# UPPER ESTIMATES OF HEAT KERNELS FOR NON-LOCAL DIRICHLET FORMS ON DOUBLING SPACES

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Abstract. In this paper, we present a new approach to obtaining the off-diagonal upper estimate of the heat kernel for any regular Dirichlet form without a killing part on the doubling space. One of the novelties is that we have obtained the weighted $L^2$-norm estimate of the survival function $1 - P_t^B 1_B$ for any metric ball $B$, which yields a nice tail estimate of the heat semigroup associated with the Dirichlet form. The parabolic $L^2$ mean-value inequality is borrowed to use.

Contents



## 1. Main results

In recent years, there has been a lot of literature devoted to the study of heat kernel estimates, see for example, Barlow and Perkins for the Sierpiński gasket [5], Fitzsimmons, Hambly and Kumagai [15] for affine nested fractals, Hambly and Kumagai [27] (see also [32]) for post-critically finite self-similar sets, Barlow and Bass [2], [3] for the Sierpiński carpets, and Kigami [30, 31] for a certain class of self-similar sets. Equivalent conditions for two-sided estimates of heat kernels for local Dirichlet forms on metric measure spaces are given by Grigor'yan, Hu, Lau [25], by Grigor'yan and Telcs [26], whilst for non-local Dirichlet forms by Bass and Levin [6], by Chen and Kumagai [9], [10], and by Chen, Kumagai and Wang [11], [12], and by Grigor'yan, Hu and Hu [19]. Equivalent conditions only for upper estimates of heat kernels for local Dirichlet forms are given by Andres, Barlow [1], by Grigor'yan, Hu [21], [22] (see also [23, Section 6], [25, Section 9] by Grigor'yan, Hu, and Lau), and by Murugan and Saloff-Coste [34], whilst for non-local Dirichlet froms, for example by Carlen, Kusuoka, and Stroock [7], and by Grigor'yan, Hu and Lau [24].

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In this paper, we are concerned with the heat kernel upper estimate for a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ without killing part on a metric space equipped with a doubling measure. Let $(M,d,\mu)$ be a metric measure space, that is, $(M,d)$ is a locally compact, separable metric space and $\mu$ is a Radon measure with full support. Assume that any open metric ball

$$B(x,r) := \{y \in M : d(y,x) < r\} \tag{1.1}$$

is *precompact*. Denote by $\lambda B := B(x,\lambda r)$. Assume that $\mu$ is *doubling* (termed *condition* (VD)), that is, there exists a constant $C \geq 1$ such that, for any $x \in M$ and any $r > 0$,

$$V(x,2r) \leq CV(x,r), \tag{1.2}$$

where $V(x,r) := \mu(B(x,r))$ is the *volume* of the open ball $B(x,r)$. For simplicity, we call the above triple $(M,d,\mu)$ a *doubling* space. Let $\overline{R}$ be the diameter of $(M,d)$, that is,

$$\overline{R} := \sup\{d(x,y) : \text{for any } x,y \text{ in } M\}.$$

The metric space $(M,d)$ considered in this paper may be bounded or unbounded so that $\overline{R} < \infty$ or $\overline{R} = \infty$.

It is known that if $\mu$ is doubling, then there exists a constant $\alpha > 0$ such that, for any $x,y \in M$ and any $0 < r \leq R < \infty$,

$$\frac{V(x,R)}{V(y,r)} \leq C\left(\frac{d(x,y)+R}{r}\right)^{\alpha}, \tag{1.3}$$

see for example [22, Proposition 5.1]. In particular, for any $x \in M$ and any $0 < r \leq R < \infty$,

$$\frac{V(x,R)}{V(x,r)} \leq C\left(\frac{R}{r}\right)^{\alpha}. \tag{1.4}$$

We say that the *reverse volume doubling* condition (RVD) holds if there exist two constants $C \geq 1$, $\alpha_1 > 0$ such that, for any $x \in M$ and any $0 < r \leq R < \overline{R}$,

$$\frac{V(x,R)}{V(x,r)} \geq C^{-1}\left(\frac{R}{r}\right)^{\alpha_1}. \tag{1.5}$$

It is known that if $(M,d)$ is connected and unbounded, then condition $(VD)$ implies condition (RVD), see for example [22, Proposition 5.2].

Let $\psi : [0,\infty) \to [0,\infty)$ be a continuous, strictly increasing function with $\psi(0) = 0$, $\psi(\infty) = \infty$. Assume that there exist some constants $C \geq 1$, $0 < \beta_1 \leq \beta_2 < \infty$ such that for any $0 < r \leq R < \infty$

$$C^{-1}\left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\psi(R)}{\psi(r)} \leq C\left(\frac{R}{r}\right)^{\beta_2}. \tag{1.6}$$

Clearly, condition (1.6) is equivalent to the following

$$C^{-1}\left(\frac{R}{r}\right)^{\frac{1}{\beta_2}} \leq \frac{\psi^{-1}(R)}{\psi^{-1}(r)} \leq C\left(\frac{R}{r}\right)^{\frac{1}{\beta_1}} \quad \text{for any } 0 < r \leq R < \infty. \tag{1.7}$$

The function $\psi$ is closely related with the *walk dimension* of a process on $M$. The typical example is

$$\psi(r) = r^{\beta}$$

for some $\beta > 0$. For instance, if $M = \mathbb{R}^n$ then $\beta = 2$, which characterizes that a Brownian motion in $\mathbb{R}^n$ has the density function satisfying the *Gaussian* estimate. If $M$ is the Sierpínski gasket in $\mathbb{R}^2$, then $\beta = \frac{\log 5}{\log 2} > 2$, which characterizes that a Brownian motion on the Sierpínski gasket has the density function satisfying the *sub-Gaussian* estimate.

Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2$. For a non-empty open $\Omega \subset M$, let $\mathcal{F}(\Omega)$ be the *closure* of $\mathcal{F} \cap C_0(\Omega)$ in the norm of $\mathcal{F}$, where $C_0(\Omega)$ is the space of all continuous functions supported in $\Omega$. It is known that if $(\mathcal{E},\mathcal{F})$ is regular, then $(\mathcal{E},\mathcal{F}(\Omega))$ is also a regular Dirichlet form in $L^2(\Omega,\mu)$. Denote by $\{P_t^{\Omega}\}_{t>0}$ the heat semigroup of $(\mathcal{E},\mathcal{F}(\Omega))$. In particular, set $\{P_t := P_t^{\Omega}\}_{t>0}$ when $\Omega = M$.

Let $U$ be any non-empty Borel subset of an open subset $\Omega$ of $M$ with $U \Subset \Omega$. For a number $\kappa \geq 1$, a function $\phi$ is called a *$\kappa$-cutoff function* of the pair $(U,\Omega)$ if $\phi \in \mathcal{F}$ and

$$0 \leq \phi \leq \kappa \text{ in } M,\ \ \phi \geq 1 \text{ in } U,\ \ \phi = 0 \text{ in } \Omega^c.$$

We denote by $\kappa$-cutoff$(U,\Omega)$ the collection of all $\kappa$-cutoff functions of the pair $(U,\Omega)$.

Any 1-cutoff function will be simply called a *cutoff function*, that is, a function $\phi \in \mathcal{F}$ such that $0 \le \phi \le 1$ in $M$, $\phi = 1$ in $U$ and $\phi = 0$ in $\Omega^c$. Denote by

$$\text{cutoff}(U, \Omega) := 1\text{-}\,\text{cutoff}(U, \Omega).$$

For any $\phi \in \kappa$-cutoff$(U, \Omega)$, we have $1 \wedge \phi \in \text{cutoff}(U, \Omega)$ by using the Markov property of $(\mathcal{E}, \mathcal{F})$.

Let $\mathcal{F}'$ be a vector space defined by

$$\mathcal{F}' = \{u + c : u \in \mathcal{F}, c \ \in \mathbb{R}\}$$

so that this space contains constants.

We list all the hypotheses to be used in this paper.

- *Condition* (C): We say that condition (C) holds if the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *conservative*, that is, if
$$P_t 1 = 1 \text{ in } M \text{ for each } t > 0. \tag{1.8}$$
- *Condition* (FK$_\nu$): There exist three constants $C \ge 1$, $\nu > 0$ and $\delta \in (0, \frac{1}{3}]$ such that, for any ball $B$ with radius $0 < R < \delta \overline{R}$ and any non-empty open subset $\Omega$ of $B$,
$$\lambda_1(\Omega) \ge \frac{C^{-1}}{\psi(R)} \left(\frac{\mu(B)}{\mu(\Omega)}\right)^\nu, \tag{1.9}$$
where $\lambda_1(U)$ is the bottom eigenvalue defined by
$$\lambda_1(U) = \inf_{u \in \mathcal{F}(U) \setminus \{0\}} \frac{\mathcal{E}(u)}{||u||_2^2}.$$

Inequality (1.9) is called the *Faber-Krahn inequality*.

We introduce conditions (Gcap$_\varepsilon$), (Gcap$_+$).

- *Condition* (Gcap$_\varepsilon$): For any $\varepsilon \in (0, 1]$ and any two concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $0 < R < R+r < \overline{R}$, there exists some $\phi \in (1+\varepsilon)$-cutoff$(B_0, B)$ such that for any measurable function $u$ with $u^2\phi \in \mathcal{F}(B)$,
$$\mathcal{E}(u^2\phi, \phi) \le \frac{C\varepsilon^{-1}}{\psi(r)} \int_B u^2 \phi d\mu, \tag{1.10}$$
where $C$ is some positive constant independent of $\varepsilon, B_0, B, u$.

Note that function $\phi$ above is independent of $u$.

- *Condition* (Gcap$_+$): For any two concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $0 < R < R + r < \overline{R}$, there exists some $\phi \in \kappa$-cutoff$(B_0, B)$ such that for any $u \in \mathcal{F}' \cap L^\infty$,
$$\mathcal{E}(u^2\phi, \phi) \le \frac{C}{\psi(r)} \int_B u^2 d\mu, \tag{1.11}$$
where $C > 0$ and $\kappa \ge 1$ are two constants independent of $B_0, B, u$.

Condition (Gcap$_+$) is slightly stronger than condition (Gcap), called the *generalized capacity condition*, which was introduced in [19], in the sense that function $\phi$ in condition (Gcap$_+$) here is independent of $u$ but it may depend on $u$ in condition (Gcap) in [19].

We introduce condition (ABB).

- *Condition* (ABB$_\zeta$): Given $0 \le \zeta < \infty$, there exists a constant $C > 0$ such that, for any three concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R+r)$ and $\Omega := B(x_0, R')$ with $0 < R < R+r < R' < \overline{R}$, there exists some $\phi \in \text{cutoff}(B_0, B)$ such that for any $u \in \mathcal{F}' \cap L^\infty$,
$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \le \zeta \int_\Omega d\Gamma_\Omega(u) + \frac{C}{\psi(r)} \int_\Omega u^2 d\mu, \tag{1.12}$$
where the measure $d\Gamma_\Omega(f)$ for any $f \in \mathcal{F}' \cap L^\infty$ is defined by (2.12) below. We say that *condition (ABB)* is satisfied if condition (ABB$_\zeta$) holds for some $\zeta \ge 0$.

Condition (ABB) is named after Andres, Barlow and Bass, who first introduced this elegant condition in [1], [4] (which was termed the *cut-off Sobolev inequality* therein). The value of constant $\zeta$ in (1.12) is important, depending on whether $\zeta = 0$ or not. When $\zeta = 0$, the energy measure $d\Gamma_\Omega(\phi)$ is absolutely continuous with respect to the measure $\mu$, and $M$ behaves like a Riemannian manifold, whilst when $\zeta \neq 0$, the energy measure $d\Gamma_\Omega(\phi)$ is singular to $\mu$, and $M$ behaves like a fractal, see [29, Theorem 2.13]. We will show that condition (ABB) implies condition (ABB$_+$) to be stated shortly. Another application of condition (ABB) is that it implies the conservativeness of $(\mathcal{E}, \mathcal{F})$, see Lemma 3.7 below.

For any ball $B_0 := B(x_0, R)$ with $0 < R < \overline{R}$, let $\Phi_{B_0}$ be a tent function sitting on $B_0$ given by

$$\Phi_{B_0}(x) = 1 \wedge \left( \frac{\mu(B_0)\psi(R)}{V(x_0, x)\psi(x_0, x)} \right)^{1/2} \quad \text{in } M \tag{1.13}$$

so that $\Phi_{B_0} = 1$ on $B_0$, where $V(x, y)$ and $\psi(x, y)$ for $(x, y) \in M \times M$ are respectively defined by

$$V(x, y) := V(x, d(x, y)) \text{ and } \psi(x, y) := \psi(d(x, y)). \tag{1.14}$$

If condition (VD) holds, then

$$\|\Phi_{B_0}\|_2^2 \leq C\mu(B_0) \tag{1.15}$$

for some positive constant $C$ independent of $B_0$, see Corollary 3.2 below.

We introduce condition (ABB$_+$).

- *Condition* (ABB$_+$): For any ball $B_0 := B(x_0, R)$ with $0 < R < \overline{R}$, there exists some $\phi \in \text{cutoff}(B_0, M)$ such that
$$C^{-1}\Phi_{B_0} \leq \phi \leq C\Phi_{B_0} \text{ in } M, \tag{1.16}$$
and, for any $u \in \mathcal{F}' \cap L^\infty$
$$\int_M u^2 d\Gamma(\phi) \leq \frac{1}{8} \int_M \phi^2 d\Gamma(u) + \frac{C}{\psi(R)} \int_M \phi^2 u^2 d\mu, \tag{1.17}$$
where $C > 0$ is some constant independent of $B_0, u$.

Note that the integrand in the last integral of (1.17) is function $\phi^2 u^2$ instead of function $u^2$ in (1.12). Condition (ABB$_+$) will be used in getting condition (SL$_2$) to be stated later on, see Lemma 3.6.

- *Condition* (J$_\leq$): The jump kernel $J(x, y)$ exists on $M \times M \setminus \text{diag}$, and there exists a constant $C > 0$ such that, for any two distinct points $x, y$ in $M$
$$J(x, y) \leq \frac{C}{V(x, y)\psi(x, y)}. \tag{1.18}$$
(For convenience, set $J(x, x) = 0$ for each $x \in M$ in the sequel.)

Recall the notions of the subcaloric, caloric functions. Let $I$ be an interval in $\mathbb{R}$. A function $u : I \to L^2$ is said to be *weakly differentiable* at $t \in I$ if for any $\varphi \in L^2$, the function $(u(\cdot), \varphi)$ is differentiable at $t$, that is, the limit

$$\lim_{\varepsilon \to 0} \left( \frac{u(t + \varepsilon) - u(t)}{\varepsilon}, \varphi \right)$$

exists. In this case, by the principle of uniform boundedness, there is some $w \in L^2$ such that

$$\lim_{\varepsilon \to 0} \left( \frac{u(t + \varepsilon) - u(t)}{\varepsilon}, \varphi \right) = (w, \varphi)$$

for any $\varphi \in L^2$. The vector $w$ is termed the *weak derivative* of $u$ at $t$, and we write $w = \frac{\partial}{\partial t}u$. For an open subset $\Omega \subset M$, a function $u : I \to \mathcal{F}$ is *subcaloric* in $I \times \Omega$ if $u$ is weakly differentiable in $L^2$ at any $t \in I$ and if for any $t \in I$ and any nonnegative $\varphi \in \mathcal{F}(\Omega)$,

$$(\frac{\partial}{\partial t}u, \varphi) + \mathcal{E}(u(t, \cdot), \varphi) \leq 0.$$

A function $u$ is said to be *caloric* in $I \times \Omega$ if the above inequality is replaced by

$$(\frac{\partial}{\partial t}u, \varphi) + \mathcal{E}(u(t, \cdot), \varphi) = 0.$$

We introduce condition (PMV$_2$), called the $L^2$-version of *parabolic mean-value inequality*.

- *Condition* ($\text{PMV}_2$): There exist three constants $\nu$, $C > 0$ and $\delta \in (0,1]$ such that, for any $B_0 := B(x_0, R)$ with $0 < R < \delta\overline{R}$ and for any function $u : (0, s] \to \mathcal{F}' \cap L^\infty$ that is nonnegative, subcaloric in $(0, s] \times B_0$ with $s = \psi(R)$, we have for any $\varepsilon > 0$

$$\underset{[\frac{3s}{4},s]\times(\frac{1}{2}B_0)}{\text{esup}} u \leq C(1 + \varepsilon^{-\frac{1+\nu}{2\nu}}) \left( \frac{1}{s\mu(B_0)} \int_{\frac{s}{2}}^{s} \int_{B_0} u^2(t, \cdot) d\mu dt \right)^{1/2} + \varepsilon T, \tag{1.19}$$

where $T$ is the *tail* arising from the jump part and given by

$$T = \begin{cases} \frac{1}{\mu(B_0)} \sup\limits_{t\in[\frac{s}{2},s]} \int_{(\frac{1}{2}B_0)^c} u_+(t, \cdot)\Phi_{B_0}^2 d\mu & \text{if } J \neq 0, \\ 0 & \text{if } J = 0, \end{cases} \tag{1.20}$$

and $\Phi_{B_0}$ is the tent function sitting on ball $B_0$ as defined in (1.13). Here the notion

$$\underset{(t,x)\in I\times\Omega}{\text{esup}} u(t, x) := \sup_{t\in I} \underset{x\in\Omega}{\text{esup}}\, u(t, x).$$

Roughly speaking, condition ($\text{PMV}_2$) says that any value of the function $u$, which is nonnegative and subcaloric in $(0, s] \times B_0$, over a smaller domain $Q_+ := [\frac{3s}{4}, s] \times (\frac{1}{2}B_0)$ can be controlled by its $L^2$-mean value over a larger domain $Q := [\frac{s}{2}, s] \times B_0$ plus a tail term, see Figure 1. The tail $T$ reflects the behavior of the positive part of function $u$ outside the half ball $\frac{1}{2}B_0$, which vanishes in the case when the process is a diffusion, that is, when $J = 0$.

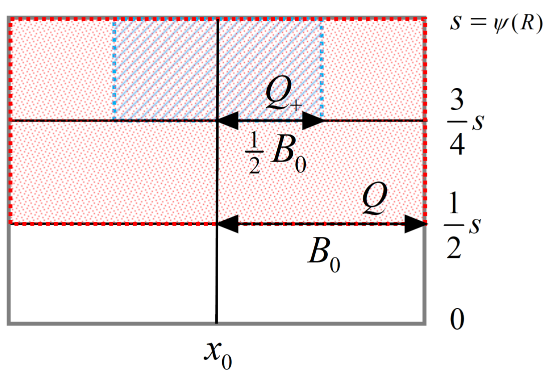


FIGURE 1. The $L^2$ parabolic mean-value inequality for ball $B_0$.

We introduce condition ($\text{PMV}_1$).

- *Condition* ($\text{PMV}_1$): There exist three constants $c_0 \in (0, \frac{1}{2})$, $C > 0$ and $\delta \in (0,1]$ such that, for any ball $B_0 := B(x_0, R)$ with $0 < R < \delta\overline{R}$ and for any function $u : (0, s] \to \mathcal{F}' \cap L^\infty$ that is nonnegative, subcaloric in $(0, s] \times B_0$ with $s = \psi(R)$, we have

$$\underset{[s-\psi(c_0R),s]\times(c_0B_0)}{\text{esup}} u \leq C \sup_{t\in[\frac{s}{2},s]} \left( \frac{1}{\mu(B_0)} \int_M u_+(t, \cdot) d\mu \right). \tag{1.21}$$

Condition ($\text{PMV}_1$) is weaker than condition ($\text{PMV}_2$), see Lemma 4.2 below. Condition ($\text{PMV}_1$) will be used to derive an on-diagonal upper bound of the heat kernel.

We introduce *condition* ($\text{SL}_2$), called the *survival estimate* in the *weighted* $L^2$*-norm*.

- *Condition* ($\text{SL}_2$): There exists some constant $C > 0$ such that, for any $t > 0$ and for any two concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, r)$ with $0 < R < r < \overline{R}$ so that $B_0 \subset B$, we have that, by abusing the symbols $\psi(B_0) := \psi(R)$ and $\psi(B) := \psi(r)$,

$$\int_B \left(1 - P_t^B 1_B\right)^2 \Phi_{B_0}^2 d\mu \leq C\mu(B_0) \left( \frac{\psi(B_0)}{\psi(B)} + \frac{t}{\psi(B)} \right) \exp\left( \frac{Ct}{\psi(B_0)} \right). \tag{1.22}$$

**Remark 1.1.** Note that (1.22) is trivially satisfied if $\frac{t}{\psi(B)} \geq C^{-1}$ or if $\frac{\psi(B_0)}{\psi(B)} \geq C^{-1}$ for a constant $C > 0$, since the integral on the left-hand side in (1.22) is bounded from above by $\|\Phi_{B_0}\|_2^2 \leq C\mu(B_0)$, whilst the term on the right-hand side is bounded from below by $\mu(B_0)$ (up to constant).

We give an explanation of inequality (1.22) from the probabilistic point of view. Let $(X_t)$ be a Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$. Denote by $\tau_B$ the *first exit time* of the process $\{X_t\}$ from a ball $B$. Then

$$1 - P_t^B 1_B(x) = 1 - \mathbb{P}_x(X_t \in B, t < \tau_B) = \mathbb{P}_x(\tau_B \leq t),$$

which is the probability of the process $(X_t)$ to leave ball $B$ before time $t$. The smaller is this probability, the higher the probability of the process staying in $B$ up to time $t$, or the higher the probability of *survival* of the process up to time $t$, assuming that the process gets killed outside $B$. Therefore, inequality (1.22) gives an upper bound of the *survival function* $1 - P_t^B 1_B$ in the norm of $L^2$ with weight $\Phi_{B_0}$ for any two concentric balls $B_0, B$ and for any time $t > 0$. As remarked above, inequality (1.22) is meaningful only when $\frac{t}{\psi(B)}$ is small and $B_0 \subset \varepsilon_0 B$ for a small number $\varepsilon_0 \in (0, 1)$, but this is enough to serve our purpose.

Condition ($\mathrm{SL}_2$) implies the conservativeness of $(\mathcal{E}, \mathcal{F})$, see Lemma 3.7 below. As a by-product, we obtain the conservativeness of the form $(\mathcal{E}, \mathcal{F})$ from conditions (ABB), ($\mathrm{J}_\leq$) only, without using the Faber-Krahn inequality. This observation was addressed in [14, Theorem 7] for the local Dirichlet form, see also [11, Theorem 7.12] for the non-local Dirichlet form, on unbounded metric spaces. The issue of conservativeness is not trivial, and was studied, for example in [18, Lemma 4.6] for a bounded or unbounded metric space and in [11, Proposition 3.1] for an unbounded metric space.

Another important application of condition ($\mathrm{SL}_2$) is that it will, together with condition ($\mathrm{PMV}_2$), yield condition ($\mathrm{S}_{1/2}$) to be stated in the following, see Lemma 3.8.

- *Condition* ($\mathrm{S}_{1/2}$): There exists a positive constant $C$ such that, for any ball $B$ of radius $R$ with $0 < R < \infty$ and any $t > 0$,

$$1 - P_t^B 1_B \leq \left(\frac{Ct}{\psi(R)}\right)^{1/2} \quad \text{in } \frac{1}{2}B. \tag{1.23}$$

Inequality (1.23) gives a nice pointwise upper estimate for the survival function $1 - P_t^B 1_B$ near the center of ball $B$. Condition ($\mathrm{S}_{1/2}$) will be used to obtain off-diagonal upper estimate of the heat kernel, see Lemma 6.6 below. Another application of condition ($\mathrm{S}_{1/2}$) is that it immediately implies the conservativeness of $(\mathcal{E}, \mathcal{F})$ by letting $\psi(R) \to \infty$ when the space $(M, d)$ is unbounded.

**Remark 1.2.** Let $\overline{R} < \infty$ so that $(M, d)$ is bounded. Then $P_t 1 = 1$ in $M$ for any $t > 0$, since $(\mathcal{E}, \mathcal{F})$ is conservative by Lemma 3.7 below. In this case, if $R > \overline{R}$ then $B = B(x_0, R) = M$, and hence $1 - P_t^B 1_B = 1 - P_t 1 = 0$ and (1.23) automatically holds for any $t > 0$. Thus, in order to verify (1.23), one needs only consider the case when $0 < R \leq \overline{R}$. On the other hand, if (1.23) holds for small $R < \delta\overline{R}$, then it also holds for large $R \geq \delta\overline{R}$ after adjusting the value of constant $C$ by using the standard covering argument. Therefore, in order to verify (1.23), one needs only to assume both $0 < R < \delta\overline{R}$ and $0 < t < \psi(\delta\overline{R})$.

We introduce conditions (DUE) and (UE).

- *Condition* (DUE): The heat kernel $p_t(x, y)$ exists pointwise on $(0, \infty) \times M \times M$, and there exists a positive constant $C$ such that

$$p_t(x, y) \leq \frac{C}{V(x, \psi^{-1}(t))} \tag{1.24}$$

  for all $x, y$ in $M$ and all $0 < t < \psi(\overline{R})$.

The above inequality (1.24) is called an *on-diagonal upper estimate* of the kernel $p_t(x, y)$.

- *Condition* (UE): The heat kernel $p_t(x, y)$ exists pointwise on $(0, \infty) \times M \times M$, and there exists a positive constant $C$ such that

$$p_t(x, y) \leq C\left(\frac{1}{V(x, \psi^{-1}(t))} \wedge \frac{t}{V(x, y)\psi(x, y)}\right) \tag{1.25}$$

  for all $x, y$ in $M$ and all $0 < t < \psi(\overline{R})$.

The above inequality (1.25) is called an *off-diagonal upper estimate* of the kernel $p_t(x,y)$. Clearly, we have (UE) $\Rightarrow$ (DUE).

For any non-empty open subset $\Omega$ of $M$ and any Borel subset $U$ of $\Omega$, recall that the capacity $\mathrm{Cap}(U,\Omega)$ for the pair $(U,\Omega)$ is defined by

$$\mathrm{Cap}(U,\Omega) := \inf\left\{\mathcal{E}(\phi,\phi) : \phi \in \mathrm{cutoff}(U,\Omega)\right\}. \tag{1.26}$$

We introduce *condition* (Cap$_\leq$).

- C*ondition* (Cap$_\leq$): There exists some constant $C > 0$ such that for any ball $B := B(x_0, R)$ with $0 < R < \overline{R}$,

$$\mathrm{Cap}\left(\frac{3}{4}B, B\right) \leq C\frac{\mu(B)}{\psi(R)}. \tag{1.27}$$

Conditions (ABB) and (J$_\leq$) imply condition (Cap$_\leq$) (cf. inequality (3.31) below).

The following is the main result of this paper.

**Theorem 1.3.** *Let* $(\mathcal{E},\mathcal{F})$ *be a regular Dirichlet form in* $L^2$ *without killing part on a doubling space* $(M,d,\mu)$. *Then the following implications are true:*

$$\begin{aligned}
(\mathrm{FK}_\nu) + (\mathrm{Gcap}_\varepsilon) + (\mathrm{J}_\leq) \quad &\Rightarrow \quad (\mathrm{FK}_\nu) + (\mathrm{Gcap}_+) + (\mathrm{J}_\leq)\\
&\Rightarrow \quad \underline{(\mathrm{FK}_\nu) + (\mathrm{ABB}) + (\mathrm{J}_\leq)}\\
&\Rightarrow \quad (\mathrm{ABB}) + (\mathrm{PMV}_2) + (\mathrm{J}_\leq)\\
&\Rightarrow \quad (\mathrm{ABB}_+) + (\mathrm{Cap}_\leq) + (\mathrm{PMV}_2) + (\mathrm{J}_\leq)\\
&\Rightarrow \quad (\mathrm{S}_{1/2}) + (\mathrm{PMV}_1) + (\mathrm{J}_\leq)\\
&\Rightarrow \quad (\mathrm{S}_{1/2}) + (\mathrm{DUE}) + (\mathrm{J}_\leq)\\
&\Rightarrow \quad (\mathrm{UE}) + (\mathrm{C})\\
&\Rightarrow \quad (\mathrm{Gcap}_\varepsilon) + (\mathrm{J}_\leq),\\
(\mathrm{DUE}) + (\mathrm{RVD}) \quad &\Rightarrow \quad (\mathrm{FK}_\nu),\\
(\mathrm{FK}_\nu) + (\mathrm{J}_\leq) \quad &\Rightarrow \quad (\mathrm{RVD}).
\end{aligned}$$

*Consequently, the following equivalence is true:*

$$(\mathrm{FK}_\nu) + (\mathrm{ABB}) + (\mathrm{J}_\leq) \Leftrightarrow (\mathrm{UE}) + (\mathrm{C}) + (\mathrm{RVD}). \tag{1.28}$$

A complete proof of Theorem 1.3 is highly non-trivial, which will follow from a series of propositions and lemmas to be addressed in detail in the following sections. Here we give a flowchart of the proof.

*Proof of Theorem 1.3.* Clearly, (Gcap$_\varepsilon$) $\Rightarrow$ (Gcap$_+$). We have the following implications:

$$\begin{aligned}
(\mathrm{Gcap}_+) \quad &\Rightarrow \quad (\mathrm{ABB}) \ \text{(see Lemma 2.3 below)}\\
(\mathrm{FK}_\nu) + (\mathrm{Gcap}_+) + (\mathrm{J}_\leq) \quad &\Rightarrow \quad (\mathrm{PMV}_2) \ \text{(see a forthcoming paper [17])}\\
(\mathrm{ABB}) + (\mathrm{J}_\leq) \quad &\Rightarrow \quad (\mathrm{ABB}_+) \ \text{(see Lemma 3.5)} \qquad (1.29)\\
(\mathrm{ABB}) + (\mathrm{J}_\leq) \quad &\Rightarrow \quad (\mathrm{Cap}_\leq) \ \text{(similar to formula (3.31))}\\
(\mathrm{ABB}_+) + (\mathrm{Cap}_\leq) + (\mathrm{J}_\leq) \quad &\Rightarrow \quad (\mathrm{SL}_2) \ \text{(see Lemma 3.6)} \qquad (1.30)\\
(\mathrm{PMV}_2) + (\mathrm{SL}_2) \quad &\Rightarrow \quad (\mathrm{S}_{1/2}) \ \text{(see Lemma 3.8)}\\
(\mathrm{PMV}_2) \quad &\Rightarrow \quad (\mathrm{PMV}_1) \ \text{(see Lemma 4.2)} \qquad (1.31)\\
(\mathrm{PMV}_1) \quad &\Rightarrow \quad (\mathrm{DUE}) \ \text{(see Lemma 4.3)}\\
(\mathrm{S}_{1/2}) + (\mathrm{J}_\leq) + (\mathrm{DUE}) \quad &\Rightarrow \quad (\mathrm{UE}) + (\mathrm{C}) \ \text{(see Lemma 6.6)} \qquad (1.32)\\
(\mathrm{UE}) + (\mathrm{C}) \quad &\Rightarrow \quad (\mathrm{Gcap}_\varepsilon) + (\mathrm{J}_\leq) \ \text{(see Lemma 6.7)}\\
(\mathrm{DUE}) + (\mathrm{RVD}) \quad &\Rightarrow \quad (\mathrm{FK}_\nu) \ \text{(see Lemma 6.8)}\\
(\mathrm{FK}_\nu) + (\mathrm{J}_\leq) \quad &\Rightarrow \quad (\mathrm{RVD}) \ \text{(see Lemma 6.9)}.
\end{aligned}$$

Finally, the equivalence (1.28) follows directly from above. The proof is complete. □

The main difficulties are to show the implications (1.29), (1.30), (1.31), and (1.32). The implication (1.30) is new. The condition ($\mathrm{SL}_2$) is invisible in the statements of Theorem 1.3. However, this condition plays an important role in our analysis.

Under conditions (VD), (RVD), the following slightly different equivalence than (1.28)

$$(\mathrm{FK}_\nu) + (\mathrm{Gcap}_+) + (\mathrm{J}_\leq) \Leftrightarrow (\mathrm{UE}) + (\mathrm{C})$$

was addressed for the purely jump Dirichlet form in $L^2$ in [11] when $(M,d)$ is unbounded with a doubling measure $\mu$ by using the probabilistic approach, and in [19] when $(M,d)$ is bounded or unbounded with an Ahlfors-regular measure $\mu$ and $\psi(r) = r^\beta$ by using the purely analytic approach, but with condition ($\mathrm{Gcap}_+$) replaced by its other variants. Here we present a new analytic approach on the bounded or unbounded metric space for any regular Dirichlet form without killing part, and therefore, it is of interest in its own right.

We further study the strongly local Dirichlet form. We introduce condition $(\mathrm{UE})_{\mathrm{exp}}$.

- *Condition* $(\mathrm{UE})_{\mathrm{exp}}$: The heat kernel $p_t(x,y)$ exists pointwise on $(0,\infty)\times M\times M$, and there exist three positive constants $C, c', c$ such that

$$p_t(x,y) \leq \frac{C}{V(x,\psi^{-1}(t))}\exp\left(-c' t\,\Psi\left(\frac{cd(x,y)}{t}\right)\right) \tag{1.33}$$

  for all $x,y$ in $M$ and all $0<t<\psi(\overline{R})$, where $\Psi$ is defined by

$$\Psi(s) := \sup_{\lambda>0}\left\{\frac{s}{\psi^{-1}(1/\lambda)} - \lambda\right\} \quad (s>0). \tag{1.34}$$

As a by-product of Theorem 1.3, we obtain the following.

**Corollary 1.4.** *Let $(\mathcal{E},\mathcal{F})$ be a regular strongly local Dirichlet form in $L^2$ on a doubling space $(M,d,\mu)$. Then the following equivalence is true:*

$$(\mathrm{FK}_\nu) + (\mathrm{Gcap}_+) \Leftrightarrow (\mathrm{UE})_{\mathrm{exp}} + (\mathrm{C}) + (\mathrm{RVD}). \tag{1.35}$$

*Proof.* Starting from implication (1.32) with $J\equiv 0$, assume that conditions ($\mathrm{S}_{1/2}$), (DUE) hold. If $(\mathcal{E},\mathcal{F})$ is strongly local and if condition ($\mathrm{S}_{1/2}$) is satisfied (thus $(\mathcal{E},\mathcal{F})$ is also conservative), we have by [22, Theorem 5.7] that, for any ball $B := B(x,R)$ with $0<R<\infty$ and any $0<t<\infty$,

$$P_t 1_{B^c} \leq C\exp\left(-c' t\,\Psi\left(\frac{cR}{t}\right)\right) \text{ in } \frac{1}{2}B.$$

From this and using condition (DUE), we obtain (1.33) by applying the standard semigroup property argument. The proof is complete. □

**Notation.** Letters $C, C'$ denote positive constants which may change at any occurrence, whilst the letters $C_i$ for $i = 1,2,\cdots$ are fixed and are randomly selected to use. The term "for *any* (or *all*) $x$" means "for an arbitrary $x$" but the statement followed is *independent* of the choice of $x$.

## 2. Condition (ABB)

In this section, we first collect the basic properties on energy measures associated with a regular Dirichlet form and then derive condition (ABB) from condition ($\mathrm{Gcap}_+$).

Recall that any regular Dirichlet form $(\mathcal{E},\mathcal{F})$ in $L^2$ admits a unique *Beurling-Deny decomposition* (cf. [16, Theorem 3.2.1 on p.120]):

$$\mathcal{E}(u,v) = \mathcal{E}^{(L)}(u,v) + \mathcal{E}^{(J)}(u,v) + \mathcal{E}^{(K)}(u,v), \tag{2.1}$$

where $\mathcal{E}^{(L)}$ is the *diffusion part* associated with a unique Radon measure $d\Gamma_L$ (the notions $\mathcal{E}^{(c)}, d\mu^c_{\langle u,v\rangle}$ are instead used in [16]):

$$\mathcal{E}^{(L)}(u,v) = \int_M d\Gamma_L(u,v), \tag{2.2}$$

whilst $\mathcal{E}^{(J)}$ is the jump part with a unique Radon measure $dj$ defined on $(M\times M)\backslash\mathrm{diag}$:

$$\mathcal{E}^{(J)}(u,v)=\iint_{(M\times M)\backslash\mathrm{diag}}(u(x)-u(y))(v(x)-v(y))dj, \tag{2.3}$$

and finally, $\mathcal{E}^{(K)}$ is the *killing part* associated with a Radon measure $dk$:

$$\mathcal{E}^{(K)}(u,v)=\int_M u(x)v(x)dk(x). \tag{2.4}$$

For simplicity, we will drop *diag* in expression $M\times M\backslash\mathrm{diag}$ in (2.3) when no confusion arises. Note that $\mathcal{E}^{(K)}\equiv 0$ in our paper and thus

$$\mathcal{E}(u,v)=\mathcal{E}^{(L)}(u,v)+\mathcal{E}^{(J)}(u,v). \tag{2.5}$$

Recall that for any $u\in\mathcal{F}$, the measure $d\Gamma_L(u):=d\Gamma_L(u,u)$ is well-defined and unique (cf. [16, lines above Lemma 3.2.3 on p.126]). Moreover, it satisfies the following properties: for any $u,v,w\in\mathcal{F}\cap L^\infty$,

- the *chain rule* ([16, Lemma 3.2.5 on p.127]):
$$d\Gamma_L(uv,w)=ud\Gamma_L(v,w)+vd\Gamma_L(u,w); \tag{2.6}$$
- the *product rule* ([16, Theorem 3.2.2 on p.129]):
$$d\Gamma_L(F(u),v)=F'(u)d\Gamma_L(u,v) \tag{2.7}$$
for any $F\in C^1(\mathbb{R})$ with $F(0)=0$;
- the *strong locality*: if $u\in\mathcal{F}$ is constant in $\Omega$ and $v\in\mathcal{F}$ is arbitrary, then
$$1_\Omega d\Gamma_L(u,v)=0 \text{ on } M \tag{2.8}$$
(cf. [16, Corollary 3.2.1 on p.128], or [33, formula (3.8) on p.387]), and
$$d\Gamma_L(u_+,v)=\mathbf{1}_{\{u>0\}}d\Gamma_L(u,v) \text{ on } M, \tag{2.9}$$
where $u_+=u\vee 0$ (cf. [33, formula (3.14) on p.390]);
- the *Cauchy-Schwarz inequality*: for any $f\in L^2(M,\Gamma_L(u))$, $g\in L^2(M,\Gamma_L(v))$
$$\int_M|fg|d\Gamma_L(u,v)\le\left(\int_M f^2d\Gamma_L(u)\right)^{1/2}\left(\int_M g^2d\Gamma_L(v)\right)^{1/2} \tag{2.10}$$
(cf. [33, on p. 390]).

By the strong locality (2.8) of $d\Gamma_L$, we can define for any constant $a\in\mathbb{R}$

$$d\Gamma_L(u+a)=d\Gamma_L(u),$$

so that the chain rule (2.6) still holds if $u,v\in\mathcal{F}'\cap L^\infty$. Moreover, for any $u\in\mathcal{F}'\cap L^\infty$, we have

$$d\Gamma_L(|u|)=d\Gamma_L(u), \tag{2.11}$$

since $d\Gamma_L(u_+,u_-)=0$ by using (2.9), (2.8), and

$$\begin{aligned}d\Gamma_L(|u|)&=d\Gamma_L(u_++u_-,u_++u_-)=d\Gamma_L(u_+)+2d\Gamma_L(u_+,u_-)+d\Gamma_L(u_-)\\&=d\Gamma_L(u_+)+d\Gamma_L(u_-)=d\Gamma_L(u).\end{aligned}$$

We define the *measure* $d\Gamma_\Omega(u,v)$ by

$$d\Gamma_\Omega(u,v)(x):=d\Gamma_L(u,v)(x)+\int_\Omega(u(x)-u(y))(v(x)-v(y))dj. \tag{2.12}$$

The measure $d\Gamma_\Omega(u,v)$ is well-defined for any $u,v\in\mathcal{F}'$ and $\Omega\subset M$. For simplicity, denote by $d\Gamma_\Omega(u):=d\Gamma_\Omega(u,u)$, and in particular, when $\Omega=M$, we denote by $d\Gamma(u):=d\Gamma_M(u)$ so that

$$\mathcal{E}(u,v)=\int_M d\Gamma(u,v) \text{ for any } u,v\in\mathcal{F}'.$$

It turns out (see for example [33, formula (3.5) on p.387]) that for any $f,u\in\mathcal{F}'\cap L^\infty$,

$$\int_M fd\Gamma(u)=\mathcal{E}(fu,u)-\frac{1}{2}\mathcal{E}(f,u^2). \tag{2.13}$$

The following is general.

**Proposition 2.1.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2$ without killing part, and $d\Gamma_\Omega$ given by (2.12) for an open subset $\Omega$ of $M$. Then, for any $f \in \mathcal{F}' \cap L^\infty$ with $f \geq 0$ in $\Omega$ and for any $\varphi \in \mathcal{F}' \cap L^\infty$ with* $\operatorname{supp}[\varphi] \subset U \subset \Omega$*, we have*

$$\int_\Omega f d\Gamma_\Omega(\varphi) = \int_U f d\Gamma_U(\varphi) + \int_{U\times(\Omega\setminus U)} (f(x) + f(y))\varphi^2(x) dj. \tag{2.14}$$

*Proof.* Since $\operatorname{supp}[\varphi] \subset U \subset \Omega$, we see that

$$\begin{aligned}
&\int_{\Omega\times\Omega} f(x)\,(\varphi(x) - \varphi(y))^2\, dj \\
&= \left(\int_{U\times U} + \int_{(\Omega\setminus U)\times U} + \int_{U\times(\Omega\setminus U)} + \int_{(\Omega\setminus U)\times(\Omega\setminus U)}\right) f(x)\,(\varphi(x) - \varphi(y))^2\, dj \\
&= \int_{U\times U} f(x)\,(\varphi(x) - \varphi(y))^2\, dj + \int_{(\Omega\setminus U)\times U} f(x)\varphi^2(y) dj + \int_{U\times(\Omega\setminus U)} f(x)\varphi^2(x) dj \\
&= \int_{U\times U} f(x)\,(\varphi(x) - \varphi(y))^2\, dj + \int_{U\times(\Omega\setminus U)} (f(x) + f(y))\varphi^2(x) dj
\end{aligned}$$

by using the symmetry of $dj$. It follows by definition (2.12) that

$$\begin{aligned}
\int_\Omega f d\Gamma_\Omega(\varphi) &= \int_\Omega f d\Gamma_L(\varphi) + \int_{\Omega\times\Omega} f(x)\,(\varphi(x) - \varphi(y))^2\, dj \\
&= \int_U f d\Gamma_U(\varphi) + \int_{U\times U} f(x)\,(\varphi(x) - \varphi(y))^2\, dj \\
&\quad + \int_{U\times(\Omega\setminus U)} (f(x) + f(y))\varphi^2(x) dj \\
&= \int_U f d\Gamma_U(\varphi) + \int_{U\times(\Omega\setminus U)} (f(x) + f(y))\varphi^2(x) dj,
\end{aligned}$$

thus showing (2.14). □

The following is needed.

**Proposition 2.2.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2$ without killing part. Then, for any open subset $\Omega$ of $M$ and for any $u \in \mathcal{F}' \cap L^\infty$, $\phi \in \mathcal{F} \cap L^\infty$ with $\operatorname{supp}[\phi] \subset \Omega$,*

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \leq 4 \int_\Omega \phi^2 d\Gamma_\Omega(u) + 2\mathcal{E}(u^2\phi, \phi), \tag{2.15}$$

*where $d\Gamma_\Omega$ is defined by (2.12). In particular, when $\Omega = M$, we have*

$$\int_M u^2 d\Gamma(\phi) \leq 4 \int_M \phi^2 d\Gamma(u) + 2\mathcal{E}(u^2\phi, \phi) \tag{2.16}$$

*for any $u \in \mathcal{F}' \cap L^\infty$, $\phi \in \mathcal{F} \cap L^\infty$.*

Inequality (2.16) was addressed in [28] but with $u \in \mathcal{F} \cap L^\infty$ (instead of $u \in \mathcal{F}' \cap L^\infty$ here). We sketch the proof for the reader's convenience.

*Proof.* For any $u \in \mathcal{F}' \cap L^\infty$, $\phi \in \mathcal{F} \cap L^\infty$, note that

$$u^2 \in \mathcal{F}' \cap L^\infty \ \text{ and } \ u^2\phi \in \mathcal{F} \cap L^\infty.$$

For $d\Gamma_L$ we shall show that

$$\int_\Omega u^2 d\Gamma_L(\phi) \leq 2\mathcal{E}^{(L)}(u^2\phi, \phi) + 4 \int_\Omega \phi^2 d\Gamma_L(u). \tag{2.17}$$

Indeed, $u \in L^2(M, d\Gamma_L(\phi))$, since

$$\int_M u^2 d\Gamma_L(\phi) \leq \|u\|_\infty^2 \int_M d\Gamma_L(\phi) = \|u\|_\infty^2\, \mathcal{E}^{(L)}(\phi) \leq \|u\|_\infty^2\, \mathcal{E}(\phi) < \infty.$$

Similarly, $\phi \in L^2(M, d\Gamma_L(u))$. By (2.6), (2.7), (2.10), it follows that

$$\begin{aligned}\int_M u^2 d\Gamma_L(\phi) &= \int_M d\Gamma_L(u^2\phi, \phi) - 2\int_M u\phi d\Gamma_L(u,\phi) \\ &\leq \mathcal{E}^{(L)}(u^2\phi,\phi) + \frac{1}{2}\int_M u^2 d\Gamma_L(\phi) + 2\int_M \phi^2 d\Gamma_L(u),\end{aligned}$$

which gives that

$$\int_M u^2 d\Gamma_L(\phi) \leq 2\mathcal{E}^{(L)}(u^2\phi,\phi) + 4\int_M \phi^2 d\Gamma_L(u). \tag{2.18}$$

Since $\phi$ is supported in $\Omega$, we see by (2.8) that $d\Gamma_L(\phi) = 0$ outside $\Omega$, and the two integrals in (2.18) are actually over $\Omega$, thus proving (2.17).

For $dj$ we shall show that

$$\int_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj \leq 2\mathcal{E}^{(J)}(u^2\phi,\phi) + 4\int_{\Omega\times\Omega} \phi^2(x)(u(x)-u(y))^2 dj. \tag{2.19}$$

Indeed, note that

$$\begin{aligned}\frac{1}{2}(u^2(x)+u^2(y))(\phi(x)-\phi(y))^2 &\leq 2(\phi(x)-\phi(y))(u^2(x)\phi(x)-u^2(y)\phi(y)) \\ &\quad +2(\phi^2(x)+\phi^2(y))(u(x)-u(y))^2,\end{aligned}$$

see [19, the inequality on lines 3-4 on p. 447] with $f = u$ and $g = \phi$. Integrating over $\Omega\times\Omega$ against $dj$ and using the symmetry of $dj$, it follows that

$$\begin{aligned}\int_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj &\leq 2\int_{\Omega\times\Omega} (\phi(x)-\phi(y))(u^2(x)\phi(x)-u^2(y)\phi(y)) dj \\ &\quad +4\int_{\Omega\times\Omega} \phi^2(x)(u(x)-u(y))^2 dj.\end{aligned} \tag{2.20}$$

On the other hand, using the fact that $\mathrm{supp}[\phi] \subset \Omega$,

$$\begin{aligned}&\int_{\Omega\times\Omega} (\phi(x)-\phi(y))(u^2(x)\phi(x)-u^2(y)\phi(y)) dj = \left(\int_{M\times M} - \int_{\Omega\times\Omega^c} - \int_{\Omega^c\times\Omega} - \int_{\Omega^c\times\Omega^c}\right)\cdots \\ &= \mathcal{E}^{(J)}(u^2\phi,\phi) - \int_{\Omega\times\Omega^c} \phi^2(x)u^2(x) dj - \int_{\Omega^c\times\Omega} \phi^2(y)u^2(y) dj \leq \mathcal{E}^{(J)}(u^2\phi,\phi).\end{aligned}$$

Plugging this into (2.20), we obtain (2.19).

Finally, summing up (2.17), (2.19), we conclude by (2.5) that

$$\begin{aligned}\int_\Omega u^2 d\Gamma_\Omega(\phi) &= \int_\Omega u^2 d\Gamma_L(\phi) + \int_{\Omega\times\Omega} u^2(x)(\phi(x)-\phi(y))^2 dj \\ &\leq 2\mathcal{E}^{(L)}(u^2\phi,\phi) + 4\int_\Omega \phi^2 d\Gamma_L(u) \\ &\quad +2\mathcal{E}^{(J)}(u^2\phi,\phi) + 4\int_{\Omega\times\Omega} \phi^2(x)(u(x)-u(y))^2 dj \\ &= 2\mathcal{E}(u^2\phi,\phi) + 4\int_\Omega \phi^2 d\Gamma_\Omega(u),\end{aligned}$$

thus proving (2.15). □

Condition (ABB) follows directly from condition ($\mathrm{Gcap}_+$) by using (2.15).

**Lemma 2.3.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2$ without killing part. Then*

$$(\mathrm{Gcap}_+) \Rightarrow (\mathrm{ABB}).$$

*Proof.* Let $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ and $\Omega := B(x_0, R')$ be any three concentric balls with $0 < R < R + r < R' < \overline{R}$. For $u \in \mathcal{F}' \cap L^\infty$, we have by condition (Gcap$_+$) that there exists some $g \in \kappa$-cutoff$(B_0, B)$ independent of $u$ where $1 \le \kappa < \infty$, such that

$$\mathcal{E}(u^2 g, g) \le \frac{C}{\psi(r)} \int_B u^2 d\mu \le \frac{C}{\psi(r)} \int_\Omega u^2 d\mu.$$

Since supp$[g] \subset B \subset \Omega$, it follows from (2.15) with $\phi$ replaced by $g$ that

$$\begin{aligned} \int_\Omega u^2 d\Gamma_\Omega(g) &\le 4 \int_\Omega g^2 d\Gamma_\Omega(u) + 2\mathcal{E}(u^2 g, g) \\ &\le 4 \int_\Omega g^2 d\Gamma_\Omega(u) + \frac{2C}{\psi(r)} \int_\Omega u^2 d\mu. \end{aligned} \tag{2.21}$$

Let $\phi = 1 \wedge g$. Clearly, $\phi \in$ cutoff$(B_0, B)$. On the other hand, since

$$d\Gamma_L(\phi) = d\Gamma_L(1 \wedge g) = 1_{\{g \le 1\}} d\Gamma_L(g) \le d\Gamma_L(g)$$

and since $g \le \kappa\phi$ in $M$ and $|\phi(x) - \phi(y)| \le |g(x) - g(y)|$ for any $x, y$ in $M$, we have

$$\begin{aligned} \int_\Omega u^2 d\Gamma_\Omega(\phi) &= \int_\Omega u^2 d\Gamma_L(\phi) + \int_{\Omega\times\Omega} u^2(x) (\phi(x) - \phi(y))^2 \, dj \\ &\le \int_\Omega u^2 d\Gamma_L(g) + \int_{\Omega\times\Omega} u^2(x) (g(x) - g(y))^2 \, dj = \int_\Omega u^2 d\Gamma_\Omega(g) \\ &\le 4 \int_\Omega g^2 d\Gamma_\Omega(u) + \frac{2C}{\psi(r)} \int_\Omega u^2 d\mu \ \ \text{(using (2.21))} \\ &\le 4\kappa^2 \int_\Omega \phi^2 d\Gamma_\Omega(u) + \frac{2C}{\psi(r)} \int_\Omega u^2 d\mu, \end{aligned}$$

thus showing (1.12). Hence, condition (ABB) is true. □

## 3. Conservativeness and condition ($S_{1/2}$)

In this section we first derive condition (ABB$_+$) from conditions (ABB), ($J_\le$). We then derive condition ($SL_2$) from conditions (ABB$_+$), ($J_\le$). We next derive the conservativeness of the form $(\mathcal{E}, \mathcal{F})$ from condition ($SL_2$). Finally, we derive condition ($S_{1/2}$) from conditions ($PMV_2$), ($SL_2$).

The following is frequently used.

**Proposition 3.1.** *Let $B_1 := B(x_0, R_1)$, $B_2 := B(x_0, R_2)$ be two concentric balls with $0 < R_1 < R_2 < \overline{R}$. If condition* (VD) *holds, then there exists some constant $C > 0$ independent of $B_1, B_2$ such that*

$$\sup_{x \in B_1} \int_{B_2^c} \frac{d\mu(y)}{V(x,y)\psi(x,y)} \le \frac{C}{\psi(R_2 - R_1)}. \tag{3.1}$$

*In particular, when $R_1 \downarrow 0$, we have for all $x$ in $M$,*

$$\int_{B(x,R)^c} \frac{d\mu(y)}{V(x,y)\psi(x,y)} \le \frac{C}{\psi(R)} \quad \text{for any } 0 < R < \overline{R}. \tag{3.2}$$

*Consequently, if conditions* (VD), ($J_\le$) *hold, then*

$$\sup_{x \in B_1} \int_{B_2^c} J(x,y) d\mu(y) \le \frac{C}{\psi(R_2 - R_1)}. \tag{3.3}$$

*Proof.* Set $r := R_2 - R_1$. Since $B_2^c \subset B(x, r)^c$ for any point $x \in B_1$, we see by (1.4), (1.6) that

$$\begin{aligned} \int_{B_2^c} \frac{d\mu(y)}{V(x,y)\psi(x,y)} &\le \int_{B(x,r)^c} \frac{d\mu(y)}{V(x,y)\psi(x,y)} = \sum_{m=0}^\infty \int_{B(x,2^{m+1}r)\setminus B(x,2^m r)} \frac{d\mu(y)}{V(x,y)\psi(x,y)} \\ &\le \sum_{m=0}^\infty \frac{V(x, 2^{m+1}r)}{V(x, 2^m r)\psi(2^m r)} \le \frac{C}{\psi(r)} \sum_{m=0}^\infty \left(\frac{2^{m+1}r}{2^m r}\right)^\alpha \left(\frac{r}{2^m r}\right)^{\beta_1} \end{aligned}$$

$$\leq \frac{C'}{\psi(r)}\sum_{m=0}^{\infty}2^{-m\beta_1}=\frac{C}{\psi(r)}$$

for a constant $C$ independent of $x_0, R_1, R_2$, thus proving (3.1). Estimate (3.2) is clear.

If conditions (VD), ($J_{\leq}$) hold, we see by (3.1) that for any $x \in B_1$

$$\int_{B_2^c} J(x,y)d\mu(y) \leq \int_{B_2^c}\frac{Cd\mu(y)}{V(x,y)\psi(x,y)} \leq \frac{C'}{\psi(R_2-R_1)},$$

thus showing (3.3). The proof is complete. □

The $L^2$-norm of the tent function $\Phi_{B_0}$ for any ball $B_0$ is controlled by $\mu(B_0)$ if condition (VD) holds.

**Corollary 3.2.** *Let $\Phi_{B_0}$ be the tent function sitting on a ball $B_0$ defined as in (1.13). If condition* (VD) *holds, then (1.15) is true.*

*Proof.* Let $B_0 = B(x_0, R)$ with $0 < R < \infty$. We have by definition (1.13)

$$\begin{aligned}\|\Phi_{B_0}\|_2^2 &= \int_{B(x_0,R)}\Phi_{B_0}^2 d\mu + \int_{B(x_0,R)^c}\Phi_{B_0}^2 d\mu\\ &= \mu(B_0) + \int_{B(x_0,R)^c}\frac{\mu(B_0)\psi(R)}{V(x_0,x)\psi(x_0,x)}d\mu \quad (\text{since } \Phi_{B_0}=1 \text{ in } B_0)\\ &\leq \mu(B_0) + \frac{C\mu(B_0)\psi(R)}{\psi(R)} = (1+C)\mu(B_0) \ \ (\text{using (3.2) with } x=x_0),\end{aligned}$$

thus showing (1.15). □

3.1. **Condition (ABB$_+$).** In this subsection, we derive condition (ABB$_+$) from condition (ABB) and condition ($J_{\leq}$).

For $n \geq 0$, let $B_n := B(x_0, R_n)$ be an increasing sequence of concentric balls with

$$0 < R_n < R_{n+1} < \infty.$$

Set $U_n := B_{n+1} \setminus B_n$, $R'_n := \frac{1}{2}(R_n + R_{n+1})$, and let

$$B'_n := B(x_0, R'_n) \tag{3.4}$$

be an intermediate ball so that $B_n \subset B'_n \subset B_{n+1}$ for each $n \geq 0$. For any $N \geq 0$, let

$$\Phi_N := \sum_{n=0}^{N} a_n\varphi_n \tag{3.5}$$

for a sequence $\{a_n\}_{n=0}^{\infty}$ of nonnegative numbers, where $\varphi_n \in \text{cutoff}(B_n, B'_n)$ if $R_{n+1} < \overline{R}$, and $\varphi_n = 1$ in $M$ if $R_{n+1} \geq \overline{R}$ (we allow $\overline{R} = \infty$). Note that $\Phi_N \in \mathcal{F} \cap L^\infty$ for any $N \geq 0$, no matter $R_{N+1} < \overline{R}$ or $R_{N+1} \geq \overline{R}$, by using the fact that $1 \in \mathcal{F}$ when $\overline{R} < \infty$ (cf. Proposition 7.1 in Appendix).

**Proposition 3.3.** *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2$ without killing part. Let $\Phi_N$ be defined by (3.5) for a sequence of nonnegative numbers $\{a_n\}_{n=0}^{\infty}$, and $\Omega$ be an open subset of $M$ with $B_N \subset \Omega$, $N \geq 1$. Then for any nonnegative $f \in \mathcal{F}' \cap L^\infty$*

$$\int_\Omega f d\Gamma_\Omega(\Phi_N) \leq \sum_{n=0}^{N} a_n^2\int_\Omega f d\Gamma_\Omega(\varphi_n) + 2\sum_{n=0}^{N-1}\sum_{m=n+1}^{N} a_n a_m \int_{B'_n\times(\Omega\setminus B_m)}(f(x)+f(y))\,dj. \tag{3.6}$$

We remark that the two sets $B'_n$ and $\Omega \setminus B_m$ appearing in the last integral in (3.6) for any $m \geq n+1$ are separated by distance $R_m - R'_n \geq \frac{1}{2}(R_{n+1} - R_n)$, and this property will be needed in order to control the jump part.

*Proof.* By the bilinearity of $d\Gamma_\Omega$,

$$d\Gamma_\Omega(\Phi_N) = d\Gamma_\Omega\Big(\sum_{n=0}^{N} a_n\varphi_n, \sum_{m=0}^{N} a_m\varphi_m\Big) = \sum_{n=0}^{N} a_n^2 d\Gamma_\Omega(\varphi_n) + 2\sum_{n=0}^{N-1}\sum_{m=n+1}^{N} a_n a_m d\Gamma_\Omega(\varphi_n, \varphi_m),$$

from which we see for any $f \in \mathcal{F}' \cap L^\infty$,

$$\int_\Omega f d\Gamma_\Omega(\Phi_N) = \sum_{n=0}^{N} a_n^2 \int_\Omega f d\Gamma_\Omega(\varphi_n) + 2 \sum_{n=0}^{N-1} \sum_{m=n+1}^{N} a_n a_m \int_\Omega f d\Gamma_\Omega(\varphi_n, \varphi_m). \tag{3.7}$$

We look at the last integral in (3.7).

For any $m \geq n+1$, since $\varphi_m = 1$ in $\mathrm{supp}[\varphi_n]$, we see that $d\Gamma_L(\varphi_n, \varphi_m) = 0$ by (2.8), and $\varphi_n \varphi_m = \varphi_n$ in $M$. By definition (2.12), we have for any $m \geq n+1$

$$\begin{aligned} d\Gamma_\Omega(\varphi_n, \varphi_m) &= d\Gamma_L(\varphi_n, \varphi_m) + \int_\Omega (\varphi_n(x) - \varphi_n(y))(\varphi_m(x) - \varphi_m(y)) dj \\ &= \int_\Omega (\varphi_n(x)(1 - \varphi_m(y)) + \varphi_n(y)(1 - \varphi_m(x))) \, dj. \end{aligned}$$

Since $\mathrm{supp}[\varphi_n] \subset B_n'$ and $1 - \varphi_m = 0$ on $B_m$, it follows that for any $m \geq n+1$

$$\begin{aligned} \int_\Omega f d\Gamma_\Omega(\varphi_n, \varphi_m) &= \int_{\Omega \times \Omega} f(x) (\varphi_n(x)(1 - \varphi_m(y)) + \varphi_n(y)(1 - \varphi_m(x))) \, dj \\ &= \int_{B_n' \times (\Omega \setminus B_m)} f(x) \varphi_n(x)(1 - \varphi_m(y)) dj + \int_{(\Omega \setminus B_m) \times B_n'} f(x) \varphi_n(y)(1 - \varphi_m(x)) dj \\ &\leq \int_{B_n' \times (\Omega \setminus B_m)} f(x) dj + \int_{(\Omega \setminus B_m) \times B_n'} f(x) dj \quad (\text{since } 0 \leq \varphi_n \leq 1 \text{ in } M) \\ &= \int_{B_n' \times (\Omega \setminus B_m)} (f(x) + f(y)) \, dj \quad (\text{using the symmetry of } dj). \end{aligned} \tag{3.8}$$

Plugging (3.8) into (3.7), we obtain (3.6). ◻

The following is elementary.

**Proposition 3.4.** *Let* $\{a_n := A^{-1} \widetilde{a}_n\}_{n \geq 0}$ *be a sequence of non-increasing numbers given by*

$$\widetilde{a}_n := \left( \frac{\mu(B_0) \psi(R)}{\mu(B_n) \psi(\lambda^n R)} \right)^{1/2} \quad \text{and} \quad A := \sum_{n=0}^{\infty} \widetilde{a}_n \tag{3.9}$$

*for balls* $B_n := \lambda^n B_0, \lambda > 1, B_0 := B(x_0, R)$ *with* $0 < R < \overline{R}$. *Let*

$$b_k := \sum_{n=k}^{\infty} a_n \quad (k \geq 0) \tag{3.10}$$

*so that* $b_0 = \sum_{n=0}^{\infty} a_n = 1$. *If condition* (VD) *holds, then*

$$\lim_{\lambda \downarrow 1} \left( \sum_{n=0}^{\infty} a_n^2 + \sup_{k \geq 0} \frac{1}{b_{k+1}^2} \sum_{n=k}^{\infty} a_n^2 \right) = 0. \tag{3.11}$$

*Moreover, there exist two positive constants* $C := C(\lambda)$ *and* $C' := C'(\lambda)$, *both of which are independent of* $B_0$, *such that*

$$\sum_{n=0}^{\infty} a_n b_{n+1} \mu(B_n) \leq C(\lambda) \mu(B_0), \tag{3.12}$$

$$\widetilde{a}_n \leq C' a_n \leq C a_{n+1} \leq C^2 b_{n+1} \leq C^2 b_n \leq C^3 \widetilde{a}_n \quad \text{for all } n \geq 0. \tag{3.13}$$

*Proof.* We show that there exist two positive constants $c_1, c_2$ independent of $B_0$ such that for all integers $n, m \geq 0$ and all $\lambda > 1$,

$$c_1 \lambda^{-(\alpha+\beta_2)n/2} \leq \frac{a_{n+m}}{a_m} = \frac{\widetilde{a}_{n+m}}{\widetilde{a}_m} \leq c_2 \lambda^{-\beta_1 n/2}. \tag{3.14}$$

Indeed, we have by definition (3.9)

$$\left( \frac{\widetilde{a}_{n+m}}{\widetilde{a}_m} \right)^2 = \frac{\mu(B_m) \psi(\lambda^m R)}{\mu(B_{n+m}) \psi(\lambda^{n+m} R)} \leq \frac{\psi(\lambda^m R)}{\psi(\lambda^{n+m} R)} \quad (\text{since } \mu(B_m) \leq \mu(B_{n+m}))$$

$$\leq \quad C^{-1}\left(\frac{\lambda^m R}{\lambda^{n+m}R}\right)^{\beta_1} = C^{-1}\lambda^{-\beta_1 n} \text{ (using (1.6))}$$

whilst by (1.4), (1.6)

$$\left(\frac{\widetilde{a}_{n+m}}{\widetilde{a}_m}\right)^2 = \frac{\mu(B_m)\psi(\lambda^m R)}{\mu(B_{n+m})\psi(\lambda^{n+m}R)} \geq C^{-1}\left(\frac{\lambda^m R}{\lambda^{n+m}R}\right)^{\alpha} \cdot C^{-1}\left(\frac{\lambda^m R}{\lambda^{n+m}R}\right)^{\beta_2} = C^{-2}\lambda^{-(\alpha+\beta_2)n},$$

thus showing (3.14) after extracting the square root.

Thus, for all $m \geq 0$ and all $\lambda > 1$,

$$\frac{c_1\widetilde{a}_m}{1-\lambda^{-(\alpha+\beta_2)/2}} \leq \sum_{n=m}^{\infty}\widetilde{a}_n \leq \frac{c_2\widetilde{a}_m}{1-\lambda^{-\beta_1/2}}, \tag{3.15}$$

where $c_1, c_2$ are the same as in (3.14); this is because we have by (3.14)

$$\sum_{n=m}^{\infty}\widetilde{a}_n = \sum_{k=0}^{\infty}\widetilde{a}_{m+k} \leq \sum_{k=0}^{\infty} c_2\lambda^{-\beta_1 k/2}\widetilde{a}_m = \frac{c_2\widetilde{a}_m}{1-\lambda^{-\beta_1/2}},$$

whilst the opposite inequality follows from

$$\sum_{n=m}^{\infty}\widetilde{a}_n = \sum_{k=0}^{\infty}\widetilde{a}_{m+k} \geq \sum_{k=0}^{\infty} c_1\lambda^{-(\alpha+\beta_2)k/2}\widetilde{a}_m = \frac{c_1\widetilde{a}_m}{1-\lambda^{-(\alpha+\beta_2)/2}}.$$

In particular, since $\widetilde{a}_0 = 1$ by definition (3.9), we see by (3.15) with $m = 0$ that

$$\frac{c_1}{1-\lambda^{-(\alpha+\beta_2)/2}} \leq A = \sum_{n=0}^{\infty}\widetilde{a}_n \leq \frac{c_2}{1-\lambda^{-\beta_1/2}}. \tag{3.16}$$

We claim that

$$\frac{c_1 a_k}{1-\lambda^{-(\alpha+\beta_2)/2}} \leq b_k = \sum_{n=k}^{\infty} a_n \leq \frac{c_2 a_k}{1-\lambda^{-\beta_1/2}} \text{ for all } k \geq 0. \tag{3.17}$$

Indeed, for all $k \geq 0$ and all $\lambda > 1$, we see by (3.15) with $m = k$ that

$$b_k = \sum_{n=k}^{\infty} a_n = \sum_{n=k}^{\infty}\frac{\widetilde{a}_n}{A} \leq \frac{c_2}{1-\lambda^{-\beta_1/2}}\frac{\widetilde{a}_k}{A} = \frac{c_2 a_k}{1-\lambda^{-\beta_1/2}},$$

whilst the opposite inequality also follows from

$$b_k = \sum_{n=k}^{\infty}\frac{\widetilde{a}_n}{A} \geq \frac{c_1}{1-\lambda^{-(\alpha+\beta_2)/2}}\frac{\widetilde{a}_k}{A} = \frac{c_1 a_k}{1-\lambda^{-(\alpha+\beta_2)/2}}.$$

This proves our claim.

We now show (3.11). Indeed, it follows from (3.14) that for all $m \geq 0$ and all $\lambda > 1$,

$$\begin{aligned}\sum_{n=m}^{\infty} a_n^2 \quad &= \quad \sum_{n=m}^{\infty}\left(\frac{\widetilde{a}_n}{A}\right)^2 = \frac{1}{A^2}\sum_{k=0}^{\infty}\widetilde{a}_{m+k}^2 \leq \frac{1}{A^2}\sum_{k=0}^{\infty}\left(c_2\lambda^{-\beta_1 k/2}\right)^2\widetilde{a}_m^2 \\ &= \quad \frac{c_2^2}{A^2(1-\lambda^{-\beta_1})}\widetilde{a}_m^2 = \frac{c_2^2}{1-\lambda^{-\beta_1}}a_m^2 \qquad (3.18) \\ &\leq \quad \frac{c_1^{-2}c_2^2\left(1-\lambda^{-(\alpha+\beta_2)/2}\right)^2}{1-\lambda^{-\beta_1}}\widetilde{a}_m^2 \quad \text{(using (3.16))}. \qquad (3.19)\end{aligned}$$

Taking $m = 0$ in (3.19) and using the fact that $\widetilde{a}_0 = 1$, we obtain

$$\lim_{\lambda\downarrow 1}\sum_{n=0}^{\infty} a_n^2 \leq \lim_{\lambda\downarrow 1}\frac{c_1^{-1}c_2^2\left(1-\lambda^{-(\alpha+\beta_2)/2}\right)^2}{1-\lambda^{-\beta_1}} = 0. \tag{3.20}$$

On the other hand, for any $k \geq 0$, we see by (3.17), (3.18) with $m = k$ that

$$\begin{aligned} \frac{1}{b_{k+1}^2}\sum_{n=k}^{\infty} a_n^2 &\leq \left(\frac{1-\lambda^{-(\alpha+\beta_2)/2}}{c_1 a_{k+1}}\right)^2 \frac{c_2^2}{1-\lambda^{-\beta_1}} a_k^2 = \left(\frac{c_2}{c_1}\right)^2 \frac{\left(1-\lambda^{-(\alpha+\beta_2)/2}\right)^2}{1-\lambda^{-\beta_1}} \left(\frac{a_k}{a_{k+1}}\right)^2 \\ &\leq \left(\frac{c_2}{c_1}\right)^2 \frac{\left(1-\lambda^{-(\alpha+\beta_2)/2}\right)^2}{1-\lambda^{-\beta_1}} \cdot \left(c_1^{-1}\lambda^{(\alpha+\beta_2)/2}\right)^2 \quad \text{(using (3.14) with } n=1), \end{aligned}$$

which implies that

$$\limsup_{\lambda\downarrow 1}\sup_{k\geq 0} \frac{1}{b_{k+1}^2}\sum_{n=k}^{\infty} a_n^2 \leq \lim_{\lambda\downarrow 1}\left(\frac{c_2}{c_1^2}\right)^2 \frac{\left(1-\lambda^{-(\alpha+\beta_2)/2}\right)^2}{1-\lambda^{-\beta_1}} \left(c_1^{-1}\lambda^{(\alpha+\beta_2)/2}\right)^2 = 0. \tag{3.21}$$

Combining (3.20), (3.21), we obtain (3.11).

We next show (3.12). Indeed, by (3.17), (3.14),

$$b_{n+1} \leq \frac{c_2 a_{n+1}}{1-\lambda^{-\beta_1/2}} \leq \frac{c_2\left(c_2\lambda^{-\beta_1/2}a_n\right)}{1-\lambda^{-\beta_1/2}} = \frac{c_2^2 a_n}{\lambda^{\beta_1/2}-1}. \tag{3.22}$$

Therefore, it follows from (3.9), (1.6) that

$$\begin{aligned} \sum_{n=0}^{\infty} a_n b_{n+1}\mu(B_n) &\leq \sum_{n=0}^{\infty} \frac{c_2^2 a_n^2}{\lambda^{\beta_1/2}-1}\mu(B_n) = \frac{c_2^2}{\lambda^{\beta_1/2}-1}\sum_{n=0}^{\infty}(A^{-1}\widetilde{a}_n)^2\mu(B_n) \\ &= \frac{c_2^2}{A^2(\lambda^{\beta_1/2}-1)}\sum_{n=0}^{\infty}\frac{\mu(B_0)\psi(R)}{\psi(\lambda^n R)} \leq C(\lambda)\mu(B_0)\sum_{n=0}^{\infty} C^{-1}\left(\frac{R}{\lambda^n R}\right)^{\beta_1} \\ &\leq C(\lambda)\mu(B_0), \end{aligned}$$

thus proving (3.12).

It remains to show (3.13). Indeed, we have the following equivalences (up to constant $C(\lambda)$)

$$\widetilde{a}_n = Aa_n \overset{\text{by (3.14)}}{\asymp} a_{n+1} \overset{\text{by (3.17)}}{\asymp} b_{n+1}$$

for all $n \geq 0$, and thus (3.13) follows. The proof is complete. □

Condition ($\text{ABB}_+$) will follow from conditions (ABB), ($\text{J}_\leq$).

**Lemma 3.5.** *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2$ without killing part. Then*

$$(\text{VD}) + (\text{ABB}) + (\text{J}_\leq) \Rightarrow (\text{ABB}_+). \tag{3.23}$$

In order to prove the implication (3.23), one needs to construct a new cutoff function $\phi$ for any ball $B_0$ in $M$ by using condition (ABB). Although the proof is quite technical, the idea of constructing such a new cutoff function $\phi$ is simple (see [1, Lemma 2.1 and Lemma 5.1] for the unbounded metric space): starting from any ball $B_0$, dividing the space $M$ into finitely many (when $M$ is bounded) or infinitely many (when $M$ is unbounded) concentric balls $\{B_n\}_{n=0}^{\infty}$, and letting $\varphi_n$ be the cutoff function for neighboring concentric balls $B_n, B_n'$ in $B_{n+1}$ by using condition (ABB), and then considering a finite or an infinite combination $\phi = \sum_{n\geq 0} a_n\varphi_n$, we will see that such a function $\phi$ is the desired by choosing suitable coefficients $\{a_n\}$ and suitable radii of balls $B_n$.

*Proof.* Let $0 < R < \overline{R}$, and

$$R_n := \lambda^n R \text{ and } R_n' := \frac{\lambda+1}{2}R_n \quad \text{for any } n \geq 0,$$

where $\lambda > 1$ is some number to be chosen later on. Set

$$B_n := B(x_0, R_n), B_n' := B(x_0, R_n') \text{ for } n \geq 0$$

so that $B_n \subset B'_n \subset B_{n+1}$. For those $n$'s with $R_{n+1} < \overline{R}$, applying condition (ABB$_\zeta$) to the triple $(B_n, B'_n, B_{n+1})$, we have for all $u \in \mathcal{F}' \cap L^\infty$

$$\begin{aligned}\int_{B_{n+1}} u^2 d\Gamma_{B_{n+1}}(\varphi_n) &\leq \zeta \int_{B_{n+1}} d\Gamma_{B_{n+1}}(u) + \frac{C}{\psi(R'_n - R_n)} \int_{B_{n+1}} u^2 d\mu \\ &\leq \zeta \int_{B_{n+1}} d\Gamma(u) + \frac{C(\lambda)}{\psi(R)} \int_{B_{n+1}} u^2 d\mu, \end{aligned} \tag{3.24}$$

for some $\varphi_n \in \text{cutoff}(B_n, B'_n)$, where we have used the fact that

$$\psi(R'_n - R_n) = \psi\Big(\frac{\lambda-1}{2} R_n\Big) \geq \psi\Big(\frac{\lambda-1}{2} R\Big) \geq c(\lambda)\psi(R) \tag{3.25}$$

for all $n \geq 0$ by using (1.6). For those $n$'s with $R_{n+1} \geq \overline{R}$ and $\overline{R} < \infty$, we take $\varphi_n = 1$ in $M$ so that (3.24) is automatically satisfied at this time.

Let $\phi$ be the function defined by

$$\phi = \sum_{n=0}^{\infty} a_n \varphi_n \tag{3.26}$$

where $\{a_n\}_{n\geq 0}$ is given in Proposition 3.4 and $\{\varphi_n\}_{n\geq 0}$ defined as above. In order to prove condition (ABB$_+$), we divide the proof into three steps.

*Step* 1. We show $\phi \in \text{cutoff}(B_0, M)$. For this, we first show $\phi \in \mathcal{F}$. Consider a partial summation

$$\Phi_N := \sum_{n=0}^{N} a_n \varphi_n.$$

It suffices to show that the sequence $\{\Phi_N\}_{N=0}^{\infty}$ converges to $\phi$ in the norm of $\mathcal{F}$. For this, it is enough to show that $\{\Phi_N\}$ is a Cauchy in $\mathcal{F}$: for any $k \geq 0$,

$$\|\Phi_{N+k} - \Phi_N\|_{\mathcal{F}} = \Big\| \sum_{n=N+1}^{N+k} a_n \varphi_n \Big\|_{\mathcal{F}} \leq \sum_{n=N+1}^{\infty} a_n \|\varphi_n\|_{\mathcal{F}} = \sum_{n=N+1}^{\infty} a_n \left(\|\varphi_n\|_2^2 + \mathcal{E}(\varphi_n)\right)^{1/2} \to 0 \tag{3.27}$$

as $N \to \infty$.

Indeed, as $0 \leq \varphi_n \leq 1$ in $M$, we see

$$\|\varphi_n\|_2^2 = \int_M \varphi_n^2 d\mu = \int_{B'_n} \varphi_n^2 d\mu \leq \mu(B'_n) \leq \mu(B_{n+1}). \tag{3.28}$$

On the other hand, letting $u = 1$ in (3.24), we have

$$\int_{B_{n+1}} d\Gamma_{B_{n+1}}(\varphi_n) \leq \frac{C(\lambda)}{\psi(R)} \int_{B_{n+1}} d\mu = \frac{C(\lambda)\mu(B_{n+1})}{\psi(R)}. \tag{3.29}$$

Applying (2.14) with $\Omega = M, f = u^2, U = B_{n+1}, \varphi = \varphi_n$ where $\text{supp}[\varphi_n] \subset B'_n \subset B_{n+1}$, we obtain for all $u \in \mathcal{F}' \cap L^\infty$

$$\begin{aligned}\int_M u^2 d\Gamma(\varphi_n) &\leq \int_{B_{n+1}} u^2 d\Gamma_{B_{n+1}}(\varphi_n) + \int_{B_{n+1}\times B_{n+1}^c} (u^2(x) + u^2(y))\varphi_n^2(x) dj \\ &\leq \int_{B_{n+1}} u^2 d\Gamma_{B_{n+1}}(\varphi_n) + \int_{B'_n\times B_{n+1}^c} (u^2(x) + u^2(y)) dj. \end{aligned} \tag{3.30}$$

In particular, when $u = 1$, it follows that

$$\begin{aligned}\mathcal{E}(\varphi_n) &= \int_M d\Gamma(\varphi_n) \leq \int_{B_{n+1}} d\Gamma_{B_{n+1}}(\varphi_n) + 2\int_{B'_n\times B_{n+1}^c} dj \\ &\leq \frac{C(\lambda)\mu(B_{n+1})}{\psi(R)} + 2\int_{B'_n\times B_{n+1}^c} dj \quad \text{(using (3.29))} \\ &\leq \frac{C'(\lambda)\mu(B_{n+1})}{\psi(R)} \end{aligned} \tag{3.31}$$

since, using (3.3) with $R_1, R_2$ being respectively replaced by $R'_n, R_{n+1}$, we have

$$\begin{aligned}\int_{B'_n\times B^c_{n+1}} dj &= \int_{B'_n}\left(\int_{B^c_{n+1}} J(x,y)d\mu(y)\right)d\mu(x) \le \int_{B'_n}\frac{C}{\psi(R_{n+1}-R'_n)}d\mu(x)\\ &\le \frac{C(\lambda)\mu\left(B'_n\right)}{\psi(R)} \le \frac{C(\lambda)\mu\left(B_{n+1}\right)}{\psi(R)}.\end{aligned}$$

Thus we see by (3.28), (3.31) that

$$\|\varphi_n\|_2^2 + \mathcal{E}(\varphi_n) \le \mu\left(B_{n+1}\right) + \frac{C'(\lambda)\mu\left(B_{n+1}\right)}{\psi(R)} \le C\mu\left(B_n\right)$$

for some constant $C > 0$ independent of $n$. Therefore, it follows from (3.27) that

$$\begin{aligned}\|\Phi_{N+k}-\Phi_N\|_{\mathcal{F}} &\le \sum_{n=N+1}^{\infty} a_n\left(\|\varphi_n\|_2^2+\mathcal{E}(\varphi_n)\right)^{1/2} \le \sum_{n=N+1}^{\infty} a_n\left(C\mu\left(B_n\right)\right)^{1/2}\\ &= \sqrt{C}\sum_{n=N+1}^{\infty}\frac{\widetilde{a}_n}{A}\mu\left(B_n\right)^{1/2} = \frac{\sqrt{C}}{A}\sum_{n=N+1}^{\infty}\left(\frac{\mu(B_0)\psi(R)}{\mu(B_n)\psi(\lambda^n R)}\right)^{1/2}\mu\left(B_n\right)^{1/2} \ \text{(using (3.9))}\\ &= \frac{\sqrt{C}}{A}\left(\mu(B_0)\psi(R)\right)^{1/2}\sum_{n=N+1}^{\infty}\frac{1}{\psi(\lambda^n R)^{1/2}} \le \frac{C'}{\psi(\lambda^{N+1}R)} \quad \text{(using (1.6))},\end{aligned}$$

which tends to 0 as $N \to \infty$, thus showing that $\{\Phi_N\}$ is a Cauchy in $\mathcal{F}$. Since $\Phi_N$ converges pointwise to $\phi$, we see that $\phi \in \mathcal{F}$, and $\|\Phi_N - \phi\|_{\mathcal{F}} \to 0$ as $N \to \infty$. Noting that $\phi = \sum_{n=0}^{\infty} a_n\varphi_n = 1$ on $B_0$, we see $\phi \in \text{cutoff}(B_0, M)$, as desired.

*Step* 2. We show that (1.17) holds for the function $\phi$ defined by (3.26) and $\{a_n\}_{n\ge 0}$ given by (3.9), if $\lambda$ is sufficiently close to 1.

Indeed, applying (3.6) with $f = u^2, \Omega = M$ and then letting $N \to \infty$, we obtain that for any $u \in \mathcal{F}'\cap L^\infty$

$$\begin{aligned}\int_M u^2 d\Gamma(\phi) &= \lim_{N\to\infty}\int_M u^2 d\Gamma(\Phi_N) \quad \text{(using (2.13) since } \Phi_N \to \phi \text{ in } \mathcal{F})\\ &\le \sum_{n=0}^{\infty} a_n^2\int_M u^2 d\Gamma(\varphi_n) + 2\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} a_n a_m\int_{B'_n\times B^c_m}\left(u^2(x)+u^2(y)\right)dj\\ &\le \sum_{n=0}^{\infty} a_n^2\left(\int_{B_{n+1}} u^2 d\Gamma_{B_{n+1}}(\varphi_n) + \int_{B'_n\times B^c_{n+1}}(u^2(x)+u^2(y))dj\right) \quad \text{(by (3.30))}\\ &\quad +2\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} a_n a_m\int_{B'_n\times B^c_m}\left(u^2(x)+u^2(y)\right)dj. \end{aligned}\tag{3.32}$$

The second term on the right-hand side in (3.32) is absorbed into the third (double summation) when $m = n+1$ because $a_n^2 \asymp a_n a_{n+1}$ by (3.13). Thus, it follows that

$$\begin{aligned}\int_M u^2 d\Gamma(\phi) &\le \sum_{n=0}^{\infty} a_n^2\int_{B_{n+1}} u^2 d\Gamma_{B_{n+1}}(\varphi_n) + C(\lambda)\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} a_n a_m\int_{B'_n\times B^c_m}\left(u^2(x)+u^2(y)\right)dj\\ &\le \sum_{n=0}^{\infty} a_n^2\left(\zeta\int_{B_{n+1}} d\Gamma(u) + \frac{C(\lambda)}{\psi(R)}\int_{B_{n+1}} u^2 d\mu\right) \quad \text{(by (3.24))}\\ &\quad +C(\lambda)\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} a_n a_m\int_{B'_n\times B^c_m}\left(u^2(x)+u^2(y)\right)dj. \end{aligned}\tag{3.33}$$

The double summation on the right-hand side in (3.33) contains the following term

$$\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} a_n a_m\int_{B'_n\times B^c_m} u^2(x)dj \le \sum_{n=0}^{\infty} a_n\left(\sum_{m=n+1}^{\infty} a_m\right)\int_{B'_n\times B^c_{n+1}} u^2(x)dj \ \text{(since } B^c_m \subset B^c_{n+1})$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} a_n b_{n+1} \int_{B'_n} u^2(x) \left( \int_{B^c_{n+1}} dj \right) d\mu(x) \ \ (\text{where } b_n = \sum_{m=n}^{\infty} a_m) \\ &\leq C(\lambda) \sum_{n=0}^{\infty} \frac{a_n^2}{\psi(R)} \int_{B_{n+1}} u^2 d\mu \end{aligned}$$

by using (3.3) and the fact that $b_{n+1} \asymp a_n$ in (3.13), and hence, it is absorbed into the second term on the right-hand side in (3.33). Rearranging the terms in (3.33), we conclude that

$$\begin{aligned} \int_M u^2 d\Gamma(\phi) &\leq \zeta \sum_{n=0}^{\infty} a_n^2 \int_{B_{n+1}} d\Gamma(u) + \frac{C_1(\lambda)}{\psi(R)} \sum_{n=0}^{\infty} a_n^2 \int_{B_{n+1}} u^2 d\mu \\ &\quad + C_2(\lambda) \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} a_n a_m \int_{B'_n \times B^c_m} u^2(y) dj \\ &=: \zeta I_1 + \frac{C_1(\lambda)}{\psi(R)} I_2 + C_2(\lambda) I_3 \end{aligned} \tag{3.34}$$

for two positive constants $C_1(\lambda), C_2(\lambda)$ depending only on $\lambda$. We estimate $I_1, I_2, I_3$ separately.

Indeed, set $U_m := B_{m+1} \setminus B_m$. For any $m \geq 1$, note that

$$\begin{aligned} \phi &= \sum_{n=0}^{\infty} a_n \varphi_n = \sum_{n=0}^{m-1} a_n \varphi_n + a_m \varphi_m + \sum_{n=m+1}^{\infty} a_n \varphi_n \\ &= a_m \varphi_m + b_{m+1} \leq a_m + b_{m+1} = b_m \ \text{ on } U_m, \end{aligned}$$

since on $U_m$, we have $\varphi_n = 0$ for any $n \leq m-1$ whilst $\varphi_n = 1$ for any $n \geq m+1$. Thus

$$b_{m+1} \leq \phi \leq b_m \text{ on } B_{m+1} \setminus B_m \text{ for all } m \geq 0. \tag{3.35}$$

For $I_1$, we have by (3.35)

$$\begin{aligned} I_1 &= \sum_{n=0}^{\infty} a_n^2 \int_{B_{n+1}} d\Gamma(u) = \sum_{n=0}^{\infty} a_n^2 \left( \int_{B_0} d\Gamma(u) + \sum_{m=0}^{n} \int_{U_m} d\Gamma(u) \right) \\ &\leq \sum_{n=0}^{\infty} a_n^2 \int_{B_0} \phi^2 d\Gamma(u) + \sum_{m=0}^{\infty} \left( \sum_{n=m}^{\infty} a_n^2 \right) \int_{U_m} \left( \frac{\phi}{b_{m+1}} \right)^2 d\Gamma(u) \ \ (\text{since } \phi = 1 \text{ on } B_0) \\ &\leq \max\left\{ \sum_{n=0}^{\infty} a_n^2, \sup_{k\geq 0} \frac{1}{b_{k+1}^2} \sum_{n=k}^{\infty} a_n^2 \right\} \int_M \phi^2 d\Gamma(u) \leq \frac{1}{8(\zeta+1)} \int_M \phi^2 d\Gamma(u) \end{aligned} \tag{3.36}$$

if $\lambda$ is close enough to 1 by using (3.11).

For $I_2$, we similarly have

$$\begin{aligned} I_2 &= \sum_{n=0}^{\infty} a_n^2 \int_{B_{n+1}} u^2 d\mu = \sum_{n=0}^{\infty} a_n^2 \left( \int_{B_0} u^2 d\mu + \sum_{m=0}^{n} \int_{U_m} u^2 d\mu \right) \\ &\leq \max\left\{ \sum_{n=0}^{\infty} a_n^2, \sup_{k\geq 0} \frac{1}{b_{k+1}^2} \sum_{n=k}^{\infty} a_n^2 \right\} \int_M \phi^2 u^2 d\mu \leq \frac{1}{8(\zeta+1)} \int_M \phi^2 u^2 d\mu \end{aligned} \tag{3.37}$$

if $\lambda$ is close enough to 1.

For $I_3$, we need more care. In fact, for any $n \geq 0$ and any $k \geq m \geq n+1$

$$\mathrm{dist}(B'_n, B^c_k) \geq R_k - R'_n = \left(1 - \frac{\lambda+1}{2}\lambda^{n-k}\right) R_k \geq \left(1 - \frac{\lambda+1}{2}\lambda^{-1}\right) R_k = \frac{1-\lambda^{-1}}{2} R_k,$$

and hence, we have by condition ($J_{\leq}$) that for any $y \in B^c_k$

$$\int_{B'_n} J(x,y) d\mu(x) \leq \int_{B'_n} \frac{C d\mu(x)}{V(x,y)\psi(x,y)} \leq \int_{B'_n} \frac{C d\mu(x)}{V(x, \frac{1-\lambda^{-1}}{2} R_k) \psi(\frac{1-\lambda^{-1}}{2} R_k)}$$

$$
\begin{aligned}
&\le \frac{C'(\lambda)\mu(B_{n+1})}{V(x_0,R_k)\psi(R_k)} = \frac{C'(\lambda)\left(\widetilde{a}_k\right)^2\mu(B_{n+1})}{\mu(B_0)\psi(R)} \text{ (using (1.3), (3.9))}\\
&\le \frac{C(\lambda)b_{k+1}^2\mu(B_n)}{\mu(B_0)\psi(R)} \text{ (using (3.13), (1.4)).}
\end{aligned}
$$

It follows by exchanging the order of summations that

$$
\begin{aligned}
I_3 &= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} a_n a_m \int_{B_n'\times B_m^c} u^2(y)dj = \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} a_n a_m \int_{B_m^c} u^2(y)\left(\int_{B_n'} J(x,y)d\mu(x)\right)d\mu(y)\\
&= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} a_n a_m \sum_{k=m}^{\infty}\int_{B_{k+1}\setminus B_k} u^2(y)\left(\int_{B_n'} J(x,y)d\mu(x)\right)d\mu(y)\\
&\le \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} a_n a_m \sum_{k=m}^{\infty}\frac{C(\lambda)b_{k+1}^2\mu(B_n)}{\mu(B_0)\psi(R)}\int_{B_{k+1}\setminus B_k}\left(\frac{\phi}{b_{k+1}}\right)^2 u^2 d\mu \text{ (using (3.35))}\\
&= \frac{C(\lambda)}{\mu(B_0)\psi(R)}\sum_{k=1}^{\infty}\int_{B_{k+1}\setminus B_k}\phi^2u^2d\mu\left(\sum_{n=0}^{k-1}a_n\mu(B_n)\sum_{m=n+1}^{k}a_m\right)\\
&\le \frac{C(\lambda)}{\mu(B_0)\psi(R)}\sum_{k=1}^{\infty}\int_{B_{k+1}\setminus B_k}\phi^2u^2d\mu\cdot C\mu(B_0) = \frac{C'(\lambda)}{\psi(R)}\int_{B_1^c}\phi^2u^2d\mu, \qquad (3.38)
\end{aligned}
$$

since by (3.12)

$$
\sum_{n=0}^{k-1}a_n\mu(B_n)\sum_{m=n+1}^{k}a_m \le \sum_{n=0}^{k-1}a_nb_{n+1}\mu(B_n) \le C\mu(B_0).
$$

Therefore, substituting (3.38), (3.37), (3.36) into (3.34), we conclude that

$$
\begin{aligned}
\int_M u^2d\Gamma(\phi) &\le \zeta I_1 + \frac{C_1(\lambda)}{\psi(R)}I_2 + C_2(\lambda)I_3\\
&\le \frac{1}{8}\int_M\phi^2d\Gamma(u) + \frac{C(\lambda)}{\psi(R)}\int_M\phi^2u^2d\mu
\end{aligned}
$$

if $\lambda$ is close enough to 1, thus proving (1.17).

*Step* 3. It remains to show (1.16). Indeed, we have by definition (1.13) that $\Phi_{B_0} = \phi = 1$ on $B_0$, whilst for any $x \in U_m = B_{m+1}\setminus B_m$ $(m \ge 0)$

$$
\begin{aligned}
\Phi_{B_0}(x) &= \left(\frac{\mu(B_0)\psi(R)}{V(x_0,x)\psi(x_0,x)}\right)^{1/2} \asymp \left(\frac{\mu(B_0)\psi(R)}{\mu(B_m)\psi(\lambda^mR)}\right)^{1/2} = \widetilde{a}_m \text{ (by definition (3.9))}\\
&\asymp b_m \asymp \phi \text{ (by (3.13), (3.35)),}
\end{aligned}
$$

thus showing (1.16). The proof is complete. □

3.2. **Condition (SL$_2$).** In this subsection, we derive condition (SL$_2$) from conditions (ABB$_+$), (J$_\le$).

**Lemma 3.6.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2$ without killing part. Then*

$$
\text{(VD)} + (\text{ABB}_+) + \left(\text{Cap}_\le\right) + (\text{J}_\le) \Rightarrow (\text{SL}_2).
$$

*Consequently, we have*

$$
\text{(VD)} + (\text{Gcap}_+) + (\text{J}_\le) \Rightarrow (\text{SL}_2).
$$

*Proof.* Let $\delta \in (0,1)$ be a number to be picked up later on. Let

$$
B_0 := B(x_0,R) \text{ and } B := B(x_0,r)
$$

be any two concentric balls with $0 < R < r < \overline{R}$ so that $B_0 \subset B$. Without loss of generality, assume that

$$
0 < 2\delta^{-1}R < r < \overline{R}. \qquad (3.39)
$$

Otherwise, condition ($SL_2$) is automatically true, since if $\frac{1}{2}\delta r \le R < r$, then

$$\frac{\psi(R)}{\psi(r)} \ge \frac{\psi(\frac{1}{2}\delta r)}{\psi(r)} \ge C^{-1} > 0,$$

and hence, inequality (1.22) is satisfied by Remark 1.1.

By condition ($ABB_+$), there exists some $\phi \in \text{cutoff}(B_0, M)$ such that both (1.16) and (1.17) are satisfied. By (1.16), the function $\phi$ decays at the same rate as the tent function $\Phi_{B_0}$ sitting on $B_0$, and hence, there exists some number $\delta \in (0,1)$ such that

$$\begin{aligned} \operatorname*{esup}_{(\delta^{-1}B_0)^c} \phi^2 &\le \operatorname*{esup}_{(\delta^{-1}B_0)^c} (C\Phi_{B_0})^2 = C^2 \operatorname*{esup}_{x\in(\delta^{-1}B_0)^c} \frac{\mu(B_0)\psi(R)}{V(x_0,x)\psi(x_0,x)} \\ &\le C^2 \frac{\mu(B_0)\psi(R)}{\mu(\delta^{-1}B_0)\psi(\delta^{-1}R)} \le C^2 \frac{\psi(R)}{\psi(\delta^{-1}R)} \le C' \left(\frac{R}{\delta^{-1}R}\right)^{\beta_1} \le \frac{1}{4} \end{aligned}$$

if $\delta$ is sufficiently small. With a choice of $\delta$, we can have

$$\operatorname*{esup}_{(\delta^{-1}B_0)^c} \phi \le \frac{1}{2}. \tag{3.40}$$

(The set $(\delta^{-1}B_0)^c$ may be empty if $(M,d)$ is bounded. In this case, we have $\operatorname{esup}_{(\delta^{-1}B_0)^c} \phi = 0$, and (3.40) is also true.)

Define a function $\phi_{B_0}$ by

$$\phi_{B_0} := (\phi - a)_+ \quad \text{with} \quad a := \operatorname*{esup}_{(\frac{1}{2}B)^c} \phi \ge 0. \tag{3.41}$$

Since $\delta^{-1}B_0 = B(x_0, \delta^{-1}R) \subset B(x_0, \frac{r}{2}) = \frac{1}{2}B$ by using (3.39), we see by (3.40)

$$a = \operatorname*{esup}_{(\frac{1}{2}B)^c} \phi \le \operatorname*{esup}_{(\delta^{-1}B_0)^c} \phi \le \frac{1}{2},$$

and hence, $\phi_{B_0} = (\phi - a)_+ \ge \frac{1}{2}$ on $B_0$, since $\phi = 1$ on $B_0$.

We claim that $\phi_{B_0} \in \mathcal{F}(\frac{1}{2}B)$, and

$$d\Gamma(\phi_{B_0}) \le d\Gamma(\phi) \quad \text{in } M \tag{3.42}$$

$$\phi_{B_0} \le \phi \le \phi_{B_0} + Cb \quad \text{in } M \text{ with } b := \left(\frac{\mu(B_0)\psi(R)}{\mu(B)\psi(r)}\right)^{1/2}, \tag{3.43}$$

where $C$ is some universal constant independent of $B_0, B$.

Indeed, by the Markov property of $(\mathcal{E},\mathcal{F})$, we see $\phi_{B_0} = (\phi - a)_+ \in \mathcal{F}$. Clearly, $\phi_{B_0} = 0$ in $(\frac{1}{2}B)^c$ by definition (3.41), thus showing $\phi_{B_0} \in \mathcal{F}(\frac{1}{2}B)$ by using [16, Corollary 2.3.1 on p. 98].

To show (3.42), noting that $|\phi_{B_0}(x) - \phi_{B_0}(y)| \le |\phi(x) - \phi(y)|$ for any points $x, y$ in $M$, we have

$$\int_M |\phi_{B_0}(x) - \phi_{B_0}(y)|^2 J(x,y) d\mu(y) \le \int_M |\phi(x) - \phi(y)|^2 J(x,y) d\mu(y),$$

whilst by (2.9)

$$d\Gamma_L(\phi_{B_0}) = d\Gamma_L((\phi - a)_+) = \mathbf{1}_{\{\phi > a\}} d\Gamma_L(\phi) \le d\Gamma_L(\phi).$$

It follows by definition (2.12) that

$$\begin{aligned} d\Gamma(\phi_{B_0})(x) &= d\Gamma_L(\phi_{B_0})(x) + \left\{\int_M (\phi_{B_0}(x) - \phi_{B_0}(y))^2 J(x,y) d\mu(y)\right\} d\mu(x) \\ &\le d\Gamma_L(\phi)(x) + \left\{\int_M (\phi(x) - \phi(y))^2 J(x,y) d\mu(y)\right\} d\mu(x) = d\Gamma(\phi)(x), \end{aligned}$$

thus showing (3.42).

By definition (3.41) and using (1.16), (1.13), we have

$$\phi_{B_0} \le \phi \le \phi_{B_0} + a = \phi_{B_0} + \operatorname*{esup}_{(\frac{1}{2}B)^c} \phi \le \phi_{B_0} + C \operatorname*{esup}_{(\frac{1}{2}B)^c} \Phi_{B_0}$$

$$\leq \quad \phi_{B_0} + C\left(\frac{\mu(B_0)\psi(R)}{\mu(\frac{1}{2}B)\psi(\frac{1}{2}r)}\right)^{1/2} \leq \phi_{B_0} + C'b,$$

thus showing (3.43). This proves our claim.

Note that by condition $\left(\mathrm{Cap}_{\leq}\right)$, there exists some $\phi_B \in \mathrm{cutoff}(\frac{3}{4}B, B)$ such that

$$\mathcal{E}(\phi_B) = \int_M d\Gamma(\phi_B) \leq \frac{C\mu(B)}{\psi(r)}. \tag{3.44}$$

We now have three functions $\phi_{B_0}, \phi, \phi_B$ with concentric balls $B_0 \subset \delta^{-1}B_0 \subset \frac{1}{2}B$, see Figure 2.

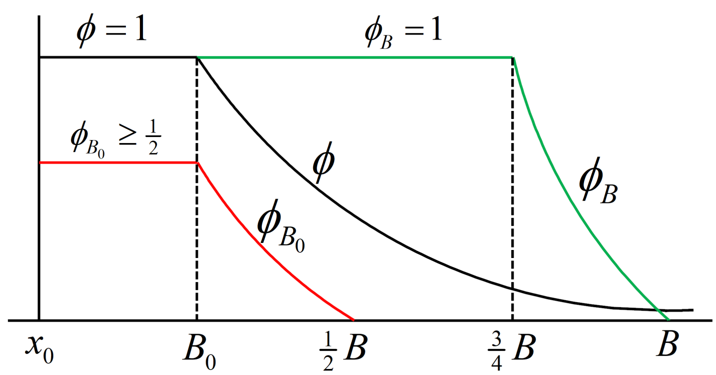


FIGURE 2. Three functions $\phi_{B_0}, \phi, \phi_B$ and concentric balls $B_0 \subset \delta^{-1}B_0 \subset \frac{1}{2}B$.

We define the integral $I(t)$ by

$$I(t) := \int_M (1 - P_t^B 1_B)^2 \, \phi_{B_0}^2 d\mu \quad (t > 0)$$

where $\phi_{B_0}$ is given by (3.41). Then for any $t > 0$

$$\begin{aligned} \frac{d}{dt}I(t) &= 2\int_M (1 - P_t^B 1_B)\,\phi_{B_0}^2 \left(-\frac{\partial}{\partial t}P_t^B 1_B\right) d\mu = 2\mathcal{E}(P_t^B 1_B, (1 - P_t^B 1_B)\,\phi_{B_0}^2) \\ &= -2\mathcal{E}(\phi_B - P_t^B 1_B, (1 - P_t^B 1_B)\,\phi_{B_0}^2) + 2\mathcal{E}(\phi_B, (1 - P_t^B 1_B)\,\phi_{B_0}^2) \\ &=: I_1(t) + I_2(t). \end{aligned} \tag{3.45}$$

To estimate $I_1(t)$, note that

$$\phi_{B_0}^2 = \phi_{B_0}^2 \phi_B \ \text{ in } M \tag{3.46}$$

since $\mathrm{supp}[\phi_{B_0}] \subset \frac{1}{2}B$ and $\phi_B = 1$ in $\frac{3}{4}B$. It follows from (2.16), with $u$ being replaced by $\phi_{B_0}$ and $\phi$ by $\phi_B - P_t^B 1_B$, that

$$\begin{aligned} I_1(t) &= -2\mathcal{E}(\phi_B - P_t^B 1_B, (1 - P_t^B 1_B)\,\phi_{B_0}^2) \\ &= -2\mathcal{E}(\phi_B - P_t^B 1_B, (\phi_B - P_t^B 1_B)\,\phi_{B_0}^2) \ \ \text{(using (3.46))} \\ &= 4\int_M (\phi_B - P_t^B 1_B)^2 d\Gamma(\phi_{B_0}) - \int_M \phi_{B_0}^2 d\Gamma(\phi_B - P_t^B 1_B). \end{aligned} \tag{3.47}$$

On the other hand, we see by condition $(\mathrm{ABB}_+)$ that

$$\begin{aligned} \int_M (\phi_B - P_t^B 1_B)^2 d\Gamma(\phi_{B_0}) &\leq \int_M (\phi_B - P_t^B 1_B)^2 d\Gamma(\phi) \ \ \text{(using (3.42))} \\ &\leq \frac{1}{8}\int_M \phi^2 d\Gamma(\phi_B - P_t^B 1_B) + \frac{C}{\psi(R)}\int_M (\phi_B - P_t^B 1_B)^2 \phi^2 d\mu. \end{aligned}$$

From this and using the fact that $\phi^2 \le 2(\phi_{B_0}^2 + C^2 b^2)$ in $M$ by (3.43), we obtain by (3.47)

$$\begin{aligned} I_1(t) &\le \int_M \left(\frac{1}{2}\phi^2 - \phi_{B_0}^2\right) d\Gamma(\phi_B - P_t^B 1_B) + \frac{4C}{\psi(R)}\int_M (\phi_B - P_t^B 1_B)^2 \phi^2 d\mu \\ &\le C^2 b^2 \int_M d\Gamma(\phi_B - P_t^B 1_B) + \frac{4C}{\psi(R)}\int_M (\phi_B - P_t^B 1_B)^2 (2\phi_{B_0}^2 + 2C^2 b^2) d\mu \\ &= C^2 b^2 \mathcal{E}(\phi_B - P_t^B 1_B) + \frac{8C}{\psi(R)}\left\{ I(t) + C^2 b^2 \int_M (\phi_B - P_t^B 1_B)^2 d\mu \right\} \\ &\le C\left\{ \frac{1}{\psi(R)} I(t) + b^2 \mathcal{E}(P_t^B 1_B) + \frac{\mu(B_0)}{\psi(r)} \right\} \end{aligned} \tag{3.48}$$

for some universal constant $C > 0$ independent of $B_0, B, t$, where we have used the facts that

$$\mathcal{E}(\phi_B - P_t^B 1_B) \le 2\left(\mathcal{E}(\phi_B) + \mathcal{E}(P_t^B 1_B)\right) \le \frac{2C\mu(B)}{\psi(r)} + 2\mathcal{E}(P_t^B 1_B) \ \ \text{(using (3.44))}$$

and that, using definition of $b$ in (3.43) and condition (VD),

$$\frac{b^2}{\psi(R)}\int_M (\phi_B - P_t^B 1_B)^2 d\mu \le \frac{b^2}{\psi(R)}\int_B d\mu = \frac{\mu(B)}{\psi(R)} \cdot \frac{\mu(B_0)\psi(R)}{\mu(B)\psi(r)} = \frac{\mu(B_0)}{\psi(r)}.$$

In order to estimate $I_2(t)$, noting that $\phi_B$ is constant on the support of function $(1 - P_t^B 1_B)\,\phi_{B_0}^2$ so that

$$\mathcal{E}^{(L)}(\phi_B, (1 - P_t^B 1_B)\,\phi_{B_0}^2) = 0,$$

it follows that

$$\begin{aligned} I_2(t) &= 2\mathcal{E}(\phi_B, (1 - P_t^B 1_B)\,\phi_{B_0}^2) = 2\mathcal{E}^{(J)}(\phi_B, (1 - P_t^B 1_B)\,\phi_{B_0}^2) \\ &= 4\int_M (1 - P_t^B 1_B(x))\,\phi_{B_0}^2(x) \left\{ \int_M (\phi_B(x) - \phi_B(y)) J(x,y) d\mu(y) \right\} d\mu(x) \\ &= 4\int_{\frac{1}{2}B} (1 - P_t^B 1_B(x))\,\phi_{B_0}^2(x) \left\{ \int_{(\frac{3}{4}B)^c} (1 - \phi_B(y)) J(x,y) d\mu(y) \right\} d\mu(x) \\ &\le \frac{C}{\psi(r)} \int_{\frac{1}{2}B} (1 - P_t^B 1_B(x))\,\phi_{B_0}^2(x) d\mu(x) \quad \text{(using (J}_\le\text{) and (3.3))} \\ &\le \frac{C}{\psi(r)} \int_{\frac{1}{2}B} \phi^2 d\mu \le \frac{C}{\psi(r)} \int_M (C\Phi_{B_0})^2 d\mu \quad \text{(using (3.43) and (1.16))} \\ &\le C' \frac{\mu(B_0)}{\psi(r)} \ \ \text{(using (1.15))}. \end{aligned} \tag{3.49}$$

Therefore, plugging (3.49), (3.48) into (3.45), we obtain that for any $t > 0$

$$\frac{d}{dt} I(t) \le I_1(t) + I_2(t) \le C\left\{ \frac{1}{\psi(R)} I(t) + b^2 \mathcal{E}(P_t^B 1_B) + \frac{\mu(B_0)}{\psi(r)} \right\}.$$

Integrating over $(0, t)$ and then using $I(0) = 0$, we have for any $t > 0$

$$\begin{aligned} \int_M (1 - P_t^B 1_B)^2\,\phi_{B_0}^2 d\mu &= I(t) \le C \exp\left(\frac{Ct}{\psi(R)}\right)\left(b^2 \int_0^t \mathcal{E}(P_s^B 1_B) ds + \frac{\mu(B_0)}{\psi(r)} t\right) \\ &\le C \exp\left(\frac{Ct}{\psi(R)}\right)\left(\frac{\mu(B_0)\psi(R)}{\psi(r)} + \frac{\mu(B_0)}{\psi(r)} t\right), \end{aligned} \tag{3.50}$$

where we have used the fact that

$$b^2 \int_0^t \mathcal{E}(P_s^B 1_B) ds \le \frac{\mu(B_0)\psi(R)}{2\psi(r)},$$

because we have $-\frac{d}{ds}\left\|P_s^B 1_B\right\|_2^2 = 2\mathcal{E}(P_s^B 1_B)$ for any $s > 0$, which implies that

$$\begin{aligned} b^2\int_0^t \mathcal{E}(P_s^B 1_B)ds &= \frac{b^2}{2}\int_0^t \left\{-\frac{d}{ds}\left\|P_s^B 1_B\right\|_2^2\right\}ds = \frac{b^2}{2}\left(\mu(B) - \left\|P_t^B 1_B\right\|_2^2\right) \\ &\le \frac{b^2}{2}\mu(B) = \frac{1}{2}\mu(B)\cdot\frac{\mu(B_0)\psi(R)}{\mu(B)\psi(r)} = \frac{\mu(B_0)\psi(R)}{2\psi(r)}. \end{aligned}$$

Finally, we derive condition ($\mathrm{SL}_2$) from (3.50). Indeed, noting that by (1.16), (3.43)

$$\Phi_{B_0} \le C\phi \le C(\phi_{B_0} + Cb) \ \text{ in } M,$$

it follows from (3.50) that for any $t > 0$,

$$\begin{aligned} \int_B \left(1 - P_t^B 1_B\right)^2 \Phi_{B_0}^2 d\mu &\le \int_B \left(1 - P_t^B 1_B\right)^2 \cdot C^2(2\phi_{B_0}^2 + 2C^2b^2)d\mu \\ &\le C\left(I(t) + b^2\int_B \left(1 - P_t^B 1_B\right)^2 d\mu\right) \\ &\le C'\mu(B_0)\exp\left(\frac{Ct}{\psi(R)}\right)\left(\frac{\psi(R)}{\psi(r)} + \frac{t}{\psi(r)}\right) + b^2\mu(B) \\ &\le C\mu(B_0)\left(\frac{\psi(R)}{\psi(r)} + \frac{t}{\psi(r)}\right)\exp\left(\frac{Ct}{\psi(R)}\right), \end{aligned}$$

thus proving (1.22). The proof is complete. □

As we have seen, in order to show condition ($\mathrm{SL}_2$), we need to consider any two concentric balls $B_0 \subset B$ and construct a new function $\phi_{B_0}$, belonging to the space $\mathcal{F}$, vanishing outside $\frac{1}{2}B$, but is comparable with the tent function $\Phi_{B_0}$ sitting on the smaller ball $B_0$.

3.3. **Conservativeness.** In this subsection, we derive the conservativeness of the form $(\mathcal{E},\mathcal{F})$ from condition ($\mathrm{SL}_2$) when $(M,d)$ is unbounded. However, if $(M,d)$ is bounded, we can get the conservativeness directly, without using condition ($\mathrm{SL}_2$).

**Lemma 3.7.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2$ without killing part. Then the following two statements are true.*

(1) *If $(M,d)$ is bounded, then $(\mathcal{E},\mathcal{F})$ is conservative.*
(2) *If $(M,d)$ is unbounded and condition ($SL_2$) is satisfied, then $(\mathcal{E},\mathcal{F})$ is conservative.*

*Consequently, if conditions* (VD), ($J_\le$), (*ABB*) *are satisfied, then $(\mathcal{E},\mathcal{F})$ is conservative.*

*Proof.* (1) Assume that $(M,d)$ is bounded. By Proposition 7.1 in Appendix, we have $1 \in \mathcal{F}$. It follows that for any $t > 0$

$$\frac{d}{dt}\|P_t 1\|_2^2 = -2\mathcal{E}(P_t 1, P_t 1) = -2\mathcal{E}(1, P_{2t}1) = 0,$$

which implies that $(P_t 1, P_t 1) = (1,1)$, and hence

$$\int_M (1 - P_{2t}1)d\mu = (1,1) - (1, P_{2t}1) = (1,1) - (P_t 1, P_t 1) = 0.$$

Thus, $P_t 1 = 1$ for any $t > 0$, showing that $(\mathcal{E},\mathcal{F})$ is conservative.

(2) Assume that $(M,d)$ is unbounded and condition ($\mathrm{SL}_2$) is satisfied. Let $B_0 := B(x_0,R) \subset B := B(x_0,r)$ be any two concentric balls. Then we have from condition ($\mathrm{SL}_2$) that, using the fact that $P_t^B 1_B \le P_t 1 \le 1$ in $M$,

$$\begin{aligned} \int_{B_0}(1 - P_t 1)^2 d\mu &\le \int_{B_0}(1 - P_t^B 1_B)^2 d\mu \le \int_B (1 - P_t^B 1_B)^2\,\Phi_{B_0}^2 d\mu \ \text{ (since } \Phi_{B_0} = 1 \text{ in } B_0) \\ &\le C\mu(B_0)\left(\frac{\psi(R)}{\psi(r)} + \frac{t}{\psi(r)}\right)\exp\left(\frac{Ct}{\psi(R)}\right) \to 0 \ \text{ as } r \to \infty. \end{aligned}$$

Thus, $P_t 1 = 1$ in $B_0$. Since $B_0$ is an arbitrary ball, we have $P_t 1 = 1$ in $M$, thus proving that $(\mathcal{E},\mathcal{F})$ is conservative. The proof is complete. □

3.4. **Condition ($\mathrm{S}_{1/2}$).** In this subsection, we derive condition ($\mathrm{S}_{1/2}$) from conditions ($\mathrm{PMV}_2$), ($\mathrm{SL}_2$).

**Lemma 3.8.** *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in $L^2$ without killing part. Then*

$$(\mathrm{VD}) + (\mathrm{PMV}_2) + (\mathrm{SL}_2) \Rightarrow (\mathrm{S}_{1/2}). \tag{3.51}$$

*Proof.* In order to show condition ($\mathrm{S}_{1/2}$), it suffices to assume that $0 < t < \psi(\delta\overline{R})$ by Remark 1.2, where $\delta$ comes from condition ($\mathrm{PMV}_2$). For a point $x_0$ in $M$, let

$$B := B(x_0, R).$$

We will show that

$$1 - P_t^B 1_B \le C\left(\frac{t}{\psi(R)}\right)^{1/2} \quad \text{in } \frac{1}{2}B \tag{3.52}$$

for any $0 < R < \overline{R}$ and any $0 < t < \psi(\delta\overline{R})$.

Indeed, if $2\psi^{-1}(t) \ge R$, then (3.52) is automatically satisfied, since in this case we have

$$\frac{t}{\psi(R)} \ge \frac{\psi(R/2)}{\psi(R)} \ge C^{-2} > 0$$

for some constant $C > 0$. In the sequel, assume that $2\psi^{-1}(t) < R < \overline{R}$.

Let $r := \psi^{-1}(t)$ and

$$B_0 := B(x_0, r)$$

so that $B_0 \subset \frac{1}{2}B$. For any open $\Omega \supset B$, let

$$u := P_t^{\Omega} 1_{\Omega} - P_t^B 1_B.$$

Then the function $u$ is nonnegative in $M$ and caloric in $(0,\infty)\times B$. Applying condition ($\mathrm{PMV}_2$) to this function $u$ over $(0,t)\times B_0$, we have

$$\begin{aligned}
\operatorname*{esup}_{\frac{1}{2}B_0} u(t,\cdot) &\le C\left(\frac{1}{t\mu(B_0)}\int_{\frac{t}{2}}^t ds\int_{B_0} u^2(s,\cdot)d\mu\right)^{1/2} + \frac{1}{\mu(B_0)}\sup_{s\in[\frac{t}{2},t]}\int_{(\frac{1}{2}B_0)^c} u(s,\cdot)\Phi_{B_0}^2 d\mu \\
&\le C'\left(\frac{1}{\mu(B_0)}\sup_{s\in[\frac{t}{2},t]}\int_M u^2(s,\cdot)\Phi_{B_0}^2 d\mu\right)^{1/2},
\end{aligned} \tag{3.53}$$

since the first term on the right-hand side is controlled by, using the fact that $\Phi_{B_0} = 1$ in $B_0$,

$$\begin{aligned}
\frac{1}{t\mu(B_0)}\int_{\frac{t}{2}}^t ds\int_{B_0} u^2(s,\cdot)d\mu &\le \frac{1}{t\mu(B_0)}\int_{\frac{t}{2}}^t ds\int_M u^2(s,\cdot)\Phi_{B_0}^2 d\mu \\
&\le \frac{1}{2\mu(B_0)}\sup_{s\in[\frac{t}{2},t]}\int_M u^2(s,\cdot)\Phi_{B_0}^2 d\mu,
\end{aligned}$$

whilst the second one is controlled by, using the Cauchy-Schwarz inequality and (1.15),

$$\begin{aligned}
\frac{1}{\mu(B_0)}\int_{(\frac{1}{2}B_0)^c} u(s,\cdot)\Phi_{B_0}^2 d\mu &\le \frac{1}{\mu(B_0)}\left\{\int_M u^2(s,\cdot)\Phi_{B_0}^2 d\mu\right\}^{1/2}\left\{\int_M \Phi_{B_0}^2 d\mu\right\}^{1/2} \\
&\le \frac{(C\mu(B_0))^{1/2}}{\mu(B_0)}\left\{\int_M u^2(s,\cdot)\Phi_{B_0}^2 d\mu\right\}^{1/2} \\
&\le \sqrt{C}\left\{\frac{1}{\mu(B_0)}\sup_{s\in[\frac{t}{2},t]}\int_M u^2(s,\cdot)\Phi_{B_0}^2 d\mu\right\}^{1/2}
\end{aligned}$$

for any $s \in [\frac{t}{2}, t]$.

On the other hand, by (3.2) and using the fact that $u \le 1$ in $(0,\infty)\times M$,

$$\int_{B^c} u^2(s,\cdot)\Phi_{B_0}^2 d\mu \le \int_{B^c}\Phi_{B_0}^2 d\mu = \int_{B(x_0,R)^c}\frac{\mu(B_0)\psi(r)}{V(x_0,x)\psi(x_0,x)}d\mu \le C\frac{\mu(B_0)\psi(r)}{\psi(R)}.$$

It follows from condition ($SL_2$) that, using the fact that $t = \psi(r)$,

$$\begin{aligned}\int_M u^2(s,\cdot)\Phi_{B_0}^2 d\mu &= \int_B u^2(s,\cdot)\Phi_{B_0}^2 d\mu + \int_{B^c} u^2(s,\cdot)\Phi_{B_0}^2 d\mu \\ &\leq \int_B \left(1 - P_s^B 1_B\right)^2 \Phi_{B_0}^2 d\mu + C\frac{\mu(B_0)\psi(r)}{\psi(R)} \\ &\leq C\mu(B_0)\left(\frac{\psi(r)}{\psi(R)} + \frac{s}{\psi(R)}\right)\exp\left(\frac{Cs}{\psi(r)}\right) + C\frac{\mu(B_0)\psi(r)}{\psi(R)} \\ &\leq 2C\mu(B_0)\left(\frac{\psi(r)}{\psi(R)} + \frac{t}{\psi(R)}\right)\exp\left(\frac{Ct}{\psi(r)}\right) \leq C'\frac{\mu(B_0)t}{\psi(R)} \qquad (3.54)\end{aligned}$$

for any $s \in [\frac{t}{2}, t]$. Therefore, we obtain from (3.53), (3.54) that

$$\begin{aligned}\underset{B(x_0,\frac{1}{2}\psi^{-1}(t))}{\operatorname{esup}} \left(P_t^\Omega 1_\Omega - P_t^B 1_B\right) &= \underset{\frac{1}{2}B_0}{\operatorname{esup}}\left(P_t^\Omega 1_\Omega - P_t^B 1_B\right) \\ &\leq C'\left(\frac{1}{\mu(B_0)} \sup_{s\in[\frac{t}{2},t]} \int_M u^2(s,\cdot)\Phi_{B_0}^2 d\mu\right)^{1/2} \\ &\leq C\left(\frac{t}{\psi(R)}\right)^{1/2} \qquad (3.55)\end{aligned}$$

for any $0 < t < \psi(\delta\overline{R})$ and for any $\Omega \supset B \supset B_0 = B(x_0, \psi^{-1}(t))$, where $C > 0$ is some universal constant independent of $t, B_0, B$.

We further extend inequality (3.55) over $\frac{1}{2}B$ (not only over $B(x_0, \frac{1}{2}\psi^{-1}(t))$) by using the standard covering argument. Namely, we show that

$$\underset{\frac{1}{2}B}{\operatorname{esup}}\left(P_t^\Omega 1_\Omega - P_t^B 1_B\right) \leq C\left(\frac{t}{\psi(R)}\right)^{1/2} \qquad (3.56)$$

for any $0 < t < \psi(\delta\overline{R})$ and any open $\Omega \supset B = B(x_0, R)$ with $2\psi^{-1}(t) < R < \overline{R}$.

Indeed, by condition (VD), there is a finite family $\{B(z_i, \frac{1}{2}r)\}_{i=1}^N$ of balls with each center $z_i$ in $\frac{1}{2}B$ such that $\frac{1}{2}B \subset \cup_i B(z_i, \frac{1}{2}r)$, where $r = \psi^{-1}(t)$. Applying (3.55) with $B_0$ being replaced by $B(z_i, r)$ and $B$ replaced by $B_i := B(z_i, \frac{R}{2})$, we have that, using the fact that $P_t^{B_i} 1_{B_i} \leq P_t^B 1_B$ in $M$,

$$\underset{B(z_i,\frac{1}{2}r)}{\operatorname{esup}} \left(P_t^\Omega 1_\Omega - P_t^B 1_B\right) \leq \underset{B(z_i,\frac{1}{2}r)}{\operatorname{esup}} \left(P_t^\Omega 1_\Omega - P_t^{B_i} 1_{B_i}\right) \leq C\left(\frac{t}{\psi(R/2)}\right)^{1/2} \leq C'\left(\frac{t}{\psi(R)}\right)^{1/2}$$

for an arbitrary point $z_i$, thus showing (3.56).

Finally, since $(\mathcal{E}, \mathcal{F})$ is conservative by Proposition 3.7, when $\Omega$ is expanding to $M$, we see that $P_t^\Omega 1_\Omega \to P_t 1 = 1$. From this and using (3.56), we obtain (3.52), as desired. The proof is complete. □

Recall that condition ($S_{1/2}$) gives a pointwise upper bound of the survival function $1 - P_t^B 1_B$, whilst condition ($SL_2$) gives an upper bound of this function in the weighted norm of $L^2$. The above lemma says that one needs the $L^2$ mean-value inequality to obtain pointwise estimate of function $1 - P_t^B 1_B$ from its $L^2$ estimate.

## 4. On-diagonal upper estimate

In this section, we first derive the $L^1$ mean-value inequality from its $L^2$-version and then obtain on-diagonal upper estimate of the heat kernel by using the $L^1$ mean-value inequality.

4.1. **Mean-value inequality.** In this subsection, we derive the $L^1$ mean-value inequality from its $L^2$ version.

**Proposition 4.1** ($L^2$-version)**.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2$ without killing part. Then*

$$(\text{Gcap}_+) + (\text{FK}_\nu) + (\text{J}_\leq) \Rightarrow (\text{PMV}_2). \tag{4.1}$$

*Consequently, for any $B_0 := B(x_0,R)$ with $0 < R < \delta\overline{R}$ where $\delta$ comes from condition* $(\text{FK}_\nu)$ *and for any function $u : (0,s] \to \mathcal{F}' \cap L^\infty$ that is nonnegative, subcaloric in $(0,s] \times B_0$ with $s = \psi(R)$, we have*

$$\operatorname*{esup}_{[\frac{3s}{4},s]\times(\frac{1}{2}B_0)} u \leq C(\nu)A^\theta \max\{A,T\}^{1-\theta}, \tag{4.2}$$

*where $\theta := \frac{2\nu}{1+3\nu}$, and $T$ is defined by (1.20) and*

$$A := \left(\frac{2}{s\mu(B_0)}\int_{\frac{s}{2}}^{s}\int_{B_0} u^2(t,\cdot)d\mu dt\right)^{1/2}. \tag{4.3}$$

*Proof.* The implication (4.1) has been obtained in the forthcoming paper [17], and so (1.19) is true. Minimizing the right-hand side of (1.19) in $\varepsilon$, for example, taking $\varepsilon = \left(\frac{A}{T}\right)^\theta$, we obtain (4.2). The proof is complete. □

We derive condition $(\text{PMV}_1)$ from condition $(\text{PMV}_2)$.

**Lemma 4.2** ($L^1$-version)**.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2$. Then*

$$(\text{PMV}_2) \Rightarrow (\text{PMV}_1).$$

*Consequently, we have*

$$(\text{Gcap}_+) + (\text{FK}_\nu) + (\text{J}_\leq) \Rightarrow (\text{PMV}_1). \tag{4.4}$$

*Proof.* Assume that the function $u : (0,s] \to \mathcal{F}' \cap L^\infty$ is nonnegative, subcaloric in $(0,\psi(R)] \times B_0$, where $B_0 := B(x_0,R)$ with $0 < R < \delta\overline{R}$ and $s = \psi(R)$. We shall show that there exist two constants $C > 0, c_0 \in (0,1)$ such that (1.21) is satisfied. The proof is divided into two steps.

*Step* 1. We will construct domains $\{D_n\}_{n\geq0}$, $\{Q_n^+\}_{n\geq1}$ contained in $[\frac{s}{2},s]\times B_0$ such that $Q_n^+ \subset D_n$ for each $n$ (see Figure 3 below), and

$$\operatorname*{esup}_{D_{n-1}} u = \operatorname*{esup}_{D_{n-1}\cap Q_n^+} u \quad \text{for any } n \geq 1. \tag{4.5}$$

Indeed, let $r_0 := c_0R$, where $c_0 \in (0,\frac{1}{2})$ is some small number to be chosen. Define $D_n$ by

$$D_n := \left[s - \frac{1}{2}\sum_{k=0}^{n}\psi(\frac{r_0}{2^k}), s\right] \times B(x_0, \sum_{k=0}^{n}\frac{r_0}{2^k}) \ \ (n \geq 0). \tag{4.6}$$

The domains $\{D_n\}_{n\geq0}$ is expanding to the domain $D_\infty$ where

$$D_\infty := \left[s - \frac{1}{2}\sum_{k=0}^{\infty}\psi(\frac{r_0}{2^k}), s\right] \times B(x_0, \sum_{k=0}^{\infty}\frac{r_0}{2^k}).$$

Note that by (1.6),

$$\sum_{k=0}^{\infty}\psi(\frac{r_0}{2^k}) = \sum_{k=0}^{\infty}\psi(\frac{c_0R}{2^k}) \leq \psi(R)\sum_{k=0}^{\infty}C\left(\frac{c_0R/2^k}{R}\right)^{\beta_1} = \frac{Cc_0^{\beta_1}\psi(R)}{1-2^{-\beta_1}} \leq \psi(R) = s \tag{4.7}$$

if $c_0$ is chosen to be sufficiently small, for example, if

$$\frac{Cc_0^{\beta_1}}{1-2^{-\beta_1}} \leq 1.$$

In this case, we have by (4.7)

$$D_\infty = \left[s - \frac{1}{2}\sum_{k=0}^{\infty}\psi(\frac{r_0}{2^k}), s\right] \times B(x_0, \sum_{k=0}^{\infty}\frac{r_0}{2^k}) \subset \left[\frac{s}{2}, s\right] \times B(x_0, 2r_0) \subset \left[\frac{s}{2}, s\right] \times B(x_0, R), \tag{4.8}$$

since $2r_0 = 2c_0R < R$ for $c_0 < \frac{1}{2}$.

Starting from domains $\{D_n\}_{n\geq 0}$, we will inductively construct domains $\{Q_n^+ \subset Q_n \subset D_n\}_{n\geq 1}$ such that (4.5) holds. Indeed, since $D_{n-1}$ can be covered by finitely many cylinders of forms

$$\left[t - \frac{1}{4}\psi(\frac{r_0}{2^n}), t\right] \times B(x, \frac{r_0}{2^{n+1}})$$

by varying points $(t, x)$ in $D_{n-1}$, we can choose one cylinder (not necessarily unique)

$$Q_n^+ := \left[s_n - \frac{1}{4}\psi(\frac{r_0}{2^n}), s_n\right] \times B(x_n, \frac{r_0}{2^{n+1}}) \text{ with } (s_n, x_n) \in D_{n-1} \tag{4.9}$$

such that the essential supremum of $u$ over $D_{n-1}$ is attained on $D_{n-1} \cap Q_n^+$, that is, equality (4.5) is satisfied. With the point $(s_n, x_n)$ chosen above, we set for $n \geq 1$

$$Q_n := \left[s_n - \frac{1}{2}\psi(\frac{r_0}{2^n}), s_n\right] \times B(x_n, \frac{r_0}{2^n}). \tag{4.10}$$

Clearly, $Q_n^+ \subset Q_n$.

We claim that $Q_n \subset D_n$ for any $n \geq 1$, so that

$$Q_n^+ \subset Q_n \subset D_n \subset D_\infty \quad (n \geq 1). \tag{4.11}$$

See Figure 3.

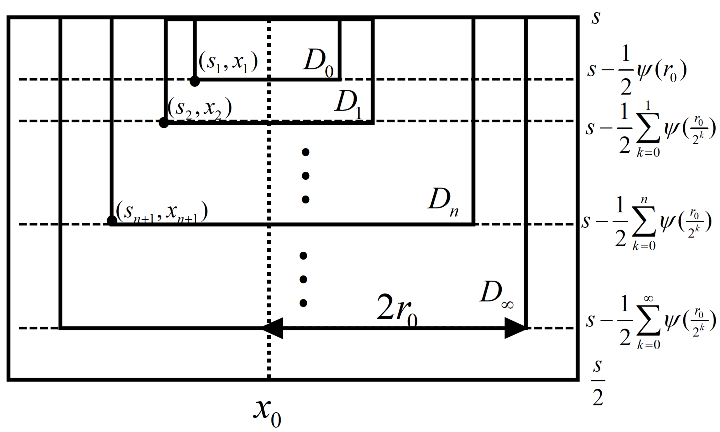


FIGURE 3. Points $(s_{n+1}, x_{n+1})$ are in domains $D_n \subset D_{n+1} \subset [\frac{s}{2}, s] \times B_0$.

Indeed, as $(s_n, x_n) \in D_{n-1}$, we see by definition (4.6) of $D_{n-1}$ that

$$s - \frac{1}{2}\sum_{k=0}^{n-1}\psi(\frac{r_0}{2^k}) \leq s_n \leq s \text{ and } d(x_0, x_n) \leq \sum_{k=0}^{n-1}\frac{r_0}{2^k} \quad (n \geq 1). \tag{4.12}$$

Thus for any $(t, x) \in Q_n$

$$\begin{aligned} s &\geq s_n \geq t \geq s_n - \frac{1}{2}\psi(\frac{r_0}{2^n}) \quad \text{(by definition (4.10) of } Q_n) \\ &\geq s - \frac{1}{2}\sum_{k=0}^{n-1}\psi(\frac{r_0}{2^k}) - \frac{1}{2}\psi(\frac{r_0}{2^n}) = s - \frac{1}{2}\sum_{k=0}^{n}\psi(\frac{r_0}{2^k}) \quad \text{(by (4.12))}, \end{aligned} \tag{4.13}$$

whilst by (4.12), (4.10), using the triangle inequality,

$$d(x_0, x) \leq d(x_0, x_n) + d(x_n, x) \leq \sum_{k=0}^{n-1}\frac{r_0}{2^k} + \frac{r_0}{2^n} = \sum_{k=0}^{n}\frac{r_0}{2^k}, \tag{4.14}$$

thus showing $(t, x) \in D_n$. This proves $Q_n \subset D_n$, as claimed.

*Step* 2. We show that (1.21) is satisfied. Indeed, let for $n \geq 1$

$$u_n(t,x) := u(t + s_n - \psi(\frac{r_0}{2^n}), x). \tag{4.15}$$

The function $u_n$ is well-defined for any $t \in (0, \psi(\frac{r_0}{2^n})]$ and any $x \in B_0$, since $t + s_n - \psi(\frac{r_0}{2^n}) \in (0, s]$ by noting that

$$\begin{aligned} s &\geq s_n \geq t + s_n - \psi(\frac{r_0}{2^n}) \geq 0 + s_n - \psi(\frac{r_0}{2^n}) \\ &\geq s - \frac{1}{2}\sum_{k=0}^{n-1} \psi(\frac{r_0}{2^k}) - \psi(\frac{r_0}{2^n}) \quad \text{(using (4.12))} \\ &> s - \sum_{k=0}^{\infty} \psi(\frac{r_0}{2^k}) \geq s - s = 0 \quad \text{(using (4.7))}. \end{aligned}$$

Note that $u_n$ is nonnegative, subcaloric in $(0, t_n] \times B_n$, where

$$B_n := B(x_n, \frac{r_0}{2^n}) \text{ and } t_n := \psi(\frac{r_0}{2^n}) \text{ for } n \geq 1. \tag{4.16}$$

Applying ($PMV_2$) to the function $u_n$ with respect to ball $B_n$, it follows from (4.2), with $B_0$ being replaced by $B_n$ and $s$ replaced by $t_n$, that for any $n \geq 1$

$$\begin{aligned} M_n := \operatorname*{esup}_{D_{n-1}} u &= \operatorname*{esup}_{D_{n-1}\cap Q_n^+} u \leq \operatorname*{esup}_{Q_n^+} u \quad \text{(using (4.5))} \qquad (4.17) \\ &= \operatorname*{esup}_{[s_n - \frac{1}{4}\psi(\frac{r_0}{2^n}), s_n]\times B(x_n, \frac{r_0}{2^{n+1}})} u \quad \text{(using definition (4.9))} \\ &= \operatorname*{esup}_{[\frac{3}{4}\psi(\frac{r_0}{2^n}), \psi(\frac{r_0}{2^n})]\times(\frac{1}{2}B_n)} u_n \quad \text{(using definition (4.15))} \\ &= \operatorname*{esup}_{[\frac{3t_n}{4}, t_n]\times(\frac{1}{2}B_n)} u_n \leq CA_n^{\theta}(A_n \vee T_n)^{1-\theta}, \qquad (4.18) \end{aligned}$$

where $A_n, T_n$ are respectively given by

$$\begin{aligned} A_n &:= \left(\frac{2}{t_n\mu(B_n)}\int_{\frac{t_n}{2}}^{t_n}\int_{B_n} u_n^2(t,\cdot)d\mu dt\right)^{1/2} = \left(\frac{1}{|Q_n|}\int_{Q_n} u^2(t,\cdot)d\mu dt\right)^{1/2} \quad \text{(by definition (4.10))}, \\ T_n &:= \frac{1}{\mu(B_n)}\sup_{t\in[\frac{t_n}{2}, t_n]}\int_{(\frac{1}{2}B_n)^c} (u_n)_+(t,\cdot)\Phi_{B_n}^2 d\mu = \frac{1}{\mu(B_n)}\sup_{t\in[s_n - \frac{1}{2}\psi(\frac{r_0}{2^n}), s_n]}\int_{(\frac{1}{2}B_n)^c} u_+(t,\cdot)\Phi_{B_n}^2 d\mu, \qquad (4.19) \end{aligned}$$

where $\Phi_{B_n}$ is the tent function sitting on ball $B_n$ defined by (1.13) but with $B_0$ replaced by $B_n$.

Note that $\{M_n\}_{n\geq1}$ defined by (4.17) is increasing since $D_{n-1} \subset D_n$. For simplicity, set $U := B(x_0, r_0)$,

$$I_n := \left[s_n - \frac{1}{2}\psi(\frac{r_0}{2^n}), s_n\right] \text{ and } U_n := \frac{1}{2}B_n = B(x_n, \frac{r_0}{2^{n+1}}) \text{ for } n \geq 1.$$

Then for any $n \geq 1$

$$\begin{aligned} T_n &= \frac{1}{\mu(B_n)}\sup_{t\in I_n}\int_{U_n^c} u_+(t,\cdot)\Phi_{B_n}^2 d\mu \\ &= \psi(\frac{r_0}{2^n})\sup_{t\in I_n}\left(\int_{U\setminus U_n} + \int_{M\setminus(U\cup U_n)}\right)\frac{u_+(t,x)d\mu(x)}{V(x_n,x)\psi(x_n,x)} =: T_n^{(1)} + T_n^{(2)}. \qquad (4.20) \end{aligned}$$

In order to estimate $T_n^{(1)}$, using the fact that $I_n \times U \subset [s_n - \frac{1}{2}\psi(\frac{r_0}{2^n}), s_n] \times B(x_0, r_0) \subset D_n$ by using (4.13) and definition (4.6), we have that for any $n \geq 1$

$$\begin{aligned} T_n^{(1)} &= \psi(\frac{r_0}{2^n})\sup_{t\in I_n}\int_{U\setminus U_n}\frac{u_+(t,x)d\mu(x)}{V(x_n,x)\psi(x_n,x)} \\ &\leq \psi(\frac{r_0}{2^n})\sup_{t\in I_n}\left(\operatorname*{esup}_{U\setminus U_n} u\right)\int_{U_n^c}\frac{d\mu(x)}{V(x_n,x)\psi(x_n,x)} \end{aligned}$$

$$\leq \quad \psi(\frac{r_0}{2^n})\Big(\operatorname*{esup}_{D_n} u\Big)\frac{C}{\psi(\frac{r_0}{2^{n+1}})} \leq C' M_{n+1} \ \ (\text{using } (3.2)). \tag{4.21}$$

For $T_n^{(2)}$, using the general fact that there exists some constant $C > 0$ such that

$$C^{-1}V(x,y) \leq V(y,x) \leq CV(x,y) \ \text{ for all } x, y \text{ in } M,$$

we have for any $x \in U_n^c$,

$$\frac{V(x_0,x)}{V(x_n,x)} \leq C\frac{V(x,x_0)}{V(x,x_n)} \leq C\left(\frac{d(x_0,x_n)+d(x_n,x)}{d(x_n,x)}\right)^{\alpha} \leq C'2^{n\alpha} \tag{4.22}$$

since $d(x_0,x_n) \leq 2r_0$ for all $n$ by (4.12) and $d(x,x_n) \geq 2^{-(n+1)}r_0$. Similarly, using the monotonicity of $\psi$ and (1.6), we have for any $x \in U_n^c$

$$\frac{\psi(x_0,x)}{\psi(x_n,x)} \leq \frac{\psi(d(x_0,x_n)+d(x_n,x))}{\psi(d(x_n,x))} \leq C2^{n\beta_2}.$$

Thus for any $x \in U_n^c$

$$\frac{V(x_0,x)\psi(x_0,x)}{V(x_n,x)\psi(x_n,x)} \leq C2^{n(\alpha+\beta_2)}.$$

From this, we obtain for any $n \geq 1$

$$\begin{aligned}
T_n^{(2)} &= \psi(\frac{r_0}{2^n})\sup_{t\in I_n}\int_{M\setminus(U\cup U_n)}\frac{u_+(t,x)d\mu(x)}{V(x_n,x)\psi(x_n,x)} \\
&\leq \psi(\frac{r_0}{2^n})\sup_{t\in I_n}\int_{M\setminus(U\cup U_n)}\frac{u_+(t,x)d\mu(x)}{V(x_0,x)\psi(x_0,x)}\cdot\operatorname*{esup}_{x\notin(U\cup U_n)}\frac{V(x_0,x)\psi(x_0,x)}{V(x_n,x)\psi(x_n,x)} \\
&\leq \psi(r_0)\sup_{t\in I_n}\int_{U^c}\frac{u_+(t,x)d\mu(x)}{V(x_0,x)\psi(x_0,x)}\cdot C2^{n(\alpha+\beta_2)} \leq C'2^{n(\alpha+\beta_2)}T
\end{aligned} \tag{4.23}$$

where $T$ is defined by

$$T := \psi(R)\sup_{t\in[\frac{s}{2},s]}\int_{U^c}\frac{u_+(t,x)d\mu(x)}{V(x_0,x)\psi(x_0,x)}. \tag{4.24}$$

Therefore, substituting (4.23), (4.21) into (4.20), we obtain

$$T_n = T_n^{(1)} + T_n^{(2)} \leq C(2^{n(\alpha+\beta_2)}T + M_{n+1}) \ \ (n \geq 1). \tag{4.25}$$

In order to estimate $A_n$ in (4.19), let $K$ be defined by

$$K = \frac{2}{s\mu(B_0)}\int_{\frac{s}{2}}^{s}\int_{B_0}u(t,\cdot)d\mu dt = \frac{1}{|Q|}\int_Q u, \tag{4.26}$$

where $Q := [\frac{s}{2}, s] \times B_0$. Then

$$\begin{aligned}
A_n^2 &= \frac{1}{|Q_n|}\int_{Q_n}u^2 \leq \frac{1}{|Q_n|}\int_{Q_n}u\cdot\operatorname*{esup}_{D_n}u \quad (\text{since } Q_n \subset D_n \text{ by } (4.11)) \\
&\leq M_{n+1}\frac{1}{|Q_n|}\int_Q u \quad (\text{since } Q_n \subset D_\infty \subset Q) \\
&= \frac{|Q|}{|Q_n|}KM_{n+1} \leq C2^{n(\alpha+\beta_2)}KM_{n+1},
\end{aligned} \tag{4.27}$$

where we have used the fact that, noting that $d(x_n,x_0) \leq 2r_0$ by (4.12) and $r_0 = c_0R$, $s = \psi(R)$,

$$\frac{|Q|}{|Q_n|} = \frac{\frac{s}{2}V(x_0,R)}{\frac{1}{2}\psi(\frac{r_0}{2^n})V(x_n,\frac{r_0}{2^n})} = \frac{\psi(R)V(x_0,R)}{\psi(\frac{c_0R}{2^n})V(x_n,\frac{c_0R}{2^n})} \leq C2^{n(\alpha+\beta_2)}.$$

Therefore, substituting (4.27), (4.25) into (4.18), we obtain

$$\begin{aligned}
M_n &\leq C\left(2^{n(\alpha+\beta_2)}KM_{n+1}\right)^{\theta/2}\left\{\left(2^{n(\alpha+\beta_2)}KM_{n+1}\right)^{1/2}\vee\left(2^{n(\alpha+\beta_2)}T+M_{n+1}\right)\right\}^{1-\theta} \\
&\leq CK^{\theta/2}2^{n(\alpha+\beta_2)}M_{n+1}^{\theta/2}\left\{(KM_{n+1})^{1/2}\vee(T+M_{n+1})\right\}^{1-\theta}.
\end{aligned} \tag{4.28}$$

We claim that there exists some constant $C > 0$ independent of $u, B_0$ such that

$$M_1 \leq C(K+T). \tag{4.29}$$

Indeed, if there exists some integer $n \geq 1$ such that $M_n \leq K$ or $M_n \leq T$, then

$$M_1 = \operatorname*{esup}_{D_0} u \leq M_n \leq K + T,$$

and hence, estimate (4.29) is true.

In the sequel, assume that $M_n \geq K \vee T$ for all $n \geq 1$. Then by (4.28)

$$M_n \leq CK^{\theta/2} b^n M_{n+1}^{1-\theta/2} \ \text{ for all } n \geq 1$$

where $b := 2^{\alpha+\beta_2}$. Iterating this inequality, we have that, setting $\gamma = 1 - \theta/2 = \frac{1+2\nu}{1+3\nu}$,

$$\begin{aligned} M_1 &\leq \left(CK^{\theta/2}b\right) M_2^{\gamma} \leq \left(CK^{\theta/2}b\right)\left(CK^{\theta/2}b^2 M_3^{\gamma}\right)^{\gamma} \leq \cdots \\ &\leq \left(CK^{\theta/2}\right)^{1+\gamma+\gamma^2+\cdots} b^{1+2\gamma+3\gamma^2+\cdots} M_{n+1}^{\gamma^n} \\ &= \left(CK^{\theta/2}\right)^{\frac{1}{1-\gamma}} b^{\frac{1}{(1-\gamma)^2}} M_{n+1}^{\gamma^n} \to C'K \ \text{ as } n \to \infty, \end{aligned}$$

since $u$ is bounded by a positive constant $C(u)$ in $(0, s] \times B_0$ and $M_{n+1}^{\gamma^n} \leq C(u)^{\gamma^n} \to 1$ as $n \to \infty$. Thus, estimate (4.29) is true again. This proves our claim.

Finally, note that by definition (4.26),

$$K = \frac{2}{s\mu(B_0)} \int_{\frac{s}{2}}^{s} \int_{B_0} u(t,x)d\mu(x)dt \leq \frac{1}{\mu(B_0)} \sup_{t \in [\frac{s}{2}, s]} \int_{B_0} u_+(t,\cdot)d\mu$$

and that, using definition (4.24) and $U = B(x_0, r_0) = B(x_0, c_0R)$,

$$T = \psi(R) \sup_{t\in[\frac{s}{2},s]} \int_{U^c} \frac{u_+(t,x)d\mu(x)}{V(x_0,x)\psi(x_0,x)} \leq \frac{C}{\mu(B_0)} \sup_{t\in[\frac{s}{2},s]} \int_M u_+(t,x)d\mu(x)$$

where $C$ is a constant $C$ independent of $B_0$. Therefore, it follows from (4.29) that

$$\operatorname*{esup}_{[s-\frac{1}{2}\psi(c_0R),s]\times B(x_0,c_0R)} u = \operatorname*{esup}_{D_0} u = M_1 \leq C(K+T) \leq \frac{C}{\mu(B_0)} \sup_{t\in[\frac{s}{2},s]} \int_M u_+(t,\cdot)d\mu,$$

thus showing (1.21). Hence, condition (PMV$_1$) is satisfied. The proof is complete. □

The result in Lemma 4.2 can be viewed as a generalization of [13, Subsection 4.6]. The idea of the proof is to construct a sequence of points such that the subcaloric function attains its maximum when the domains are expanding, which yields an iteration inequality by using the $L^2$-version of parabolic mean-value inequality, and then we get the desired by solving this iteration inequality.

4.2. **Condition (DUE).** In this subsection, we derive condition (DUE) from (PMV$_1$).

**Lemma 4.3.** *The following implication is true:*

$$(\text{PMV}_1) \Rightarrow (\text{DUE}).$$

*Proof.* Let $x_0 \in M$ and $0 < t < \psi(\overline{R})$. For any nonnegative $f \in L^1 \cap L^\infty$, let $u(s,x) = P_{s+t-t'}f(x)$ for $(s,x) \in (0,\infty) \times M$, where $t' = t \wedge \psi(\delta\overline{R})$ and the constant $\delta$ comes from condition (PMV$_1$). Clearly,

$$t' \geq C^{-1}t$$

for all $0 < t < \psi(\overline{R})$, where $C$ is a constant independent of $t, \overline{R}$. Then $u$ is nonnegative, caloric in $(0,\infty) \times M$. Applying condition (PMV$_1$) to the function $u$ over cylinder $(0,t'] \times B(x_0, \psi^{-1}(t'))$, we have by (1.21) that for $\mu$-almost all $x \in B(x_0, c_0\psi^{-1}(C^{-1}t)) \subset B(x_0, c_0\psi^{-1}(t'))$,

$$\begin{aligned} P_t f(x) &= u(t',x) \leq \frac{C}{V(x_0,\psi^{-1}(t'))} \sup_{\frac{t'}{2}\leq s\leq t'} \int_M u(s,x)d\mu(x) \\ &\leq \frac{C'}{V(x_0,\psi^{-1}(t))} \|f\|_{L^1} \ (\text{since } \|P_s f\|_1 \leq \|f\|_1). \end{aligned} \tag{4.30}$$

We shall show that for $\mu$-almost all $x \in B_0 := B(x_0, R)$ with $0 < R < \overline{R}$

$$P_t f(x) \le \frac{C}{V(x_0, \psi^{-1}(t))} \left(1 + \frac{R}{\psi^{-1}(t)}\right)^{\alpha} \|f\|_{L^1} \tag{4.31}$$

for all nonnegative $f \in L^1 \cap L^\infty$, where $C > 0$ is a constant independent of $B_0$, $f$, $t$, $x$.

Indeed, by the doubling property, we can cover the ball $B_0$ by a finite number of balls $\{B(\xi_i, c_0\psi^{-1}(C^{-1}t))\}_{i=1}^N$ with each center $\xi_i$ in $B_0$. It follows from (4.30) with $x_0$ being replaced by $\xi_i$ that, for $\mu$-almost all $z \in B(\xi_i, c_0\psi^{-1}(C^{-1}t))$,

$$\begin{aligned} P_t f(z) &\le \frac{C}{V(\xi_i, \psi^{-1}(t))}\|f\|_{L^1} = \frac{C}{V(x_0, \psi^{-1}(t))}\frac{V(x_0, \psi^{-1}(t))}{V(\xi_i, \psi^{-1}(t))}\|f\|_{L^1} \\ &\le \frac{C'}{V(x_0, \psi^{-1}(t))}\left(1 + \frac{R}{\psi^{-1}(t)}\right)^{\alpha} \|f\|_{L^1} \ \text{(using } d(x_{0,}\xi_i) < R \text{ and (1.3)).} \end{aligned}$$

Varying $z$ in the ball $B_0$, we obtain (4.31), as desired.

Applying [20, Theorem 2.2], we conclude from (4.31) that there exists a pointwise defined heat kernel $p_t(x, y)$ on $(0, \infty) \times M \times M$ such that for all $x \in B_0$, all $0 < t < \psi^{-1}(\overline{R})$ and all $y \in M$,

$$p_t(x, y) \le \frac{C}{V(x_0, \psi^{-1}(t))}\left(1 + \frac{R}{\psi^{-1}(t)}\right)^{\alpha}, \tag{4.32}$$

where $C > 0$ is some constant independent of $x, y, t, R$. In particular, letting $R \to 0$, we obtain that for all $x, y$ in $M$ and all $0 < t < \psi(\overline{R})$,

$$p_t(x, y) \le \frac{C}{V(x, \psi^{-1}(t))}, \tag{4.33}$$

thus showing that condition (DUE) is true. The proof is complete. □

## 5. Truncated Dirichlet form

In order to obtain off-diagonal upper estimate of the heat kernel $p_t(x, y)$, we need to truncate the form $(\mathcal{E}, \mathcal{F})$ by any number $\rho > 0$, and this truncated bilinear form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ is also a regular Dirichlet form in $L^2$. In this section, we will show the existence and the upper bound of the heat kernel $q_t(x, y)$ associated with the truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ for any $\rho > 0$.

For any $\rho > 0$, let

$$\mathcal{E}^{(\rho)}(u, v) = \mathcal{E}^{(L)}(u, v) + \int_M \int_{B(x,\rho)} (u(x) - u(y))(v(x) - v(y))J(x, y)d\mu(y)d\mu(x). \tag{5.1}$$

By (2.5), we see that

$$\mathcal{E}(u, v) = \mathcal{E}^{(\rho)}(u, v) + \int_M \int_M (u(x) - u(y))(v(x) - v(y))J_\rho(x, y)d\mu(y)d\mu(x), \tag{5.2}$$

where $J_\rho(x, y)$ is defined by

$$J_\rho(x, y) := 1_{\{d(x,y)\ge\rho\}} J(x, y) \ \text{ for any } (x, y) \in M \times M. \tag{5.3}$$

By conditions (VD), ($J_\le$), we have by (3.2)

$$\sup_{x\in M} \int_M J_\rho(x, y)d\mu(y) = \sup_{x\in M} \int_{B(x,\rho)^c} J(x, y)d\mu(y) \le \frac{C}{\psi(\rho)} \tag{5.4}$$

for some constant $C > 0$ independent of $\rho$. It follows from [24, Proposition 4.2] that the form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ is closable, and its closure $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ is also a regular Dirichlet form on $L^2$. Let $\{Q_t := Q_t^{(\rho)}\}_{t\ge 0}$ be the heat semigroup corresponding to $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$. The form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ is $\rho$-*local*, that is, $\mathcal{E}^{(\rho)}(u, v) = 0$ for any $u, v \in \mathcal{F}^{(\rho)}$ such that $u$ is constant in some $\rho$-*neighbourhood* of supp[$v$].

**Proposition 5.1.** *Let* $\{Q_t := Q_t^{(\rho)}\}$ *be the heat semigroup associated with* $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ *defined by (5.1). If conditions* (VD), $(J_{\leq})$, *(DUE) hold, then* $\{Q_t\}$ *admits the heat kernel* $q_t(x,y)$ *pointwise defined on* $(0,\infty)\times M \times M$ *for any* $\rho > 0$*. Moreover,*

$$q_t(x,y) \leq \frac{C}{V(x,\psi^{-1}(t))} \exp\left(\frac{Ct}{\psi(\rho)}\right) \tag{5.5}$$

*for all* $x, y$ *in* $M$ *and all* $0 < t < \psi(\overline{R})$*, where* $C > 0$ *is some constant independent of* $t, x, y$ *and* $\rho$*.*

*Proof.* For any $\rho > 0$, we see by (5.4) that

$$K_\rho := \sup_{x\in M} \int_M J_\rho(x,y)d\mu(y) \leq \frac{C}{\psi(\rho)}, \tag{5.6}$$

which implies that $\mathcal{F} = \mathcal{F}^{(\rho)}$, since by (5.2)

$$\begin{aligned}\left|\mathcal{E}(u) - \mathcal{E}^{(\rho)}(u)\right| &= \left|\int_{M\times M}(u(x)-u(y))^2 J_\rho(x,y)d\mu(y)d\mu(x)\right| \\ &\leq 4\,\|u\|_2^2 \sup_{x\in M}\int_M J_\rho(x,y)d\mu(y) \leq 4K_\rho\,\|u\|_2^2\,.\end{aligned}$$

Let $f \in L^2$ be nonnegative in $M$ and $u(t,x) := Q_t f(x)$. Then $u$ is caloric with respect to $\mathcal{E}^{(\rho)}$ in $(0,\infty)\times M$, that is, for any $t > 0$ and any nonnegative function $\varphi \in \mathcal{F}^{(\rho)}$

$$(\frac{\partial}{\partial t}u(t,\cdot),\varphi) + \mathcal{E}^{(\rho)}(u(t,\cdot),\varphi) = 0.$$

For $t > 0, x \in M$, we let

$$v(t,x) := \exp(-2tK_\rho)u(t,x).$$

Then $v$ is subcaloric with respect to $\mathcal{E}$ in $(0,\infty)\times M$, since for any nonnegative function $\varphi \in \mathcal{F}$

$$\begin{aligned}(\frac{\partial}{\partial t}v(t,\cdot),\varphi) + \mathcal{E}(v(t,\cdot),\varphi) &= \exp(-2tK_\rho)\Big\{-2K_\rho(u(t,\cdot),\varphi) + (\frac{\partial}{\partial t}u(t,\cdot),\varphi) + \mathcal{E}^{(\rho)}(u(t,\cdot),\varphi) \\ &\quad + \int_M\int_M (u(t,x)-u(t,y))(\varphi(x)-\varphi(y))J_\rho(x,y)d\mu(y)d\mu(x)\Big\} \\ &\leq \exp(-2tK_\rho)\Big\{-2K_\rho(u(t,\cdot),\varphi) \\ &\quad + \int_M\int_M \left[u(t,x)\varphi(x) + u(t,y)\varphi(y)\right] J_\rho(x,y)d\mu(y)d\mu(x)\Big\} \\ &\leq 0.\end{aligned}$$

By the parabolic maximum principle (cf. [21, Lemma 4.16]), we have

$$v(t,x) = \exp(-2tK_\rho)Q_t f(x) \leq P_t f(x) \ \text{ for } (t,x) \in (0,\infty)\times M,$$

which combines with (5.6) to yield that, for any $t > 0$,

$$Q_t f \leq \exp(2tK_\rho)P_t f \leq \exp\left(\frac{Ct}{\psi(\rho)}\right)P_t f \ \text{ in } M. \tag{5.7}$$

Therefore, it follows from condition (DUE) that

$$Q_t f(x) \leq \exp\left(\frac{Ct}{\psi(\rho)}\right)P_t f(x) \leq \exp\left(\frac{Ct}{\psi(\rho)}\right)\frac{C}{V(x,\psi^{-1}(t))}\,\|f\|_1 \tag{5.8}$$

for all $0 < t < \psi(\overline{R})$ and $\mu$-almost all $x$ in $M$, where $C > 0$ is independent of $t, x, f, \rho$. Applying [20, Theorem 2.2], we conclude from (5.8) that there exists a pointwise defined heat kernel $q_t(x,y)$ on $(0,\infty)\times M\times M$ such that for all $x, y \in M$ and all $0 < t < \psi^{-1}(\overline{R})$,

$$q_t(x,y) \leq \frac{C}{V(x,\psi^{-1}(t))}\exp\left(\frac{Ct}{\psi(\rho)}\right),$$

thus showing (5.5). The proof is complete. □

Proposition 5.1 gives an on-diagonal upper estimate of $q_t(x,y)$. In order to obtain an off-diagonal upper estimate, we need to estimate the tail $Q_t 1_{B^c}$ for any ball $B$.

For any $t>0$ and any point $x$ in $M$, we replace $Q_t f$ for any $f \in L^2$ by its pointwise realization:

$$Q_t f(x) := \int_M q_t(x,y) f(y) d\mu(y). \tag{5.9}$$

This pointwise definition makes sense since $q_t(x,y)$ is pointwise defined for each $x,y$ in $M$ and $t>0$.

**Lemma 5.2.** *Let $\{Q_t := Q_t^{(\rho)}\}$ be the heat semigroup associated with $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ defined by (5.1). If conditions $(S_{1/2})$, $(J_\le)$ hold, then for any $t>0$ and any $0<\rho\le \frac{R}{4}$ with $0<R<\infty$,*

$$Q_t 1_{B(x_0,R)^c} \le \left(\frac{C_2 t}{\psi(\rho)}\right)^{\frac{1}{2}\left(\left[\frac{R}{4\rho}\right]-1\right)} \quad \text{pointwise in } B(x_0,2\rho), \tag{5.10}$$

*where $C_2\ge 1$ is some universal constant independent of $t, x_0, R$ and $\rho$, and $[b]$ denotes the integer part of a number $b$.*

*Proof.* Let $B := B(x_0,R)$ be any ball with $0<R<\infty$. By condition $(S_{1/2})$, we have that for all $t>0$,

$$1 - P_t^B 1_B \le \left(\frac{Ct}{\psi(R)}\right)^{1/2} \quad \text{in } \frac{1}{2}B, \tag{5.11}$$

where $C>0$ is some constant independent of $t$ and ball $B$.

Applying Lemma 7.2 with $\Omega = B$, $f = 1_B$ in Appendix, we have by (5.4) that for any $t>0$

$$P_t^B 1_B - Q_t^B 1_B \le 2t \operatorname*{esup}_{x\in M} \int_{B(x,\rho)^c} J(x,y) d\mu(y) \le \frac{Ct}{\psi(\rho)} \quad \text{in } B.$$

From this, we have by (5.11) that

$$1 - Q_t^B 1_B \le 1 - P_t^B 1_B + \frac{Ct}{\psi(\rho)} \le C_1 \left\{ \left(\frac{t}{\psi(R)}\right)^{1/2} + \frac{t}{\psi(\rho)} \right\} \quad \text{in } \frac{1}{2}B \tag{5.12}$$

for all $t>0$, where $C_1 \ge 1$ is some constant independent of $t, B$ and $\rho$.

We show (5.10). Without loss of generality, assume that $\frac{C_2 t}{\psi(\rho)} \le 1$ with constant $C_2\ge 1$ to be determined below; otherwise (5.10) is automatically true since $Q_t 1_{B(x_0,R)^c} \le 1$ in $M$. For $t>0$, let

$$\phi(R,t) := C_1 \left\{ \left(\frac{t}{\psi(R)}\right)^{1/2} + \frac{t}{\psi(\rho)} \right\},$$

which is non-decreasing in $t\in[0,\infty)$ for any $R$. Let

$$r = 2\rho \quad \text{and} \quad k = \left[\frac{R}{4\rho}\right] \ge 1.$$

Since $B^c = B(x_0,R)^c \subset B(x_0,kr)^c$ and $\frac{t}{\psi(\rho)} \le \frac{1}{C_2} \le 1$, applying Lemma 7.3 in Appendix and using (5.12), we have that in $B(x_0,2\rho) = B(x_0,r)$,

$$\begin{aligned} Q_t 1_{B^c} &\le Q_t 1_{B(x_0,kr)^c} \le \phi(r-\rho,t)^{k-1} = \phi(\rho,t)^{k-1} = \left(C_1 \left\{ \left(\frac{t}{\psi(\rho)}\right)^{1/2} + \frac{t}{\psi(\rho)} \right\}\right)^{k-1} \\ &\le \left(2C_1 \left(\frac{t}{\psi(\rho)}\right)^{1/2}\right)^{k-1} = \left(\frac{\sqrt{2C_1} t}{\psi(\rho)}\right)^{\frac{1}{2}(k-1)} = \left(\frac{C_2 t}{\psi(\rho)}\right)^{\frac{1}{2}\left(\left[\frac{R}{4\rho}\right]-1\right)} \end{aligned}$$

where $C_2 := \sqrt{2C_1} \ge 1$, thus showing (5.10) with such a constant $C_2$. The proof is complete. □

We now derive off-diagonal upper estimate of $q_t(x,y)$ by using the on-diagonal upper bound (5.5) and the tail estimate (5.10).

**Proposition 5.3.** *Let $q_t(x,y)$ be the heat kernel of the $\rho$-local Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ in $L^2$ for any $\rho>0$. If conditions (DUE), $(S_{1/2})$, $(J_\leq)$ hold, then*

$$q_t(x,y)\leq C\left\{\frac{1}{V(x,\psi^{-1}(t))}+\frac{1}{V(y,\psi^{-1}(t))}\right\}\left(\frac{C_2t}{2\psi(\rho)}\right)^{\frac{1}{2}\left(\left[\frac{d(x,y)}{8\rho}\right]-1\right)}\exp\left(\frac{Ct}{\psi(\rho)}\right) \tag{5.13}$$

*for any $0<t<\psi(\overline{R})$ and any $x,y\in M$ with $d(x,y)\geq 8\rho$, where $C>0$ is a constant independent of $t,x,y$ and $\rho$, and $C_2$ is the same as in (5.10).*

*Proof.* Let $\rho>0$ and $0<t<\psi(\overline{R})$. Let $x,y$ be any two distinct points in $M$ such that

$$8\rho\leq r:=d(x,y)<\overline{R}.$$

By the semigroup property, we have

$$\begin{aligned}q_t(x,y) &= \int_M q_{t/2}(x,z)q_{t/2}(z,y)d\mu(z)\\ &\leq \int_{B(x,\frac{r}{2})^c} q_{t/2}(x,z)q_{t/2}(z,y)d\mu(z)+\int_{B(y,\frac{r}{2})^c} q_{t/2}(x,z)q_{t/2}(z,y)d\mu(z)\\ &=: I_1+I_2.\end{aligned}$$

For $I_1$, we have by (5.5), (5.10) that

$$\begin{aligned}I_1 &= \int_{B(x,\frac{r}{2})^c} q_{t/2}(x,z)q_{t/2}(z,y)d\mu(z)\leq \operatorname*{esup}_{z\in M} q_{t/2}(z,y)\cdot Q_{\frac{t}{2}}1_{B(x,\frac{r}{2})^c}(x)\\ &\leq \frac{C}{V(y,\psi^{-1}(t/2))}\exp\left(\frac{Ct/2}{\psi(\rho)}\right)\cdot\left(\frac{C_2t/2}{\psi(\rho)}\right)^{\frac{1}{2}\left(\left[\frac{r}{8\rho}\right]-1\right)}\\ &\leq \frac{C}{V(y,\psi^{-1}(t))}\exp\left(\frac{Ct}{\psi(\rho)}\right)\cdot\left(\frac{C_2t}{2\psi(\rho)}\right)^{\frac{1}{2}\left(\left[\frac{r}{8\rho}\right]-1\right)},\end{aligned}$$

since $0<\rho\leq\frac{r}{8}$.

For $I_2$, we similarly have that

$$I_2=\int_{B(y,\frac{r}{2})^c} q_{t/2}(x,z)q_{t/2}(z,y)d\mu(z)\leq\frac{C}{V(x,\psi^{-1}(t))}\exp\left(\frac{Ct}{\psi(\rho)}\right)\cdot\left(\frac{C_2t}{2\psi(\rho)}\right)^{\frac{1}{2}\left(\left[\frac{r}{8\rho}\right]-1\right)}.$$

Therefore, it follows that

$$q_t(x,y)\leq I_1+I_2\leq C\left\{\frac{1}{V(x,\psi^{-1}(t))}+\frac{1}{V(y,\psi^{-1}(t))}\right\}\exp\left(\frac{Ct}{\psi(\rho)}\right)\cdot\left(\frac{C_2t}{2\psi(\rho)}\right)^{\frac{1}{2}\left(\left[\frac{r}{8\rho}\right]-1\right)},$$

thus proving (5.13). The proof is complete. □

## 6. Proof of Theorem 1.3

In this section, we shall finish proving Theorem 1.3. The main task is to derive condition (UE) from conditions $(S_{1/2})$, (DUE), $(J_\leq)$.

6.1. **Off-diagonal upper bound.** In this subsection we will derive condition (UE). We need the following result about a generator perturbated by a bounded operator in a Banach space.

**Proposition 6.1.** [35, Theorem 3.5 and formula (13)] *Let $\Delta$ be the (non-positive definite) infinitesimal generator of a strongly continuous semigroup $\{Q_t\}_{t\geq0}$ on a Banach space $\mathcal{H}$, and let $A$ be a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$. Then the semigroup $\{P_t\}_{t\geq0}$ generated by $\Delta+A$ can be expressed by*

$$P_t=\sum_{n=0}^{\infty}Q_t^{(n)}, \tag{6.1}$$

*where* $Q_t^{(0)} = Q_t$, *and*

$$Q_t^{(n)} = \int_0^t Q_{t-s} A Q_s^{(n-1)} ds \ \text{ for each } n \geq 1 \tag{6.2}$$

*is well-defined, strongly continuous on* $\mathcal{H}$. *If* $\{Q_t\}_{t\geq 0}$ *is further contractive on* $\mathcal{H}$, *then*

$$\|Q_t^{(n)}\| \leq \frac{(t\|A\|)^n}{n!} \ \text{ for each } n \geq 0. \tag{6.3}$$

For any $\rho > 0$, we define the operator $A^{(\rho)}$ by

$$A^{(\rho)} f(x) = 2 \int_{B(x,\rho)^c} (f(y) - f(x)) J(x,y) d\mu(y) \ \text{ for } x \in M. \tag{6.4}$$

The following says that $A^{(\rho)}$ is well-defined on $L^2$ if conditions (VD), ($J_\leq$) hold.

**Proposition 6.2.** *If conditions* (VD), ($J_\leq$) *hold, then the operator* $A^{(\rho)}$ *defined by (6.4) is bounded from* $L^2$ *to* $L^2$ *for each* $\rho > 0$, *that is*

$$\|A^{(\rho)}\|_{L^2\to L^2} := \sup_{\|f\|_2=1} \|A^{(\rho)} f\|_2 \leq \frac{C_3}{\psi(\rho)} \tag{6.5}$$

*for some positive constant* $C_3$ *independent of* $\rho$.

*Proof.* By the Cauchy-Schwarz inequality, we have by (5.4) that for any $f \in L^2$,

$$\begin{aligned}
\|A^{(\rho)} f\|_2^2 &= 4 \int_M \Big( \int_M (f(y) - f(x)) J_\rho(x,y) d\mu(y) \Big)^2 d\mu(x) \\
&\leq 4 \int_M \Big( \int_M |f(y) - f(x)|^2 J_\rho(x,y) d\mu(y) \cdot \int_M J_\rho(x,y) d\mu(y) \Big) d\mu(x) \\
&\leq \frac{4C}{\psi(\rho)} \int_M \int_M 2(f(x)^2 + f(y)^2) J_\rho(x,y) d\mu(y) d\mu(x) \\
&= \frac{16C}{\psi(\rho)} \int_M \int_M f(x)^2 J_\rho(x,y) d\mu(y) d\mu(x) \\
&= 16 \left( \frac{C}{\psi(\rho)} \right)^2 \int_M f(x)^2 dx,
\end{aligned}$$

thus showing (6.5) with $C_3 = 4C$ independent of $\rho$. □

The following gives the relationship between two semigroups $\{P_t\}$ and $\{Q_t\}$.

**Lemma 6.3.** *If conditions* (VD), ($J_\leq$) *hold,, then for any* $\rho > 0$ *and any nonnegative* $f \in L^2$,

$$P_t f(x) \leq Q_t f(x) + 2 \int_0^t ds \int_M Q_{t-s} J_\rho(\cdot, z)(x) \cdot P_s f(z) d\mu(z) \tag{6.6}$$

*for any* $t > 0$ *and* $\mu$*-almost all* $x \in M$, *where* $J_\rho(x,y)$ *is defined by (5.3).*

*Proof.* Note that

$$\mathcal{L} = \mathcal{L}^{(\rho)} + A^{(\rho)}, \tag{6.7}$$

where $\mathcal{L}$, $\mathcal{L}^{(\rho)}$ are the infinitesimal generators of $\{P_t\}_{t\geq 0}, \{Q_t\}_{t\geq 0}$ respectively, and $A^{(\rho)}$ is given by (6.4). In fact, we have for any $f \in \text{dom}(\mathcal{L})$ and any $g \in \mathcal{F}$

$$\begin{aligned}
(-\mathcal{L} f, g) &= \mathcal{E}(f,g) = \mathcal{E}^{(\rho)}(f,g) + \int_{M\times M} (f(x) - f(y))(g(x) - g(y)) J_\rho(x,y) d\mu(y) d\mu(x) \\
&= (-\mathcal{L}^{(\rho)} f, g) - 2 \int_M g(x) \left\{ \int_M (f(y) - f(x)) J_\rho(x,y) d\mu(y) \right\} d\mu(x) \\
&= (-\mathcal{L}^{(\rho)} f, g) - (A^{(\rho)} f, g).
\end{aligned}$$

Since $A^{(\rho)}$ is bounded from $L^2$ to $L^2$ by Proposition 6.2, it follows that $\text{dom}(\mathcal{L}) = \text{dom}(\mathcal{L}^{(\rho)})$, and so (6.7) is true.

Therefore, it follows from Proposition 6.1 with $\Delta = \mathcal{L}^{(\rho)}$, $A = A^{(\rho)}$ that

$$P_t = \sum_{n=0}^{\infty} Q_t^{(n)}, \tag{6.8}$$

where $Q_t^{(0)} = Q_t$, and

$$Q_t^{(n)} = \int_0^t Q_{t-s} A^{(\rho)} Q_s^{(n-1)} ds \ \text{ for each } n \geq 1. \tag{6.9}$$

We show (6.6). Indeed, the series $\sum_{n=0}^{\infty} Q_t^{(n)}$ is absolutely convergent in the norm of $\|\cdot\|_{L^2\to L^2}$, since for any $t > 0$

$$\begin{aligned}
\int_0^t \left\| Q_{t-s} A^{(\rho)} Q_s^{(n)} \right\|_{L^2\to L^2} ds &\leq \int_0^t \left\| A^{(\rho)} Q_s^{(n)} \right\|_{L^2\to L^2} ds \ \text{ (since } Q_t \text{ is contractive in } L^2) \\
&\leq \int_0^t \|A^{(\rho)}\|_{L^2\to L^2} \cdot \|Q_s^{(n)}\|_{L^2\to L^2} ds \\
&\leq \int_0^t \|A^{(\rho)}\|_{L^2\to L^2} \cdot \frac{(s\|A^{(\rho)}\|_{L^2\to L^2})^n}{n!} ds \quad \text{(using (6.3))} \\
&= \frac{(\|A^{(\rho)}\|_{L^2\to L^2})^{n+1}}{n!} \int_0^t s^n ds \leq \frac{1}{(n+1)!} \left( \frac{C_3}{\psi(\rho)} t \right)^{n+1} \quad \text{(using (6.5))},
\end{aligned}$$

which yields that

$$\sum_{n=0}^{\infty} \int_0^t \left\| Q_{t-s} A^{(\rho)} Q_s^{(n)} \right\|_{L^2\to L^2} ds \leq \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left( \frac{C_3}{\psi(\rho)} t \right)^{n+1} = \exp\left( \frac{C_3}{\psi(\rho)} t \right) - 1.$$

Exchanging the order of summation and integration, we obtain from (6.8) that for any $f \in L^2$ and any $t > 0$,

$$\begin{aligned}
P_t f = \sum_{n=0}^{\infty} Q_t^{(n)} f &= Q_t f + \sum_{n=1}^{\infty} \int_0^t Q_{t-s} A^{(\rho)} Q_s^{(n-1)} f ds \\
&= Q_t f + \int_0^t Q_{t-s} A^{(\rho)} \left\{ \sum_{n=1}^{\infty} Q_s^{(n-1)} f \right\} ds \\
&= Q_t f + \int_0^t Q_{t-s} A^{(\rho)} P_s f ds \ \text{ (using (6.8) again).}
\end{aligned} \tag{6.10}$$

On the other hand, we see by (6.4) that for any $t, s > 0$ and any nonnegative $f, g \in L^2$

$$\begin{aligned}
\left( Q_t A^{(\rho)} P_s f, g \right) &= \left( A^{(\rho)} P_s f, Q_t g \right) \ \text{ (using the symmetry of } Q_t) \\
&= \int_M \Big( 2 \int_M (P_s f(y) - P_s f(x)) J_\rho(x,y) d\mu(y) \Big) Q_t g(x) d\mu(x) \\
&\leq \int_M \Big( 2 \int_M P_s f(y) J_\rho(x,y) d\mu(y) \Big) Q_t g(x) d\mu(x) \\
&= 2 \int_M P_s f(y) \Big( \int_M Q_t g(x) J_\rho(x,y) d\mu(x) \Big) d\mu(y) \\
&= 2 \int_M P_s f(y) \Big( \int_M g(x) Q_t J_\rho(\cdot, y)(x) d\mu(x) \Big) d\mu(y) \\
&= 2 \int_M g(x) \Big( \int_M P_s f(y) Q_t J_\rho(\cdot, y)(x) d\mu(y) \Big) d\mu(x),
\end{aligned}$$

which gives that for $\mu$-almost every $x$ in $M$,

$$Q_t A^{(\rho)} P_s f(x) \leq 2 \int_M P_s f(y) Q_t J_\rho(\cdot, y)(x) d\mu(y). \tag{6.11}$$

Finally, plugging (6.11) into (6.10), we obtain (6.6). □

**Remark 6.4.** Observe that a slightly sharper inequality with respect to (6.6) was obtained by using the complicated Meyer decomposition in [11, formula (4.34)] wherein the factor "2" in (6.6) is replaced by "1". Here we give a simpler analytic proof by using an early result in year 1953 by Phillips [35].

The following gives the relationship between the two heat kernels $p_t(x,y)$ and $q_t(x,y)$.

**Lemma 6.5.** *Let $p_t(x,y)$ be the heat kernel of a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ in $L^2$ without killing part, and $q_t(x,y)$ be the heat kernel of the $\rho$-local Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ in $L^2$ for any $\rho>0$. If conditions* (VD), $(J_\le)$, $(S_{1/2})$ *hold, then*

$$p_t(x,y)\le q_t(x,y)+\frac{Ct}{V(x,\rho)\psi(\rho)} \tag{6.12}$$

*for all $0<t\le \frac{1}{4C_2}\psi(\rho)$ and $\mu$-almost all $x,y\in M$, where $C>0$ is some constant independent of $t,x,y$ and $\rho$, and constant $C_2$ is the same as in (5.10).*

*Proof.* Let $\{Q_t=Q_t^{(\rho)}\}$ be a pointwise realization of the heat semigroup of $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ as defined in (5.9). We first show that there exists a constant $C>0$ such that for all $x,z\in M,\rho>0$ and all $0<t\le\frac{1}{4C_2}\psi(\rho)$,

$$Q_tJ_\rho(\cdot,z)(x)\le\frac{C}{V(x,\rho)\psi(\rho)}, \tag{6.13}$$

where $J_\rho(x,y)$ is defined by (5.3).

Indeed, noting that for any $y,z\in M$,

$$J_\rho(z,y)=1_{\{d(y,z)\ge\rho\}}J(y,z)\le\frac{C}{V(y,\rho)\psi(\rho)},$$

we see that for any $t>0$ and any $x,z\in M$

$$\begin{aligned} Q_tJ_\rho(\cdot,z)(x) &= \int_M q_t(x,y)J_\rho(y,z)d\mu(y)\le\int_M q_t(x,y)\frac{C}{V(y,\rho)\psi(\rho)}d\mu(y)\\ &= \frac{C}{\psi(\rho)}\sum_{k=0}^{\infty}\int_{B(x,(k+1)\rho)\setminus B(x,k\rho)}\frac{q_t(x,y)}{V(y,\rho)}d\mu(y)\\ &\le \frac{C'}{V(x,\rho)\psi(\rho)}\sum_{k=0}^{\infty}(k+2)^\alpha Q_t1_{B(x,k\rho)^c}(x), \end{aligned} \tag{6.14}$$

where we have used the fact that for any $y$ in $B(x,(k+1)\rho)\setminus B(x,k\rho)$

$$\frac{1}{V(y,\rho)}=\frac{1}{V(x,\rho)}\frac{V(x,\rho)}{V(y,\rho)}\le\frac{C}{V(x,\rho)}\left(\frac{d(x,y)+\rho}{\rho}\right)^\alpha\le\frac{C'}{V(x,\rho)}(k+2)^\alpha$$

by virtue of (1.3).

On the other hand, we have from inequality (5.10), which follows from condition $(\mathrm{S}_{1/2})$ and condition $(\mathrm{J}_\le)$, that for any $k\ge4$,

$$Q_t1_{B(x,k\rho)^c}(x)\le\left(\frac{C_2t}{\psi(\rho)}\right)^{\frac12\left(\left[\frac{k\rho}{4\rho}\right]-1\right)}\le\left(\frac14\right)^{\frac12\left(\frac{k}{4}-1\right)}=2\left(\frac12\right)^{\frac{k}{4}} \tag{6.15}$$

if $\frac{C_2t}{\psi(\rho)}\le\frac14$, that is, if

$$\frac{t}{\psi(\rho)}\le\frac{1}{4C_2}. \tag{6.16}$$

(Note that $B(x,k\rho)^c$ may be empty but (6.15) still holds in this case since $Q_t1_{B(x,k\rho)^c}=0$ in $M$.)

Thus, plugging (6.15) into (6.14) and using the fact that $Q_t1_{B(x,k\rho)^c}\le1$ in $M$, we obtain for any $x,z$ in $M$ and any $\rho>0$,

$$Q_tJ_\rho(\cdot,z)(x)\le\frac{C'}{V(x,\rho)\psi(\rho)}\left\{\sum_{0\le k\le3}(k+2)^\alpha Q_t1_{B(x,k\rho)^c}(x)+\sum_{k\ge4}(k+2)^\alpha Q_t1_{B(x,k\rho)^c}(x)\right\}$$

$$\leq \frac{C}{V(x,\rho)\psi(\rho)}\left(1+\sum_{k\geq 4}(k+2)^{\alpha}\cdot 2\left(\frac{1}{2}\right)^{\frac{k}{4}}\right) \leq \frac{C'}{V(x,\rho)\psi(\rho)}$$

provided that $\frac{t}{\psi(\rho)} \leq \frac{1}{4C_2}$, thus proving (6.13).

Therefore, it follows from (6.6), (6.13) that for all $0 < t \leq \frac{1}{4C_2}\psi(\rho)$ and for $\mu$-almost all $x \in M$,

$$\begin{aligned} P_t f(x) &\leq Q_t f(x) + 2\int_0^t ds \int_M Q_{t-s}J_\rho(\cdot,z)(x)\cdot P_s f(z)d\mu(z) \\ &\leq Q_t f(x) + 2 \operatorname*{esup}_{s\in(0,t],z\in M} Q_{t-s}J_\rho(\cdot,z)(x)\int_0^t \|P_s f\|_1\, ds \\ &\leq Q_t f(x) + \frac{2C't}{V(x,\rho)\psi(\rho)}\|f\|_1 \end{aligned}$$

for any nonnegative $f \in L^2 \cap L^1$, thus proving (6.12). The proof is complete. □

We remark that inequality (6.12) was obtained in [11, the proof of Proposition 5.3] on the unbounded metric space by using the probabilistic approach.

We are now in a position to derive an off-diagonal upper bound of the heat kernel $p_t(x,y)$.

**Lemma 6.6.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2$ without killing part. Then*

$$(\mathrm{VD}) + (\mathrm{S}_{1/2}) + (\mathrm{J}_{\leq}) + (\mathrm{DUE}) \Rightarrow (\mathrm{UE}).$$

*Proof.* We shall use (6.12) to derive condition (UE). Indeed, let $x,y$ be any two distinct fixed points in $M$ and $0 < t < \psi(\overline{R})$. Let $r := d(x,y)$ and

$$\rho = \frac{r}{4k}, \tag{6.17}$$

where $k \geq 2$ is an integer to be determined below. Without loss of generality, assume that

$$r \geq 4k\psi^{-1}(4C_2 t) \quad\Leftrightarrow\quad \psi(\rho) = \psi(\frac{r}{4k}) \geq 4C_2 t, \tag{6.18}$$

where $C_2$ is the same as in (5.10); otherwise, condition (UE) follows directly from (DUE).

By condition (6.18), we see that

$$d(x,y) = 4k\rho > \rho \geq \psi^{-1}(4C_2 t) > \psi^{-1}(t) \tag{6.19}$$

so that $t \leq \frac{1}{4C_2}\psi(\rho)$. It follows from condition (VD) and (1.7) that

$$\frac{V(x,y)}{V(x,\psi^{-1}(t))} = \frac{V(x,d(x,y))}{V(x,\psi^{-1}(t))} \leq C'\left(\frac{d(x,y)}{\psi^{-1}(t)}\right)^{\alpha} \leq C\left(\frac{\psi(x,y)}{t}\right)^{\alpha/\beta_1}. \tag{6.20}$$

Exchanging the order of $x$ and $y$ and noting that $V(x,y) \asymp V(y,x)$, we similarly have

$$\frac{V(x,y)}{V(y,\psi^{-1}(t))} \leq C\left(\frac{\psi(x,y)}{t}\right)^{\alpha/\beta_1}. \tag{6.21}$$

On the other hand, we have by (6.17), (1.6) that

$$\psi(\rho) = \psi(\frac{r}{4k}) \geq C(k)\psi(r) = C(k)\psi(x,y). \tag{6.22}$$

Therefore, using (6.20)-(6.22), it follows from (5.13) that

$$\begin{aligned} q_t(x,y) &\leq \frac{C}{V(x,y)}\left(\frac{\psi(x,y)}{t}\right)^{\alpha/\beta_1}\left(\frac{C_2 t}{2C(k)\psi(x,y)}\right)^{\frac{1}{2}(\frac{k}{2}-1)}\exp\left(\frac{Ct}{C(k)\psi(x,y)}\right) \\ &\leq \frac{C'(k)}{V(x,y)}\left(\frac{t}{\psi(x,y)}\right)^{\frac{1}{2}(\frac{k}{2}-1)-\frac{\alpha}{\beta_1}} \end{aligned} \tag{6.23}$$

since $\frac{t}{\psi(x,y)} \leq 1$ by using (6.19). From this, we conclude from (6.12) that

$$p_t(x,y) \leq q_t(x,y) + \frac{Ct}{V(x,\rho)\psi(\rho)}$$

$$\begin{aligned} &\leq \frac{C'(k)}{V(x,y)}\left(\frac{t}{\psi(x,y)}\right)^{\frac{1}{2}(\frac{k}{2}-1)-\frac{\alpha}{\beta_1}} + \frac{C(k)t}{V(x,y)\psi(x,y)} \\ &\leq \frac{C(k)t}{V(x,y)\psi(x,y)} \end{aligned} \tag{6.24}$$

for any $0 < t < \psi(\overline{R})$ and $\mu$-almost all points $x, y$ in $M$, provided that

$$\frac{1}{2}(\frac{k}{2}-1) - \frac{\alpha}{\beta_1} \geq 1 \quad \Leftrightarrow \quad k \geq 6 + \frac{4\alpha}{\beta_1}. \tag{6.25}$$

Condition (6.25) will be guaranteed if we take a large integer $k$, for example, if $k = 7 + \left[\frac{4\alpha}{\beta_1}\right]$.

Finally, for any $0 < t < \psi(\overline{R})$, by using [20, Theorem 2.2], there exists an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ on $M$ independent of $t$ such that

$$p_t(x, \cdot) \in C(\{F_n\})$$

for any point $x$ in $M$, and that $p_t(x, y) = 0$ whenever one of points $x, y$ lies outside the set

$$M_0 := \bigcup_{n=1}^{\infty} F_n.$$

Therefore, it follows from (6.24) that

$$p_t(x, y) \leq \frac{Ct}{V(x,y)\psi(x,y)}$$

for any $0 < t < \psi(\overline{R})$ and any two points $x, y$ in $M$, if

$$t \leq \frac{1}{4C_2}\psi(\rho) = \frac{1}{4C_2}\psi\left(\frac{d(x,y)}{4k}\right).$$

Hence, condition (UE) is true. The proof is complete. □

As we have seen, in order to obtain off-diagonal upper estimate of the heat kernel $p_t(x, y)$, we need to truncate the jump part of the biliear form $\mathcal{E}$ by any positive number $\rho$ and then derive upper estimate of the heat kernel $q_t^{(\rho)}(x, y)$ associated with the truncated form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$, and finally use the relationship (6.12) between the original heat semigroups $\{P_t\}$ and the truncated heat semigroup $\{Q_t^{(\rho)}\}$.

### 6.2. Conditions (Gcap$_\varepsilon$) and (FK$_\nu$).

In this subsection we will derive conditions (Gcap$_\varepsilon$) and (J$_\leq$) from conditions (UE) and (C).

**Lemma 6.7.** *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2$. Then*

$$\text{(UE)} + \text{(C)} \Rightarrow (\text{J}_\leq) + (\text{Gcap}_\varepsilon).$$

*Proof.* We first show the implication (UE) $\Rightarrow$ (J$_\leq$). The proof is standard.

Indeed, let $A, B$ be any two disjoint compact subsets of $M$ and $f, g \in \mathcal{F} \cap C_0(M)$ be any two functions supported on $A, B$ respectively. By condition (UE), we see that for any $x \in A, y \in B$ and any $0 < t < \psi(\overline{R})$

$$p_t(x, y) \leq \frac{Ct}{V(x,y)\psi(x,y)}.$$

It follows that

$$\begin{aligned} 2\int_A\int_B f(x)g(y)dj(x,y) = -\mathcal{E}(f,g) &= \lim_{t\to 0}\frac{1}{t}(P_t f - f, g) = \lim_{t\to 0}\frac{1}{t}(P_t f, g) \\ &= \lim_{t\to 0}\frac{1}{t}\int_A\int_B f(x)g(y)p_t(x,y)d\mu(y)d\mu(x) \\ &\leq \limsup_{t\to 0}\frac{1}{t}\int_A\int_B f(x)g(y)\frac{Ct}{V(x,y)\psi(x,y)}d\mu(y)d\mu(x) \\ &= \int_A\int_B f(x)g(y)\frac{C}{V(x,y)\psi(x,y)}d\mu(y)d\mu(x). \end{aligned}$$

Since $(\mathcal{E},\mathcal{F})$ is regular, the functions $\sum_{i=1}^n f_i(x)g_i(y)$ with $f_i, g_i \in \mathcal{F}\cap C_0(M)$, $\mathrm{supp}[f_i]\cap\mathrm{supp}[g_i]=\emptyset$, $1\le i\le n$, constitute a dense subalgebra of $C_0(M\times M\setminus \mathrm{diag})$, see for example [16, Lemma 1.4.2 on p.29]. Thus, the measure $dj$ has a density function $J$ on $(M\times M)\setminus\mathrm{diag}$, which satisfies

$$J(x,y)\le \frac{C}{2V(x,y)\psi(x,y)},$$

thus showing that condition ($J_\le$) is true.

It remains to show the implication (UE) + (C) ⇒ ($\mathrm{Gcap}_\varepsilon$). In fact, we will show

$$\text{(UE)} + \text{(C)} \Rightarrow (\mathrm{S}_+) \Rightarrow (\mathrm{Gcap}_\varepsilon), \tag{6.26}$$

where *condition* ($\mathrm{S}_+$) means that: there exists some constant $C_5\ge 1$ such that for any $0<\delta\le\frac12$, any ball $B' := B(x_0,R')$ with $0<R'<\overline{R}$ and any $0<t<\psi(\overline{R})$, $x\in(1-\delta)^2B'$,

$$1-P_t^{B'}1_{B'}(x)\le \frac{C_5 t}{\psi(\delta R')}, \tag{6.27}$$

where $\{P_t^\Omega\}$ for any ball $\Omega$ is understood to be a pointwise realization of the heat semigroup, that is

$$P_t^\Omega f(x)=\int_M p_t^\Omega(x,y)f(y)d\mu(y)$$

for any $t>0$ and any point $x$ in $M$, whose existence is guaranteed by condition (DUE).

Indeed, for any $x\in(1-\delta)B'$, $\delta\in(0,1]$ and any $0<t<\psi(\overline{R})$, we see by condition (UE) that

$$\begin{aligned} P_t 1_{(B')^c}(x) &= \int_{B(x_0,R')^c} p_t(x,y)d\mu(y)\le \int_{B(x,\delta R')^c} p_t(x,y)d\mu(y)\\ &= \int_{B(x,\delta R')^c}\frac{Ct}{V(x,y)\psi(x,y)}d\mu(y)\le\frac{Ct}{\psi(\delta R')} \end{aligned} \tag{6.28}$$

by using (3.2) with $R$ replaced by $\delta R'$, where $C$ is some positive constant independent of $t, B', \delta$.

We will derive (6.27) by using (6.28). Indeed, applying (7.2) in Appendix with $\Omega=M$, $U=B'$, $K=(1-\frac{\delta}{2})\overline{B'}$ and $f=1_{(1-\delta)B'}$, we obtain

$$P_t1_{(1-\delta)B'}-P_t^{B'}1_{(1-\delta)B'}\le \sup_{s\in(0,t]}\ \operatorname*{esup}_{\left((1-\frac{\delta}{2})\overline{B'}\right)^c} P_s1_{(1-\delta)B'}\ \text{ in } M,$$

which yields that, using condition (C),

$$\begin{aligned} 1-P_t^{B'}1_{(1-\delta)B'} &\le 1-P_t1_{(1-\delta)B'}+\sup_{s\in(0,t]}\ \operatorname*{esup}_{\left((1-\frac{\delta}{2})\overline{B'}\right)^c} P_s1_{(1-\delta)B'}\\ &= P_t1_{((1-\delta)B')^c}+\sup_{s\in(0,t]}\ \operatorname*{esup}_{z\in\left((1-\frac{\delta}{2})\overline{B'}\right)^c} P_s1_{(1-\delta)B'}(z)\ \text{ in } M. \end{aligned} \tag{6.29}$$

We look at the two terms on the right-hand side of (6.29).

Indeed, applying (6.28) with $B'$ replaced by $(1-\delta)B'$, we see that

$$P_t1_{((1-\delta)B')^c}(x)\le\frac{Ct}{\psi(\delta(1-\delta)R')}\le\frac{C't}{\psi(\delta R')} \tag{6.30}$$

for any $x\in(1-\delta)^2B'$ and any $0<t<\psi(\overline{R})$ if $\delta\in(0,\frac12]$, where $C'$ is independent of $x,t,\delta,B'$.

On the other hand, for any $z\in\left((1-\frac{\delta}{2})\overline{B'}\right)^c$, we have $(1-\delta)B'\subset B(z,\frac{\delta}{2}R')^c$, and so using (6.28) again,

$$P_s1_{(1-\delta)B'}(z)\le P_s1_{B(z,\frac{\delta}{2}R')^c}(z)\le\frac{Cs}{\psi(\frac{\delta}{2}R')}\le\frac{C's}{\psi(\delta R')}. \tag{6.31}$$

Substituting (6.31), (6.30) into (6.29), we have

$$1-P_t^{B'}1_{B'}(x)\le 1-P_t^{B'}1_{(1-\delta)B'}(x)\le\frac{C't}{\psi(\delta R')}+\sup_{s\in(0,t]}\frac{C's}{\psi(\delta R')}\le\frac{Ct}{\psi(\delta R')}$$

for any $x\in(1-\delta)^2B'$, any $0<t<\psi(\overline{R})$ and any $\delta\in(0,\frac12]$, thus proving condition ($\mathrm{S}_+$).

We will show that for any $\lambda > 0$, any $\delta \in (0, \frac{1}{2}]$ and any ball $B'$ of radius $R'$ with $0 < R' < \overline{R}$, we have

$$1 \geq \sup_M(\lambda h) \geq \inf_{(1-\delta)^2 B'}(\lambda h) \geq w(s), \tag{6.32}$$

where $w(s)$ is given by

$$w(s) := 1 - C_5 s^{-1} + (C_5 - 1 + C_5 s^{-1})e^{-s} \geq 1 - C_5 s^{-1} \tag{6.33}$$

with $s = \lambda\psi(\delta R')$ and constant $C_5 \geq 1$ given by (6.27), and the function $h$ is defined by

$$h := \int_0^\infty e^{-\lambda t} P_t^{B'} 1_{B'} dt. \tag{6.34}$$

This can be done by using (6.27). Indeed, since $P_t^{B'} 1_{B'} \leq 1$ in $M$, we see by (6.34)

$$h \leq \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda} \text{ in } M,$$

thus showing the leftmost inequality of (6.32).

On the other hand, we have by (6.34), (6.27) that, using $s = \lambda\psi(\delta R')$,

$$\begin{aligned} \inf_{(1-\delta)^2 B'}(\lambda h) &= \lambda \inf_{(1-\delta)^2 B'} \int_0^\infty e^{-\lambda t} P_t^{B'} 1_{B'} dt \geq \lambda \int_0^{\psi(\delta R')} e^{-\lambda t} \left(1 - \frac{C_5 t}{\psi(\delta R')}\right) dt \\ &= \int_0^s e^{-x} \left(1 - C_5 s^{-1} x\right) dx = (1 - e^{-s}) - C_5 s^{-1} (1 - e^{-s} - s e^{-s}) = w(s), \end{aligned}$$

thus showing the rightmost inequality of (6.32).

Finally, we show that condition (Gcap$_\varepsilon$) is true by using (6.32). To do this, we need to construct some $(1+\varepsilon)$-cutoff$(B_0, B)$ for any $\varepsilon \in (0,1)$ and for any two concentric balls $B_0, B$.

Indeed, let $h$ be defined by (6.34). Then $h \in \mathcal{F}(B') \cap L^\infty$, which satisfies

$$\mathcal{E}(h, \varphi) + \lambda(h, \varphi) = (1_{B'}, \varphi) = \|\varphi\|_1 \tag{6.35}$$

for any nonnegative $\varphi \in \mathcal{F}(B')$. Define the function

$$\phi := \frac{\lambda h}{w(s)} = \frac{\lambda h}{1 - C_5 s^{-1} + (C_5 - 1 + C_5 s^{-1})e^{-s}}, \tag{6.36}$$

where $s = \lambda\psi(\delta R')$. Clearly, such a function $\phi \in \mathcal{F}(B')$ because so is $h$, and moreover, $h \geq 1$ on $(1-\delta)^2 B'$ by (6.32),

$$\phi = \frac{\lambda h}{w(s)} \leq \frac{1}{w(s)} \leq 1 + \varepsilon \text{ in } M$$

if $s \geq \frac{1+\varepsilon}{\varepsilon} C_5$, since in this case we have by (6.33)

$$w(s) \geq 1 - C_5 s^{-1} \geq (1+\varepsilon)^{-1}.$$

Thus, $\phi$ is one $(1+\varepsilon)$-cutoff function for any pair $((1-\delta)^2 B', B')$ if we choose $\lambda$ such that

$$\lambda\psi(\delta R') = s = \frac{1+\varepsilon}{\varepsilon} C_5. \tag{6.37}$$

It remains to show that (1.10) is true for any measurable function $u$ with $u^2\phi \in \mathcal{F}(B')$. Indeed, we have by (6.35) with $\varphi = u^2\phi \in \mathcal{F}(B')$ that

$$\begin{aligned} \mathcal{E}(u^2\phi, \phi) &= \frac{\lambda}{w(s)} \mathcal{E}(u^2\phi, h) = \frac{\lambda}{w(s)} \left\{ \left\|u^2\phi\right\|_1 - \lambda(h, u^2\phi) \right\} \\ &\leq \frac{\lambda}{w(s)} \left\|u^2\phi\right\|_1 = \frac{(1+\varepsilon)\varepsilon^{-1} C_5}{w(s)\psi(\delta R')} \int_M u^2\phi d\mu \ \text{ (using (6.37))} \\ &\leq \frac{(1+\varepsilon)^2 \varepsilon^{-1} C_5}{\psi(\delta R')} \int_M u^2\phi d\mu \ \text{ (since } w(s) \geq (1+\varepsilon)^{-1}) \\ &\leq \frac{4 C_5 \varepsilon^{-1}}{\psi(\delta R')} \int_M u^2\phi d\mu \ \text{ (since } 0 < \varepsilon \leq 1). \end{aligned} \tag{6.38}$$

We shall use (6.38) to derive (1.10).

Indeed, for any two concentric balls $B_0 = B(x,R)$, $B := B(x_0, R+r)$ with $0 < R < R+r < \overline{R}$, we choose $\delta = \frac{r}{2(R+r)} \in (0, \frac{1}{2})$ so that $(1-\delta)^2 \geq 1 - 2\delta = \frac{R}{R+r}$. Replacing $B'$ by $B(x_0, R+r)$ in (6.38) so that

$$(1-\delta)^2 B' = B(x_0, (1-\delta)^2(R+r)) \supset B(x_0, R) = B_0,$$

we conclude that $\phi \in (1+\varepsilon)$-cutoff $(B_0, B)$, and

$$\mathcal{E}(u^2\phi, \phi) \leq \frac{4C_5\varepsilon^{-1}}{\psi(\delta(R+r))} \int_{B(x_0,R+r)} u^2 \phi d\mu \leq \frac{C\varepsilon^{-1}}{\psi(r)} \int_M u^2 \phi d\mu$$

for some positive constant $C$ independent of $\varepsilon, B_0, B, u$. This proves that condition ($\text{Gcap}_\varepsilon$) holds for any $\varepsilon \in (0,1)$. The proof is complete. □

We derive the Faber-Krahn inequality from conditions (DUE), (VD), (RVD).

**Lemma 6.8.** *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2$. Then*

$$(\text{DUE}) + (\text{VD}) + (\text{RVD}) \Rightarrow (\text{FK}_\nu).$$

*Proof.* We will show the following

$$\frac{\mathcal{E}(u)}{\|u\|_2^2} \geq \frac{C^{-1}}{\psi(R)} \left(\frac{\mu(B)}{\mu(\Omega)}\right)^\nu \tag{6.39}$$

for any $B := B(x_0, R)$ with $0 < R < \delta\overline{R}$ and any non-empty open subset $\Omega$ of $B$, where $\delta \in (0, \frac{1}{3}]$ is a small number to be picked up.

Indeed, by the spectral resolution, we see for any $s > 0$, $f \in \mathcal{F}$

$$\mathcal{E}(f, P_s f) = \int_0^\infty \lambda e^{-\lambda s} d(E_\lambda f, f) \leq \int_0^\infty \lambda d(E_\lambda f, f) = \mathcal{E}(f).$$

From this, we have for any $t > 0$, $f \in \mathcal{F}$

$$\|f\|_2^2 - t\mathcal{E}(f) \leq \|f\|_2^2 - \int_0^t \mathcal{E}(f, P_s f) ds = \|f\|_2^2 + \int_0^t \frac{d}{ds}(f, P_s f) ds = (f, P_t f). \tag{6.40}$$

On the other hand, we see by condition (DUE) that for any nonnegative $f \in \mathcal{F}(\Omega)$

$$\begin{aligned} (f, P_t f) &= \int_{\Omega\times\Omega} f(x) f(y) p_t(x,y) d\mu(y) d\mu(x) \leq \operatorname*{esup}_{x,y\in\Omega} p_t(x,y) \|f\|_1^2 \\ &\leq \operatorname*{esup}_{x\in\Omega} \frac{C}{V(x, \psi^{-1}(t))} \|f\|_1^2 \quad \text{for any } 0 < t < \psi(\overline{R}). \end{aligned} \tag{6.41}$$

If $0 < t \leq \psi(2R)$, then for any $x \in \Omega \subset B$

$$\begin{aligned} \frac{\mu(B)}{V(x, \psi^{-1}(t))} &\leq \frac{V(x, 2R)}{V(x, \psi^{-1}(t))} \leq C \left(\frac{2R}{\psi^{-1}(t)}\right)^\alpha \quad \text{(using (1.4))} \\ &\leq C' \left(\frac{\psi(2R)}{t}\right)^{\alpha/\beta_1} \quad \text{(using (1.7))}, \end{aligned}$$

which combines with (6.41) to yield that

$$(f, P_t f) \leq \frac{C_4}{\mu(B)} \left(\frac{\psi(2R)}{t}\right)^{\alpha/\beta_1} \|f\|_1^2$$

for some universal constant $C_4$ independent of $B, \Omega, t, f$.

From this and (6.40), it follows that for any nonnegative $f \in \mathcal{F}(\Omega)$

$$\begin{aligned} \|f\|_2^2 - t\mathcal{E}(f) &\leq \frac{C_4}{\mu(B)} \left(\frac{\psi(2R)}{t}\right)^{\alpha/\beta_1} \|f\|_1^2 \\ &\leq C_4 \frac{\mu(\Omega)}{\mu(B)} \left(\frac{\psi(2R)}{t}\right)^{\alpha/\beta_1} \|f\|_2^2 \quad \text{for any } 0 < t \leq \psi(2R) \end{aligned} \tag{6.42}$$

since $\|f\|_1^2 \leq \mu(\Omega) \|f\|_2^2$ for any $f \in L^2$ supported on $\Omega$ by the Cauchy-Schwarz inequality.

On the other hand, if $\psi(2R) \le t < \psi(\overline{R})$, then for any $x \in \Omega$

$$\begin{aligned}\frac{\mu(B)}{V(x,\psi^{-1}(t))} &\le \frac{V(x,2R)}{V(x,\psi^{-1}(t))} \le C\left(\frac{2R}{\psi^{-1}(t)}\right)^{\alpha_1} \quad (\text{using } (1.5))\\ &\le C'\left(\frac{\psi(2R)}{t}\right)^{\alpha_1/\beta_2} \quad (\text{using } (1.7)),\end{aligned}$$

which combines with (6.41) to yield that

$$(f, P_t f) \le \frac{C_4}{\mu(B)}\left(\frac{\psi(2R)}{t}\right)^{\alpha_1/\beta_2} \|f\|_1^2 .$$

From this and (6.40), it follows that for any nonnegative $f \in \mathcal{F}(\Omega)$

$$\|f\|_2^2 - t\mathcal{E}(f) \le C_4\frac{\mu(\Omega)}{\mu(B)}\left(\frac{\psi(2R)}{t}\right)^{\alpha_1/\beta_2} \|f\|_2^2 \quad \text{for any } \psi(2R) \le t < \psi(\overline{R}). \tag{6.43}$$

We distinguish two cases.

*Case* 1 when $\frac{\mu(\Omega)}{\mu(B)} \le \frac{1}{2C_4}$. In this case, we have

$$t_1 := \psi(2R)\left(2C_4\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta_1/\alpha} \le \psi(2R).$$

Applying (6.42) with $t = t_1$, we obtain

$$\|f\|_2^2 - t_1\mathcal{E}(f) \le C_4\frac{\mu(\Omega)}{\mu(B)}\left(\frac{\psi(2R)}{t_1}\right)^{\alpha/\beta_1} \|f\|_2^2 = \frac{1}{2}\|f\|_2^2$$

and hence,

$$\frac{\mathcal{E}(f)}{\|f\|_2^2} \ge \frac{1}{2t_1} = \frac{1}{2\psi(2R)}\left(2C_4\frac{\mu(\Omega)}{\mu(B)}\right)^{-\beta_1/\alpha} \ge \frac{C^{-1}}{\psi(R)}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^{\beta_1/\alpha} .$$

This proves that (6.39) holds with $\nu = \beta_1/\alpha$.

*Case* 2 when $\frac{\mu(\Omega)}{\mu(B)} \ge \frac{1}{2C_4}$. In this case, we have

$$t_2 := \psi(2R)\left(2C_4\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta_2/\alpha_1} \ge \psi(2R).$$

To secure condition $t_2 < \psi(\overline{R})$, we need further to restrict the range of $R$. In fact, if $\overline{R} < \infty$ and if $R < \delta\overline{R}$ with $\delta \le \frac{1}{2}$ to be chosen, then

$$\begin{aligned}t_2 &= \psi(2R)\left(2C_4\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta_2/\alpha_1} \le \psi(2\delta\overline{R})\,(2C_4)^{\beta_2/\alpha_1} \quad (\text{since } \mu(\Omega) \le \mu(B))\\ &\le C\left(\frac{2\delta\overline{R}}{\overline{R}}\right)^{\beta_1}(2C_4)^{\beta_2/\alpha_1}\,\psi(\overline{R}) = C\,(2\delta)^{\beta_1}\,(2C_4)^{\beta_2/\alpha_1}\,\psi(\overline{R}) < \psi(\overline{R})\end{aligned}$$

provided that $\delta$ is sufficiently small, for example,

$$C\,(2\delta)^{\beta_1}\,(2C_4)^{\beta_2/\alpha_1} = \frac{1}{2} \quad \Leftrightarrow \quad \delta = \frac{1}{2}\left(\frac{1}{2C}\,(2C_4)^{-\beta_2/\alpha_1}\right)^{1/\beta_1} . \tag{6.44}$$

With this choice of $\delta$, applying (6.43) with $\psi(2R) \le t = t_2 < \psi(\overline{R})$, we obtain

$$\|f\|_2^2 - t_2\mathcal{E}(f) \le C_4\frac{\mu(\Omega)}{\mu(B)}\left(\frac{\psi(2R)}{t_2}\right)^{\alpha_1/\beta_2} \|f\|_2^2 = \frac{1}{2}\|f\|_2^2$$

and hence,

$$\frac{\mathcal{E}(f)}{\|f\|_2^2} \ge \frac{1}{2t_2} = \frac{1}{2\psi(2R)}\left(2C_4\frac{\mu(\Omega)}{\mu(B)}\right)^{-\beta_2/\alpha_1} \ge \frac{C^{-1}}{\psi(R)}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^{\beta_1/\alpha}$$

since $\frac{\mu(B)}{\mu(\Omega)} \ge 1$ and $\frac{\beta_2}{\alpha_1} \ge \frac{\beta_1}{\alpha}$. Thus (6.39) holds again with $\nu = \beta_1/\alpha$ and with $\delta$ defined by (6.44).

Therefore, condition (FK$_\nu$) is true with $\nu = \beta_1/\alpha$. □

6.3. **The reverse volume doubling condition.** In this subsection, we derive the reverse volume doubling property from the Faber-Krahn inequality.

**Lemma 6.9.** *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2$. Then*

$$(\mathrm{FK}_\nu) + (\mathrm{J}_\leq) + (\mathrm{VD}) \Rightarrow (\mathrm{RVD}).$$

*Proof.* We divide the proof into three steps.

*Step* 1. We show that there exists a constant $\varepsilon_0 \in (0, \frac{1}{2})$ such that the space $(M, d)$ is $(1 - \varepsilon_0)$-annulus connected, that is, for any ball $B := B(x_0, R)$ with $0 < R < \overline{R}$, the annulus

$$B \setminus \varepsilon_0 B \neq \emptyset. \tag{6.45}$$

To show (6.45), we distinguish two cases according to whether $R < \delta\overline{R}$ or not, where constant $\delta$ comes from condition (FK$_\nu$).

*Case* 1 when $0 < R < \delta\overline{R}$. Indeed, let $\varepsilon \in (0, 1)$ be any number such that

$$B \setminus \varepsilon B = \emptyset. \tag{6.46}$$

Then $\Omega := \varepsilon B = B$ is both open and closed, and so it is compact. By Proposition 7.1 in Appendix, the function $1_\Omega \in \mathcal{F}(\Omega)$. Applying condition (FK$_\nu$) with respect to the pair $(\Omega, B) = (\varepsilon B, B)$, we see that

$$\frac{\mathcal{E}(1_\Omega)}{\|1_\Omega\|_2^2} \geq \lambda_1(\Omega) \geq \frac{C^{-1}}{\psi(R)} \left(\frac{\mu(B)}{\mu(\Omega)}\right)^\nu = \frac{C^{-1}}{\psi(R)}.$$

From this and using $\Omega = \varepsilon B = B$, we have

$$\begin{aligned}
\frac{C^{-1}\mu(\Omega)}{\psi(R)} &\leq \mathcal{E}(1_\Omega) = \mathcal{E}^{(L)}(1_\Omega) + \mathcal{E}^{(J)}(1_\Omega) = \mathcal{E}^{(J)}(1_\Omega) \\
&= \int_{M\times M} (1_\Omega(x) - 1_\Omega(y))^2 dj = 2\int_{\Omega\times\Omega^c} dj = 2\int_{(\varepsilon B)\times B^c} dj \\
&\leq \frac{C\mu(\Omega)}{\psi((1-\varepsilon)R)} \quad \text{(using (3.3))} \\
&= \frac{C\mu(\Omega)}{\psi(R)} \frac{\psi(R)}{\psi((1-\varepsilon)R)} \leq \frac{(1-\varepsilon)^{-\beta_2} C'\mu(\Omega)}{\psi(R)} \quad \text{(using (1.6))},
\end{aligned}$$

which implies that $\varepsilon \geq 1 - C_6^{1/\beta_2}$ for some universal constant $C_6 > 1$. Choose $\varepsilon_1 \in (0, 1)$ to be some number such that $\varepsilon_1 < 1 - C_6^{1/\beta_2}$, for example,

$$\varepsilon_1 := 1 - \frac{1}{2} C_6^{1/\beta_2}. \tag{6.47}$$

It follows that

$$B(x_0, R) \setminus B(x_0, \varepsilon R) \neq \emptyset \tag{6.48}$$

for any ball $B$ of radius $R$ with $0 < R < \delta\overline{R}$ and for any $0 \leq \varepsilon \leq \varepsilon_1$ with $\varepsilon_1$ given by (6.47).

*Case* 2 when $\overline{R} < \infty$ and $\delta\overline{R} \leq R < \overline{R}$. Then, noting that $B = B(x_0, R) \supset B\big(x_0, 3\delta\overline{R}/4\big)$ and

$$B\Big(x_0, \varepsilon_1 \left(3\delta\overline{R}/4\right)\Big) \supset B\Big(x_0, \varepsilon_1 3\delta R/4\Big) = \frac{\varepsilon_1(3\delta)}{4} B,$$

we see from (6.48), with $R$ being replaced by $3\delta\overline{R}/4$, that

$$B \setminus \Big(\frac{\varepsilon_1(3\delta)}{4} B\Big) \supset B\Big(x_0, 3\delta\overline{R}/4\Big) \setminus B\Big(x_0, \varepsilon_1 \left(3\delta\overline{R}/4\right)\Big) \neq \emptyset.$$

Therefore, letting $\varepsilon_0 := \frac{\varepsilon_1(3\delta)}{4} < \varepsilon_1$, we conclude that (6.45) holds with this $\varepsilon_0$ for any ball $B$ of radius $R$ with $0 < R < \overline{R}$.

*Step* 2. We show that there exist two constants $\delta_0 \in (0, \frac{1}{4}), C_0 > 1$ such that, for any ball $B$ of radius $R$ with $0 < R < \overline{R}$, we have

$$\frac{\mu(B)}{\mu(\delta_0 B)} \geq C_0 \tag{6.49}$$

by using condition (6.45).

Indeed, let $B := B(x_0, R)$. By condition (6.45), the set $B \setminus \varepsilon_0 B$ is not empty, and so there exists a point $y$ in $B \setminus \varepsilon_0 B$ such that the two balls $B(x_0, \frac{\varepsilon_0}{2} R)$ and $B(y, \frac{\varepsilon_0}{2} R)$ are disjoint, but both of which are contained in ball $B(x_0, (1 + \frac{\varepsilon_0}{2})R)$. From this, we see that

$$V\Big(x_0, (1 + \frac{\varepsilon_0}{2})R\Big) \geq V\Big(x_0, \frac{\varepsilon_0}{2} R\Big) + V\Big(y, \frac{\varepsilon_0}{2} R\Big).$$

On the other hand, using (1.3) and the fact that $d(x_0, y) < R$, we have

$$\frac{V\Big(x_0, \frac{\varepsilon_0}{2} R\Big)}{V\Big(y, \frac{\varepsilon_0}{2} R\Big)} \leq C \left(\frac{R + \frac{\varepsilon_0}{2} R}{\frac{\varepsilon_0}{2} R}\right)^{\alpha} = C \left(1 + 2\varepsilon_0^{-1}\right)^{\alpha},$$

from which, it follows that

$$\begin{aligned} V\Big(x_0, (1 + \frac{\varepsilon_0}{2})R\Big) &\geq V\Big(x_0, \frac{\varepsilon_0}{2} R\Big) + C^{-1} \left(1 + 2\varepsilon_0^{-1}\right)^{-\alpha} V\Big(x_0, \frac{\varepsilon_0}{2} R\Big) \\ &= [1 + C^{-1} \left(1 + 2\varepsilon_0^{-1}\right)^{-\alpha} ] V\Big(x_0, \frac{\varepsilon_0}{2} R\Big). \end{aligned}$$

Letting $R' = (1 + \frac{\varepsilon_0}{2})R$, we obtain from above that

$$V(x_0, R') \geq [1 + C^{-1} \left(1 + 2\varepsilon_0^{-1}\right)^{-\alpha} ] V\Big(x_0, \frac{\varepsilon_0}{2 + \varepsilon_0} R'\Big).$$

Thus, by letting

$$\delta_0 = \frac{\varepsilon_0}{2 + \varepsilon_0} < 1 \text{ and } C_0 = 1 + C^{-1} \left(1 + 2\varepsilon_0^{-1}\right)^{-\alpha} > 1$$

and by renaming $R'$ by $R$, we see that (6.49) holds for any ball $B$ of radius $R$ with $0 < R < \overline{R}$.

*Step* 3. Finally, we show that condition (RVD) is satisfied by using condition (6.49). Indeed, for any $0 < r < R < \overline{R}$, there exists some integer $N \geq 1$ such that

$$\delta_0^N \leq \frac{r}{R} < \delta_0^{N-1}.$$

It follows from (6.49) that

$$\frac{V(x_0, R)}{V(x_0, r)} \geq \frac{V(x_0, R)}{V(x_0, \delta_0^{N-1} R)} \geq C_0^{N-1} \geq C_0^{-1} (C_0)^{\frac{\ln(\frac{R}{r})}{\ln(\delta_0^{-1})}} = C_0^{-1} \left(\frac{R}{r}\right)^{\frac{\ln(C_0)}{\ln(\delta_0^{-1})}},$$

thus showing (1.5) with $\alpha_1 = \frac{\ln(C_0)}{\ln(\delta_0^{-1})} > 0$, and so condition (RVD) is satisfied. The proof is complete. □

We mention that a similar argument in the proof of Lemma 6.9 was addressed by Carron in [8], see also [13, Lemma 2.2(2)] by Coulhon and Grigor'yan.

## 7. Appendix

In this Appendix, we first give the following result and then collect the known results that have been used in this paper.

**Proposition 7.1.** *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2$ on a metric measure space $(M, d, \mu)$. If $\Omega$ is a non-empty open compact subset of $M$, then the indicator $1_\Omega \in \mathcal{F}(\Omega)$. In particular, if $M$ is bounded and every metric ball is assumed to be precompact, then* $\mathrm{cutoff}(M, M) = \{1\}$.

*Proof.* Since $\Omega$ is open and compact, the indicator function $1_\Omega \in C_0(M)$, the space of all continuous functions with compact support in $M$. By [16, Lemma 1.4.2 on p.29], there exists a sequence $\{u_n\}$ of functions from $\mathcal{F} \cap C_0(M)$ with $\mathrm{supp}[u_n] \subset \Omega$ such that $u_n \to 1_\Omega$ uniformly on $M$ as $n \to \infty$. Therefore, there is some integer $n$ such that $u_n \geq \frac{1}{2}$ in $\Omega$, thus showing that $1_\Omega = 2(\frac{1}{2} \wedge u_n) \in \mathcal{F}$. Since the function $1_\Omega$ vanishes outside $\Omega$, we see that $1_\Omega \in \mathcal{F}(\Omega)$ by using [16, Corollary 2.3.1 on p. 98]).

If $M$ is bounded, then $M$ is compact, since $M$ is the closure of a ball $B$ and every metric ball is assumed to be precompact. Thus $1 \in \mathcal{F}$ and so $\mathrm{cutoff}(M, M) = \{1\}$. □

The following results are known.

**Lemma 7.2** ([24, Proposition 4.6])**.** *Let $\Omega$ be a non-empty open set in $M$ and $f \in L^2 \cap L^\infty$ be nonnegative in $M$. Then for any $t > 0$ and $\mu$-almost every $x \in \Omega$,*

$$|P_t^\Omega f(x) - Q_t^\Omega f(x)| \leq 2t\|f\|_\infty \operatorname*{esup}_{x\in M} \int_{B(x,\rho)^c} J(x,y)d\mu(y). \tag{7.1}$$

**Lemma 7.3** ([24, Theorem 3.1])**.** *Let $\{Q_t := Q_t^{(\rho)}\}_{t\geq 0}$ be the heat semigroup of some $\rho$-local Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ in $L^2$. Let $\phi(r, \cdot)$ be a non-decreasing function in $(0, \infty)$ for any $r > 0$. Assume that for any ball $B := B(x,r)$ and for any $t \in (0, T_0)$ where $T_0 \in (0, \infty]$,*

$$1 - Q_t^B 1_B \leq \phi(r,t) \ \text{ in } \frac{1}{4}B.$$

*Then for any ball $B(x,r)$ with $r > \rho$ and $t \in (0, T_0)$, for any integer $k \geq 1$,*

$$Q_t 1_{B(x,kr)^c} \leq \phi(r-\rho,t)^{k-1} \ \text{ in } \ B(x,r).$$

**Lemma 7.4** ([21, Lemma 4.18])**.** *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2$. Then for any two open subsets $U \subset \Omega$ of $M$, for any compact set $K \subset U$, for any $0 \leq f \in L^2(M)$ and all $t > 0$,*

$$\operatorname*{esup}_{\Omega}\left(P_t^\Omega f - P_t^U f\right) \leq \sup_{s\in(0,t]} \operatorname*{esup}_{\Omega\setminus K} P_s^\Omega f. \tag{7.2}$$

*In particular, when $\Omega = M$, $U = B$ for any metric ball $B$, we have for any $t > 0$*

$$P_t^B f(x) \geq P_t f(x) - \sup_{s\in(0,t]} \operatorname*{esup}_{x\in\left(\frac{1}{2}\overline{B}\right)^c} P_s f(x). \tag{7.3}$$

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