Gluing quasiconformal mappings in the plane

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Abstract

In this paper, several versions of gluing theorems for quasiconformal mappings in the plane are obtained. The best possibility of gluing quasiconformal mappings is investigated. As an application, we provide a new short proof of the gluing theorem obtained by Jiang and Qi.

1 . Introduction

It is a very well known fact that one can not extend any two arbitrary analytic functions defined on two disjoint domains into an analytic function on the complex plane. However, in the study of complex dynamics, one would like to extend two conformal maps into one map defined on the complex plane. To overcome the difficulty caused by the rigidity of an analytic mapping, such a map can be only quasiconformal. This becomes a gluing problem in complex dynamics. Moreover, one would like to control the best quasiconformal dilatation in the gluing problem. It used to be a difficult problem. However, Jiang showed that it is possible for n different conformal maps each of which fixes a point in its domain (they are called conformal germs). This becomes Jiang's gluing theorem [3].

Theorem A. *Let $\{z_k\}_{k=1}^m$ be a set of distinct points in $\mathbb{C}$ and U_k be a neighborhood of z_k for every $k = 1, 2, \cdots, m$. Suppose $\{U_k\}_{k=1}^m$ are pairwise disjoint and $f_k(z)$ is a conformal mapping defined on U_k which fixes z_k for every $k = 1, 2, \cdots, m$. Then for every $\epsilon > 0$, there exists a number $r > 0$ and a $(1 + \epsilon)-$quasiconformal self-homeomorphism f of $\mathbb{C}$ such that*

$$f|_{U(z_k,r)} = f_k|_{U(z_k,r)},$$

where $U(z_k, r)$ is the open disk of radius r centered at z_k for $k = 1, 2, \cdots, m$.

The method he used to show this gluing theorem is by so-called holomorphic motions. Jiang further showed that as long as one understands the holomorphic motion

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theorem, the proof of the gluing problem is not very difficult. However, the mathematical mechanism of the gluing is hinted in the holomorphic motion theorem. Therefore, Jiang asked for a new proof in the point of view of Teichmüller theory. Jiang and Qi [4] studied this question and then proved a more general theorem for quasiconformal germs by using Reich and Strebel's results in Teichmüller theory.

Theorem B. *Let $\{z_k\}_{k=1}^m$ be a set of distinct points in $\mathbb{C}$ and U_k be a neighborhood of z_k for every $k = 1, 2, \cdots, m$. Suppose $\{U_k\}_{k=1}^m$ are pairwise disjoint and $f_k(z)$ is a $K-$quasiconformal mapping defined on U_k which fixes z_k for every $k = 1, 2, \cdots, m$. Then for every $\epsilon > 0$, there exists a number $r > 0$ and a $(K + \epsilon)-$quasiconformal self-homeomorphism f of $\mathbb{C}$ such that*

$$f|_{U(z_k,r)} = f_k|_{U(z_k,r)},$$

where $U(z_k, r)$ is the open disk of radius r centered at z_k for $k = 1, 2, \cdots, m$.

In the theorem, the condition that $f_k(z)$ fixes z_k should be a strong restriction so that the possible application of the theorem is circumscribed to a narrow range. In this paper, we will loosen the restriction and extend Jiang's gluing theorem into n quasiconformal mappings (not germs), again by Reich and Strebel's results in Teichmüller theory. Besides these, a better estimate of the quasiconformal dilatation is given.

Actually, we give several versions of gluing theorems according to the domains that the gluing mappings depend on. Only for the gluing theorem in the unit disk, we give a detailed proof. The rest cases can be obtained in an extremely similar way. Nevertheless, these versions play their own irreplaceable roles. Theorem B will be a direct consequence of one of these versions. To illuminate the mechanism hidden in these theorems, we also show that the best possible gluing mappings are those extremal quasiconformal mappings with reduced boundary condition.

2 . Some preliminaries

Let S be a plane domain with at least two boundary points. The Teichmüller space $T(S)$ is the space of equivalence classes of quasiconformal maps f from S to a variable domain $f(S)$. Two quasiconformal maps f from S to $f(S)$ and g from S to $g(S)$ are equivalent if there is a conformal map c from $f(S)$ onto $g(S)$ and a homotopy through quasiconformal maps h_t mapping S onto $g(S)$ such that $h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c \circ f(p) = g(p)$ for every $t \in [0, 1]$ and every p in the boundary of S. Denote by $[f]$ the Teichmüller equivalence class of f; also sometimes denote the equivalence class by $[\mu]$ where μ is the Beltrami differential of f. The constants

$$K(f) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty},\ K([f]) = \inf\{K(g) :\ g \in [f]\}$$

are called the maximal dilatation of f and the extremal maximal dilatation of $[f]$ respectively. If $K([f])$ is attained by f, then f is called an extremal quasiconformal mapping in $[f]$.

Denote by $Bel(S)$ the Banach space of Beltrami differentials $\mu = \mu(z)d\bar{z}/dz$ on S with finite L^∞-norm and by $M(S)$ the open unit ball in $Bel(S)$.

The boundary dilatation of f is defined as

$$H^*(f) = \inf\{K(f|_{S\backslash E}) : \ E \text{ is a compact subset of } S\},$$

where $K(f|_{S\backslash E})$ is the maximal dilatation of $f|_{S\backslash E}$. The boundary dilatation of $[f]$ is defined as

$$H([f]) = \inf\{H^*(g) : \ g \in [f]\}.$$

It is obvious that $H([f]) \leq K([f])$. Following [1], $[f]$ is called a Strebel point if $H([f]) < K([f])$. By the frame mapping theorem of Strebel [9], if $[f]$ is a Strebel point, then the uniquely determined extremal mapping in $[f]$ is a Teichmüller mapping whose Beltrami differential is in the form $\mu = k\overline{\varphi}/|\varphi|$ $(0 < k < 1)$, where φ is an integrable holomorphic quadratic differential on S.

Let D and $\mathfrak{D}$ be two quasidisks in the complex $\mathbb{C}$. Suppose $h : \partial D \to \partial\mathfrak{D}$ be a quasisymmetric homeomorphism. Let $Q(h)$ be the collection of all quasiconformal mappings from D to $\mathfrak{D}$ with the boundary value h. Define

$$K(h) := \inf\{K(f)|\ f \in Q(h)\}.$$

Let $\{z_k\}_{k=1}^m$, $\{w_k\}_{k=1}^m$ be two sets of distinct points in D and in $\mathfrak{D}$ respectively. Let $QE(h; z_1, \cdots, z_m; w_1, \cdots, w_m)$ be the collection of all quasiconformal mappings from D to $\mathfrak{D}$ with the boundary value h and the condition $f(z_k) = w_k,\ k = 1, \cdots, m$. Define

$$K(h; z_1, \cdots, z_m; w_1, \cdots, w_m) := \inf\{K(f)|\ f \in QE(h; z_1, \cdots, z_m; w_1, \cdots, w_m)\}.$$

Suppose $\{X_k\}_{k=1}^m$ is a set of pairwise disjoint quasidisks in D and $\{Y_k\}_{k=1}^m$ is another set of pairwise disjoint quasidisks in $\mathfrak{D}$. Let h_k be the quasisymmetric homeomorphism from ∂X_k to ∂Y_k for $k = 1, 2, \cdots, m$.

The following lemma can be deduced from Kelingos' result in [5] by induction (the reader may refer to [4] for more details).

Lemma 2.1. *Use the denotations above. Then, there exists a quasiconformal mapping from* $D\backslash\{X_k\}_{k=1}^m$ *to* $\mathfrak{D}\backslash\{Y_k\}_{k=1}^m$ *such that* $f|_{\partial X_k} = h_k$ *for* $k = 1, 2, \cdots, m$.

The following main inequality (see [2, 8]) is a key tool in our argument.

Theorem C. *Suppose* $f,\ g : R \to R'$ *are two quasiconformal homeomorphisms from a Riemann surface* R *to another surface* R' *which are homotopic modulo the boundary. Then for every integrable holomorphic quadratic differential* $\varphi = \varphi dz^2$ *on* R, *we have*

$$\|\varphi\| \leq \iint_R |\varphi(z)| \frac{|1 - \mu_f(z)\frac{\varphi(z)}{|\varphi(z)|}|^2}{1 - |\mu_f(z)|^2} D_{g^{-1}}(f(z))dxdy, \tag{2. 1}$$

where $D_{g^{-1}}(w) = \frac{1+|\mu_{g^{-1}}(z)|}{1-|\mu_{g^{-1}}(z)|}$ *is the dilatation of* g^{-1} *at* $w = g(z)$ *and* μ_f, μ_g *and* $\mu_{g^{-1}}$ *are the Beltrami differentials of the quasiconformal homeomorphisms* f, g *and* g^{-1} *respectively.*

Let $d(z_1, z_2)$ denote the hyperbolic distance between two points z_1, z_2 in Δ, i.e.,

$$d(z_1, z_2) = \frac{1}{2} \log \frac{1 + |\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}|}{1 - |\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}|}.$$

Lemma 2.2. *Let σ be a $K-$quasiconformal mapping from Δ onto itself with $\sigma|_{\partial\Delta} = id$. Then for any given point $z \in \Delta$, we have*

$$d(z, \sigma(z)) \leq \frac{\pi^2}{8}\sqrt{K}.$$

Proof. In terms of the notion of the Teichmüller shift mapping, the lemma follows from Theorem 1 in [12] immediately. □

3. Gluing of quasiconformal mappings in the unit disk

In this section, we glue more than one quasiconformal mapping in the unit disk together to obtain a new quasiconformal mapping whose maximal dilatation can be controlled properly.

Theorem 1. *Denote by $\Delta = \{|z| < 1\}$ the unit disk. Let $\{z_k\}_{k=1}^m$ be a set of distinct points in Δ and $U_k \subset \Delta$ be a neighborhood of z_k for every $k = 1, 2, \cdots, m$. Suppose $\{U_k\}_{k=1}^m$ are pairwise disjoint and $f_k(z)$ is a $K-$quasiconformal mapping defined on U_k which sends z_k to $w_k \in \Delta$ for every $k = 1, 2, \cdots, m$. Suppose h is a quasisymmetric self-homeomorphism of $\partial\Delta$. Then for every $\epsilon > 0$, there exists a number $r > 0$ and a quasiconformal self-homeomorphism f of Δ such that*
(1) $f|_{\partial\Delta} = h$;
(2)

$$f|_{U(z_k,r)} = f_k|_{U(z_k,r)},$$

where $U(z_k, r)$ is the open disk of radius r centered at z_k for $k = 1, 2, \cdots, m$;
(3) $K(f) \leq \max\{K(h; z_1, \cdots, z_m; w_1, \cdots, w_m), K\} + \epsilon$.

Proof. Choose a sequence $\{r_n\}_{n=1}^\infty$ such that (1) $r_n > r_{n+1} > 0$, (2) $U(z_k, r_1) \subset U_k$ for $k = 1, 2, \cdots, m$, (3) $r_n \to 0$ as $n \to \infty$. Set $J_n = \bigcup_{k=1}^m U(z_k, r_n)$ and $V_n = \Delta \backslash \bar{J}_n$. Then V_n is a $(m + 1)-$connected domain in Δ.

By Lemma 2.1, there is a quasiconformal self-homeomorphism g of Δ with the property: (1) $g = h$ on $\partial\Delta$, (2) $g|_{U(z_k,r_1)} = f_k|_{U(z_k,r_1)}$ for $k = 1, 2, \cdots, m$. Restrict g on V_n and regard $[g|_{V_n}]$ as a point in the Teichmüller space $T(V_n)$. Then there is an extremal quasiconformal mapping f_n in $[g|_{V_n}]$ such that $f_n = g$ on ∂V_n and $K(f_n|_{V_n}) = K([g|_{V_n}])$. It is obvious that $K(f_n|_{V_n}) \leq K(g|_{V_n})$. For brevity, let $K_m(h) = K(h; z_1, \cdots, z_m; w_1, \cdots, w_m)$. If for some n, $K(f_n|_{V_n}) \leq \max\{K_m(h), K\} + \epsilon$, then

$$\widetilde{f}_n(z) := \begin{cases} f_n(z), & z \in V_n, \\ g(z), & z \in \bar{J}_n \end{cases} \tag{3. 1}$$

is the required quasiconformal mapping.

Now, we assume that $K(f_n|_{V_n}) > \max\{K_m(h), K\} + \epsilon$ holds for all $n \in \mathbb{N}$. Then $K([g|_{V_n}]) > H([g|_{V_n}])$ and $[g|_{V_n}]$ is a Strebel point in the Teichmüller space $T(V_n)$. Consequently, the extremal quasiconformal mapping f_n is a Teichmüller mapping with the Beltrami differential $\mu_n = k_n\overline{\varphi_n}/|\varphi_n|$ $(0 < k_n < 1)$, where φ_n is the associated holomorphic quadratic differential on V_n with L^1- norm

$$\|\varphi_n\| = \iint_{V_n} |\varphi_n(z)|\, dxdy = 1.$$

Claim. φ_n converges to 0 uniformly on any compact subset of Δ_m as $n \to \infty$ where $\Delta_m = \Delta \backslash \{z_k\}_{k=1}^m$.

Note the condition $\lim_{n\to\infty} V_n = \Delta_m$. We can assume, by contradiction, that there is φ_0 holomorphic in Δ_m, $\varphi_0 \not\equiv 0$ and a subsequence $\{n_j\}$ of $\mathbb{N}$ with $n_j < n_{j+1}$ such that $\varphi_{n_j} \to \varphi_0$ as $j \to \infty$. We may choose a subsequence of μ_{n_j}, also denoted by itself, such that $k_{n_j} \to k_0$ as $j \to \infty$. Thus, the Teichmüller differential μ_{n_j} converges to $\mu_0 = k_0\overline{\varphi_0}/|\varphi_0|$ in Δ.

We now show that

$$\iint_{\Delta_m} |\varphi_0(z)|\, dxdy < \infty,$$

which indicates that φ_0 has at most poles of first order at $\{z_k\}_{k=1}^m$. In fact, by the Fatou lemma, we have

$$\iint_{\Delta_m} |\varphi_0(z)|\, dxdy = \iint_{\Delta_m} \lim_{j\to\infty} |\varphi_{n_j}(z)|\, dxdy \le \liminf_{j\to\infty} \iint_{\Delta_m} |\varphi_{n_j}(z)|\, dxdy = 1,$$

where we prescribe that $\varphi_{n_j}(z) = 0$ when $z \in \Delta\backslash V_{n_j}$.

Observe that $K(\widetilde{f}_{n_j}) \le K(g)$ for all $j > 0$. Therefore, by the classical result on convergence of quasiconformal mappings, we see that $\widetilde{f}_{n_j}$ converges uniformly to f_0 on $\overline{\Delta}$, where $f_0 \in QE(h; z_1, \cdots, z_m; w_1, \cdots, z_m)$ with the Beltrami differential μ_0. f_0 is an extremal Teichmüller mapping in $QE(h; z_1, \cdots, z_m; w_1, \cdots, z_m)$ and $K(f_0) = K_m(h)$. On the other hand, the assumption that $K(f_n|_{V_n}) > \max\{K_m(h), K\} + \epsilon$ holds for all $n \in \mathbb{N}$ implies $K_m(h) \ge K_m(h) + \epsilon$. This gives rise to a contradiction. The proof of the Claim is completed.

Fix a positive integer N. By the definition of boundary dilatation, we have

$$H([g|_{V_N}]) \le \max\{K_m(h), K\}.$$

Hence, there exists a quasiconformal mapping $\widetilde{g} \in [g|_{V_N}]$ such that (1) $\widetilde{g} = g$ on ∂V_N (2) $H^*(\widetilde{g}) = H([g|_{V_N}])$. Therefore, there is a compact subset $E \subset V_N$ such that

$$K(\widetilde{g}|_{V_N\backslash E}) \le \max\{K_m(h), K\} + \frac{\epsilon}{2}. \tag{3. 2}$$

For any $n > N$, let

$$F_n(z) := \begin{cases} \widetilde{g}(z), & z \in V_N, \\ g(z), & z \in V_n\backslash V_N. \end{cases}$$

Notice that $f_n^{-1} \circ F_n$ is the identity map on ∂V_n. We apply the main inequality (2. 1) on V_n and get

$$\begin{aligned} 1 =& \|\varphi_n\| \le \iint_{V_n} |\varphi_n(z)| \frac{|1-\mu_n(z)\frac{\varphi_n(z)}{|\varphi_n(z)|}|^2}{1-|\mu_n(z)|^2} D_{F_n^{-1}}(f_n(z))dxdy \\ =& \iint_{V_n} \frac{|\varphi_n(z)|}{K(f_n)} D_{F_n^{-1}}(f_n(z))dxdy. \end{aligned}$$

Thus,

$$\begin{aligned} K(f_n) \le& \iint_{V_n} |\varphi_n(z)| D_{F_n^{-1}}(f_n(z))dxdy \\ =& \iint_{f_n^{-1}\circ F_n(E)} |\varphi_n(z)| D_{F_n^{-1}}(f_n(z))dxdy + \iint_{f_n^{-1}\circ F_n(V_n\backslash E)} |\varphi_n(z)| D_{F_n^{-1}}(f_n(z))dxdy. \end{aligned}$$

Define

$$\sigma_n = \begin{cases} f_n^{-1} \circ F_n, & z \in V_n \\ id, & z \in \Delta \backslash V_n. \end{cases}$$

Then $\sigma_n = id$ on $\partial\Delta$. As restricted on V_n

$$K(f_n) \le K(F_n) \le \max\{K(g),\ K(\widetilde{g}|_{V_N})\} := M,$$

$K(\sigma_n) \le M^2$ for all $n > N$. By Lemma 2.2, for any $z \in E$ and all $n > N$, it holds that

$$d(z, \sigma_n(z)) \le \frac{\pi^2}{8} M.$$

Therefore, all $\sigma_n(E)$ are contained in a fixed compact subset of V_n.

By the degenerating property of $\{\varphi_n\}$,

$$\iint_{f_n^{-1}\circ F_n(E)} |\varphi_n(z)| D_{F_n^{-1}}(f_n(z))dxdy \le \frac{\epsilon}{2}$$

holds for all sufficiently large n. By the definition of F_n and the inequality (3. 2),

$$\iint_{f_n^{-1}\circ F_n(V_n\backslash E)} |\varphi_n(z)| D_{F_n^{-1}}(f_n(z))dxdy \le \max\{K_m(h), K\} + \frac{\epsilon}{2}$$

and consequently,

$$K(f_n) \le \max\{K_m(h), K\} + \epsilon.$$

The proof of Theorem 1 is completed.

□

Use the denotations in Theorem 1. Let $QE(r; h; f_1, \cdots, f_m)$ be the set of quasiconformal homeomorphisms f in $Q(h)$ satisfying the condition $f|_{U(z_k,r)} = f_k|_{U(z_k,r)}$ $(k = 1, \cdots, m)$. Choose an extremal mapping denoted by f^r from $QE(r; h; f_1, \cdots, f_m)$ such that

$$K(f^r) = K(r) := \inf\{K(f) : f \in QE(r; h; f_1, \cdots, f_m)\}.$$

Then $K(r)$ is an increasing function with respect to small $r > 0$. Notice that $K(r) \geq \max\{K_m(h), K(r; f_1, \cdots, f_m)\}$ where

$$K(r; f_1, \cdots, f_m) = \max\{K(f_1|_{U(z_1,r)}, \cdots, f_m|_{U(z_m,r)}\}.$$

When $K_m(h) \geq K$, by Theorem 1 we have

$$K(r) \to K(h; z_1, \cdots, z_m; w_1, \cdots, w_m), \text{ as } r \to 0.$$

Then the limit mapping $\lim_{r\to 0} f^r$ is actually an extremal mapping in $QE(h; z_1, \cdots, z_m)$. These extremal mappings are the best possible gluing quasiconformal mappings. In particular, when $h = id$, the extremal mapping is unique, known as a Teichmüller mapping; moreover if $m = 1$, the extremal mapping is so-called Teichmüller shift mapping (see [6, 11]).

4. Gluing of quasiconformal mappings in the plane

Denote by $QC(z_1, \cdots, z_m; w_1, \cdots, w_m)$ the collection of all quasiconformal mappings from $\mathbb{C}$ to itself with the condition $f(z_k) = w_k,\ k = 1, \cdots, m$. Define

$$K(z_1, \cdots, z_m; w_1, \cdots, w_m) := \inf\{K(f)|\ f \in QC(z_1, \cdots, z_m; w_1, \cdots, w_m)\}.$$

Taking almost words by words from the proof of Theorem 1, we can get the version for gluing of quasiconformal mappings in the plane.

Theorem 2. *Let $\{z_k\}_{k=1}^m$ be a set of distinct points in $\mathbb{C}$ and U_k be a neighborhood of z_k for every $k = 1, 2, \cdots, m$. Suppose $\{U_k\}_{k=1}^m$ are pairwise disjoint and $f_k(z)$ is a $K-$quasiconformal mapping defined on U_k which sends z_k to $w_k \in \mathbb{C}$ for every $k = 1, 2, \cdots, m$. Then for every $\epsilon > 0$, there exists a number $r > 0$ and a quasiconformal self-homeomorphism f of $\mathbb{C}$ such that*
(1)

$$f|_{U(z_k,r)} = f_k|_{U(z_k,r)},$$

where $U(z_k, r)$ is the open disk of radius r centered at z_k for $k = 1, 2, \cdots, m$;
(2) $K(f) \leq max\{K(z_1, \cdots, z_m; w_1, \cdots, w_m), K\} + \epsilon$.

If $m \leq 2$, then $K(z_1, \cdots, z_m; w_1, \cdots, w_m) = 1$ and $K(f)$ can be less than $K + \epsilon$ in the theorem. In particular, the theorem is trivial when $m = 1$.

Use the denotations in Theorem 2. Let $QC(r; f_1, \cdots, f_m)$ be the set of quasiconformal self-homeomorphisms f of $\mathbb{C}$ satisfying the condition $f|_{U(z_k,r)} = f_k|_{U(z_k,r)}$. Choose an extremal mapping denoted by f^r from $QC(r; f_1, \cdots, f_m)$ such that

$$K(f^r) = K[r] := \inf\{K(f) : f \in QE(r; f_1, \cdots, f_m)\}.$$

Then $K[r]$ is an increasing function.

When $K(z_1, \cdots, z_m; w_1, \cdots, w_m) \geq K$, Theorem 2 implies

$$K[r] \to K(z_1, \cdots, z_m; w_1, \cdots, w_m), \text{ as } r \to 0,$$

and the limit mapping $\lim_{r\to 0} f^r$ is an extremal mapping in $QC(z_1, \cdots, z_m; w_1, \cdots, w_m)$ which is a Teichmüller mapping when $m \geq 3$. The associated holomorphic quadratic differential φdz^2 has simple poles at z_k, $k = 1, \cdots, m$. The extremal Teichmüller mapping is the best possible gluing quasiconformal mapping.

Theorem B is the main result, Theorem 1 of [4]. Here we give a new simple proof.

Proof of Theorem B. Since $w_k = z_k$, $k = 1, 2, \cdots, m$, we have

$$K(z_1, \cdots, z_m; w_1, \cdots, w_m) = 1.$$

The theorem follows from Theorem 2 immediately. We can also derive Theorem B from Theorem 1 in another way. Choose a sufficiently large circle T such that all U_k and $f_k(U_k)$ are encircled in T. Let $h : T \to T$ be the identity map id. Let the gluing mapping f be the identity map outside T. Since $K(id; z_1, \cdots, z_m; z_1, \cdots, z_m) = 1$, for small $r > 0$ the gluing quasiconformal mapping f inside T can be chosen so that $K(f) \leq K + \epsilon$ by Theorem 1.

Due to the condition that $f_k(z)$ fixes z_k, Jiang and Qi [4] can glue quasiconformal mapping locally on every $U_k(z_k, r)$ with an identity map outside and then obtain a required gluing quasiconformal mapping by compositions of these mappings. Once the condition is removed, it seems that their method no longer provides a best possible gluing mapping globally.

5. Gluing quasiconformal mappings along $\partial\Delta$

Using the same technique, we can obtain the following version for gluing quasiconformal mappings along $\partial\Delta$.

Theorem 3. *Let $\{z_k\}_{k=1}^m$ be a set of distinct points on $\partial\Delta$ and $U_k \subset \Delta$ be a neighborhood of z_k in Δ for every $k = 1, 2, \cdots, m$. Suppose $\{U_k\}_{k=1}^m$ are pairwise disjoint and $f_k(z)$ is a $K-$quasiconformal mapping defined on U_k which sends z_k to $w_k \in \partial\Delta$ for every $k = 1, 2, \cdots, m$. Suppose h is a quasisymmetric self-homeomorphism of $\partial\Delta$ satisfying $h = f_k$ on $\partial\Delta \cap \partial U_k$, $k = 1, 2, \cdots, m$. Then for every $\epsilon > 0$, there exists a number $r > 0$ and a quasiconformal self-homeomorphism f of Δ such that*
(1) $f|_{\partial\Delta} = h$;
(2)

$$f|_{U(z_k,r)} = f_k|_{U(z_k,r)},$$

where $U(z_k, r) = \Delta \cap \{z : |z - z_k| < r\}$, $k = 1, 2, \cdots, m$;
(3) $K(f) \leq \max\{K(h), K\} + \epsilon$.

As done in the previous sections, f^r can be chosen in the same way. Comparing the version with the previous two versions, one can find the conclusion (3) $K(f) \leq \max\{K(h), K\} + \epsilon$ is irrelevant to those points $\{z_k\}_{k=1}^m$, $\{w_k\}_{k=1}^m$. The reason is that every $U(z_k, r)$ is reduced to the boundary point z_k when $r \to 0$. When $K(h) \geq K$, the extremal mappings in $Q(h)$ are the best possible gluing quasiconformal mappings.

At last, it should be noted that the gluing theorem can even be obtained when the unit disk or the plane replaced by a more general domain. As a demonstration, we give a gluing theorem in a multiply-connected domain.

Suppose $\{X_k\}_{k=1}^m$ is a set of pairwise disjoint quasidisks in Δ and $\{Y_k\}_{k=1}^m$ is another set of pairwise disjoint quasidisks in Δ. Let $\{z_k\}_{k=1}^m$ be a set of distinct points in $\Delta\backslash\{\overline{X}_k\}_{k=1}^m$ and $\{w_k\}_{k=1}^m$ a set of distinct points in $\Delta\backslash\{\overline{Y}_k\}_{k=1}^m$. Suppose h is a quasisymmetric self-homeomorphism of $\partial\Delta$ and h_k is a quasisymmetric homeomorphism from ∂X_k to ∂Y_k for $k = 1, 2, \cdots, m$. Let $QE(h, h_1, \cdots, h_m; z_1, \cdots, z_m; w_1, \cdots, w_m)$ be the set of quasiconformal mappings from $\Delta\backslash\{X_k\}_{k=1}^m$ to $\Delta\backslash\{Y_k\}_{k=1}^m$ with the conditions (1) $f|_{\partial\Delta} = h$, (2) $f|_{\partial X_k} = h_k$, $f(z_k) = w_k$, for every $k = 1, 2, \cdots, m$. Define

$$\begin{aligned} &K(h, h_1 \cdots, h_m; z_1, \cdots, z_m; w_1, \cdots, w_m) = K_m(h) \\ &:= \inf\{K(g)|\ g \in QE(h, h_1 \cdots, h_m; z_1, \cdots, z_m; w_1, \cdots, w_m)\}. \end{aligned}$$

Theorem 4. *Let $\{z_k\}_{k=1}^m$ be a set of distinct points in $\Delta\backslash\{\overline{X}_k\}_{k=1}^m$ and $U_k \subset \Delta\backslash\{\overline{X}_k\}_{k=1}^m$ be a neighborhood of z_k for every $k = 1, 2, \cdots, m$. Suppose $\{U_k\}_{k=1}^m$ are pairwise disjoint and $f_k(z)$ is a $K-$quasiconformal mapping defined on U_k which sends z_k to $w_k \in \Delta\backslash\{\overline{Y}_k\}_{k=1}^m$ for every $k = 1, 2, \cdots, m$. Suppose h is a quasisymmetric self-homeomorphism of $\partial\Delta$ and h_k is a quasisymmetric homeomorphism from ∂X_k to ∂Y_k for $k = 1, 2, \cdots, m$. Then for every $\epsilon > 0$, there exists a number $r > 0$ and a quasiconformal self-homeomorphism f from $\Delta\backslash\{X_k\}_{k=1}^m$ to $\Delta\backslash\{Y_k\}_{k=1}^m$ such that*
(1) $f \in QE(h, h_1, \cdots, h_m; z_1, \cdots, z_m; w_1, \cdots, w_m)$;
(2)

$$f|_{U(z_k,r)} = f_k|_{U(z_k,r)},$$

where $U(z_k, r)$ is the open disk of radius r centered at z_k for $k = 1, 2, \cdots, m$;
(3) $K(f) \le \max\{K_m(h), K\} + \epsilon$.

When $K_m(h) \ge K$, the best possible gluing quasiconformal mappings are those extremal mappings in $QE(h, h_1 \cdots, h_m; z_1, \cdots, z_m; w_1, \cdots, w_m)$.

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