Novel Suboptimal Filter via Higher Order Central Moments

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Dedicate to Peter Caine on the occasion of his 70th birthday

Abstract

In this paper, we construct a new suboptimal filter by deriving the Ito’s stochastic differential equations of the estimation of higher order central moments satisfy and imposing some conditions to form a closed system. The essentially infinite-dimensional cubic sensor problem has been investigated in detail numerically to illustrate the reasonableness of the imposed conditions, and the numerical experiments support our discussion. A 2-dimensional polynomial filtering problem has also been experimented.

Index Terms

nonlinear filtering, suboptimal method, higher central moments

I. INTRODUCTION

The nonlinear filtering (NLF) problem involves the estimation of a stochastic process (called the signal or state process) that cannot be observed directly. Information containing the state is obtained from observations of a related process, i.e., the observation process. The main goal of NLF is to determine the conditional expectations, or perhaps even to compute the entire conditional density of the state, given the observation history. For an excellent introduction to NLF theory, we refer the readers to the book by Jazwinski [13].

In 1960, Kalman [14] published a historically important paper on linear filtering that are highly influential in modern industry. It is the so-called Kalman filter (KF). One year later,

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the continuous version of KF has been investigated by Kalman and Bucy [15]. Since then, the
Kalman-Bucy filter has been widely used in science and engineering, for example in navigation
and guidance systems, radar tracking, sonar ranging, satellite and airplane orbit determination,
and forecasting in weather, econometrics and finance. However, the Kalman-Bucy filter has
limited application due to the linearity assumptions of the drift term, the observation term and
the Gaussian assumption of the initial value.

The success of KF for the linear Gaussian estimation problems encouraged many researchers
to generalize the Kalman’s results to nonlinear dynamical systems. However, the NLF problem
is an essentially more difficult problem since the resulting optimal filter is, in general, infinite-
dimensional, i.e., the conditional density depends on all its moments. Those methods which
attempt to compute the density function directly or numerically are called the global approaches,
see the survey paper [18] for detail.

Although the global ones can completely solve the NLF problems, the heavy computation
is one of the major obstacles in their real-time applications. Another way-out is to use the
approximate method to construct a suboptimal filter. The existing approximate filters for the
NLF problems include the extended Kalman filter (EKF), the unscented Kalman filter (UKF),
the ensemble Kalman filter (EnKF), particle filters (PF), and splitting up method, see [29], [10],
[6] and [16]. All of these methods have their own weakness. UKF and EnKF assume that the
probability density of the state vector is Gaussian. PF could be inefficient and is sensitive to
outliers. Resampling step is applied at every iteration, which results in a rapid loss of diversity in
particles. Furthermore, PF are more applicable at low- and moderate high-dimensional systems,
see [3] for the obstacles to high dimensional cases. The splitting up method requires $g$ and $h$
in the model (1) to be bounded, which even excludes the linear case. Recently, Germani, et. al. [8],
[9] developed a suboptimal method, so-called Carleman approach, based on the alogrithm for
the bilinear system [5]. However, recently the first and the last author found that the Carleman
approach can fail completely in some 1-d NLF problem and developed a suboptimal method
via Hermite polynomials [22]. The use of higher central moments to improve the performance
of NLF has been studied by many researchers, see [23] and references therein. In fact, the
cumulants can be a better choice than the central moments, and the study on the cumulants for
NLF can be found in [30].

In this paper, we shall propose a new suboptimal filter by investigating the Ito’s stochastic
differential equation (SDE) which the higher central moments satisfy. Although the use of the
higher central moments for NLF problems has been attempted for a long time and the second
order EKF has been standard in the literature, see [13], the detailed derivation has never been
clearly written down for NLF, especially the polynomial filtering problems, which can be viewed
as the truncation of Taylor expansion of any nonlinear smooth functions. When arrived at an
infinite dimensional system, the higher central moments are conventionally truncated to form a
closed system as in [19]. No one doubts the reasonableness of the truncation. It is in this paper
that for the first time we investigate other options to form a closed system, say condition (12).
The numerical experiments support the condition. Also we compare our methods with some
existing ones. Our method works in nearly perfect agreement with theory.

An outline of this paper is as follows. In section II, we introduce the continuous-time model
in this paper. Our method is derived and described in section III. Section IV is devoted to two numerical experiments, which validate our method. Our method is more flexible by choosing different truncation mode \( \tilde{N} \). The Conclusion is in section V.

II. PRELIMINARIES

The model we consider in this paper is the continuous-state-continuous-observation one:

\[
\begin{cases}
    dx_t = f(x_t, t)dt + g(x_t, t)dv_t \\
    dy_t = h(x_t, t)dt + dw_t,
\end{cases}
\]

where \( x_t, v_t, y_t, \) and \( w_t \) are \( \mathbb{R}^n -, \mathbb{R}^r -, \mathbb{R}^m - \), and \( \mathbb{R}^m - \) valued processes, respectively, and \( f: \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n, g: \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}, h: \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^m \) are possibly nonlinear function of \( x \). Assume that \( \{v_t, t \geq 0\} \) and \( \{w_t, t \geq 0\} \) are Brownian motion processes with \( \text{Var}[dv_t] = Q(t)dt \) and \( \text{Var}[dw_t] = R(t)dt \), respectively. Moreover, \( \{v_t, t \geq 0\}, \{w_t, t \geq 0\} \) and \( x_0 \) are independent. The initial observation is assumed to be \( y_0 = 0 \).

Without loss of generality, we assume \( Q(t) \) is a diagonal matrix, \( Q(t) = \text{diag}(q^2_1, \ldots, q^2_n) \). In fact, if it is not, we have spectral decomposition of \( Q(t) = PP' \), where \( PP' = \Lambda, \Lambda \) is diagonal matrix. By letting \( g^* = gP, P^* = P'dv, \) then \( \text{Var}[dv^*] = \Lambda dt \). We could further assume that \( Q(t) = I, \) due to the function \( g \) in front (replacing \( g \) by \( gQ^{1/2} \)).

Let us clarify the notations we shall use in this paper. Let \( p \equiv p(x, t \mid Y_t) \) be the conditional probability density function of the state \( x_t \), given the observation history \( Y_t \equiv \{y_s, 0 \leq s \leq t\} \), then the conditional expectation of \( x_t \) is defined as

\[
\hat{x}_t \equiv E^t[x_t] \equiv E[x_t \mid Y_t].
\]

For conciseness, we may use the vector notations, denoted as \( \vec{k} = (k_1, k_2, \ldots, k_n) \). We say \( \vec{k} \leq \vec{a} \), if \( k_i \leq a_i \), for all \( 1 \leq i \leq n \). The strict inequality holds, if \( k_i < a_i \), for some \( 1 \leq i \leq n \). We denote \( P_k \) as

\[
P_k \equiv E^t\left[ (x_1 - \vec{x}_1)^{k_1} \ldots (x_n - \vec{x}_n)^{k_n} \right] \equiv E \left[ (x_1 - \vec{x}_1)^{k_1} \ldots (x_n - \vec{x}_n)^{k_n} \mid Y_t \right].
\]

Say \( P_k \) is the lower order of \( P_{\vec{a}} \) if \( \vec{k} < \vec{a} \). By convention, \( \vec{0} = (0, 0, \cdots, 0) \) and \( \vec{e}_i \) denotes 1 for the \( i-\)th component, 0 otherwise. \( P_{\vec{0}} = 1 \) and \( P_{\vec{e}_i} = 0 \), for \( 1 \leq i \leq n \).

Furthermore, \( \min \left\{ \vec{k}, \vec{l} \right\} = (\min\{k_1, l_1\}, \min\{k_2, l_2\}, \cdots, \min\{k_n, l_n\}) \), \( \vec{k} + \vec{l} = (k_1 + l_1, \cdots, k_n + l_n) \), \( \left| \vec{k} \right|_1 = \sum_{i=1}^n k_i \) and \( \left| \vec{k} \right|_{\infty} = \max_{1 \leq i \leq n} k_i \).

III. NEW SUBOPTIMAL FILTER

Let \( f_i(x, t), g_{ij}(x, t) \) and \( h_i(x, t), 1 \leq i \leq n, 1 \leq j \leq m, \) be some smooth nonlinear functions in \( x \). They can be approximated by their truncated Taylor expansions:

\[
f_i(x, t) \approx \sum_{\left| m \right|_1 \leq M_f} f_{i;m}(t) \prod_{a=1}^n x_a^{m_a},
\]

\[
g_{ij}(x, t) \approx \sum_{\left| m \right|_1 \leq M_g} g_{ij;m}(t) \prod_{a=1}^n x_a^{m_a}
\]
\[ h_i(x, t) \approx \sum_{|\vec{m}| \leq M_h} h_{i;\vec{m}}(t) \prod_{a=1}^{n} x_a^{m_a} \]  

(5)

where \( M_f \), \( M_g \) and \( M_h \) are the highest degrees kept in the expansions of \( \{f_i\}_{1 \leq i \leq n}, \{g_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq p} \) and \( \{h_i\}_{1 \leq i \leq m} \), respectively.

In the sequel, we shall focus on the derivation of the method for the polynomial filtering problems.

Proposition 3.1: For continuous filtering problem given by the system (1) with \( f_i(x, t), g_{ij}(x, t), h_i(x, t) \) approximated by (3)-(5), the conditional mean \( \hat{x}_i \) satisfies the following Ito’s SDE

\[
d\hat{x}_i = \sum_{|\vec{m}| \leq M_f} \sum_{\vec{a} \leq \vec{k} \leq \vec{m}} f_{i;\vec{m}}(\prod_{a=1}^{n} \left( \frac{m_a}{k_a} \right) (\hat{x}_a)^{m_a-k_a}) P_{\vec{k}} dt \\
+ \sum_{1 \leq j, s \leq m} r^{js} \left( dy_j - \sum_{|\vec{m}| \leq M_h} \sum_{\vec{a} \leq \vec{k} \leq \vec{m}} h_{j;\vec{m}}(\prod_{a=1}^{n} \left( \frac{m_a}{k_a} \right) (\hat{x}_a)^{m_a-k_a}) P_{\vec{k}} dt \right) \\
\cdot \left( \sum_{|\vec{m}| \leq M_h} \sum_{\vec{a} \leq \vec{k} \leq \vec{m}} h_{s;\vec{m}}(\prod_{a=1}^{n} \left( \frac{m_a}{k_a} \right) (\hat{x}_a)^{m_a-k_a}) P_{\vec{k}+\vec{a}} \right),
\]

(6)

where \( (r^{js})_{m \times m} \) is the matrix \( R^{-1} \).

Proof: According to [13], the conditional mean \( \hat{x}_i \) satisfies

\[
d\hat{x}_i = \hat{f}_i dt + (dy - \hat{h} dt)^T R^{-1} (\hat{x}_i - \hat{\hat{x}}_i).
\]

(7)

Using binomial expansion, we have

\[
\prod_{a=1}^{n} x_a^{m_a} = \prod_{a=1}^{n} (x_a - \hat{x}_a + \hat{x}_a)^{m_a} = \prod_{a=1}^{n} \sum_{\vec{a} \leq \vec{k} \leq \vec{m}} \left( \frac{m_a}{k_a} \right) (x_a - \hat{x}_a)^{k_a} (\hat{x}_a)^{m_a-k_a}
\]

= \sum_{\vec{a} \leq \vec{k} \leq \vec{m}} \prod_{a=1}^{n} \left( \frac{m_a}{k_a} \right) (x_a - \hat{x}_a)^{k_a} (\hat{x}_a)^{m_a-k_a}
\]

(8)

then

\[
\prod_{a=1}^{n} (x_a - \hat{x}_a)^{\alpha_a} h_s = \sum_{|\vec{m}| \leq M_h} \sum_{\vec{a} \leq \vec{k} \leq \vec{m}} h_{s;\vec{m}}(\prod_{a=1}^{n} \left( \frac{m_a}{k_a} \right) (\hat{x}_a)^{m_a-k_a}) P_{\vec{a}+\vec{k}}.
\]

(9)

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Similarly, we have
\[ E^{t}[\prod_{a=1}^{n}(x_a - \hat{x}_a)^{\alpha_a}f_i] = \sum_{|\hat{m}_1|\leq M_f} \sum_{0\leq k\leq \hat{m}} f_{i;\hat{m}}(\prod_{a=1}^{n} (m_a k_a)((\hat{x}_a)^{m_a-k_a})P_{\alpha+k}. \tag{10} \]

Especially,
\[ \hat{f}_i = E^{t}[\prod_{a=1}^{n}(x_a - \hat{x}_a)^{0}f_i] = \sum_{|\hat{m}_1|\leq M_f} \sum_{0\leq k\leq \hat{m}} f_{i;\hat{m}}(\prod_{a=1}^{n} (m_a k_a)((\hat{x}_a)^{m_a-k_a})P_{\alpha+k}, \]
\[ \hat{h}_j = E^{t}[\prod_{a=1}^{n}(x_a - \hat{x}_a)^{0}h_j] = \sum_{|\hat{m}_1|\leq M_h} \sum_{0\leq k\leq \hat{m}} h_{j;\hat{m}}(\prod_{a=1}^{n} (m_a k_a)((\hat{x}_a)^{m_a-k_a})P_{\alpha+k}, \]

and
\[ \hat{h}_s \hat{x}_i - \hat{h}_s \hat{x}_i = E^{t}[(x_i - \hat{x}_i)h_s] = \sum_{|\hat{m}_1|\leq M_h} \sum_{0\leq k\leq \hat{m}} h_{s;\hat{m}}(\prod_{a=1}^{n} (m_a k_a)((\hat{x}_a)^{m_a-k_a})P_{\alpha+k}. \tag{9} \]

Equation (6) is followed immediately by plugging the above three equalities into equation (7) with the fact that \((dy - \hat{h}dt)R^{-1}(\hat{h}x_i - \hat{h}\hat{x}_i) = \sum_{j=1}^{m}(dy_j - \hat{h}j dt)\left[ \sum_{s=1}^{m} r^{js}(\hat{h}x_i - \hat{h}\hat{x}_i) \right]. \tag{10} \]

It is clear to see that in (6), the central moments \(P_{\bar{k}+\bar{e}_i}\) for \(\bar{k} \leq \bar{m}\), with \(|\hat{m}_1| \leq M_h\) and \(P_{\bar{k}}\) for \(\bar{k} \leq \bar{m}\), with \(|\hat{m}_1| \leq M_f\) are needed to compute \(\hat{x}_i\). Thus, let us give the Ito’s SDE for \(P_{\alpha}\) with \(|\alpha| \geq 2\) in the following proposition.

**Proposition 3.2:** For continuous filtering problem given by the system (1) with \(f_i(x,t), g_{ij}(x,t)\) and \(h_i(x,t)\) approximated by (3)-(5), the SDE for \(P_{\alpha}\) is
\[
dP_{\alpha} = \left( - \sum_{a=1}^{n} \alpha_a \sum_{|\hat{m}_1|\leq M_f} \sum_{0\leq k\leq \hat{m}} f_{a;\hat{m}}(\prod_{b=1}^{n} (m_b k_b)((\hat{x}_b)^{m_b-k_b})P_{\alpha+k} \right. \]
\[ \quad + \frac{1}{2} \sum_{a=1}^{n} \alpha_a(\alpha_a - 1) \left( \sum_{1 \leq i, j \leq n} \sum_{|\hat{m}_1|\leq M_h} \sum_{0\leq k\leq \hat{m}} h_{i;j;\hat{m}}(\prod_{b=1}^{n} (m_b k_b)((\hat{x}_b)^{m_b-k_b})P_{\alpha+k} \right. \]
\[ \quad \cdot \left. \left. \left( \sum_{|\hat{m}_1|\leq M_h} \sum_{0\leq k\leq \hat{m}} h_{j;\hat{m}}(\prod_{b=1}^{n} (m_b k_b)((\hat{x}_b)^{m_b-k_b})P_{\alpha+k} \right) \right) \right) \right) \]
\[ \left. \left. + \sum_{i=1}^{n} \sum_{|\hat{m}_1|\leq M_f} \sum_{0\leq k\leq \hat{m}} \alpha_i f_{i;\hat{m}}(\prod_{a=1}^{n} (m_a k_a)((\hat{x}_a)^{m_a-k_a})P_{\alpha+k} \right) \right) \]

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Than its first order approximation – EKF.

Our idea is to cleverly impose some conditions to eliminate the terms $\vec{\alpha}$.

We motivate by observing the last term of (11) for $\vec{\alpha}$.

With equation (6) and (11) by hand, we are ready to propose our new suboptimal method.

\[
+ \sum_{1 \leq i < j \leq n, |\vec{m}|_1 \leq M_y, 0 \leq k \leq m_1 + m_2} \alpha_i \alpha_j g_i; \tilde{m}_1 g_j; \tilde{m}_2 \left( \prod_{a=1}^{n} \left( m_a^1 + m_a^2 \right) \left( \tilde{x}_a \right)^{m_a^1 + m_a^2 - k_a} \right) P_{\vec{a} + k - \vec{e}_i - \vec{e}_j}
\]

\[
+ \frac{1}{2} \sum_{i, l=1, |\vec{m}|_1 \leq M_y, 0 \leq k \leq m_1 + m_2} \alpha_i (\alpha_i - 1) g_i; \tilde{m}_1 g_l; \tilde{m}_2 \left( \prod_{a=1}^{n} \left( m_a^1 + m_a^2 \right) \left( \tilde{x}_a \right)^{m_a^1 + m_a^2 - k_a} \right) P_{\vec{a} + k - 2\vec{e}_i}
\]

\[
+ \sum_{a < b} \left( \alpha_a \alpha_b P_{\vec{a} - \vec{e}_a - \vec{e}_b} \left( \sum_{1 \leq i, j \leq n, |\vec{m}|_1 \leq M_y, 0 \leq k \leq m_1 + m_2} \right)
\]

\[
\cdot \left( \sum_{|\vec{m}|_1 \leq M_y, 0 \leq k \leq m_1 + m_2} h_{i, \vec{m}} \left( \prod_{c=1}^{n} \left( m_c^1 \right) \left( \tilde{x}_c \right)^{m_c^1 - k_c} \right) P_{\vec{k} + \vec{e}_a} \right) dt
\]

\[- \left( dy - \tilde{x}_t dt \right)^{-1} \left( \sum_{|\vec{m}|_1 \leq M_y, 0 \leq k \leq m_1 + m_2} \left[ h_{i, \vec{m}} \right]_{n \times 1} \left( \prod_{b=1}^{n} \left( m_b^1 \right) \left( \tilde{x}_b \right)^{m_b^1 - k_b} \right)
\]

\[
\cdot \left( \sum_{a=1}^{n} \alpha_a P_{\vec{k} + \vec{e}_a} P_{\vec{a} - \vec{e}_a} - P_{\vec{a} + \vec{e}_a} + P_{\vec{a} + \vec{k}} \right) \right).
\]

(11)

With equation (6) and (11) by hand, we are ready to propose our new suboptimal method. Our idea is to cleverly impose some conditions to eliminate the terms $P_{\vec{a}}$ in (6) and (11), for $|\vec{a}|_\infty > |\vec{N}|_\infty$, for some given truncation $\vec{N}$, such that the equations of $\tilde{x}_t$ and $P_{\vec{a}}, \vec{a} \leq \vec{N}$, form a closed system. Thus, it is solvable and provides, generally speaking, more accurate approximation than its first order approximation – EKF.

We motivate by observing the last term of (11) for $\vec{a} > \vec{e}_i$, for some $1 \leq i \leq n$. That is, we exclude two trivial cases: (a) $P_{\vec{e}_i} = 0$, for some $1 \leq i \leq n$; (b) $P_{\vec{a}} = 1$. It turns out that the last term vanishes if we impose the condition that

\[
P_{\vec{a} + \vec{k}} = \sum_{a=1}^{n} \alpha_a P_{\vec{k} + \vec{e}_a} P_{\vec{a} - \vec{e}_a} + P_{\vec{a}} P_{\vec{k}}.
\]

(12)
Notice that \( P_{\tilde{a} - \tilde{e}_i}, P_{\tilde{k} + \tilde{e}_i}, P_{\tilde{a}} \) and \( P_{\tilde{k}} \) on the right-hand side of (12) are of lower order of \( P_{\tilde{a} + \tilde{k}} \).

Let us state our conditions more precisely. Given the truncation mode \( \tilde{N} > \tilde{e}_i \), for some \( 1 \leq i \leq n \), we shall form a closed system of equations for \( \hat{x}_{it}, 1 \leq i \leq n \), and \( P_{\tilde{a}}, \tilde{a} \leq \tilde{N} \). For arbitrary \( \tilde{a} > \tilde{e}_i \), for some \( 1 \leq i \leq n \), there are three cases:

**Case 1:** \( \tilde{a} \leq \tilde{N} \). Keep as it is, i.e. \( P_{\tilde{a}} \);

**Case 2:** There exist \( 1 \leq i \neq j \leq n \) such that \( \alpha_i \leq \tilde{N}_i \) and \( \alpha_j > \tilde{N}_j \). We impose the condition (12) to \( P_{\tilde{a}} = P_{\tilde{\beta} + \tilde{k}} \), where \( \tilde{\beta} = \min\{\tilde{a}, \tilde{N}\} \) and \( \tilde{k} = \tilde{a} - \tilde{\beta} \);

**Case 3:** \( \tilde{a} > \tilde{N} \). Condition (12) is imposed to \( P_{\tilde{a}} = P_{\tilde{N} + \tilde{k}} \), where \( \tilde{k} = \tilde{a} - \tilde{N} \).

**Remark 3.3:** Given any \( \tilde{a} \) in case 2 or 3, we shall impose the condition accordingly until it reduces to the combination of \( P_{\tilde{l}} \)s, where all \( \tilde{l} \)s belong to case 1. Hence, the condition (12) may be imposed more than once to reduce certain \( P_{\tilde{a}} \) in case 2 or 3 to case 1.

**Algorithm of our method** For continuous filtering problem given by system (1) with \( f_i(x, t), g_{ij}(x, t), \) and \( h_i(x, t) \) approximated by (3)-(5), then a closed system of equations of \( \hat{x}_i, 1 \leq i \leq n \), and \( P_{\tilde{a}}, \tilde{a} \leq \tilde{N} \) is derived, if the condition (12) is imposed accordingly. Specifically, the closed system of the equations is given by: equation (6) for conditional mean \( \hat{x}_i, 1 \leq i \leq n \); SDE (11) for \( P_{\tilde{a}} \), for \( \tilde{a} < \tilde{N} \); ordinary differential equation (11) for \( P_{\tilde{N}} \) (the last term of (11) vanishes here) and all the \( P_{\tilde{a}} \) with \( \tilde{a} \) in case 2 or 3 are properly reduced to \( P_{\tilde{a}}, \tilde{a} \) in case 1 by condition (12).

**Remark 3.4:** By examining term-by-term in (6) and (11) with \( |\tilde{a}|_1 = 2 \), we see that when \( M_f, M_g \) and \( M_h \leq 1 \), they form a closed system under the condition (12), which yields exactly the Kalman-Bucy filter. Indeed, if \( f(x, t) = F(t)x, g(x, t) = G(t), \) and \( h(x, t) = H(t)x \) in (1) for arbitrary \( n \geq 1 \), and the condition (12) is imposed, then our method gives

\[
\begin{align*}
    d\hat{x} &= F\hat{x}dt + PH^T R^{-1}(dy - H\hat{x}dt) \\
    \frac{dP}{dt} &= FP + PF^T + gQg^T - PH^T R^{-1}HP,
\end{align*}
\]

where \( \hat{x} = [\hat{x}_1, \cdots, \hat{x}_n] \), \( P = [P_l]_{l, |\tilde{a}|_1 = 1} \).

**Remark 3.5:** When \( n = 1 \), the lower bounds for some \( P_{k_l} \), \( k \geq 2 \), can be obtained by Jensen’s inequality and Hölder’s inequality, see details in Lemma 3.6 below. These lower bounds will be used to check the reasonableness of the conditions (12) imposed in cubic sensor problem in the next section.

**Lemma 3.6 (Lower bound of \( P_{k_l} \)):** Let \( P_k = E^t \left[ (x - \hat{x})^k \right] \), with convention that \( P_0 = 1 \), we have

1) \( P_k \geq P_l^k \), for all \( k \geq l \geq 1 \) and \( k, l \) are even integers greater than 2;

2) If \( k, l \) and \( \frac{(k-l)p}{1-p} \) are all even integers, then \( P_k \leq P_l^\frac{p}{1-p} P_{\frac{k-l}{p-1}}^{\frac{1-p}{p}} \), where \( p \geq 1 \), for all \( k \geq l \geq 0 \).
Proof: 1) It is trivial to see that when \( k = l \) the equality holds. So let us assume that \( k > l \) and look at \( P_{2k} \):

\[
P_k = \int_{\mathbb{R}} (x - \hat{x})^k p(x|Y_t)dx \geq \left[ \int_{\mathbb{R}} (x - \hat{x})^l p(x|Y_t)^{\frac{k}{l}} dx \right]^{\frac{l}{k}} P_{l}^{\frac{k}{l}},
\]

as long as \( k \geq l \geq 1 \), where the first inequality is due to the fact that \( 0 \leq p(x|Y_t) \leq 1 \) and the second one follows from Jensen’s inequality. It is Jensen’s inequality that requires that \( \frac{k}{l} \) is an even integer greater than 2, so that \( x^\frac{k}{l} \) is convex in \( \mathbb{R} \).

2) Similar as before, we have

\[
P_k = \left[ \int_{\mathbb{R}} (x - \hat{x})^l p(x|Y_t)^{m(1-l)} dx \right]^{\frac{1}{m}} P_{l}^{\frac{m}{1-m}},
\]

for all \( p \geq 1 \) and \( 0 \leq l \leq k \). The conclusion follows by letting \( mp = 1 \).

Remark 3.7: Lemma 3.6 indicates that, in general, the moment sequence \( P_k \)'s satisfy the following lower bounds: \( P_4 \geq P_2^2 \) (by 1)); \( P_6 \geq P_2^3 \) (by 1)) or \( P_6 \geq \frac{P_2^2}{P_2} \) (by 2)), and etc. The lower bounds for \( P_k \)'s with \( n \geq 2 \) are not clear [17].

IV. NUMERICAL EXPERIMENTS

In this section, we shall illustrate our method applied to two different filtering problems: cubic sensor problem and a polynomial filtering problem with 2-dimensional state. In the cubic sensor problem, we compare our method with \( N = 2, 3 \) with EKF and PF with 50 particles. Further, we formulate and implement our method to a polynomial filtering problem with 2-dimensional state. The numerical result has been also compared with EKF, UKF and EnKF with 20 ensembles.

A. Cubic sensor problem

This problem is modeled by SDE (1) with \( f(x, t) = 0, g(x, t) = 1, \) and \( h(x, t) = x^3 \), which has been shown rigorously that it is essentially infinite-dimensional in [11] and has been studied by many authors, refer to [2], [25] and [28]. In order to get a fair comparison with EKF in computational complexity, we first propose to pick \( N = 2 \). Intuitively, the larger \( N \) is, the more accurate approximation is obtained for the state. Hence, we also pick \( N = 3 \) in our method for comparison.
Notice that $M_h = 3$, $M_f = M_g = 0$. On the right-hand sides of (6) and (11) with $\alpha \leq 2$, $P_3 - P_5$ show up and need to be reduced to some functions of $P_2$, $P_1 = 0$ and $P_0 = 1$. The conditions we imposed are:

\[ P_3 = P_{2+1} \quad \text{(12)} \]
\[ P_4 = P_{2+2} \quad \text{(12)} \]
\[ P_5 = P_{2+3} \quad \text{(12)} \]

The condition on $P_4$ satisfies the lower bound in Remark 3.7. Our method for $\hat{x}_t$ and $P_2$ gives

\[
\begin{cases}
    d\hat{x}_t = \frac{1}{R} (P_2^2 + 3P_2(\hat{x}_t^2)) (dy - (3P_2\hat{x}_t + (\hat{x}_t)^3) dt) \\
    \frac{dP_2}{dt} = 1 - \frac{1}{R} (P_2^2 + 3P_2(\hat{x}_t^2))^2
\end{cases}
\]  (14)

When choosing $N = 3$ in our method, the conditions imposed are:

\[ P_4 = P_{3+1} \quad \text{(12)} \]
\[ P_5 = P_{3+2} \quad \text{(12)} \]
\[ P_6 = P_{3+3} \quad \text{(12)} \]

Again from Remark 3.7, the condition on $P_4$, $P_6$ are also reasonable, in the sense that $P_4 \geq P_2^2$ and $P_6 \geq \frac{P_2^2}{P_2} = 9P_2^2$. The SDE given by our method for $\hat{x}_t$, $P_2$ and $P_3$ is:

\[
\begin{cases}
    d\hat{x}_t = \frac{1}{R} [dy - (\hat{x}_t^3 + 3\hat{x}_tP_2 + P_3)dt] \cdot (3\hat{x}_t^2P_2 + 3\hat{x}_tP_3 + 3P_2^2) \\
    \frac{dP_2}{dt} = 1 - \frac{1}{R} (3\hat{x}_t^2P_2 + 3\hat{x}_tP_3 + 3P_2^2)^2 \\
    \frac{dP_3}{dt} = -\frac{3}{R} (3\hat{x}_t^2P_2 + 3\hat{x}_tP_3 + 3P_2^2) \cdot (3\hat{x}_t^2P_3 + 6\hat{x}_tP_2^2 + 3P_2P_3)
\end{cases}
\]  (16)

We randomly generate 100 sample paths (except those EKF explodes before $T$) with $Q = R = 1$ and $P_0 = 0.01$, and apply EKF, PF with 50 particles, our method with $N = 2$ (14) and $N = 3$ (16) to estimate the real state. The PF used in our experiment is the SIR algorithm, see Algorithm 4, [1]. It is worth to note that there has been much progress in PF since the SIR algorithm, including: regularised PFs [24], auxiliary PFs [26], particle flow filters [7], Gaussian PFs [4], transport PFs [27], various MCMC methods (e.g., Metropolis adjusted Langevin or MALA, hybrid Monte Carlo, Girolami’s geodesic flow on Riemannian manifolds, etc.). The SDEs of EKF and our methods are numerically solved by Euler-Maruyama scheme [12]. The
<table>
<thead>
<tr>
<th>Filters</th>
<th>Variance of the errors</th>
<th>Average CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>PF with 500 particles</td>
<td>0.4566</td>
<td>4.146493s</td>
</tr>
<tr>
<td>EKF</td>
<td>4.4487</td>
<td>0.002505s</td>
</tr>
<tr>
<td>our method with $N = 2$</td>
<td>0.4562</td>
<td>0.002325s</td>
</tr>
<tr>
<td>our method with $N = 3$</td>
<td>0.3425</td>
<td>0.003405s</td>
</tr>
</tbody>
</table>

TABLE I: Variance of the estimation errors and average CPU time of different filters applied to the cubic sensor problem.

<table>
<thead>
<tr>
<th>Number of particles</th>
<th>Variance of estimation errors</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.5167</td>
<td>0.465100s</td>
</tr>
<tr>
<td>100</td>
<td>0.4246</td>
<td>0.909719s</td>
</tr>
<tr>
<td>200</td>
<td>0.4493</td>
<td>2.642290s</td>
</tr>
<tr>
<td>500</td>
<td>0.3596</td>
<td>4.251382s</td>
</tr>
<tr>
<td>1000</td>
<td>0.4765</td>
<td>8.555768s</td>
</tr>
<tr>
<td>5000</td>
<td>0.4461</td>
<td>37.203790s</td>
</tr>
</tbody>
</table>

TABLE II: Number of particles v.s. variance of estimation error

The total experimental time is $T = 10$ and the time step is $dt = 0.01$. The averaged mean and variance of the 100 experiments using EKF, PF and our methods have been displayed in Fig. 1. The figure shows that our method with $N = 3$ is superior than the other three. The variance of the estimation errors and the average CPU time has been list in Table I.

To explain why in Table I the number of particles is chosen to be 500 in PF, we experiment the cubic sensor problem by generating the sample path using $\text{randn('state',100)}$, with $T = 10$ and $dt = 0.01$. The performance is measured by variance of estimation errors. In Table II, we display the errors and the CPU times with different number of particles from 50 to 5000. It shows that using 500 particles the PF accuracy is roughly the same as our method. Presumably, this is the optimal accuracy, which explains why the performance stops to be improved by using more particles.

Remark 4.1: The condition (12) on $P_{\alpha}$ can’t be shown rigorously. It is just like no one can show that the truncation (conventionally operation to form a close system) yields the theoretically best approximation of $P_{\alpha}$.

In the sequel, we shall use the global method proposed in [20], [21] to numerically compute the $P_k$s of cubic sensor problem. This investigation will give us some indication on the reasonableness of our condition (12). [20], [21] introduced a method to directly approximate the conditional density function $\rho(x, t)$, and then we can obtain the approximate higher central.
moment of the states by

$$P_l = E^t[(x - \hat{x})^l] = \int_R (x - \hat{x})^l \rho(x, t) \, dx,$$

where $l \geq 2$, for the one-dimensional state. We apply the method in [20], [21] with appropriately chosen parameters ($\alpha = 2.5$, truncation modes $N_f = 45$) to 10 randomly generated real states. All the real states are generated with $Q = R = 1$ and the initial density function is assumed to be $u_0(x) = e^{-\frac{x^2}{2}}$. The total experimental time is $T = 10$, and time step is $dt = 0.001$. The approximate higher central moments are computed numerically by Gaussian-Hermite quadrature rule. The averaged higher central moments $P_2^k - P_6^k$ obtained by method in [20], [21] have been plotted in Fig. 2. It indicates that we probably should impose $P_{2k+1}^k \approx 0$ and $P_{2k}^k \neq 0$, which matches the condition (13) and (15).

B. Polynomial filtering problem with 2-dimensional state

In this subsection, we shall illustrate our method formulated for polynomial filtering problems of higher dimensional states. Let us take the following example:

$$\begin{align*}
    f_1 &= 0, \\
    f_2 &= x_1^2, \\
    h_1 &= x_1 x_2, \\
    h_2 &= x_2^2, \\
    g &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \\
    R &= I_2,
\end{align*}$$

and the initial state

$$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \sim N \begin{pmatrix} 1.1 \\ 1.1 \end{pmatrix}, \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.$$
Fig. 2: The averaged higher central moments for cubic sensor problem are displayed.

Let us choose $\vec{N} = (2, 2)$ in our method. Notice that $M_f = 2$ and $M_h = 2$. Observing the right-hand side of (6) and (11) for $P_{\vec{a}}$ with $\vec{a} \leq \vec{N}$, it contains all $P_{\vec{a}}$, $\vec{a} \leq \vec{N} + \vec{k}$, for $|\vec{k}|_1 \leq M_h$. We need to reduce all $P_{\vec{a}}$, $\vec{a}$ in case 2 or 3 by condition (12).

\[
P_{30} = P_{(2,0)+(1,0)} \overset{(12)}{=} 2P_{20}P_{10} + P_{20}P_{10} = 3P_{20}P_{10} = 0; \tag{19}
\]

\[
P_{31} = P_{(2,1)+(1,0)} \overset{(12)}{=} 2P_{20}P_{11} + P_{11}P_{20} + P_{21}P_{10} = 3P_{20}P_{11};
\]

\[
P_{32} = P_{(2,2)+(1,0)} \overset{(12)}{=} 2P_{20}P_{12} + 2P_{11}P_{21} + P_{22}P_{10} = 2P_{20}P_{12} + 2P_{11}P_{21};
\]

\[
P_{33} = P_{(2,2)+(1,1)} \overset{(12)}{=} 2P_{12}P_{21} + 2P_{21}P_{12} + P_{22}P_{11} = P_{12}P_{21} + P_{22}P_{11};
\]

\[
P_{40} = P_{(2,0)+(2,0)} \overset{(12)}{=} 2P_{30}P_{10} + P_{20}^2 = P_{20}^2;
\]

\[
P_{41} = P_{(2,1)+(2,0)} \overset{(12)}{=} 2P_{30}P_{11} + P_{21}P_{20} + P_{21}P_{20} = 2P_{20}P_{21};
\]

\[
P_{42} = P_{(2,2)+(2,0)} \overset{(12)}{=} 2P_{12}P_{30} + 2P_{21}P_{21} + P_{22}P_{20} = P_{20}P_{22}.
\]

Similar arguments could be used to obtain $P_{03} = 0$, $P_{13} = 3P_{02}P_{11}$, $P_{23} = 2P_{02}P_{21} + 2P_{11}P_{21}$, $P_{04} = P_{02}^2$, $P_{14} = 2P_{02}P_{12}$ and $P_{24} = 2P_{12}^2 + P_{02}P_{22}$. According to (6) and (11), our method yields a SDE of $\dot{x}_1$, $\dot{x}_2$, $P_{02}$, $P_{11}$, $P_{20}$, $P_{12}$, $P_{21}$ and $P_{22}$. We don’t write down the lengthy expression here due to the page limitation.

Numerical results for this example are displayed in Fig. 3. In this example, we generate 20
Fig. 3: NSF compared with EKF, UKF and EnKF with 20 ensembles are displayed for the 2D polynomial filtering problem (17), (18). The upper one in each subfigure is the trajectory of \( \hat{x}_1 \), while the lower one is that of \( \hat{x}_2 \).
sample paths randomly. The total experimental time is $T = 10$, and the time step is $dt = 0.001$. The figures are the average of 20 runs. One can see that our method tracks as well as EKF and UKF. But EnKF with 20 ensembles performs not very well. As to the efficiency, our method takes 15.4s while it costs 163.4s for UKF to obtain the similar result.

V. CONCLUSIONS

In this paper, given a truncation $\vec{N}$, starting from equation (11) for $P_{\vec{\alpha}}$, we construct our method by imposing some conditions (12) to reduce all the higher order central moments to the combination of the lower order ones $P_{\vec{\alpha}}$, $\vec{\alpha} \leq \vec{N}$. After the reduction, our method arrives at a closed system of equations (6) for $\dot{x}_{it}$, $1 \leq i \leq n$ and (11) for $P_{\vec{\alpha}}$, $\vec{\alpha} \leq \vec{N}$. This is completely new and different from the conventional operation–truncation. Since no one can show the truncation yields the best approximation, our procedure provides another reasonable way to form a closed system. Our method is a natural generalization of EKF. It is also more flexible by choosing the truncation $\vec{N}$ according to the desired accuracy and the demand of computational complexity. The imposed condition (12) in our method satisfies the lower bounds of $P_{k,s}$, and it is justified numerically for the cubic sensor problem by using the higher central moments obtained from Yau-Yau’s method [20]. Our method has also been formulated and implemented for the filtering problems with 2-dimensional state. Numerical results verifies that our method works in nearly perfect agreement with theory.

REFERENCES


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The captions of all the figures in this manuscript:
1) Fig. 1’s caption: Our method with \( N = 2, 3 \) for cubic sensor problem are compared with the EKF and the PF with 50 particles. Left: the averaged mean v.s. time; Right: the averaged variance v.s. time.
2) Fig. 2’s caption: The averaged higher central moments for cubic sensor problem are displayed.
3) Fig. 3’s caption: NSF compared with EKF, UKF and EnKF with 20 ensembles are displayed for the 2D polynomial filtering problem (17), (18). The upper one in each subfigure is the trajectory of \( \hat{x}_1 \), while the lower one is that of \( \hat{x}_2 \).