

A VOLUME PRESERVING FLOW AND THE ISOPERIMETRIC PROBLEM IN WARPED PRODUCT SPACES

PENGFEI GUAN, JUNFANG LI, AND MU-TAO WANG

ABSTRACT. In this article, we continue the work in [6] and study a normalized hypersurface flow in the more general ambient setting of warped product spaces. This flow preserves the volume of the bounded domain enclosed by a graphical hypersurface, and monotonically decreases the hypersurface area. As an application, the isoperimetric problem in warped product spaces is solved for such domains.

1. INTRODUCTION

Let $(\mathbf{B}^n, \tilde{g})$ be a closed Riemannian manifold. Let $\phi = \phi(r)$ be a smooth positive function defined on the interval $[r_0, \bar{r}]$ for some $r_0 < \bar{r}$. We consider a Riemannian manifold $(\mathbf{N}^{n+1}, \bar{g})$ (possibly with boundary) with the warped product structure,

$$(1.1) \quad \bar{g} = dr^2 + \phi^2 \tilde{g}, \quad r \in [r_0, \bar{r}]$$

where \tilde{g} is the metric of the manifold \mathbf{B}^n . \mathbf{N}^{n+1} is naturally equipped with a conformal Killing field $X = \phi(r)\partial_r$. Let M be a smooth closed embedded hypersurface in \mathbf{N}^{n+1} , which is parametrized by an embedding F_0 . We consider the following evolution equation for a family of embeddings of hypersurfaces with F_0 as an initial data, i.e. $F(\cdot, t) = F_0$:

$$(1.2) \quad \frac{\partial F}{\partial t} = (n\phi' - uH)\nu,$$

where ν is the outward unit normal vector field, H is the mean curvature, and $u = \langle X, \nu \rangle$ is the support function of the hypersurface defined by $F(\cdot, t)$. A hypersurface M is said to be *graphical* if it is defined by $r = \rho(p), p \in \mathbf{B}^n$ for a smooth function ρ on \mathbf{B}^n . When $(\mathbf{B}^n, \tilde{g})$ is the standard unit sphere \mathbb{S}^n in \mathbb{R}^{n+1} and $\phi(r) = \sin(r), r, \sinh(r)$, $(\mathbf{N}^{n+1}, \bar{g})$ represents $\mathbb{S}^{n+1}, \mathbb{R}^{n+1}, \mathbb{H}^{n+1}$ respectively. In these special cases, flow (1.2) was studied in [6] in connection with the isoperimetric problem. In this article, we consider (1.2) in more general ambient setting of warped product spaces.

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Below are our main theorems.

Theorem 1.1. *Let M_0 be a smooth graphical hypersurface in $(\mathbf{N}^{n+1}, \bar{g})$ with $n \geq 2$ and \bar{g} in (1.1). If $\phi(r)$ and \tilde{g} satisfy the following conditions:*

$$(1.3) \quad \begin{aligned} \tilde{Ric} &\geq (n-1)K\tilde{g}, \\ 0 &\leq (\phi')^2 - \phi''\phi \leq K \text{ on } [r_0, \bar{r}] \end{aligned}$$

where $K > 0$ is a constant and \tilde{Ric} is the Ricci curvature of \tilde{g} , then the evolution equation (1.2) with M_0 as the initial data has a smooth solution for $t \in [0, \infty)$. Moreover, the solution hypersurfaces converge exponentially to a level set of r as $t \rightarrow \infty$.

As an application, we obtain a solution to the isoperimetric problem for warped product spaces. Let $S(r)$ be a level set of r and $B(r)$ be the bounded domain enclosed by $S(r)$ and $S(r_0)$. The volume of $B(r)$ and surface area of $S(r)$, both positive functions of r , are denoted as $V(r)$ and $A(r)$, respectively. Note that $V = V(r)$ is strictly increasing function of r . Consider the single variable function $\xi(x)$ that satisfies

$$(1.4) \quad A(r) = \xi(V(r)),$$

for any $r \in [r_0, \bar{r}]$. The function $\xi(x)$ is well-defined.

Theorem 1.2. *Let $\Omega \subset \mathbf{N}^{n+1}$ be a domain bounded by a smooth graphical hypersurface M and $S(r_0)$. We assume $\phi(r)$ and \tilde{g} satisfy the conditions (1.3) in Theorem 1.1, then*

$$(1.5) \quad \text{Area}(M) \geq \xi(\text{Vol}(\Omega)),$$

where $\text{Area}(M)$ is the area of M and $\text{Vol}(\Omega)$ is the volume of Ω , and function ξ is defined in (1.4). If, in addition to (1.3), $(\phi')^2 - \phi''\phi < K$ on $[r_0, \bar{r}]$ then “=” is attained in (1.5) if and only if M is a level set of r .

Some remarks are in order.

- Remark 1.3.**
- (i) *The upper bound condition $(\phi')^2 - \phi''\phi \leq K$ is needed for the monotonicity properties of the flow, see Theorem 2.8. In regards to the recent results of Li-Wang [9], this condition is necessary for solving isoperimetric problems. More details of these conditions can be found in Section 5. Indeed, the condition in this setting implies the corresponding level set of r is a stable CMC, which locally minimizes areas subject to the constraint of fixing enclosed volumes.*
 - (ii) *The lower bound condition $(\phi')^2 - \phi''\phi \geq 0$ is needed for the gradient estimate and this condition is closely related to the notion of “photon sphere” in general relativity, see more details in Section 5.*
 - (iii) *The function $A(r)$ is given explicitly by $A(r) = \phi^n(r)\text{Area}(\mathbf{B}^n)$ and $V(r)$ is characterized by the ODE: $\frac{dV}{dr} = A(r), V(r_0) = 0$. To determine the function ξ , one can first solve r in terms of V and then plug into the formula of $A(r)$. For example, when $n = 1$ and*

$\phi(r) = \sin(r)$, we deduce that $A(V) = \sqrt{V(4\pi - V)}$ or $\xi(x) = \sqrt{x(4\pi - x)}$. When $n = 1$ and $\phi(r) = \sinh(r)$, we deduce that $A(V) = \sqrt{V(4\pi + V)}$ or $\xi(x) = \sqrt{x(4\pi + x)}$.

The hypersurface flow (1.2) is a local flow preserving volume. This seems to be a novel feature comparing with the known flows in the literature. More specifically, equation (1.2) is a pointwise defined parabolic PDE and it preserves the enclosed volume along the flow. To the authors' knowledge, most of hypersurface flows in the literature is either local but can not preserve integral geometric quantities, such as volume, surface area, etc., or after renormalization, the point-wise DE turns to a PDE which contains some integral terms. For example, in Huisken's famous work on mean curvature flow [7], the original local flow is not volume preserving. If one rescales the hypersurface so that the volume is preserved along the flow, then an extra integral term which involves total squared mean curvature and surface area has to be included. On the other hand, in another paper of Huisken [8], a volume preserving mean curvature flow was discussed. The definition of this flow already contains a global quantity related to total mean curvature and surface area. More details of these comparisons can be found in previous work [6].

Another advantage of the flow (1.2) is that the existence and exponential convergence do not depend on any convexity condition of the domain. This also seems to be a surprising property for hypersurface flows.

We note that part of the work in this article has been announced on the PDE workshop at Oberwolfach [5] in 2013.

The rest of the paper is organized as follows. In section 2, we discuss hypersurfaces in warped product spaces, and prove a Minkowski identity and monotonic properties along the normalized flow. In section 3, we convert the flow into a parabolic PDE for a graphical hypersurface and prove the C^0 estimate, the main gradient estimate and the exponential convergence of the flow under conditions in Theorem 1.1. In section 4, we derive evolution equations for support function and for mean curvature, then obtain bounds for these geometric quantities. In section 5, we discuss the conditions imposed on the warping function ϕ and also discuss the case when $K = 0$ in Condition (1.3).

2. A MINKOWSKI IDENTITY AND THE MONOTONICITY

Throughout this paper, we use Einstein convention for repeated indexes. We use \bar{g} , g , and \tilde{g} to denote the metrics of the ambient warped product space \mathbf{N}^{n+1} , hypersurface M^n , and the base \mathbf{B}^n respectively. Consequently, we use $\bar{\nabla}$, ∇ , and $\tilde{\nabla}$ to denote gradient with respect to the metrics \bar{g} , g , and \tilde{g} respectively. Similarly, we have notations such as Laplacian $\bar{\Delta}$, Δ , and $\tilde{\Delta}$ in different contexts.

Let $(\mathbf{N}^{n+1}, \bar{g})$ be a Riemannian manifold with warped product structure,

$$(2.1) \quad \bar{g} = ds^2 = dr^2 + \phi^2 \tilde{g},$$

where \tilde{g} is the metric of the base manifold \mathbf{B}^n and $\phi = \phi(r)$ is a smooth positive function on (r_0, \bar{r}) for some $\bar{r} \leq \infty$.

Lemma 2.1. *Let $Y = \eta(r)\partial_r$ be a vector field. Then the Lie derivative of Y is given by $\mathcal{L}_Y \bar{g} = 2\eta'(dr^2 + \frac{\eta\phi'}{\eta'\phi}\phi^2\tilde{g})$. In particular, the vector field $X = \phi(r)\partial_r$ is a conformal Killing field, i.e., $\mathcal{L}_X \bar{g} = 2\phi'(r)\bar{g}$. Moreover, $D_i X_j = \phi'(r)\bar{g}_{ij}$.*

Proof. Recall the Lie derivatives for differential forms are

$$\begin{aligned} \mathcal{L}_Y dy^\beta &= \frac{\partial \eta^\beta}{\partial y^\alpha} dy^\alpha \\ \mathcal{L}_Y f &= Y(f). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}_Y dr &= \eta'(r)dr \\ \mathcal{L}_Y dr \otimes dr &= 2\eta'(r)dr \otimes dr. \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_Y \phi^2 \tilde{g} &= 2\phi\phi'\eta\tilde{g} \\ \mathcal{L}_Y \bar{g} &= 2\eta'(r)dr \otimes dr + 2\phi\phi'\eta\tilde{g} \\ &= 2\eta'(r)(dr \otimes dr + \frac{\phi'\eta}{\phi\eta'}\phi^2\tilde{g}). \end{aligned}$$

Let $\eta = \phi$, then $Y = X$. The second part of the lemma follows immediately. \square

By direct computations, see for example [1] and [2], we have the Ricci tensor with respect to the metric \bar{g} .

Lemma 2.2. *The Ricci curvature tensor of $(\mathbf{N}^{n+1}, \bar{g})$ is given by*

$$(2.2) \quad \bar{Ric} = -n\frac{\phi''}{\phi}dr^2 + [(n-1)(K - \phi'^2) - \phi\phi'']\tilde{g} + \tilde{Ric} - (n-1)K\tilde{g}.$$

Let $M^n \subset \mathbf{N}^{n+1}$ be a smooth hypersurface in the warped product space. Under local coordinates on M^n , denote by g_{ij} , h_{ij} , $h_j^i = g^{ik}h_{kj}$, and $H = h_j^j$, the induced metric, the second fundamental form, the Weingarten tensor, and the mean curvature respectively for $i, j = 1, \dots, n$. We also let σ_l denote the l -th elementary symmetric functions of the principal curvatures, i.e., the eigenvalues of the Weingarten tensor for $1 \leq l \leq n$.

We need the following lemma.

Lemma 2.3. *Let $\sigma_2^{ij} = \frac{\partial \sigma_2}{\partial h_{ij}} = Hg^{ij} - h^{ij}$ be the cofactor tensor. Then the trace of its covariant derivative is*

$$(2.3) \quad \sigma_2^{ij}(h)_j = -\bar{R}_{i\nu},$$

where ν is the unit outward normal of the hypersurface.

Proof. In this proof, we will not use Einstein convention temporarily. For convenience, we use orthonormal coordinates and do not distinguish upper and lower indexes. By definition of σ_l , see e.g. [10], we have

$$(2.4) \quad \begin{aligned} \sum_{j=1}^n \sigma_2^{ij}(h)_j &= \sigma_2^{ii}(h)_i + \sum_{j \neq i} \sigma_2^{ij}(h)_j \\ &= \sum_{l=1}^n h_{ll,i} - h_{ii,i} - \sum_{j \neq i} h_{ji,j} \end{aligned}$$

By the Codazzi equation, we obtain

$$(2.5) \quad h_{il,j} - h_{jl,i} = \langle \bar{R}(\partial_j, \partial_i)\nu, \partial_l \rangle = \bar{R}_{ji\nu l},$$

and

$$(2.6) \quad h_{il,l} - h_{ll,i} = \bar{R}_{i\nu}.$$

Thus,

$$(2.7) \quad \sum_{j=1}^n \sigma_2^{ij}(h)_j = -\bar{R}_{i\nu}$$

□

The following lemma is well-known, for example, can be found in [2] which follows from Lemma 2.1 and the Gauss equation directly.

Proposition 2.4. *Let $X = \phi(r)\partial_r$ be the conformal vector field and $\Phi'(r) = \phi(r)$. Then on a hypersurface $M \subset \mathbf{N}^{n+1}$,*

$$(2.8) \quad \begin{aligned} \Phi_{ij} &= \phi'(r)g_{ij} - uh_{ij} \\ \Delta\Phi &= n\phi'(r) - Hu, \end{aligned}$$

where $u = \langle X, \nu \rangle$, Φ_{ij} is the Hessian of the function Φ , $\Delta\Phi$ is the Laplacian of the function Φ , both with respect to the induced metric g on M .

By direct computations, we have

Lemma 2.5. *Let $M^n \subset \mathbf{N}^{n+1}$ be a graphical hypersurface, which is defined by a function $r = \rho(p)$, $p \in \mathbf{B}^n$. Then*

$$(2.9) \quad \begin{aligned} \bar{R}_{\nu\nu} + n\frac{\phi''}{\phi} &= (n-1)\frac{K - \phi'^2 + \phi\phi''}{\phi^2} \frac{|\tilde{\nabla}\rho|^2}{\omega^2} + (\tilde{R}_{ij} - (n-1)K\tilde{g}_{ij})\frac{\rho_i\rho_j}{\phi^2\omega^2}, \\ g^{ij}\bar{R}_{\nu e_i}\nabla_j\Phi &= -(n-1)(K - \phi'^2 + \phi\phi'')\frac{|\tilde{\nabla}\rho|^2}{\omega^3} - (\tilde{R}_{ij} - (n-1)K\tilde{g}_{ij})\frac{\rho_i\rho_j}{\omega^3}, \end{aligned}$$

where $\nu = \frac{\phi}{\sqrt{\phi^2 + |\tilde{\nabla}\rho|^2}}(\partial_r - \frac{\rho_i}{\phi^2}\partial_i)$ is the unit outward normal vector and $e_i = \rho_i\partial_r + \partial_i$ are the tangent vector fields.

Proof. Using Lemma 2.2, we first compute

$$(2.10) \quad \begin{aligned} \bar{R}_{\nu\nu} &= -(n-1)(K - \phi'^2 + \phi\phi'')\frac{u^2}{\phi^4} + ((n-1)(K - \phi'^2) - \phi\phi'')\frac{1}{\phi^2} \\ &\quad + (\tilde{R}_{ij} - (n-1)K\tilde{g}_{ij})\frac{\rho_i\rho_j}{\phi^2\omega^2}. \end{aligned}$$

The first identity of the lemma follows immediately after simplifications.

Using Lemma 2.2 again, we have

$$(2.11) \quad \bar{R}_{\nu e_i} = - (n-1) \frac{K - \phi'^2 + \phi\phi''}{\phi\omega} \rho_i - (\tilde{R}_{ik} - (n-1)K\tilde{g}_{ik}) \frac{\rho_k}{\phi\omega}.$$

Combining (3.1) and (2.11), we finish the proof of the second identity. \square

Now we derive a Minkowski identity.

Lemma 2.6. *Let $X = \phi(r)\partial_r$ be the conformal vector field and $\Phi(r) = \phi(r)$. Then on a hypersurface $M \subset \mathbf{N}^{n+1}$,*

$$(2.12) \quad (n-1) \int_M \phi' \sigma_1 d\mu = 2 \int_M \sigma_2 u d\mu + \int_M \bar{R}_{i\nu} \Phi_i,$$

where $u = \langle X, \nu \rangle$.

Proof. Applying Proposition 2.4 and contracting the cofactor tensor σ_2^{ij} with the hessian of Φ , we have

$$(2.13) \quad \sigma_2^{ij} \Phi_{ij} = (n-1)\phi' \sigma_1 - 2\sigma_2 u.$$

Integrate equation (2.13) over M and after integration by parts, we have

$$(2.14) \quad \begin{aligned} & (n-1) \int_M \phi' \sigma_1 d\mu - 2 \int_M \sigma_2 u d\mu \\ &= \int_M \sigma_2^{ij} \Phi_{ij} d\mu \\ &= - \int_M \sigma_2^{ij} (h^j)_j \Phi_i \\ &= \int_M \bar{R}_{i\nu} \Phi_i, \end{aligned}$$

where the last inequality follows from (2.3). \square

Let $M(t)$ be a smooth family of closed hypersurfaces in \mathbf{N}^{n+1} . Let $F(\cdot, t)$ denote a point on $M(t)$. We consider the flow (1.2) in $(\mathbf{N}^{n+1}, \bar{g})$ where \bar{g} is given as in (1.1).

Proposition 2.7. *Under flow $\partial_t F = f\nu$ of closed hypersurfaces in a Riemannian manifold, suppose Ω_t is the domain enclosed by the evolving hypersurface $M(t)$ and a fixed hypersurface, we have the following evolution equations.*

$$(2.15) \quad \begin{aligned} \partial_t g_{ij} &= 2fh_{ij} \\ \partial_t h_{ij} &= -\nabla_i \nabla_j f + f(h^2)_{ij} - fR_{\nu ij\nu} \\ \partial_t h_i^j &= -g^{jk} \nabla_k \nabla_j f - g^{jk} f(h^2)_{ki} - fg^{jk} R_{\nu ik\nu} \end{aligned}$$

Moreover, we have

$$\begin{aligned} A'(t) &= \int_M f H d\mu_g, \\ V'(t) &= \int_M f d\mu_g, \end{aligned}$$

where $A(t)$ is the area of $M(t)$ and $V(t)$ is the volume of Ω_t .

Using Proposition 2.7, we obtain the following monotonicity formulae.

Theorem 2.8. *Let $M(t)$ be a smooth one parameter family of closed graphical hypersurfaces in \mathbf{N}^{n+1} with $M(0) = \partial\Omega$ which solves the parabolic equations (1.2) on $[0, T)$. Then the enclosed volume is a constant, and if $K - \phi'^2 + \phi\phi'' \geq 0$, surface area is non-increasing along the flow.*

Proof. The proof is a consequence of Proposition 2.7 and the Minkowski identity from Lemma 2.6.

$$\begin{aligned} V'(t) &= \int (n\phi' - Hu) d\mu_g = 0 \\ A'(t) &= \int (n\phi' - Hu) H d\mu_g \\ &= \int (n\phi' H - \frac{2n}{n-1} \sigma_2 u) d\mu_g + \int (\frac{2n}{n-1} \sigma_2 - H^2) u d\mu_g \\ &= \int \bar{n} R_{i\nu} \nabla_i \Phi d\mu_g + \int (\frac{2n}{n-1} \sigma_2 - H^2) u d\mu_g \\ (2.16) \quad &\leq 0, \end{aligned}$$

where we have used Lemma 2.6, (2.9), and Newton-McLaurin inequality. \square

Note that above proof fails for case $n = 1$. This case is treated by different argument in [3].

3. GRAPHICAL HYPERSURFACE AND C^1 ESTIMATE

We now focus only on those hypersurfaces that are graphical. Let M be the graph of a smooth and positive function ρ on \mathbf{B}^n . Let $\partial_1, \dots, \partial_n$ be a local frame along M and ∂_ρ be the vector field along radial direction. For simplicity, all the covariant derivatives are with respect to the metric \tilde{g}_{ij} and denoted as $\tilde{\nabla}$ when there is no confusion in the context.

Denote

$$\omega := \sqrt{\phi^2 + |\tilde{\nabla}\rho|^2},$$

then the outward unit normal is $\nu = \frac{\phi}{\omega}(1, -\frac{\rho_1}{\phi^2}, \dots, -\frac{\rho_n}{\phi^2})$. The support function, induced metric, inverse metric matrix, second fundamental form

can be expressed as follows.

$$(3.1) \quad \begin{aligned} u &= \frac{\phi^2}{\omega} \\ g_{ij} &= \frac{\phi^2}{\omega^2} \tilde{g}_{ij} + \rho_i \rho_j, \quad g^{ij} = \frac{1}{\phi^2} (\tilde{g}^{ij} - \omega^{-2} \rho_i \rho_j) \\ h_{ij} &= \omega^{-1} (-\phi \tilde{\nabla}_i \tilde{\nabla}_j \rho + 2\phi' \rho_i \rho_j + \phi^2 \phi' \tilde{g}_{ij}) \\ h_j^i &= \frac{1}{\phi^2 \omega} (\tilde{g}^{ik} - \omega^{-2} \rho_i \rho_k) (-\phi \tilde{\nabla}_k \tilde{\nabla}_j \rho + 2\phi' \rho_k \rho_j + \phi^2 \phi' \tilde{g}_{kj}) \end{aligned}$$

where all the covariant derivatives $\tilde{\nabla}$ and ρ_i are w.r.t. the base metric \tilde{g}_{ij} .

For convenience, we let

$$(3.2) \quad \begin{aligned} b_{ij} &= -\phi \omega^2 \rho_{ij} + \phi \rho_i \left(\frac{|\tilde{\nabla} \rho|^2}{2} \right)_j + \phi' \phi^2 \rho_i \rho_j + \phi' \phi^2 \omega^2 \tilde{g}_{ij} \\ h_j^i &= \frac{1}{\phi^2 \omega^3} b_j^i \\ \tilde{H} &= b_i^i = -\phi \omega^2 \tilde{\Delta} \rho + \phi \rho_i \left(\frac{|\tilde{\nabla} \rho|^2}{2} \right)_i + \phi' \phi^2 |\tilde{\nabla} \rho|^2 + n \phi' \phi^2 \omega^2 \end{aligned}$$

Thus, $\tilde{H} = \phi^2 \omega^3 H$ and $H = \frac{1}{\phi^2 \omega^3} \tilde{H}$

We now consider the flow equation (1.2) of graphical hypersurfaces in \mathbf{N}^{n+1} . It is known that if a closed hypersurface is graphical and satisfies

$$\partial_t F = f \nu,$$

then the evolution of the scalar function $\rho = \rho(F(z, t), t)$ satisfies

$$\partial_t \rho = f \frac{\omega}{\phi}.$$

Thus it suffices to consider the following parabolic initial value problem on \mathbf{B}^n ,

$$(3.3) \quad \begin{cases} \partial_t \rho = (n\phi' - Hu) \frac{\omega}{\phi}, \rho = \rho(p, t) \text{ for } (p, t) \in \mathbf{B}^n \times [0, \infty) \\ \rho(\cdot, 0) = \rho_0, \end{cases}$$

where ρ_0 is the radial function of the initial hypersurface.

We next show that the radial function ρ is uniformly bounded from above and below.

Proposition 3.1. *Let M_0 be a graphical hypersurface defined by function ρ_0 in \mathbf{N}^{n+1} . If $\rho(p, t)$ solves the initial value problem (3.3), then for any $(p, t) \in \mathbf{B}^n \times [0, T)$,*

$$\min_{p \in \mathbf{B}} \rho(p, 0) \leq \rho(p, t) \leq \max_{p \in \mathbf{B}} \rho(p, 0).$$

Proof. At critical points of ρ , the following conditions hold,

$$\tilde{\nabla} \rho = 0, \omega = \phi.$$

It follows from (3.1) that, at critical points of ρ , $\tilde{H} = -\phi^3 \tilde{\Delta} \rho + n\phi' \phi^4$. Together with (3.3), at critical points,

$$\rho_t = \frac{1}{\phi} \tilde{\Delta} \rho.$$

By the standard maximum principle, this proves the uniform upper and lower bounds for ρ . \square

We now consider gradient estimate. Throughout the rest of this section, the covariant derivatives will be with respect to the metric \tilde{g} on \mathbf{B}^n .

Theorem 3.2. (Gradient estimate and exponential convergence.)
 Let $\rho(\cdot, t)$ be a solution to the flow (1.2) on $[0, T]$. If $(\phi')^2 - \phi''\phi \geq 0$, then

$$(3.4) \quad \max_{M(t)} e^{\alpha t} |\tilde{\nabla}\rho|^2 \leq \max_{M(0)} |\tilde{\nabla}\rho|^2,$$

for some $\alpha > 0$ which is independent of t .

Proof. Recall the evolution of $\partial_t \rho$,

$$(3.5) \quad \partial_t \rho = n \frac{\phi'}{\phi} \omega - \frac{1}{\phi \omega^3} \tilde{H},$$

where \tilde{H} was defined as in (3.2). We derive the evolution of $\frac{|\tilde{\nabla}\rho|^2}{2}$ below. Throughout the proof, we will work at a maximum point of the test function $\frac{|\tilde{\nabla}\rho|^2}{2}$, so that the following critical point conditions will hold,

$$(3.6) \quad \tilde{\nabla}\omega^2 = \tilde{\nabla}\phi^2, \text{ or } \tilde{\nabla}\omega = \frac{\phi\phi'}{\omega} \tilde{\nabla}\rho.$$

First we have, at critical points of the test function,

$$\begin{aligned} \tilde{\nabla}\rho \tilde{\nabla}\tilde{H} &= \tilde{\nabla}\rho \tilde{\nabla} \left[-\phi\omega^2 \tilde{\Delta}\rho + \phi\rho_i \left(\frac{|\tilde{\nabla}\rho|^2}{2} \right)_i + \phi'\phi^2 |\tilde{\nabla}\rho|^2 + n\phi'\phi^2 \omega^2 \right] \\ &= -\phi\omega^2 \tilde{\nabla}\rho \tilde{\nabla}\tilde{\Delta}\rho + \phi\rho_i \rho_k \left(\frac{|\tilde{\nabla}\rho|^2}{2} \right)_{ik} \\ &\quad - \rho_k (\phi\omega^2)_k \tilde{\Delta}\rho + \rho_k (\phi'\phi^2)_k |\tilde{\nabla}\rho|^2 + n\rho_k (\phi'\phi^2)_k \omega^2 + n\phi'\phi^2 \rho_k (\phi^2)_k \end{aligned}$$

Note that

$$\begin{aligned} -\phi\omega^2 \tilde{\nabla}\rho \tilde{\nabla}\tilde{\Delta}\rho &= -\phi\omega^2 \rho_k (\tilde{\nabla}_i \rho_{ik} - \tilde{R}_{ik} \rho_i) \\ &= -\phi\omega^2 \tilde{\Delta} \frac{|\tilde{\nabla}\rho|^2}{2} + \phi\omega^2 |\rho_{ij}|^2 + \phi\omega^2 \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho). \end{aligned}$$

Thus

$$(3.7) \quad \begin{aligned} \tilde{\nabla}\rho \tilde{\nabla}\tilde{H} &= -\phi(\omega^2 e^{ik} - \rho^i \rho^k) \left(\frac{|\tilde{\nabla}\rho|^2}{2} \right)_{ik} + \phi\omega^2 |\rho_{ij}|^2 + \phi\omega^2 \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) \\ &\quad - \rho_k (\phi\omega^2)_k \tilde{\Delta}\rho + \rho_k (\phi'\phi^2)_k (|\tilde{\nabla}\rho|^2 + n\omega^2) + n\phi'\phi^2 \rho_k (\phi^2)_k \end{aligned}$$

Now we have,

$$\begin{aligned}
(3.8) \quad \partial_t \frac{|\tilde{\nabla}\rho|^2}{2} &= \tilde{\nabla}\rho \tilde{\nabla}\rho_t \\
&= n\left(\frac{\phi'}{\phi}\right)' |\tilde{\nabla}\rho|^2 \omega + n\left(\frac{\phi'}{\phi}\right) \tilde{\nabla}\rho \tilde{\nabla}\omega - \tilde{H} \tilde{\nabla} \frac{1}{\phi\omega^3} \tilde{\nabla}\rho - \frac{1}{\phi\omega^3} \tilde{\nabla}\rho \tilde{\nabla}\tilde{H} \\
&= \frac{1}{\omega^3} (\omega^2 e^{ik} - \rho^i \rho^k) \left(\frac{|\tilde{\nabla}\rho|^2}{2}\right)_{ik} - \frac{1}{\omega} |\rho_{ij}|^2 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) \\
&\quad + \frac{1}{\phi\omega^3} \rho_k (\phi\omega^2)_k \tilde{\Delta}\rho - \frac{1}{\phi\omega^3} \rho_k (\phi'\phi^2)_k (|\tilde{\nabla}\rho|^2 + n\omega^2) + \frac{n}{\phi\omega^3} \phi' \phi^2 \rho_k (\phi^2)_k \\
&\quad + n\left(\frac{\phi'}{\phi}\right)' |\tilde{\nabla}\rho|^2 \omega + n\left(\frac{\phi'}{\phi}\right) \tilde{\nabla}\rho \tilde{\nabla}\omega - \tilde{H} \tilde{\nabla} \frac{1}{\phi\omega^3} \tilde{\nabla}\rho
\end{aligned}$$

Let $\mathcal{L}(\psi) := \partial_t \psi - \frac{1}{\omega^3} (\omega^2 e_{ij} - \rho^i \rho^j) \psi_{ij}$ be a parabolic operator for any function ψ defined on \mathbf{B}^n . Then applying the critical point conditions,

$$\begin{aligned}
(3.9) \quad \mathcal{L}\left(\frac{|\tilde{\nabla}\rho|^2}{2}\right) &= -\frac{1}{\omega} |\rho_{ij}|^2 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) \\
&\quad + \frac{1}{\phi\omega^3} \rho_k (\phi\omega^2)_k \tilde{\Delta}\rho - \frac{1}{\phi\omega^3} \rho_k (\phi'\phi^2)_k (|\tilde{\nabla}\rho|^2 + n\omega^2) - \frac{n}{\phi\omega^3} \phi' \phi^2 \rho_k (\phi^2)_k \\
&\quad + n\left(\frac{\phi'}{\phi}\right)' |\tilde{\nabla}\rho|^2 \omega + n\left(\frac{\phi'}{\phi}\right) \tilde{\nabla}\rho \tilde{\nabla}\omega - \tilde{H} \tilde{\nabla} \frac{1}{\phi\omega^3} \tilde{\nabla}\rho \\
&= -\frac{1}{\omega} |\rho_{ij}|^2 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) + n\left(\frac{\phi'}{\phi}\right)' |\tilde{\nabla}\rho|^2 \omega \\
&\quad - \frac{(\phi'\phi^2)'}{\phi\omega^3} |\tilde{\nabla}\rho|^2 (|\tilde{\nabla}\rho|^2 + n\omega^2) - \frac{n}{\phi\omega^3} \phi' \phi^2 (\phi^2)' |\tilde{\nabla}\rho|^2 + n\frac{(\phi')^2}{\omega} |\tilde{\nabla}\rho|^2 \\
&\quad - \tilde{H} \tilde{\nabla} \frac{1}{\phi\omega^3} \tilde{\nabla}\rho + \frac{1}{\phi\omega^3} \rho_k (\phi\omega^2)_k \tilde{\Delta}\rho \\
&= -\frac{1}{\omega} |\rho_{ij}|^2 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) + n\left(\frac{\phi'}{\phi}\right)' |\tilde{\nabla}\rho|^2 \omega \\
&\quad - \frac{(\phi'\phi^2)'}{\phi\omega^3} |\tilde{\nabla}\rho|^2 (|\tilde{\nabla}\rho|^2 + n\omega^2) - \frac{n}{\phi\omega^3} \phi' \phi^2 (\phi^2)' |\tilde{\nabla}\rho|^2 + n\frac{(\phi')^2}{\omega} |\tilde{\nabla}\rho|^2 \\
&\quad + \tilde{H} \left(\frac{\phi'}{\phi^2\omega^3} + \frac{3\phi'}{\omega^5}\right) |\tilde{\nabla}\rho|^2 + \left(\frac{\phi'}{\phi\omega} + \frac{2\phi\phi'}{\omega^3}\right) |\tilde{\nabla}\rho|^2 \tilde{\Delta}\rho
\end{aligned}$$

At critical points,

$$(3.10) \quad \tilde{H} = -\phi\omega^2 \tilde{\Delta}\rho + \phi'\phi^2 |\tilde{\nabla}\rho|^2 + n\phi'\phi^2 \omega^2$$

and

$$\begin{aligned}
(3.11) \quad \mathcal{L}\left(\frac{|\tilde{\nabla}\rho|^2}{2}\right) &= -\frac{1}{\omega} |\rho_{ij}|^2 - \frac{\phi\phi'}{\omega^3} \tilde{\Delta}\rho |\tilde{\nabla}\rho|^2 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) + n\left(\frac{\phi'}{\phi}\right)' |\tilde{\nabla}\rho|^2 \omega \\
&\quad - \frac{(\phi'\phi^2)'}{\phi\omega^3} (|\tilde{\nabla}\rho|^2 + n\omega^2) |\tilde{\nabla}\rho|^2 - \frac{n}{\omega^3} \phi' \phi (\phi^2)' |\tilde{\nabla}\rho|^2 + n\frac{(\phi')^2}{\omega} |\tilde{\nabla}\rho|^2 \\
&\quad + (\phi')^2 \left(\frac{1}{\omega^3} + \frac{3\phi^2}{\omega^5}\right) (|\tilde{\nabla}\rho|^2 + n\omega^2) |\tilde{\nabla}\rho|^2 \\
&= -\frac{1}{\omega} |\rho_{ij}|^2 - \frac{\phi\phi'}{\omega^3} \tilde{\Delta}\rho |\tilde{\nabla}\rho|^2 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) + n\left(\frac{\phi'}{\phi}\right)' |\tilde{\nabla}\rho|^2 \omega \\
&\quad - \frac{n}{\omega^3} \phi' \phi (\phi^2)' |\tilde{\nabla}\rho|^2 + n\frac{(\phi')^2}{\omega} |\tilde{\nabla}\rho|^2 + \left(\frac{3\phi'^2 \phi^2}{\omega^5} - \frac{\phi''\phi + \phi'^2}{\omega^3}\right) (|\tilde{\nabla}\rho|^2 + n\omega^2) |\tilde{\nabla}\rho|^2.
\end{aligned}$$

Recall the critical point conditions, by rotating the coordinates, we can pick $\rho_1 = |\tilde{\nabla}\rho|$, thus

$$(3.12) \quad \rho_{11} = 0, \text{ and } \rho_{1j} = 0, \forall j = 2, \dots, n.$$

Moreover, we can diagonalize ρ_{jk} for $j, k = 2, \dots, n$ at the crical point and $\tilde{\Delta}\rho := \sum_{j \geq 2} \rho_{jj}$. By completing the square, we have

$$(3.13) \quad \begin{aligned} \mathcal{L}\left(\frac{|\tilde{\nabla}\rho|^2}{2}\right) &= -\frac{1}{\omega} \sum_{j \geq 2} \left(\rho_{jj} + \frac{1}{2} \frac{\phi\phi'}{\omega^2} |\tilde{\nabla}\rho|^2 \right)^2 + \frac{n-1}{4} \frac{\phi^2\phi'^2}{\omega^5} |\tilde{\nabla}\rho|^4 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) + n \left(\frac{\phi'}{\phi}\right)' |\tilde{\nabla}\rho|^2 \omega \\ &\quad - \frac{n}{\omega^3} \phi' \phi (\phi^2)' |\tilde{\nabla}\rho|^2 + n \frac{(\phi')^2}{\omega} |\tilde{\nabla}\rho|^2 + \left(\frac{3\phi'^2\phi^2}{\omega^5} - \frac{\phi''\phi + \phi'^2}{\omega^3} \right) (|\tilde{\nabla}\rho|^2 + n\omega^2) |\tilde{\nabla}\rho|^2 \\ &= -\frac{1}{\omega} \sum_{j \geq 2} \left(\rho_{jj} + \frac{1}{2} \frac{\phi\phi'}{\omega^2} |\tilde{\nabla}\rho|^2 \right)^2 \\ &\quad + \frac{n-1}{4\omega^5} (\phi\phi')^2 |\tilde{\nabla}\rho|^4 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) + n \left(\frac{\phi'}{\phi}\right)' |\tilde{\nabla}\rho|^2 \omega \\ &\quad - \frac{2n}{\omega^3} (\phi')^2 \phi^2 |\tilde{\nabla}\rho|^2 + n \frac{(\phi')^2}{\omega} |\tilde{\nabla}\rho|^2 + \left(\frac{3\phi'^2\phi^2}{\omega^5} - \frac{\phi''\phi + \phi'^2}{\omega^3} \right) (|\tilde{\nabla}\rho|^2 + n\omega^2) |\tilde{\nabla}\rho|^2 \\ &\quad + \left(\frac{3\phi'^2\phi^2}{\omega^5} - \frac{\phi''\phi + \phi'^2}{\omega^3} \right) (|\tilde{\nabla}\rho|^2 + n\omega^2) |\tilde{\nabla}\rho|^2 \\ &= -\frac{1}{\omega} \sum_{j \geq 2} \left(\rho_{jj} + \frac{1}{2} \frac{\phi\phi'}{\omega^2} |\tilde{\nabla}\rho|^2 \right)^2 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) \\ &\quad + \frac{|\tilde{\nabla}\rho|^2}{\omega^5} \left[n \left(\frac{\phi'}{\phi}\right)' \omega^6 + n\phi'^2 \omega^4 + \frac{n-1}{4} \phi^2 \phi'^2 |\tilde{\nabla}\rho|^2 - 2n\phi'^2 \phi^2 \omega^2 \right. \\ &\quad \left. + (2\phi'^2 \phi^2 - \phi'^2 |\tilde{\nabla}\rho|^2 - \phi''\phi \omega^2) (|\tilde{\nabla}\rho|^2 + n\omega^2) \right] \end{aligned}$$

Notice that

$$(3.14) \quad \begin{aligned} &n \left(\frac{\phi'}{\phi}\right)' \omega^6 + (2\phi'^2 \phi^2 - \phi'^2 |\tilde{\nabla}\rho|^2 - \phi''\phi \omega^2) (|\tilde{\nabla}\rho|^2 + n\omega^2) \\ &= n \left(\frac{\phi'}{\phi}\right)' \omega^6 - (\phi''\phi - \phi'^2) (|\tilde{\nabla}\rho|^2 + n\omega^2) \omega^2 + (\phi'^2 \phi^2 - 2\phi'^2 |\tilde{\nabla}\rho|^2) (|\tilde{\nabla}\rho|^2 + n\omega^2) \\ &= n \frac{\phi''\phi - \phi'^2}{\phi^2} |\tilde{\nabla}\rho|^4 \omega^2 + (n-1) (\phi''\phi - \phi'^2) |\tilde{\nabla}\rho|^2 \omega^2 + (\phi'^2 \phi^2 - 2\phi'^2 |\tilde{\nabla}\rho|^2) (|\tilde{\nabla}\rho|^2 + n\omega^2) \end{aligned}$$

Thus

$$(3.15) \quad \begin{aligned} \mathcal{L}\left(\frac{|\tilde{\nabla}\rho|^2}{2}\right) &= -\frac{1}{\omega} \sum_{j \geq 2} \left(\rho_{jj} + \frac{1}{2} \frac{\phi\phi'}{\omega^2} |\tilde{\nabla}\rho|^2 \right)^2 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) \\ &\quad + \frac{|\tilde{\nabla}\rho|^2}{\omega^5} \left[n \frac{\phi''\phi - \phi'^2}{\phi^2} |\tilde{\nabla}\rho|^4 \omega^2 + (n-1) (\phi''\phi - \phi'^2) |\tilde{\nabla}\rho|^2 \omega^2 \right. \\ &\quad \left. - (n+2) \phi'^2 |\tilde{\nabla}\rho|^4 - \frac{3}{4} (n-1) \phi^2 \phi'^2 |\tilde{\nabla}\rho|^2 \right] \end{aligned}$$

By the assumption $\tilde{Ric} \geq (n-1)K\tilde{g}$ with $K > 0$. As far as $\phi'^2 - \phi''\phi \geq 0$, we have

$$\mathcal{L}\left(\frac{|\tilde{\nabla}\rho|^2}{2}\right) \leq 0.$$

By the maximum principle, there is a uniform upper bound for $|\tilde{\nabla}\rho|$. Moreover, with the uniform C^0 and gradient estimates, we now have

$$(3.16) \quad \mathcal{L}\left(\frac{|\tilde{\nabla}\rho|^2}{2}\right) \leq -\frac{1}{\omega}\tilde{Ric}(\tilde{\nabla}\rho, \tilde{\nabla}\rho) \leq -\alpha|\tilde{\nabla}\rho|^2,$$

where $\alpha > 0$ is a uniform constant depending on the upper bound of $|\tilde{\nabla}\rho|^2$. This implies the exponential convergence. \square

4. EVOLUTION OF SUPPORT FUNCTION AND THE MEAN CURVATURE

In this section, we prove a uniform upper bound estimate for the mean curvature H along the flow. We also prove a uniform positive lower bound for support function u under condition $(\phi')^2 - \phi\phi'' > 0$. Although the results in this section is not needed for proving the main theorem, as important properties of the flow itself, we include them here for completeness and future interests.

Suppose that the metric on (N^{n+1}, \bar{g}) is a warped product of the form (1.1). We denote the Riemannian metric and the Levi-Civita connection of (N^{n+1}, \bar{g}) by $\langle \cdot, \cdot \rangle$ and $\bar{\nabla}$, respectively. The conformal Killing field is $X = \bar{\nabla}\Phi$ (recall $\Phi'(r) = \phi(r)$) and X satisfies

$$(4.1) \quad \langle \bar{\nabla}_Y X, Z \rangle = \bar{\nabla}_Y \bar{\nabla}_Z \Phi = \phi' \langle Y, Z \rangle,$$

where $\bar{\nabla}\bar{\nabla}\Phi$ is the Hessian of Φ with respect to \bar{g} , and Y and Z are any two vector fields on N^{n+1} .

Let M_t by a family of hypersurfaces evolves by (1.2):

$$\frac{\partial F}{\partial t} = f\nu,$$

where f is given by

$$(4.2) \quad f = n\phi' - uH.$$

The outward unit normal ν of M_t evolves by

$$(4.3) \quad \frac{\partial \nu}{\partial t} = -\nabla^{M_t} f,$$

where we use ∇^{M_t} and Δ^{M_t} to denote the gradient and Laplace operators on M_t , with respect to the induced metric.

We first compute the evolution equation of u . Note that in view of the evolution equation the relevant parabolic operator for any geometric quantity defined on M_t is $\partial_t - u\Delta_{M_t}$.

We compute using (4.1) and (4.3):

$$(4.4) \quad \begin{aligned} \partial_t \langle X, \nu \rangle &= f \langle \bar{\nabla}_\nu X, \nu \rangle + \langle X, \nabla^{M_t}(uH) \rangle - n \langle X, \nabla^{M_t} \phi' |_{M_t} \rangle \\ &= f \phi' |_{M_t} + \langle X, \nabla^{M_t}(uH) \rangle - n \langle X, \nabla^{M_t} \phi' |_{M_t} \rangle. \end{aligned}$$

Choosing the orthonormal frame $\{e_i\}_{i=1\dots n}$ to M_t such that $\nabla_{e_i}^{M_t} e_j = 0$ at a point where the following calculation is conducted:

$$(4.5) \quad \Delta^{M_t} u = e_i \langle \bar{\nabla}_{e_i} X, \nu \rangle + e_i \langle X, \bar{\nabla}_{e_i} \nu \rangle.$$

The first term vanishes by (4.1). Recall that $\bar{\nabla}_{e_i} \nu = h_{ij} e_j$ ($\nabla_{e_i}^{M_t} e_j = 0$), where h_{ij} is the second fundamental form of M_t and the second term is equal to

$$\begin{aligned} e_i (h_{ij} \langle X, e_j \rangle) &= e_i (h_{ij}) \langle X, e_j \rangle + h_{ij} \langle \nabla_{e_i} X, e_j \rangle + h_{ij} \langle X, \nabla_{e_i} e_j \rangle \\ &= (\nabla_i^{M_t} h_{ij}) \langle X, e_j \rangle + \phi' H - \sum h_{ij}^2 \langle X, \nu \rangle. \end{aligned}$$

Plugging this back to (4.5) and multiplying each term by $u = \langle X, \nu \rangle$, we obtain

$$(4.6) \quad u \Delta^{M_t} u = u (\nabla_i^{M_t} h_{ij}) \langle X, e_j \rangle + u \phi' H - u^2 \sum h_{ij}^2.$$

Combining (4.4) and (4.6), we obtain (we use ϕ' to denote $\phi'|_{M_t}$ in the following)

$$\begin{aligned} \partial_t u - u \Delta^{M_t} u &= u^2 \sum h_{ij}^2 - 2\phi' H u + n(\phi')^2 + H \langle X, \nabla^{M_t} u \rangle \\ &\quad + u [\langle X, \nabla^{M_t} H \rangle - (\nabla_i^{M_t} h_{ij}) \langle X, e_j \rangle] - n \langle X, \nabla^{M_t} \phi' \rangle \\ (4.7) \quad &= \left(\sum h_{ij}^2 - \frac{H^2}{n} \right) u^2 + \frac{1}{n} (H u - n\phi')^2 + H \langle X, \nabla^{M_t} u \rangle \\ &\quad + u [\langle X, \nabla^{M_t} H \rangle - (\nabla_i^{M_t} h_{ij}) \langle X, e_j \rangle] - n \langle X, \nabla^{M_t} \phi' \rangle. \end{aligned}$$

Note that $\langle X, \nabla^{M_t} H \rangle - (\nabla_i^{M_t} h_{ij}) \langle X, e_j \rangle$ can be expressed in terms of $\bar{Ric}(X^\top, \nu)$ where $X^\top = \langle X, e_i \rangle e_i$ is the component of X that is tangential to M_t . This is the same as the term that appears in the monotonicity formula. However, in this case, if we only want to prove that $u > 0$ is preserved along the flow, the sign of the term $\bar{Ric}(X^\top, \nu)$ does not matter.

On the other hand, we compute

$$\nabla^{M_t} \phi' = \bar{\nabla} \phi' - \langle \bar{\nabla} \phi', \nu \rangle \nu$$

and $\bar{\nabla} \phi' = \phi'' \bar{\nabla} r = \phi'' \partial_r = \frac{\phi''}{\phi} X$. Therefore,

$$\langle X, \nabla^{M_t} \phi' \rangle = \frac{\phi''}{\phi} \langle X, X - u\nu \rangle = \frac{\phi''}{\phi} (\phi^2 - u^2).$$

Plugging this into (4.7), we obtain

$$\begin{aligned} \partial_t u - u \Delta^{M_t} u &= \left(\sum h_{ij}^2 - \frac{H^2}{n} \right) u^2 + \frac{1}{n} (H u - n\phi')^2 + H \langle X, \nabla^{M_t} u \rangle \\ (4.8) \quad &\quad + u [\langle X, \nabla^{M_t} H \rangle - (\nabla_i^{M_t} h_{ij}) \langle X, e_j \rangle] - n \frac{\phi''}{\phi} (\phi^2 - u^2). \end{aligned}$$

Recall in (2.6) and (2.9), we have derived $R_{\nu e_i}$ and

$$\begin{aligned} & [\langle X, \nabla^{M_t} H \rangle - (\nabla_i^{M_t} h_{ij}) \langle X, e_j \rangle] = -g^{ij} \bar{\nabla}_i \Phi \bar{R}_{j\nu} \\ & = (n-1)(K - \phi'^2 + \phi\phi'') \frac{|\bar{\nabla}\rho|^2}{\omega^3} + (\tilde{R}ic_{ik} - (n-1)K\tilde{g}_{ik}) \frac{\rho_k \rho_i}{\omega^3} \geq 0 \end{aligned}$$

Thus (4.8) can be simplified as

$$(4.9) \quad \partial_t u - u\Delta^{M_t} u \geq H\langle X, \nabla^{M_t} u \rangle + n(\phi'^2 - \phi''\phi) - 2\phi' Hu.$$

We now switch to the evolution of mean curvature H along the flow.

Proposition 4.1. *Along the flow (1.2), the mean curvature of a graphical hypersurface evolves as the follows*

$$(4.10) \quad \begin{aligned} \partial_t H &= u\Delta H + H\nabla H\nabla\Phi + 2\nabla H\nabla u + \phi'(H^2 - n|A|^2) \\ &\quad - \frac{n}{\phi^2} \left[(n-1)\phi'(K - \phi'^2) + (n-2)\phi\phi'\phi'' + \phi^2\phi''' \right] \left(1 - \frac{u^2}{\phi^2}\right) \\ &\quad - n\frac{\phi'}{\phi^6} (\tilde{R}_{ij} - (n-1)K\tilde{g}_{ij}) \rho_i \rho_j u^2. \end{aligned}$$

Proof. Using Proposition 2.7 and replacing the general f by $n\phi' - Hu$, we have

$$\begin{aligned} \partial_t H &= -\Delta_g f - f|A|^2 - f\bar{R}_{\nu\nu} \\ &= -\Delta_g(n\phi' - Hu) - f|A|^2 - f\bar{R}_{\nu\nu} \\ &= u\Delta H + H\Delta u + 2\nabla H\nabla u - n\Delta\phi' - f|A|^2 - f\bar{R}_{\nu\nu} \\ &= u\Delta H + H\Delta u + 2\nabla H\nabla u - n\Delta\phi' - f|A|^2 - f\bar{R}_{\nu\nu} \end{aligned}$$

Using Proposition 2.4, we have

$$\begin{aligned} \Delta\phi' &= \frac{\phi''}{\phi}\Delta\Phi + \nabla\frac{\phi''}{\phi}\nabla\Phi \\ &= \frac{\phi''}{\phi}f + \frac{1}{\phi}\left(\frac{\phi''}{\phi}\right)'|\nabla\Phi|^2, \end{aligned}$$

where $f = n\phi' - Hu$.

Combining with Proposition 2.4, we have

$$\begin{aligned} \partial_t H &= u\Delta H + H\left[\nabla H\nabla\Phi + \bar{R}_{\nu i}\nabla_i\Phi + H\phi' - |A|^2 u\right] \\ &\quad + 2\nabla H\nabla u - n\frac{\phi''}{\phi}f - n\frac{1}{\phi}\left(\frac{\phi''}{\phi}\right)'|\nabla\Phi|^2 - f|A|^2 - f\bar{R}_{\nu\nu} \\ &= u\Delta H + H\nabla H\nabla\Phi + 2\nabla H\nabla u + H\bar{R}_{\nu i}\nabla_i\Phi + \phi'(H^2 - n|A|^2) \\ &\quad - (n\frac{\phi''}{\phi} + \bar{R}_{\nu\nu})f - n\frac{1}{\phi}\left(\frac{\phi''}{\phi}\right)'|\nabla\Phi|^2. \end{aligned}$$

Replacing f by $n\phi' - Hu$, we get

$$(4.11) \quad \begin{aligned} \partial_t H &= u\Delta H + H\nabla H\nabla\Phi + 2\nabla H\nabla u + \phi'(H^2 - n|A|^2) \\ &\quad + H\left[\bar{R}_{\nu i}\nabla_i\Phi + u\left(n\frac{\phi''}{\phi} + \bar{R}_{\nu\nu}\right)\right] \\ &\quad - n\left[\left(n\frac{\phi''}{\phi} + \bar{R}_{\nu\nu}\right)\phi' + \frac{1}{\phi}\left(\frac{\phi''}{\phi}\right)'|\nabla\Phi|^2\right]. \end{aligned}$$

By (2.9) and the definition of $u = \frac{\phi^2}{\omega}$, we find out the mean curvature term in the second line of (4.11) vanishes, i.e.,

$$(4.12) \quad \bar{R}_{\nu i} \nabla_i \Phi + u \left(n \frac{\phi''}{\phi} + \bar{R}_{\nu\nu} \right) = 0.$$

On the other hand, since $|\nabla \Phi|^2 = \frac{\phi^2 |\tilde{\nabla} \rho|^2}{\omega^2}$, using (2.9) again, we can compute the last line in (4.11),

$$(4.13) \quad \begin{aligned} & \left[\left(n \frac{\phi''}{\phi} + \bar{R}_{\nu\nu} \right) \phi' + \frac{1}{\phi} \left(\frac{\phi''}{\phi} \right)' |\nabla \Phi|^2 \right] \\ &= \left[(n-1) \frac{K - \phi'^2 + \phi \phi''}{\phi^2} \phi' + \phi \left(\frac{\phi''}{\phi} \right)' \right] \frac{|\tilde{\nabla} \rho|^2}{\omega^2} + (\tilde{R}_{ij} - (n-1)K\tilde{g}_{ij}) \frac{\rho_i \rho_j}{\phi^2 \omega^2} \phi'. \end{aligned}$$

This yields that

$$(4.14) \quad \begin{aligned} \partial_t H &= u \Delta H + H \nabla H \nabla \Phi + 2 \nabla H \nabla u + \phi' (H^2 - n|A|^2) \\ &\quad - n \left[(n-1) \frac{K - \phi'^2 + \phi \phi''}{\phi^2} \phi' + \phi \left(\frac{\phi''}{\phi} \right)' \right] \frac{|\tilde{\nabla} \rho|^2}{\omega^2} - n (\tilde{R}_{ij} - (n-1)K\tilde{g}_{ij}) \frac{\rho_i \rho_j}{\phi^2 \omega^2} \phi'. \end{aligned}$$

Plugging the definition of u into (4.14), we finished the proof. \square

Remark 4.2. *By direct computations, if the ambient space is substatic, namely satisfying conditions H1-H4 in Brendle's work [2], the last two lines in (4.10) are nonnegative, which immediately implies a uniform upper bound for the mean curvature by the maximum principle.*

Next, we show that the mean curvature H has a uniform upper bound in general without assuming the substatic conditions for the ambient metric.

Theorem 4.3. *Suppose*

$$(4.15) \quad \phi'(r) > 0, \quad \forall r \in (r_0, \bar{r}).$$

The mean curvature of the graphical hypersurface evolving along the flow (1.2) has a uniform upper bound,

$$\max_{t>0} H(\cdot, t) \leq C,$$

where C is a uniform constant which is independent of t .

Proof. From the evolution equations of H and Φ , we obtain

$$\begin{aligned}
(4.16) \quad \mathcal{L}(H + \Phi) &\leq H\nabla(H + \Phi)\nabla\Phi + 2\nabla(H + \Phi)\nabla u + \phi'(H^2 - n|A|^2) \\
&\quad - H|\nabla\Phi|^2 - 2\nabla\Phi\nabla u \\
&\quad - \frac{n}{\phi^2} \left[(n-1)\phi'(K - \phi'^2) + (n-2)\phi\phi'\phi'' + \phi^2\phi''' \right] \left(1 - \frac{u^2}{\phi^2}\right) \\
&= H\nabla(H + \Phi)\nabla\Phi + 2\nabla(H + \Phi)\nabla u + \phi'(H^2 - n|A|^2) \\
&\quad - H|\nabla\Phi|^2 - 2h(\nabla\Phi, \nabla\Phi) \\
&\quad - \frac{n}{\phi^2} \left[(n-1)\phi'(K - \phi'^2) + (n-2)\phi\phi'\phi'' + \phi^2\phi''' \right] \left(1 - \frac{u^2}{\phi^2}\right)
\end{aligned}$$

Since $|\nabla\Phi|^2 = \phi^2(1 - \frac{u^2}{\phi^2})$ and functions related to ϕ are all uniformly bounded from above and below, we conclude that the term in the last line of equation (4.16) is uniformly bounded by $C_1|\nabla\Phi|^2$ where C_1 is a uniform constant which does not depend on t . Thus

$$\begin{aligned}
(4.17) \quad \mathcal{L}(H + \Phi) &\leq H\nabla(H + \Phi)\nabla\Phi + 2\nabla(H + \Phi)\nabla u + \phi'(H^2 - n|A|^2) \\
&\quad - H|\nabla\Phi|^2 - 2h(\nabla\Phi, \nabla\Phi) + C_1|\nabla\Phi|^2
\end{aligned}$$

At a maximum point of the test function $H + \Phi$, we can choose normal coordinates, so that the metric tensor at the point is the identity matrix. If we choose a coordinate system so that the x_1 axis direction is the direction of $\nabla\Phi$, then $\Phi_i = 0$, for all $i = 2, \dots, n$.

$$(4.18) \quad \mathcal{L}(H + \Phi) \leq -\phi'(n|A|^2 - H^2) - (H + 2h_{11} - C_1)|\nabla\Phi|^2$$

where we have used the critical point condition to eliminate the gradient terms.

Without loss of generality, we assume $\frac{H}{2} - C_1 \geq 0$, otherwise the test function $H + \Phi$ is uniformly bounded from above by a constant and the proof is done. This reduces (4.18) to

$$(4.19) \quad \mathcal{L}(H + \Phi) \leq -\phi'(n|A|^2 - H^2) - \left(\frac{H}{2} + 2h_{11}\right)|\nabla\Phi|^2$$

Next we consider two different cases.

Case I: Suppose at the maximum point, $\frac{H}{2} + 2h_{11} > 0$, then (4.18) is reduced to

$$\mathcal{L}(H + \Phi) \leq 0,$$

and by the maximum principle $H + \Phi$ is bounded, and so is H .

Case II: Suppose that at the maximum point, $\frac{H}{2} + 2h_{11} \leq 0$. Let $\lambda_i := h_{ii}$, $|A|^2 = \lambda_1^2 + |\tilde{A}|^2$, and $H = \lambda_1 + \tilde{H}$. Then

$$(4.20) \quad -\lambda_1 \geq \frac{1}{4}H.$$

and

$$(4.21) \quad \begin{aligned} n|A|^2 - H^2 &= (n|\tilde{A}|^2 - \tilde{H}^2) + (n+1)\lambda_1^2 - 2\lambda_1 H \\ &\geq (n+1)\lambda_1^2 - 2\lambda_1 H \end{aligned}$$

where we used Cauchy-Schwartz inequality in the inequality.

Recall $\phi' \geq C_2 > 0$ and $0 \leq |\nabla\Phi|^2 \leq C_3$. Applying (4.21), (4.19) yields

$$\begin{aligned} \mathcal{L}(H + \Phi) &\leq -C_2 \left[(n+1)\lambda_1^2 - 2\lambda_1 H \right] - C_3 \left(\frac{H}{2} + 2\lambda_1 \right) \\ &= -C_2(n+1)\lambda_1^2 - C_3 \frac{H}{2} - 2(-\lambda_1)(C_2 H - C_3). \end{aligned}$$

We assume that $H > \frac{C_3}{C_2} \geq 0$, otherwise H has an upper bound. This yields

$$\mathcal{L}(H + \Phi) \leq 0.$$

By the maximum principle, we conclude that the test function $H + \Phi$ has a uniform upper bound. \square

By Theorem 4.3, we know that the mean curvature H has a uniform upper bound. If we further assume the strict inequality $\phi'^2 - \phi''\phi > 0$, then by the uniform C^0 estimate and uniform upper bound for H , the second term on the right hand side of (4.9) is the dominant term as $u \rightarrow 0$. Thus by the maximum principle, we have shown the uniform positivity of the support function u which in turn yields the needed gradient estimate.

Proposition 4.4. *Let $M_0 \subset \mathbf{N}^{n+1}$ be a smooth graphical hypersurface with support function $u > 0$. Assume condition (4.15) and*

$$(\phi')^2 - \phi''\phi > 0.$$

then there exists a uniform constant $C > 0$ independent of t , such that

$$u(\cdot, t) \geq C > 0$$

as long as the solution of the flow (1.2) exists.

5. CONDITIONS ON THE WARPING FUNCTION ϕ

Since equation (3.3) is a quasilinear parabolic equation, it follows Proposition 3.1 and Theorem 3.2, the flow is uniformly parabolic. The longtime existence and regularity follow from the standard parabolic theory. The solution converges exponentially to a slice $\rho = \text{constant}$ by Theorem 3.2. This proves Theorem 1.1. Theorem 1.2 follows from the following proposition.

Proposition 5.1. *Let $\Omega \subset \mathbf{N}^{n+1}$ be a domain bounded by a smooth graphical hypersurface M and $S(r_0)$. We assume $\phi(r)$ and \tilde{g} satisfy the conditions (1.3) in Theorem 1.1, then*

$$(5.1) \quad \text{Area}(M) \geq \text{Area}(S(r^*)),$$

where r^ is the unique real number in $[r_0, \bar{r}]$ such that volume of $B(r^*)$ enclosed by $S(r^*)$ and $S(r_0)$ is equal to $\text{Vol}(\Omega)$. If equality in (5.1) holds, then M must be umblic. If, in addition to (1.3), $(\phi')^2 - \phi''\phi < K$ on $[r_0, \bar{r}]$ then “=” is attained in (1.5) if and only if M is a level set of r .*

Proof. By Theorem 2.8, area if evolving hypersurfaces along flow 1.2 is decreasing and enclosed volume is preserved. By Theorem 1.1, flow converges to a slice $S(r^*)$. Then we must have $Vol(B(r^*)) = Vlo(\Omega)$. Note that $Vol(B(r))$ is strictly increasing in r , thus r^* is unique. This proves inequality (5.1). If equality holds there, inequality (2.16) must be equality. Therefore $\sigma_2 = \frac{n-1}{2n}\sigma_1^2$ at every point, this implies M is umbilic. If $(\phi')^2 - \phi''\phi < K$ on $[r_0, \bar{r}]$, (2.16) implies $\tilde{\nabla}\rho \equiv 0$. This is, M is a slice. \square

We will illustrate that both the lower bound and upper bound in (1.3) are necessary in certain sense and have geometric or physics interpretations.

The lower bound of $(\phi')^2 - \phi''\phi$ is closely related to the notion of “photon spheres”. For each warped product space with a Riemannian metric $dr^2 + \phi^2(r)\tilde{g}_{ij}du^i du^j$, there is an associated static spacetime \mathfrak{S} with the spacetime metric

$$(5.2) \quad -(\phi'(r))^2 dt^2 + dr^2 + \phi^2(r)\tilde{g}_{ij}du^i du^j.$$

Such a spacetime has a natural conformal Killing-Yano two form and is of great interest in general relativity, see [11, Remark 3.7]. For the Schwarzschild manifold with the Riemannian metric $\frac{1}{1-\frac{m}{s}}ds^2 + s^2\tilde{g}_{ij}du^i du^j$ (after a change of coordinates $s = \phi(r)$), the associated static spacetime is the Schwarzschild spacetime with the spacetime metric

$$-(1 - \frac{m}{s})dt^2 + \frac{1}{1 - \frac{m}{s}}ds^2 + s^2\tilde{g}_{ij}du^i du^j.$$

We recalled that a hypersurface is said to be *totally umbilical* if the second fundamental form is proportional the induced metric (the first fundamental form). A totally umbilical timelike hypersurface \mathfrak{T} of a spacetime \mathfrak{S} is called a *photon sphere* [4], where null geodesics are trapped (i.e. a null geodesic which is initially tangent to \mathfrak{T} remains within the hypersurface \mathfrak{T}). This is easily seen from the following relation: for any null vector field X ,

$$\nabla_X^{\mathfrak{S}} X = \nabla_X^{\mathfrak{T}} X,$$

where $\nabla^{\mathfrak{S}}$ and $\nabla^{\mathfrak{T}}$ are the covariant derivatives of \mathfrak{S} and \mathfrak{T} , respectively.

We claim that the equation $(\phi')^2 - \phi\phi'' = 0$ characterizes exactly the location of the photon sphere.

Proposition 5.2. *For a spacetime \mathfrak{S} with metric of the form (5.2), $(\phi')^2 - \phi\phi'' = 0$ at r_0 if and only if $r = r_0$ is a photon sphere.*

Proof. It suffices to prove that the hypersurface \mathfrak{T} defined by $r = r_0$ is totally umbilical. The induced metric of \mathfrak{T} is

$$(5.3) \quad -(\phi')^2 dt^2 + \phi^2\tilde{g}_{ij}du^i du^j$$

and the unit outward normal of \mathfrak{T} is $\frac{\partial}{\partial r}$. We compute

$$\langle \nabla_{\partial_t}^{\mathfrak{S}} \partial_r, \partial_t \rangle = \frac{1}{2} \partial_r (-(\phi'))^2 = -\phi' \phi''.$$

and

$$\langle \nabla_{\partial_i}^{\mathfrak{S}} \partial_r, \partial_j \rangle = \frac{1}{2} \partial_r (\phi^2 \tilde{g}_{ij}) = \phi \phi' \tilde{g}_{ij},$$

therefore the second fundamental form of \mathfrak{X} is

$$-\phi' \phi'' dt^2 + \phi \phi' \tilde{g}_{ij} du^i du^j.$$

Comparing this with (5.3) yields the desired conclusion. \square

On the other hand, the upper bound for $(\phi')^2 - \phi''\phi$ is needed for the proof of the monotonicity properties of the flow, see Theorem 2.8. In [9], it is shown that the isoperimetric problem may not hold when this condition is violated.

To end the paper, we discuss the convergence of flow (1.2) when $K = 0$ in (1.3). For example, then base manifold \mathbf{B}^n is Ricci flat. As we assume $0 \leq (\phi')^2 - \phi\phi'' \leq K$, this forces $(\phi'(r))^2 = \phi(r)\phi''(r), \forall r \in [r_0, \bar{r}]$. Note that Proposition 3.1 still holds in this case.

Proposition 5.3. *Assume $K = 0$ in (1.3) and $(\phi'(r))^2 = \phi(r)\phi''(r), \forall r \in [r_0, \bar{r}]$. Then flow (1.2) exists for all $t \in [0, \infty)$ and converges to a slice $\rho = c$ in $C^k, \forall k$ as $t \rightarrow \infty$.*

Proof. Identity $(\phi'(r))^2 = \phi(r)\phi''(r)$ is equivalent to $(\log \phi(r))'' = 0$. That is,

$$(5.4) \quad \phi(r) = ae^{br},$$

for some constants $a > 0, b \in \mathbb{R}$. In the proof of Theorem 3.2, at the critical point of $|\nabla \rho|^2$, (3.15) holds,

$$(5.5) \quad \begin{aligned} \mathcal{L}\left(\frac{|\tilde{\nabla} \rho|^2}{2}\right) = & -\frac{1}{\omega} \sum_{j \geq 2} \left(\rho_{jj} + \frac{1}{2} \frac{\phi \phi'}{\omega^2} |\tilde{\nabla} \rho|^2 \right)^2 - \frac{1}{\omega} \tilde{Ric}(\tilde{\nabla} \rho, \tilde{\nabla} \rho) \\ & + \frac{|\tilde{\nabla} \rho|^2}{\omega^5} \left[n \frac{\phi'' \phi - \phi'^2}{\phi^2} |\tilde{\nabla} \rho|^4 \omega^2 + (n-1)(\phi'' \phi - \phi'^2) |\tilde{\nabla} \rho|^2 \omega^2 \right. \\ & \left. - (n+2)\phi'^2 |\tilde{\nabla} \rho|^4 - \frac{3}{4}(n-1)\phi^2 \phi'^2 |\tilde{\nabla} \rho|^2 \right] \end{aligned}$$

It follows that

$$\frac{|\tilde{\nabla} \rho|^2}{2} \leq \max_{t=0} \frac{|\tilde{\nabla} \rho|^2}{2}.$$

Therefore, by the standard theory of parabolic quasilinear PDE, we have regularity estimates in C^k for all $k \geq 1$ and the flow exists all time.

To show the convergence, we only need to show $\|\tilde{\nabla} \rho\|_{L^\infty} \rightarrow 0$ as $t \rightarrow \infty$. First, if $b \neq 0$ in (5.4), by the C^0 estimate, we have

$$(5.6) \quad \mathcal{L}\left(\frac{|\tilde{\nabla} \rho|^2}{2}\right) \leq -\frac{|\tilde{\nabla} \rho|^2}{\omega^5} \left[(n+2)\phi'^2 |\tilde{\nabla} \rho|^4 + \frac{3}{4}(n-1)\phi^2 \phi'^2 |\tilde{\nabla} \rho|^2 \right] \leq -C \left(\frac{|\tilde{\nabla} \rho|^2}{2}\right)^2,$$

for some $C > 0$. By standard ODE comparison to the equation $f'(t) = -Cf^2(t)$, we get

$$(5.7) \quad \max_{M(t)} |\tilde{\nabla} \rho|^2 \leq \frac{\max_{M(0)} |\tilde{\nabla} \rho|^2}{Ct \max_{M(0)} |\tilde{\nabla} \rho|^2 + 1},$$

for some $C > 0$ which is independent of t .

If $b = 0$ in (5.4), then $\phi = a > 0$ is a constant function. In this case, $H = -\frac{1}{a}\tilde{\nabla}(\frac{\tilde{\nabla}\rho}{\sqrt{a^2+|\tilde{\nabla}\rho|^2}})$. Evolution equation (1.2) becomes

$$\rho_t = \tilde{\nabla}\left(\frac{\tilde{\nabla}\rho}{\sqrt{a^2 + |\tilde{\nabla}\rho|^2}}\right).$$

Multiply ρ in above equation, then integrate over $[0, t] \times B^n$,

$$\frac{1}{2}\left(\int_{B^n} \rho^2(t, \cdot) - \int_{B^n} \rho^2(0, \cdot)\right) = -\int_0^t \int_{B^n} \frac{|\tilde{\nabla}\rho(t, \cdot)|^2}{\sqrt{a^2 + |\tilde{\nabla}\rho(t, \cdot)|^2}} dt.$$

The left hand side is bounded as $t \rightarrow \infty$, by regularity estimate, we must have

$$(5.8) \quad \int_{B^n} \frac{|\tilde{\nabla}\rho(t, \cdot)|^2}{\sqrt{a^2 + |\tilde{\nabla}\rho(t, \cdot)|^2}} \rightarrow 0, \quad t \rightarrow \infty.$$

That is $\tilde{\nabla}\rho(t, \cdot) \rightarrow 0$ in L^2 , regularity estimates imply $\tilde{\nabla}\rho(t, \cdot) \rightarrow 0$ in L^∞ .

In conclusion, evolution equation (1.2) with M as the initial data has a smooth solution for $t \in [0, \infty)$ and the solution hypersurfaces converge to a level set of r as $t \rightarrow \infty$. \square

As a consequence, in the case of $K = 0$ in (1.3), if $\Omega \subset \mathbf{N}^{n+1}$ is a domain bounded by a smooth graphical hypersurface M and $S(r_0)$, then there exist a function ξ such that

$$(5.9) \quad \text{Area}(M) \geq \xi(\text{Vol}(\Omega)).$$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTREAL,
 QUEBEC H3A 0B9, CANADA
E-mail address: `guan@math.mcgill.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM,
 AL 35294
E-mail address: `jfli@uab.edu`

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW
 YORK, NY 10027
E-mail address: `mtwang@math.columbia.edu`