.

On the other hand, recall $g^{ij}u_{ij}^{n+1} = 0$,

$$\Delta\phi = g^{ij}\phi_{ij} = \delta_{ij}\frac{1}{1+\lambda_i^2}(-2)\delta_{ij} = -2\sum_i \frac{1}{1+\lambda_i^2}.$$

Therefore

$$-2\sum_i \frac{1}{1+\lambda_i^2} \leq -\mu\left(\frac{1}{4b^2}\frac{\lambda_1^2}{1+\lambda_1^2} - \frac{2}{b}\frac{\lambda_1}{1+\lambda_1^2}\right)$$

and

$$\frac{1}{4b^2}\frac{\lambda_1^2}{1+\lambda_1^2} - \frac{2}{b}\frac{\lambda_1}{1+\lambda_1^2} \leq \frac{2n}{\mu}$$

or

$$\frac{1}{4b^2}\lambda_1^2 - \frac{2}{b}\lambda_1 \leq \frac{2n}{\mu}(1+\lambda_1^2).$$

Choose $\mu = 16nb^2$, then

$$\frac{1}{8b^2}\lambda_1^2 - \frac{2}{b}\lambda_1 \leq \frac{1}{8b^2}.$$

Therefore

$$\lambda_1 \leq 16b + 1$$

at the point p and

$$v(p) = \sqrt{\prod_i (1+\lambda_i^2)} \leq [1 + (16b+1)^2]^{\frac{n}{2}}.$$

Thus

$$\eta(0)v^{\frac{1}{n}}(0) \leq \max \eta v^{\frac{1}{n}} \leq \eta(p)(v(p))^{\frac{1}{n}} \leq e^{16nb^2}(16b+2).$$

Now $\eta(0) = e^{16nb^2(\frac{1}{2b}u^{n+1}(0)+1)^+} - 1$. Set $b = \max\{1, -u^{n+1}(0)\}$, then if $-u^{n+1}(0) \geq 1$ we have

$$v(0)^{\frac{1}{n}} \leq (e^{8n(u^{n+1}(0))^2} - 1)^{-1} e^{16n(u^{n+1}(0))^2}[16(-u^{n+1}(0)) + 2] \leq C_1 e^{C_2|u(0)|^2}.$$

Otherwise, i.e. $-u^{n+1}(0) < 1$, we still have the same estimate by changing the constants C_1 and C_2. Therefore, an estimate of the following form is proved by rescaling:

$$|du(x_0)| \leq C_1 \exp\{C_2|u(x_0)|^2/d\}$$

where C_1 and C_2 are constants depending only on n.

5. Applications. We combine a result in [21] to obtain the following Bernstein type theorem.

THEOREM 5.1. *If* $u : \mathbb{R}^n \to \mathbb{R}^m$ *is an entire solution to the minimal surface system and if* $u^\alpha \geq 0$ *for each* α *and if there exists an* $\epsilon > 0$ *such that* $|\lambda_i \lambda_j| \leq 1 - \epsilon, i \neq j$ *for the singular values of du, the each* u^α *is a linear function.*

Proof. This theorem is proved in [21] under the assumption that u has bounded gradient. Now we let $d \to \infty$ in Theorem A and we see $|du| \leq C_1$. □

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