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On Turán densities of small triple graphs ☆

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ABSTRACT

For a family of k-graphs $\mathcal{F}$, the Turán number $T(n, \mathcal{F})$ is the maximum number of edges in a k-graph of order n that does not contain any member of $\mathcal{F}$. The Turán density $t(\mathcal{F}) = \lim_{n\to\infty} T(n, \mathcal{F})/\binom{n}{k}$. Let $\mathcal{K}_4$ be the tetrahedron that is the complete triple graph of order four. Let the triple graph $\mathcal{F}_{p,q}$ defined on the vertex set $P \cup Q$ with $|P| = p$ and $|Q| = q$ consist of those edges which intersect P in either one or three vertices. Let $V = \{1, 2, 3, 4, 5\}$ and let $\mathcal{F}_5$ be defined on V with $E(\mathcal{F}_5) = \{123, 145, 245\}$. Let $\mathcal{F}_5^c$ denote the complement of $\mathcal{F}_5$, and let $\mathcal{P}_5$ be the weak pentagon obtained from $\mathcal{F}_5$ by adding the edge 134 and let $\mathcal{C}_5$ be the pentagon obtained from $\mathcal{P}_5$ by adding one more edge 235. We prove that

- $1/2 \le t(\mathcal{F}_{1,4}) \le 2/3$,
- $2\sqrt{3} - 3 \le t(\mathcal{K}_4, \mathcal{F}_5^c) \le 2 - \sqrt{2}$,
- $1/4 \le t(\mathcal{F}_{1,4}, \mathcal{P}_5) \le 6\sqrt{2} - 8$,
- $2/9 \le t(\mathcal{F}_{1,3}, \mathcal{F}_{3,2}, \mathcal{C}_5) \le 1/\sqrt{11}$.

The first result relates to a conjecture of Mubayi and Markström–Talbot that $t(\mathcal{F}_{1,3}) = 2/7$. The best known bounds are $2/7 \le t(\mathcal{F}_{1,3}) < 0.32908$. The second result relates to an old conjecture of Turán that $t(\mathcal{K}_4) = 5/9$. The best bounds are $5/9 \le t(\mathcal{K}_4) \le (3 + \sqrt{17})/12$. The last two results relate to a result of Mubayi and Rödl that $t(\mathcal{F}_{1,3}, \mathcal{C}_5) \le 10/31$.

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1. Introduction

Given a family of k-uniform hypergraphs (or k-graphs) $\mathcal{F}$, we say that a k-graph $\mathcal{H}$ is $\mathcal{F}$-free if $\mathcal{H}$ contains no subgraph isomorphic to any member in $\mathcal{F}$. The *Turán number* $T(n,\mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free k-graph of order n. It is well known [8] that the ratio $T(n,\mathcal{F})/\binom{n}{k}$ is decreasing in n. Therefore, the limit $t(\mathcal{F}) := \lim_{n\to\infty} T(n,\mathcal{F})/\binom{n}{k}$ exists, which is called the *Turán density* of $\mathcal{F}$. It is also well known that determining the Turán density of any complete hypergraph is the most fundamental open problem in extremal combinatorics.

Let p and q be two positive integers, and P and Q be two disjoint sets with $|P| = p$ and $|Q| = q$. Then the triple graph $\mathcal{F}_{p,q}$ is defined on the vertex set $P\cup Q$ and consists of those edges which intersect P in either one or three vertices. Let $\mathcal{K}_4$ be the tetrahedron that is the complete triple graph of order four. Let $\mathcal{F}_5$ be the cycle obtained from $\mathcal{F}_{3,2}$ by deleting an edge which has only one vertex in P. Thus if $P = \{u, v, w\}$ and $Q = \{x, y\}$, then $E(\mathcal{F}_5) = \{uvw, uxy, vxy\}$. Let $\mathcal{F}_5^c$ denote the complement of $\mathcal{F}_5$, and let $\mathcal{P}_5$ be the weak pentagon obtained from $\mathcal{F}_5$ by adding the edge uwx and let $\mathcal{C}_5$ be the pentagon obtained from $\mathcal{P}_5$ by adding one more edge vwy.

Recently many attempts are toward to evaluate positive Turán densities of small triple graphs. The best current results for triple graphs of order 4 and 5 are listed in the following.

- $2/7 \le t(\mathcal{F}_{1,3}) = t(\mathcal{F}_{2,3}) < 0.32908$, by Frankl and Füredi [3], and Markström and Talbot [10] respectively;
- $5/9 \le t(\mathcal{K}_4) \le (3+\sqrt{17})/12 = 0.593592\ldots$, by Turán [14], and Chung and Lu [1] respectively;
- $t(\mathcal{F}_5) = 2/9$, by Frankl and Füredi [2];
- $t(\mathcal{F}_{3,2}) = 4/9$, by Füredi, Pikhurko and Simonovits [6,5];
- $2\sqrt{3}-3 \le t(\mathcal{C}_5) \le 2-\sqrt{2} = 0.5857\ldots$, by Mubayi and Rödl [12];
- $t(\mathcal{F}_{1,3}, \mathcal{C}_5) \le 10/31 = 0.32258\ldots$, by Mubayi and Rödl [12].

Very recently, Razborov [13] applied a semi-definite program with numerical computations and improved many previously known bounds among which is the following: $t(\mathcal{F}_{1,3}) = t(\mathcal{F}_{2,3}) \le 0.2978$, $t(\mathcal{K}_4) \le 0.561666$, $t(\mathcal{C}_5) < 0.4683$ and $t(\mathcal{F}_{1,3}, \mathcal{C}_5) \le 0.2546$. However, he did not feel motivated enough to try to convert the floating-point computation into a rigorous mathematical proof.

It was conjectured by Mubayi [11] and Markström and Talbot [10], Turán [14] and Razborov [13] respectively that the lower bounds are the true values for $\mathcal{F}_{1,3}$, the tetrahedron and the pentagon respectively.

Among a few known positive Turán densities, the triple graphs $\mathcal{F}_5$ and $\mathcal{F}_{3,2}$ are the only two of order five. Though unable to improve any of the above results, we prove rigorously that density exceeding $2-\sqrt{2}$ forces either a tetrahedron or a copy of $\mathcal{F}_5^c$, and density exceeding $0.3015\cdots$ forces either a pentagon or a copy of either $\mathcal{F}_{1,3}$ or $\mathcal{F}_{3,2}$. More precisely, we obtain the following results.

Theorem 1.1. $1/2 \le t(\mathcal{F}_{1,4}) \le 2/3$.

Theorem 1.2. $2\sqrt{3}-3 \le t(\mathcal{K}_4, \mathcal{F}_5^c) \le 2-\sqrt{2}$.

Theorem 1.3. $1/4 \le t(\mathcal{F}_{1,4}, \mathcal{P}_5) \le 6\sqrt{2}-8$.

Theorem 1.4. $2/9 \le t(\mathcal{F}_{1,3}, \mathcal{F}_{3,2}, \mathcal{C}_5) \le 1/\sqrt{11}$.

The proofs combine the induction and an application of the Cauchy–Schwarz inequality. These techniques were also used by Mubayi and Rödl [12]. We remark that a $\{\mathcal{K}_4, \mathcal{F}_5^c\}$-free triple graph is exactly a $\mathcal{K}_4$-free triple graph with every five vertices spanning at most six edges, and we believe that the upper bound in Theorems 1.2 may be improved.

Conjecture. $t(\mathcal{K}_4, \mathcal{F}_5^c) = 2\sqrt{3}-3$.

2. The triple graph $\mathcal{F}_{1,4}$

Definition. Let $\mathcal{H}$ be a triple graph and $S \subset V(\mathcal{H})$. The *link multigraph* of S in $\mathcal{H}$ is the multigraph G with $V(G) = V(\mathcal{H}) - S$, and $E(G) = \{uv : uvw \in E(\mathcal{H}) \text{ for some } w \in S\}$.

Proof of Theorem 1.1. The lower bound follows from a construction of Goldwasser [7]. Let $\mathcal{H}$ be the complement of a triple graph induced by the Fano plane. It is clear that $\mathcal{H}$ is $\mathcal{F}_{1,4}$-free with 7 vertices and 28 edges. Suppose $|V| = n$ and V is partitioned into $V = V_1 \cup V_2 \cup \cdots \cup V_7$. Define a triple graph $\mathcal{H}' = (V, E)$, where

$$E = \{v_{i_1}v_{i_2}v_{i_3} : 1 \le i_1 < i_2 < i_3 \le 7,\ v_{i_j} \in V_{i_j},\ i_1i_2i_3 \in E(\mathcal{H})\}.$$

It is clear that $\mathcal{H}'$ is also $\mathcal{F}_{1,4}$-free. It is still possible to add edges to $\mathcal{H}'$ keeping the property that it is $\mathcal{F}_{1,4}$-free. Indeed, we can add any $\mathcal{F}_{1,4}$-free triple graph $\mathcal{H}''$ within V_i for $i = 1, 2, \ldots, 7$. Consider a set S of five vertices in the resulting triple graph. If $|S \cap V_i| < 3$ for all $i = 1, \ldots, 7$, then edges of S are all of $\mathcal{H}'$, and thus S contains no copy of $\mathcal{F}_{1,4}$. If there is some i such that $|S \cap V_i| = 3$, then S has at most four edges in total, and thus contains no copy of $\mathcal{F}_{1,4}$. If there is some i such that $|S \cap V_i| > 3$, then edges of S are all of $\mathcal{H}''$, and thus S again contains no copy of $\mathcal{F}_{1,4}$. We define $\mathcal{H}'$ with parts V_i for $i = 1, \ldots, 7$ of size as equal as possible and repeat this construction recursively. This results in a triple graph with $[a + o(1)]\binom{n}{3}$ edges, where a satisfies

$$28(n/7)^3 + 7a\binom{n/7}{3} = [a + o(1)]\binom{n}{3}.$$

Solving gives $a = 1/2$.

For the upper bound, suppose that $\mathcal{H}$ is a triple graph of order n and size at least $\frac{2}{3}\binom{n}{3} + n^2$. We prove by induction on n that $\mathcal{H}$ contains a copy of $\mathcal{F}_{1,4}$. It thus suffices to find a vertex in $\mathcal{H}$ of degree at most $\frac{2}{3}\binom{n-1}{2} + 2n - 1$. Take a vertex $v \in V(\mathcal{H})$ and let G be the link graph of v. If $\mathcal{H}$ is $\mathcal{F}_{1,4}$-free, then G is K_4-free. Thus by Turán's Theorem [14],

$$d(v) = |E(G)| \le (n-1)^2/3 < \frac{2}{3}\binom{n-1}{2} + 2n - 1.$$

This completes the proof. □

Remark. The following general result can be obtained by applying the same technique.

Theorem 2.1. $t(\mathcal{F}_{1,k}) \le (k-2)/(k-1)$ *for* $k > 1$.

3. Either tetrahedron or $\mathcal{F}_5^c$

Lemma 3.1 (*[12]*). *Let G be a multigraph with vertex partition $A \cup B$ and maximum multiplicity two. For $S \subset V(G)$, let $m(S)$ be the number of edges (counting multiplicities) induced by S. Suppose that, for all S of size three,*

1. *if $|A \cap S| \ge 2$, then $m(S) \le 4$, and*
2. *if $A \cap S \ne \emptyset$, then $m(S) \le 5$.*

Then $|E(G)| \le \binom{|A|}{2} + 2\binom{|B|}{2} + |A|\,|B| + |A|$.

Proof of Theorem 1.2. The lower bound follows from a construction of Mubayi and Rödl [12]. Let $\mathcal{H}$ be a triple graph on vertices partitioned into two sets A and B. Let $E(\mathcal{H})$ consist of all triples uvw, where $u, v \in A$ and $w \in B$. Note that $\mathcal{C}_5$ is a subgraph of $\mathcal{F}_5^c$. It is easily checked that $\mathcal{H}$ is $\{\mathcal{K}_4, \mathcal{F}_5^c\}$-free. If we add any $\{\mathcal{K}_4, \mathcal{F}_5^c\}$-free triple graph $\mathcal{H}'$ within B, then it is also easily checked that the resulting graph keeps $\{\mathcal{K}_4, \mathcal{F}_5^c\}$-free. Repeating this construction recursively and choosing proper sizes of A and B to maximize produces a triple graph of size $[2\sqrt{3} - 3 + o(1)]\binom{n}{3}$.

For the upper bound, let $c = 2 - \sqrt{2}$ and $b \geq 2$, and suppose that $\mathcal{H}$ is a triple graph of order n and size at least $c\binom{n}{3} + bn^2$. We prove by induction on n that $\mathcal{H}$ contains either a tetrahedron or a copy of $\mathcal{F}_5^c$. It thus suffices to find a vertex in $\mathcal{H}$ of degree at most $c\binom{n-1}{2} + b(2n-1)$. Since the average codegree of $\mathcal{H}$ is at least cn, there is a pair uv with $d(uv) \geq cn$. Let $A \subset N(uv)$ be of size cn, and let $B = V(\mathcal{H}) - A$. Let $G(u)$ and $G(v)$ be the link graphs in $\mathcal{H} - \{u, v\}$ of u and v, respectively. Consider the multigraph $G = G(u) \cup G(v)$. If $\mathcal{H}$ is $\mathcal{K}_4$-free, then $G[A]$ must be simple. Moreover if there is a triple $S = \{w, x, y\}$ with $w \in A$ and $m(S) = 6$ or $w, x \in A$ and $m(S) \geq 5$, then clearly $S \cup \{u, v\}$ contains a copy of $\mathcal{F}_5^c$. Thus G, A and B satisfy the condition of Lemma 3.1. Consequently,

$$\begin{aligned} |E(G)| &< \binom{|A|}{2} + |A|\,|B| + |B|^2 + |A| = \binom{cn}{2} + c(1-c)n^2 + (1-c)^2 n^2 + cn \\ &\leq [c^2 + 2c(1-c) + 2(1-c)^2]\binom{n-1}{2} + 2b(2n-1) - 2n \\ &\leq 2c\binom{n-1}{2} + 2b(2n-1) - 2n. \end{aligned}$$

The last inequality holds since the quadratic function $x^2 + 2(1-x) - 2x$ has roots $x = 2 \pm \sqrt{2}$ and opens upward. Thus one of the graphs $G(u)$ and $G(v)$ has at most $c\binom{n-1}{2} + b(2n-1) - n$ edges, and the vertex corresponding to this link graph has degree in $\mathcal{H}$ at most $c\binom{n-1}{2} + b(2n-1)$. This completes the proof. □

4. Either weak pentagon or $\mathcal{F}_{1,4}$

The following lemma is a special case of a result of Füredi and Kündgen [4].

Lemma 4.1. *Let G be a triangle-free multigraph of order $n > 2$ with maximum multiplicity two which has only isolated multiple edges. Then G has at most $n^2/4$ edges.*

Proof. We use induction on n. This is clearly true for $n = 3$. If G is simple, then Mantel's theorem [9] implies that $|E(G)| \leq n^2/4$. So we may assume that there is an isolated multiple edge. Then deleting the multiple edge along with its pair of vertices, we obtain a subgraph H of G. Now by induction, we have $|E(G)| = |E(H)| + 2 \leq (n-2)^2/4 + 2 \leq n^2/4$. □

The following property is an asymmetric variation of the above lemma.

Lemma 4.2. *Let G_1 and G_2 be two K_4-free simple graphs on the same vertex set V, and let $G = G_1 \cup G_2$. Let $V = A \cup B$, and $a := |A| > 2$ and $b := |B|$ respectively. For $S \subset V(G)$, let $m(S)$ be the number of edges (counting multiplicities) induced by S. Suppose that $G[A]$ is triangle-free and has only isolated multiple edges, and moreover*

1. *for each S of size three, if $A \cap S \neq \emptyset$, then $m(S) \leq 4$, and*
2. *for each vertex $v \in B$, if there is a vertex $u \in A$ such that $m(uv) = 2$, then u is the only neighbor of v in A.*

Then $|E(G)| \leq a^2/4 + ab + 2b^2/3$.

Proof. The inequality holds trivially for $b \leq 1$ and it is easy to see that $|E(G)| \leq 2 + 2b + 2b^2/3$ for $a = 2$. So we may assume that $b \geq 2$ and firstly consider $a = 3$. Let $G[A, B]$ denote the bipartite subgraph of G induced by all edges with one end in A and the other in B. If $G[A, B]$ is simple, then we have

$$|E(G)| \leq m(A) + ab + m(B) \leq 2 + 3b + 2b^2/3.$$

We may thus assume that an edge uv has multiplicity two, where $u \in A$ and $v \in B$. Then by Condition 2, u is the only neighbor of v in A. Note that if there is a vertex $w \in A$ such that uw is a multiple edge then $d_A(u) = 2$ since $G[A]$ has only isolated multiple edges. Also note that Condition 1 implies that every vertex of B has at most two edges to $\{u, v\}$. Let $H = G - \{u, v\}$. Then we have

$$\begin{aligned}|E(G)| &\le |E(H)| + 2 + 2(b-1) + 2\\ &\le 2 + 2(b-1) + 2(b-1)^2/3 + 2 + 2b\\ &< 2 + 3b + 2b^2/3.\end{aligned}$$

This proves the case when $a = 3$. Next we consider $a > 3$ and Lemma 4.1 is applicable to $G[A]$. We use induction on $a + b$. Again if $G[A, B]$ is simple, then by Lemma 4.1,

$$|E(G)| \le m(A) + ab + m(B) \le a^2/4 + ab + 2b^2/3.$$

So we may assume that an edge uv has multiplicity two, where $u \in A$ and $v \in B$. Then as above let $H = G - \{u, v\}$ and the induction hypothesis applied to H implies that

$$\begin{aligned}|E(G)| &\le |E(H)| + a - 1 + 2(b-1) + 2\\ &\le (a-1)^2/4 + (a-1)(b-1) + 2(b-1)^2/3 + a + 2b - 1\\ &= a^2/4 + ab + 2b^2/3 - a/2 - b/3 + 11/12\\ &< a^2/4 + ab + 2b^2/3.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3. For the lower bound, consider the complete three partite triple graph $\mathcal{H} = (V, E)$ with $V = V_1 \cup V_2 \cup V_3$ and $E = \{v_1v_2v_3 : v_i \in V_i,\ i = 1, 2, 3\}$. It contains neither a weak pentagon nor a copy of $\mathcal{F}_{1,4}$. It is still possible to add edges to $\mathcal{H}$ keeping this property. Indeed, we can add any $\{\mathcal{F}_{1,4}, \mathcal{P}_5\}$-free triple graph $\mathcal{H}'$ within V_i for $i = 1, 2, 3$. Consider a set S of five vertices in the resulting triple graph. If $|S \cap V_i| < 3$ for all $i = 1, 2, 3$, then edges of S are all in $\mathcal{H}$, and thus S contains neither a weak pentagon nor a copy of $\mathcal{F}_{1,4}$. If there is some i such that $|S \cap V_i| = 3$, then S has at most four edges in total, and the four edges form a $\mathcal{F}_{3,2}$ in S. Thus S contains neither a weak pentagon nor a copy of $\mathcal{F}_{1,4}$. If there is some i such that $|S \cap V_i| > 3$, then edges of S are all of $\mathcal{H}'$, and thus S again contains neither a weak pentagon nor a copy of $\mathcal{F}_{1,4}$. We choose the triple graph $\mathcal{H}$ with parts V_i for $i = 1, 2, 3$ of size as equal as possible and repeat this construction recursively. This results in a triple graph with $[a + o(1)]\binom{n}{3}$ edges, where a is given by

$$(n/3)^3 + 3a\binom{n/3}{3} = [a + o(1)]\binom{n}{3}.$$

Solving gives $a = 1/4$.

For the upper bound, let $c = 6\sqrt{2} - 8$, and suppose that $\mathcal{H}$ is a triple graph of order n and size at least $c\binom{n}{3} + n^2$. As before, we prove by induction on n that $\mathcal{H}$ contains either a weak pentagon or a copy of $\mathcal{F}_{1,4}$. It thus suffices to find a vertex in $\mathcal{H}$ of degree at most $c\binom{n-1}{2} + 2n - 1$. Since the average codegree of $\mathcal{H}$ is at least cn, there is a pair uv with $d(uv) \ge cn$. Let $A \subset N(uv)$ be of size cn, and let $B = V(\mathcal{H}) - A$. Let $G(u)$ and $G(v)$ be the link graphs in $\mathcal{H} - \{u, v\}$ of u and v, respectively. Consider the multigraph $G = G(u) \cup G(v)$.

Claim 1. *If $\mathcal{H}$ is $\{\mathcal{F}_{1,4}, \mathcal{P}_5\}$-free, then $G[A]$ satisfies the hypotheses of Lemma 4.1.*

Assume to the contrary that A contains a triple $S = \{w, x, y\}$ spanning at least three edges. Since $\mathcal{H}$ is $\mathcal{F}_{1,4}$-free, neither $G(u)$ nor $G(v)$ contains any triangle in A. Thus one of the edges of S is in $G(u)$ and another in $G(v)$. By symmetry, we may assume that $wx \in G(u)$ and $wy \in G(v)$, then the two triples uwx and vwy along with the two triples uvx and uvy form a weak pentagon, which is a contradiction.

Claim 2. *If $\mathcal{H}$ contains no weak pentagon, then for each S of size three, $A \cap S \neq \emptyset$ implies $m(S) \le 4$.*

Suppose to the contrary that there is such an $S = \{w, x, y\}$ that $A \cap S \neq \emptyset$ and $m(S) \geq 5$. Then S contains a triangle. Assume $w \in A \cap S$. Note that one of the two edges wx and wy has multiplicity two. Thus by symmetry, we may assume that $wx \in G(u)$ and $wy \in G(v)$. Then uvw, uwx and vwy together with either uxy or vxy form a weak pentagon, which is a contradiction.

Claim 3. *If $\mathcal{H}$ contains no weak pentagon, then for each vertex $w \in B$, if there is a vertex $x \in A$ such that wx is a multiple edge, then x is the only neighbor of w in A.*

Assume to the contrary that there is another vertex $y \in A$ such that $wy \in G$, then either $wy \in G(u)$ or $G(v)$. If $wy \in G(u)$, then uvx and uvy along with uwy and vwx form a weak pentagon, which is a contradiction. If $wy \in G(v)$, then uvx and uvy along with uwx and vwy form a weak pentagon, which is also a contradiction.

By the claims, we may assume that G satisfies the conclusion of Lemma 4.2. Thus

$$\begin{aligned} |E(G)| &\leq |A|^2/4 + |A|\,|B| + 2|B|^2/3 = (cn)^2/4 + c(1-c)n^2 + 2(1-c)^2n^2/3 \\ &\leq [c^2/2 + 2c(1-c) + 4(1-c)^2/3]\binom{n-1}{2} + 2n \\ &\leq 2c\binom{n-1}{2} + 2n. \end{aligned}$$

The last inequality holds since the quadratic function $4(1-x)^2/3 - 3x^2/2$ has the greater root $x = 6\sqrt{2} - 8$ and opens downward. Thus one of the graphs $G(u)$ and $G(v)$ has at most $c\binom{n-1}{2} + n$ edges, and the vertex corresponding to this link graph has degree in $\mathcal{H}$ at most $c\binom{n-1}{2} + 2n - 1$. This completes the proof. □

5. Either pentagon, $\mathcal{F}_{1,3}$ or $\mathcal{F}_{3,2}$

Proof of Theorem 1.4. For the lower bound, note that the complete three partite triple graph with parts of size as equal as possible contains neither pentagon nor a copy of $\mathcal{F}_{1,3}$ or $\mathcal{F}_{3,2}$. This triple graph has density $2/9$.

For the upper bound, let $c = 1/\sqrt{11}$, and suppose that $\mathcal{H}$ is a triple graph with at least $c\binom{n}{3} + n^2$ edges. We will prove by induction on n that $\mathcal{H}$ contains either a pentagon or a copy of either $\mathcal{F}_{1,3}$ or $\mathcal{F}_{3,2}$. It thus suffices to find a vertex in $\mathcal{H}$ of degree at most $c\binom{n-1}{2} + 2n - 1$.

Let $V = V(\mathcal{H})$. Given vertices u and v in V, let $N(uv) = \{w : uvw \in E(\mathcal{H})\}$, and let $d(uv) = |N(uv)|$. For an edge $e = uvw$, let

$$s(e) = d(uv) + d(uw) + d(vw).$$

If $s(e) > n$, then there is a vertex x in at least two of the sets $N(uv), N(uw), N(vw)$, and $S = \{u, v, w, x\}$ contains a copy of $\mathcal{F}_{1,3}$. We may thus assume that $s(e) \leq n$ for every edge e. Define $\epsilon > 0$ by

$$(1-\epsilon)n = \max_{e \in E(\mathcal{H})} s(e). \tag{1}$$

Using $\sum_{u,v \in V} d(uv) = 3|E(\mathcal{H})|$, the Cauchy–Schwarz inequality and the upper bound from (1) on $s(e)$, we obtain

$$9|E(\mathcal{H})|^2\binom{n}{2}^{-1} \leq \sum_{u,v \in V} d^2(uv) = \sum_{e \in E(\mathcal{H})} s(e) \leq (1-\epsilon)n|E(\mathcal{H})|.$$

It follows that

$$c\binom{n}{3} + n^2 \leq |E(\mathcal{H})| \leq \frac{1-\epsilon}{3}\left[\binom{n}{3} + \frac{n(n-1)}{3}\right],$$

and thus

$$c < (1-\epsilon)/3. \tag{2}$$

Let $e = uvw$ be an edge with $s(e) = (1-\epsilon)n$. Let $G(u)$ be the link graph of u in $\mathcal{H} - \{u, v, w\}$, $G(v)$ and $G(w)$ are similarly defined. Let $G := G(u) \cup G(v) \cup G(w)$, $A = N(uv) \cup N(uw) \cup N(vw)$ and let $B = V(\mathcal{H}) - A$. From now on, we also suppose that $\mathcal{H}$ contains neither pentagon nor a copy of $\mathcal{F}_{1,3}$ or $\mathcal{F}_{3,2}$, and aim at a contradiction. Recall that a set of vertices is *stable* if no pair of the vertices are adjacent. Since $\mathcal{H}$ is $\{\mathcal{F}_{1,3}, \mathcal{F}_{3,2}\}$-free, we obtain that

- the three link graphs $G(u)$, $G(v)$ and $G(w)$ are all triangle-free,
- the multiplicity of each edge of G is at most two, and
- the three sets $N(uv)$, $N(uw)$ and $N(vw)$ are all stable and disjoint from each other in G (and thus $G[A]$ is tripartite).

Claim 1 in the proof of Theorem 1.8 [12] claims that $G[A]$ is simple. Therefore, by Mantel's theorem and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
d(u)+d(v)+d(w) &\le |E(G(u))| + |E(G(v))| + |E(G(w))| + 2s(e) = |E(G)| + 2s(e) \\
&\le |E(G[A])| + 2|A|\,|B| + |E(G[B])| + 2n \\
&\le d(uv)d(uw) + d(uv)d(vw) + d(uw)d(vw) \\
&\quad + 2\epsilon(1-\epsilon)n^2 + 3|B|^2/4 + 2n \\
&= \{s^2(e) - [d^2(uv) + d^2(uw) + d^2(vw)]\}/2 \\
&\quad + 2\epsilon(1-\epsilon)n^2 + 3\epsilon^2 n^2/4 + 2n \\
&\le [s^2(e) - s^2(e)/3]/2 + \epsilon(2 - 5\epsilon/4)n^2 + 2n \\
&= (1-\epsilon)^2 n^2/3 + \epsilon(2 - 5\epsilon/4)n^2 + 2n \\
&\le [2(1-\epsilon)^2/3 + 2\epsilon(2 - 5\epsilon/4)]\binom{n-1}{2} + 3(2n-1).
\end{aligned}$$

Thus one of u, v, w has degree at most

$$[2(1-\epsilon)^2/9 + 2\epsilon(2 - 5\epsilon/4)/3]\binom{n-1}{2} + (2n-1).$$

If this is at most $c\binom{n-1}{2} + 2n - 1$, then we may apply induction, so we may assume that

$$c < 2(1-\epsilon)^2/9 + 2\epsilon(2 - 5\epsilon/4)/3. \tag{3}$$

Inequalities (2) and (3) yield

$$1/\sqrt{11} = c < \min\{(1-\epsilon)/3,\ 2(1-\epsilon)^2/9 + 2\epsilon(2 - 5\epsilon/4)/3\}.$$

This is impossible since

$$\max_{\epsilon \in (0,1)} \min\{(1-\epsilon)/3,\ 2(1-\epsilon)^2/9 + 2\epsilon(2 - 5\epsilon/4)/3\} = c,$$

with the maximum of the minimum of these two functions of ϵ occurring at $\epsilon = 1 - 3/\sqrt{11}$. This contradiction completes the proof. □

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