On nonuniqueness of geodesic disks in infinite-dimensional Teichmüller spaces

GUOWU YAO

Dedicated to Professor Zhong Li on his 80th birthday

Abstract

Let $T(S)$ be the Teichmüller space over a hyperbolic Riemann surface S. A geodesic disk in $T(S)$ is defined the image of an isometric embedding of the Poincaré disk into $T(S)$. In this paper, it is shown that for any non-Strebel point $\tau \in T(S)\backslash\{[0]\}$, there are infinitely many geodesic disks containing the straight line $\{[t\mu] : t \in (-1/k, 1/k)\}$ where μ is an extremal representative of τ with $\|\mu\|_\infty = k$. An infinitesimal version is also obtained.

1. Introduction

Let S be a hyperbolic Riemann surface, that is, it is covered by a holomorphic map: $\pi : \mathbb{D} \to S$, where $\mathbb{D} = \{|z| < 1\}$ is the open unit disk. Then S can be expressed as a quotient space $\mathbb{D}/\Gamma$, where Γ is a Fuchsian group acting on $\mathbb{D}$. Denote by $Bel(S)$ the Banach space of Beltrami differentials $\mu = \mu(z)d\bar{z}/dz$ on S with finite L^∞-norm and by $M(S)$ the open unit ball in $Bel(S)$.

For each element $\mu \in M(S)$ there exists a Riemann surface S^μ and a quasiconformal mapping $f^\mu : S \to S^\mu$ such that the complex dilatation of f^μ is μ. We denote by $\widetilde{f}^\mu : \mathbb{D} \to \mathbb{D}$ the lift of f^μ with the points 1, i and -1 fixed. Then f^μ is uniquely determined by μ. Two elements μ_1 and μ_2 are said to be equivalent if and only if $\widetilde{f}^{\mu_1}|_{\partial\mathbb{D}} = \widetilde{f}^{\mu_2}|_{\partial\mathbb{D}}$. The Teichmüller space $T(S)$ of S is defined as the quotient space of $M(S)$ under the equivalence relation. The point $[0]$, a Teichmüller equivalence class of the trivial Beltrami differential $\mu = 0$, is called the basepoint of $T(S)$. The Teichmüller metric between two points τ and σ is defined as follows:

$$d(\tau, \sigma) = \inf_{\mu\in\tau,\ \nu\in\sigma} \log \frac{1 + \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty}{1 - \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty},$$

Define

$$k_0(\tau) = \inf\{\|\mu\|_\infty : \mu \in \tau\}.$$

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We say that μ is extremal in τ if $\|\mu\|_\infty = k_0(\tau)$ (accordingly, f^μ is called extremal), uniquely extremal if $\|\nu\|_\infty > k_0([\mu])$ for any other $\nu \in [\mu]$. We call that a Beltrami differential μ is of constant modulus if $|\mu|$ is a constant a.e. on S.

For any μ, define $h^*(\mu)$ to be the infimum over all compact subsets E contained in R of the essential supremum norm of the Beltrami differential $\mu(z)$ as z varies over $S\backslash E$. Define $h(\tau)$ to be the infimum of $h^*(\mu)$ taken over all representatives ν of the class τ. It is obvious that $h(\tau) \leq k_0(\tau)$. τ is called a Strebel point if $h(\tau) < k_0(\tau)$; otherwise, τ is called a non-Strebel point.

When S is a compact Riemann surface, or generally speaking, when S is of finite analytic type, $T(S)$ is a finite-dimensional complex manifold. Otherwise, it is infinite-dimensional.

The Poincaré distance $d_{\mathbb{D}}(\cdot,\cdot)$ on $\mathbb{D}$ is defined by

$$d_{\mathbb{D}}(t_1,t_2) = \log\frac{1+|(t_1-t_2)/(1-\bar{t}_1t_2)|}{1-|(t_1-t_2)/(1-\bar{t}_1t_2)|},\ t_1,\ t_2 \in \mathbb{D}.$$

We recall some geometric terminologies adapted from [1] by Busemann. Let X and Y be metric spaces. An isometry of X into Y is a distance preserving map. A straight line in Y is a (necessarily closed) subset L that is an isometric image of the real line $\mathbb{R}$. A geodesic in Y is an isometric image of a nontrivial compact interval of $\mathbb{R}$. Its endpoints are the images of the endpoints of the interval, and we say that the geodesic joins its endpoints.

Geodesics play an important role in the geometry of Teichmüller spaces. Unlike in a finite-dimensional Teichmüller space, the geodesic geometry is much more complicated in an infinite-dimensional Teichmüller space. It is possible that there are infinitely many geodesics passing through two points in an infinite-dimensional Teichmüller space.

A geodesic disk is the image of a map $\Psi : \mathbb{D} \hookrightarrow T(S)$ which is an isometric embedding with respect to the Poincaré metric on $\mathbb{D}$ and the Teichmüller metric on $T(S)$. If Ψ is also holomorphic, we call such a geodesic disk to be a holomorphic geodesic disk.

For any point $\tau \neq [0]$, let $\mu \in \tau$ be an extremal differential and $\|\mu\|_\infty = k < 1$ unless otherwise specialized. Then the embedding

$$\begin{aligned}\Psi_\mu : \mathbb{D} &\hookrightarrow T(S),\\ t &\longmapsto [t\mu/k],\end{aligned}$$

is a holomorphic isometry and hence there is at least a holomorphic geodesic disk containing $[0]$ and τ. Let $G[\mu]$ denote the standard geodesic disk $\Psi_\mu(\mathbb{D})$.

In [2], Earle, Kra and Krushkal proved that if τ contains an extremal differential of nonconstant modulus, then there are infinitely many holomorphic isometries and holomorphic geodesic disks with the above properties. By a lengthy computation, Li [8] proved the following theorem.

Theorem A. *Let $\tau \neq [0]$ be a non-Strebel point in $T(S)$. Then there are infinitely many isometries $\Psi : \mathbb{D} \hookrightarrow T(S)$ with $\Psi(0) = [0]$ and $\Psi(k_0(\tau)) = \tau$.*

It is known [9, 10] that τ contains an extremal differential of nonconstant modulus unless μ is uniquely extremal and with constant modulus. Therefore, if the extremal in the non-Streble point τ is not unique, then Theorem A is covered by the result in [2]. In particular, when μ is a unique extremal with constant modulus, by Theorem 6 in [2] Ψ_μ is the unique holomorphic isometry with $\Psi(0) = [0]$ and $\Psi(k_0([\mu])) = [\mu]$, in other words, there is a unique holomorphic geodesic disk containing [0] and $[\mu]$. In such a case, Theorem A has its independent interest because it says that there are still infinitely many isometries Ψ with $\Psi(0) = [0]$ and $\Psi(k_0([\mu])) = [\mu]$.

Li's result tells that there are infinitely many geodesic disks containing a non-Strebel point and the basepoint. The motivation of this paper is to construct different geodesic disks such that they contain a fixed straight line. Before stating the main result, we introduce Li's construction in [8] as follows.

Suppose E is a compact subset of S. Define the Beltrami differential $\mu_t(z)$ with a parameter $t \in \mathbb{D}$ by the following equations.

$$\mu_t(z) := \begin{cases} \mu(z)/k, & as\ z \in S,\ |t| \leq k, \\ \mu(z)/k, & as\ z \in S\backslash E,\ |t| > k, \\ \mu(z)/|t|, & as\ z \in E,\ |t| > k. \end{cases} \tag{1.1}$$

Li proved that the map

$$\begin{aligned} \Psi_E : \mathbb{D} &\hookrightarrow T(S), \\ t &\longmapsto [t\mu_t], \end{aligned}$$

is an isometry where Ψ_E is regarded as a map with respect to E. Let $\Delta_k = \{t \in \mathbb{D} : |t| < k\}$. One might observe the phenomenon that $\Psi_\mu = \Psi_E$ on Δ_k and all geodesic disks share a common small disk $\Psi_\mu(\Delta_k)$ and that the straight line $L_\mu := \{[t\mu] : t \in (-1/k, 1/k)\}$ is not contained in the geodesic disks $\Psi_E(\mathbb{D})$ generally.

If μ is a unique extremal of constant modulus, the geodesic connecting [0] with τ is unique [2, 7] and all the geodesic disks pass through the unique geodesic. Perhaps one expects that the phenomenon emerging in Li's construction is subject to an inherent rigidity and hence there should be a unique geodesic disk containing the straight line $L_\mu := \{[t\mu] : t \in (-1/k, 1/k)\}$. However, the following theorem shows that the expectation is not true.

Theorem 1. *Let $\tau = [\mu] \neq [0]$ be a non-Strebel point in $T(S)$. Then there are infinitely many geodesic disks containing the straight line $L_\mu := \{[t\mu] : t \in (-1/k, 1/k)\}$.*

Theorem 1 is proved in Section 3. Our construction is self-contained and more directly than Li's. To make the demonstration more readable, we prefer to proving the infinitesimal version of Theorem 1 in advance in Section 2.

2 . Infinitesimal version of Theorem 1

We need to introduce some basic concepts at first. The cotangent space to $T(S)$ at the basepoint is the Banach space $Q(S)$ of integrable holomorphic quadratic differentials on S with L^1-norm

$$\|\varphi\| = \iint_S |\varphi(z)|\, dxdy < \infty. \tag{2.1}$$

The dimension of $Q(S)$ is finite if and only if that of $T(S)$ is finite. In what follows, let $Q^1(S)$ denote the unit sphere of $Q(S)$.

Two Beltrami differentials μ and ν in $Bel(S)$ are said to be infinitesimally equivalent if

$$\iint_S \mu\varphi\, dxdy = \iint_S \nu\varphi\, dxdy, \text{ for any } \varphi \in Q(S).$$

The tangent space $B(S)$ of $T(S)$ at the basepoint is defined as the set of the quotient space of $Bel(S)$ under the equivalence relation. Denote by $[\mu]_B$ the equivalence class of μ in $B(S)$. The set of all Beltrami differentials equivalent to zero is called the $N-$class in $Bel(S)$.

$B(S)$ is a Banach space with its standard sup norm

$$\|[\mu]_B\| = \|\mu\| := \sup_{\varphi\in Q^1(S)} \left|\iint_S \mu\varphi\, dxdy\right|$$

and infinitesimal metric

$$\begin{aligned} &d([\mu]_B, [\nu]_B) := \|\mu - \nu\| \\ &= \sup_{\varphi\in Q^1(S)} |\iint_S (\mu - \nu)\varphi\, dxdy|, \ [\mu]_B, \ [\nu]_B \in B(S). \end{aligned}$$

We say that μ is infinitesimally extremal (in $[\mu]_B$) if $\|\mu\|_\infty = \|\mu\|$, infinitesimally uniquely extremal if $\|\nu\|_\infty > \|\mu\|$ for any other $\nu \in [\mu]_B$.

In a parallel manner we can define the boundary dilatation for the infinitesimal $[\mu]_B$. The boundary dilatation $b([\mu]_B)$ is the infimum over all elements in the equivalence class $[\mu]_B$ of the quantity $b^*(\nu)$. Here $b^*(\nu)$ is the infimum over all compact subsets E contained in S of the essential supremum of the Beltrami differential ν as z varies over $S - E$. It is obvious that $b^*(\mu) \le \|\mu\|$. $[\mu]_B$ is called a Strebel point if $b([\mu]_B) < \|\mu\|$.

As is well known, μ is extremal if and only if it has a so-called Hamilton sequence, namely, a sequence $\{\phi_n \in Q^1(S) : \ n \in \mathbb{N}\}$, such that

$$\lim_{n\to\infty} \iint_S \mu\phi_n(z)dxdy = \|\mu\|_\infty. \tag{2.2}$$

$\{\phi_n\}$ is called degenerating if it converges to 0 uniformly on compact subset of S. In particular, $[\mu]$ (or $[\mu]_B$) is a non-Strebel point if and only if μ has a degenerating Hamilton sequence (cf. [3]).

A geodesic plane in $B(S)$ is the image of a map $\Upsilon : \mathbb{C} \hookrightarrow B(S)$ which is an isometric embedding with respect to the Euclidean metric on $\mathbb{C}$ and the infinitesimal metric on $B(S)$. Then the embedding

$$\begin{aligned} \Upsilon_\mu : \mathbb{C} &\hookrightarrow B(S), \\ t &\longmapsto [t\mu/k]_B, \end{aligned}$$

is a holomorphic isometry.

It is convenient to prescribe μ an extremal with $k = \|\mu\|_\infty = 1$ in this section. The following theorem is the counterpart of Theorem 1 in the infinitesimal setting.

Theorem 2. *Let $[\mu]_B \neq [0]_B$ be a non-Strebel point in $B(S)$. Then there are infinitely many geodesic planes containing the straight line $l_\mu := \{[t\mu]_B : \ t \in \mathbb{R}\}$.*

Proof. Let E be a non-empty compact subset of S and $p(t,z):\mathbb{C}\times S\to\mathbb{C}$ be the function related to E with the following properties:
(1) $|p(t_1,z)-p(t_2,z)|\le|t_1-t_2|$, (t_1,z), $(t_2,z)\in\mathbb{C}\times S$,
(2) $p(t,z)=t$, $(t,z)\in\mathbb{C}\times(S\backslash E)$,
(3) $p(t,z)=t$, $t\in\mathbb{R}$.

$p(t,z)$ is clearly continuous with respect to $t\in\mathbb{C}$. Denote by F_E the function space of all $p(t,z)$ with the above properties over $\mathbb{C}\times S$. F_E is not void since F_E contains at least two elements. One is $I(t,z)=t$ for $(t,z)\in\mathbb{C}\times S$. The other is

$$\widetilde{I}(t,z)=\begin{cases}\bar{t}, & z\in E,\\ t, & z\in S\backslash E.\end{cases}$$

On the one hand, it is evident that F_E is convex. That is, given $p_1(t,z)$ and $p_2(t,z)$ in F_E, then $ap_1(t,z)+(1-a)p_2(t,z)\in F_E$ for any $a\in(0,1)$. On the other hand, it is easy to verify that F_E has the associative law, i.e., $p_2\circ p_1(t,z):=p_2(p_1(t,z),z)\in F_E$. Thus, F_E has sufficiently many elements by the convex combination and associative operation.

Now we construct geodesic planes. Given $p(t,z)\in F_E$, let $\mu_t(z)=p(t,z)\mu(z)$ for $t\in\mathbb{C}$. Define

$$\begin{aligned}\Upsilon_{p,E}:\mathbb{C}&\hookrightarrow B(S),\\ t&\longmapsto[\mu_t]_B.\end{aligned}$$

Then $\Upsilon_{p,E}$ is an isometric embedding. We need to check the equality:

$$d([\mu_t]_B,[\mu_s]_B)=|t-s|,\ t,s\in\mathbb{C}. \tag{2.3}$$

Since μ has a degenerating Hamilton sequence $\{\phi_n\}$, we have

$$\begin{aligned}&\lim_{n\to\infty}\iint_S(\mu_t-\mu_s)e^{i\beta}\phi_n(z)dxdy=\lim_{n\to\infty}\iint_{S\backslash E}(t-s)\mu(z)e^{i\beta}\phi_n(z)dxdy\\ &=\lim_{n\to\infty}\iint_S(t-s)\mu(z)e^{i\beta}\phi_n(z)dxdy=|t-s|,\end{aligned}$$

where $\beta=-\arg(t-s)$ when $t\neq s$. On the other hand, the equality $\|\mu_t-\mu_s\|_\infty=|t-s|$ follows from the properties (1)$\sim$(3) readily. Thus, $\{e^{i\beta}\phi_n\}$ is a Hamilton sequence for $\mu_t-\mu_s$ and (2.3) follows immediately.

The geodesic planes $\Upsilon_{p,E}(\mathbb{C})$ vary over two parameters $E\subset\mathbb{D}$ and $p\in F_E$, and all of them contain the straight line l_μ by the property (3) of $p(t,z)$.

We now show that there are infinitely many geodesic planes containing the line l_μ when E varies. Actually, let

$$p(t,z)=aI(t,z)+(1-a)\widetilde{I}(t,z),\ a\in(0,1),$$

then $p(t,z)\in F_E$. In particular, let $a=\frac{1}{2}$, we have

$$p(t,z)=\frac{I(t,z)+\widetilde{I}(t,z)}{2}=\begin{cases}Re(t), & z\in E,\\ t, & z\in S\backslash E.\end{cases}$$

and

$$\mu_t(z) = \begin{cases} Re(t)\mu(z), & z \in E, \\ t\mu(z), & z \in S\backslash E. \end{cases}$$

Fix $t_0 = \lambda i$ where $\lambda > 0$. We have $\mu_{t_0}(z) = Re(t_0)\mu(z) \equiv 0$ for $z \in E$. Take a sequence $\{E_n : n \in \mathbb{N}\}$ of compact subsets of S, such that $E_n \subset E_{n+1}$ $(n \in \mathbb{N})$ and $S = \bigcup_{n=0}^{\infty} E_n$. For every E_n, let $\mu_{t_0}^n(z)$ denote the Beltrami differential $p(t_0, z)\mu(z)$.

It remains to show that $\{[\mu_{t_0}^n]_B : n \in \mathbb{N}\}$ contains infinitely many elements. Notice that

$$\mu_{t_0}^n(z) = \begin{cases} 0, & z \in E_n, \\ t_0\mu(z), & z \in S\backslash E_n. \end{cases}$$

Let $n_0 = 0$. By a simple argument, there is a sufficiently large n_1, such that $[\mu_{t_0}^{n_0}]_B \neq [\mu_{t_0}^{n_1}]_B$. Furthermore, a similar argument yields a number n_2, such that $[\mu_{t_0}^{n_2}]_B \neq [\mu_{t_0}^{n_j}]_B$ $(j = 0, 1)$. Repeating the same argument, we can get a sequence $\{n_j\}$, such that $[\mu_{t_0}^{n_j}]_B \neq [\mu_{t_0}^{n_m}]_B$ whenever $j \neq m$. The proof of Theorem 2 is completed. □

3. Proof of Theorem 1

We modify some technique used in the last section to prove Theorem 1. Let E be a non-empty compact subset of S and $g(t, z) : \mathbb{D} \times S \to \mathbb{D}$ be the function related to E with the following properties:
(1) $d_{\mathbb{D}}(g(t_1, z), g(t_2, z)) \leq d_{\mathbb{D}}(t_1, t_2),\ (t_1, z),\ (t_2, z) \in \mathbb{D} \times S$,
(2) $g(t, z) = t,\ (t, z) \in \mathbb{D} \times (S\backslash E)$,
(3) $g(t, z) = t,\ t \in (-1, 1)$.

$g(t, z)$ is clearly continuous with respect to $t \in \mathbb{D}$. Denote by G_E the function space of all $g(t, z)$ with the above properties over $\mathbb{D} \times S$. G_E contains at least two elements. One is $I(t, z) = t$ for $(t, z) \in \mathbb{D} \times S$. The other is

$$\widetilde{I}(t, z) = \begin{cases} \bar{t}, & z \in E, \\ t, & z \in S\backslash E. \end{cases}$$

Certainly, observing the equation (1.1) and Li's proof in [8], one can check that the function

$$L(t, z) = \begin{cases} \frac{t}{k}, & z \in S,\ |t| \leq k, \\ \frac{t}{k}, & z \in S\backslash E,\ |t| > k, \\ \frac{t}{|t|}, & z \in E,\ |t| > k, \end{cases}$$

has the property (1) while it does not belong to G_E.

Claim 1: G_E is convex in the Euclidean sense. Precisely, given $g_1(t, z)$ and $g_2(t, z)$ in G_E, then $ag_1(t, z) + (1 - a)g_2(t, z) \in G_E$ for any $a \in (0, 1)$.

On the one hand, it is evident that $ag_1(t, z) + (1 - a)g_2(t, z)$ has the properties (2) and (3). On the other hand, by Lemma 6.4 on page 75 in [5], $ag_1(t, z) + (1 - a)g_2(t, z)$ has the property (1). The gives Claim 1.

Given two points $t, s \in \mathbb{D}$, denote by $< t, s >$ the hyperbolic geodesic segment in $\mathbb{D}$. For any $a \in [0,1]$, there exists exactly one point r in $< t, s >$ such that $d_{\mathbb{D}}(t,r) = (1-a)d_{\mathbb{D}}(t,s)$ and $d_{\mathbb{D}}(r,s) = ad_{\mathbb{D}}(t,s)$. By an analogy with the standard convex combination we write $r = at \oplus (1-a)s$ in the hyperbolic sense. In particular, $\frac{1}{2}t \oplus \frac{1}{2}s$ is called the hyperbolic metric center of $< t, s >$.

Claim 2: G_E is convex in the hyperbolic sense. Precisely, given $g_1(t,z)$ and $g_2(t,z)$ in G_E, then $ag_1(t,z) \oplus (1-a)g_2(t,z) \in G_E$ for any $a \in (0,1)$.

At first, it is evident that $ag_1(t,z) \oplus (1-a)g_2(t,z)$ has the properties (2) and (3). Secondly, by Lemma 6.8 in [5], we see that $ag_1(t,z) \oplus (1-a)g_2(t,z)$ has the property (1). Claim 2 is proved.

Claim 3: G_E has the associative law, i.e., $g_2 \circ g_1(t,z) := g_2(g_1(t,z),z) \in G_E$.

The claim follows from a simple computation.

Thus, G_E has sufficiently many elements by the associative operation and two kinds of convex combination as described in Claims 1 and 2.

Now we construct geodesic disks. For brevity, let $\alpha(z) = \mu(z)/k$ and then $\|\alpha\|_\infty = 1$. Given $g(t,z) \in G_E$, let $\alpha_t(z) = g(t,z)\alpha(z)$ for $t \in \mathbb{D}$. Define

$$\begin{aligned} \Psi_{g,E} : \mathbb{D} &\hookrightarrow T(S), \\ t &\longmapsto [\alpha_t]. \end{aligned}$$

We show that $\Psi_{g,E}$ is an isometric embedding. It suffices to check the equality:

$$d([\alpha_t],[\alpha_s]) = d_{\mathbb{D}}(t,s),\ t,s \in \mathbb{D}. \tag{3.1}$$

Let $f_t : S \to S^{\alpha_t}$ and $f_s : S \to S^{\alpha_s}$ be quasiconformal mappings with Beltrami differentials α_t and α_s respectively. It is convenient to assume that $s \neq 0$ and $t \neq s$. Set $f = f_t \circ f_s^{-1}$ and assume that the Beltrami differential of f is ν. Then a simple computation shows,

$$\nu \circ f_s(z) = \frac{1}{\tau}\frac{\alpha_t(z) - \alpha_s(z)}{1 - \overline{\alpha_s(z)}\alpha_t(z)} = \frac{1}{\tau}\frac{g(t,z) - g(s,z)}{1 - \overline{g(s,z)}g(t,z)}\alpha(z),$$

where $z = f_s^{-1}(w)$ for $w \in S^{\alpha_s}$ and $\tau = \overline{\partial f_s}/\partial f_s$.

At first, by the property (1) we see that on $f_s(S)$,

$$\|\nu\|_\infty \leq \left|\frac{t-s}{1-st}\right|, \tag{3.2}$$

Furthermore, the equality $\|\nu\|_\infty = \left|\frac{t-s}{1-\bar{s}t}\right|$ follows from the property (2).

Since μ has a degenerating Hamilton sequence $\{\phi_n\}$, α_s is necessarily extremal due to the following equality:

$$\begin{aligned} &\lim_{n\to\infty} \iint_S \alpha_s e^{i\arg s}\phi_n(z)dxdy = \lim_{n\to\infty} \iint_{S\setminus E} \alpha_s e^{i\arg s}\phi_n(z)dxdy \\ &= \lim_{n\to\infty} \iint_{S\setminus E} s\mu e^{i\arg s}\phi_n(z)dxdy = \lim_{n\to\infty} \iint_S s\mu e^{i\arg s}\phi_n(z)dxdy = |s|. \end{aligned}$$

Thus f_s is extremal on S as well as f_s^{-1} extremal on $S^{\alpha_s} = f_s(S)$. Therefore, the Beltrami differential $\widetilde{\alpha}_s = -\alpha_s(f_s^{-1})\overline{\partial f_s^{-1}}/\partial f_s^{-1}$ of f_s^{-1} is extremal with $\|\widetilde{\alpha}_s\| = |s|$. Moreover, since $[\widetilde{\alpha}_s]$ is also a non-Strebel point in $T(S^{\alpha_s})$, it has a degenerating Hamilton sequence $\{\widetilde{\phi}_n\}$ in $Q^1(S^{\alpha_s})$ such that

$$\lim_{n\to\infty} \iint_{S^{\alpha_s}} \widetilde{\alpha}_s \widetilde{\phi}_n(w) dudv = |s|.$$

Furthermore, due to the degenerateness of $\{\widetilde{\phi}_n\}$, it is easy to verify

$$\lim_{n\to\infty} \iint_{S^{\alpha_s}} \nu(w) e^{i\theta} \widetilde{\phi}_n(w) dudv = \lim_{n\to\infty} \iint_{S^{\alpha_s}\backslash f_s(E)} \nu(w) e^{i\theta} \widetilde{\phi}_n(w) dudv$$
$$= \lim_{n\to\infty} \iint_{S^{\alpha_s}\backslash f_s(E)} \frac{(t-s)\widetilde{\alpha}_s(w)/s}{1-\bar{s}t|\widetilde{\alpha}_s(w)|^2/|s|^2} e^{i\theta}\widetilde{\phi}_n(w) dudv = \left|\frac{t-s}{1-\bar{s}t}\right|,$$

where $\theta = \arg s - \arg \frac{t-s}{1-\bar{s}t}$. Thus, $e^{i\theta}\widetilde{\phi}_n$ is a Hamilton sequence for ν and (3.1) follows immediately.

The geodesic disks $\Psi_{g,E}(\mathbb{D})$ vary over two parameters $E \subset \mathbb{D}$ and $g \in G_E$, and all of them contain the straight line L_μ due to the property (3) of $g(t,z)$. By the same technique as used in last section, it derives that there are infinitely many geodesic disks containing the line L_μ when E varies. This completes the proof of Theorem 1.

Generally, if $\mu(z)$ is not equivalent to 0 on E, $\Psi_{g,E}(\mathbb{D})$ share no small disk when g varies over G_E. To see this, let

$$g(t,z) = aI(t,z) \oplus (1-a)\widetilde{I}(t,z),\ a \in (0,1).$$

Then $g(t,z) \in G_E$. In particular, when $a = \frac{1}{2}$,

$$\alpha_t(z) = g(t,z)\alpha(z) = \begin{cases} \frac{2Re(t)\alpha(z)}{|t|^2+1+\sqrt{(|t|^2+1)^2-4(Re(t))^2}}, & z \in E, \\ t\alpha(z), & z \in S\backslash E. \end{cases}$$

It is easy to see that $\alpha_t(z) \equiv 0$ on E when $Re(t) = 0$.

Remark. *In the construction, the set E can be chosen to be non-compact on the condition that $\partial S \bigcap (S\backslash E)$ has a substantial boundary point of τ. For more knowledge of substantial boundary points, the reader may refer to [4, 6].*

4. Two problems

We call a straight line in the form $\{[t\mu] : t \in (-1/k, 1/k)\}$ to be an amenable straight line. Theorem 1 indicates that, the geodesic disk passing through an amenable straight line with non-Strebel points can not be unique. A straight line might not be in the amenable form $\{[t\mu] : t \in (-1/k, 1/k)\}$. Anyway, a geodesic can be embedded into a straight line. However, we do not know the answer to the following problem.

Problem 1. *Is there always a geodesic disk passing through a given geodesic?*

By Theorem 6 in [2], it is possible that there is no holomorphic geodesic disks passing through certain geodesics.

So far, we know little information on the geodesic disk containing a Strebel point and the basepoint. The paper is concluded by the following problem.

Problem 2. *Suppose* $[\mu]$ *is a Strebel point. Is the standard geodesic disk* $G[\mu]$ *the unique candidate containing the straight line* L_μ*?*

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Guowu Yao
Department of Mathematical Sciences
Tsinghua University
Beijing, 100084, People's Republic of China
E-mail: `gwyao@math.tsinghua.edu.cn`