



## Decision Support

## Duopoly game of callable products in airline revenue management

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## ABSTRACT

This paper studies the capacity allocation game between duopolistic airlines which could offer callable products. Previous literature has shown that callable products provide a riskless source of additional revenue for a monopolistic airline. We examine the impact of the introduction of callable products on the revenues and the booking limits of duopolistic airlines. The analytical results demonstrate that, when there is no spill of low-fare customers, offering callable products is a dominant strategy of both airlines and provides Pareto gains to both airlines. When customers of both fare classes spill, offering callable products is no longer a dominant strategy and may harm the revenues of the airlines. Numerical examples demonstrate that whether the two airlines offer callable products and whether offering callable products is beneficial to the two airlines mainly depends on their loads and capacities. Specifically, when the difference between the loads of the airlines is large, the loads of the airlines play the most important role. When the difference between the loads of the airlines is small, the capacities of the airlines play the most important role. Moreover, numerical examples show that the booking limits of the two airlines in the case with callable products are always higher than those in the case without callable products.

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## 1. Introduction

During the last three decades, the technology of revenue management has been used more and more widely around the world, and has played a significant role in improving the profits of corporations. As a result, this field has attracted much attention from scholars (e.g., Steinhardt & Gönsch, 2012; Hu, Caldentey, & Vulcano, 2013; Otero & Akhavan-Tabatabaei, 2015). In the airline industry in which the technology of revenue management is most widely used, airlines usually have to face two types of customers: low-valuation customers who accept low-fare tickets only but are willing to book in advance and high-valuation customers who are willing to buy expensive tickets but arrive just before the plane takes off. Usually, the airlines cannot forecast the demand from high-valuation customers with certainty or convince them to book earlier than the low-valuation customers. Thus, “despite heavy investment in sophisticated revenue management systems, airlines lose millions of dollars a year in potential revenue; both when low-fare bookings displace higher than expected high-fare bookings (‘cannibalization’) and when airlines fly empty seats protected for high-fare bookings that do not materialize (‘spoilage’)”

(Gallego, Kou, & Phillips, 2008). Many kinds of mechanisms are proposed to hedge against demand uncertainty from high-valuation customers, e.g., overbooking (Aydın, Birbil, Frenk, & Noyan, 2012; Karaesmen & Van Ryzin, 2004), last-minute discounts (Ovchinnikov & Milner, 2012), flexible products (Gallego & Phillips, 2004), etc. All these mechanisms have shortcomings: overbooking adds operational complexity to management; last-minute discounts may induce the customers to wait rather than to book early; flexible products require that the customers are indifferent among the alternative flights. To avoid the above shortcomings, Gallego et al. (2008) proposed the concept of “callable products”, which refers to units of capacity sold to self-selected low-fare customers who willingly grant the airline the option to “call” the capacity at a pre-specified recall price. The concept of callable products does not add operational complexity and can be used together with other mechanisms.

Gallego et al. (2008) showed that callable products provide a riskless source of additional revenue for a monopolistic airline. In practice, airlines usually have to face other competitors. Seat allocation among different fare classes by one airline affects the demand and the optimal seat allocation of other airlines. Therefore, there are several questions to be addressed. In a competitive environment, does offering callable products still provide a riskless source of additional revenues? How does the introduction of callable products affect the capacity allocation decisions of the airlines? What is the order relationship between the booking limits

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under competition and those under monopoly? This paper aims to answer these questions.

This paper studies the capacity allocation game between duopolistic airlines which could offer callable products. We examine how the introduction of callable products affects the booking limits and the revenues of duopolistic airlines. It is shown that when the low-fare customers do not spill, offering callable products is a dominant strategy of both airlines and provides Pareto gains to both (In this paper, the word “spill” means that if either type of customer cannot be satisfied by one airline, the customers go to the other airline and can be recaptured by the other airline). When customers of both fare classes spill, offering callable products is no longer a dominant strategy and may harm the revenues of the airlines. Numerical examples demonstrate that whether the two airlines offer callable products and whether offering callable products is beneficial to the two airlines mainly depends on their loads and capacities (the load of an airline is the ratio of the average total demand of the airline to the capacity of the airline). Specifically, when the difference between the loads of the airlines is large, the loads of the airlines play the most important role. When the difference between the loads of the airlines is small, the capacities of the airlines play the most important role. Moreover, numerical examples demonstrate that the booking limits of the two airlines in the case with callable products are always higher than those in the case without callable products.

The rest of the paper is organized as follows. [Section 2](#) reviews the literature on callable products and on revenue management game. [Section 3](#) describes the key elements of the model. [Section 4](#) presents a comprehensive model analysis. Specifically, [Section 4.1](#) gives sufficient conditions for the existence and uniqueness of the Nash equilibrium; [Section 4.2](#) examines the impact of callable products on the revenues of the airlines when there is no low-fare spill; [Section 4.3](#) compares the booking limits under competition with those under monopoly; [Section 4.4](#) conducts a sensitivity analysis of the booking limits with respect to the price parameters. In [Section 5](#), we run numerical examples to examine the impact of offering callable products on the revenues and the booking limits of the two airlines, where both the low-fare and high-fare customers spill. [Section 6](#) concludes the paper and points out directions for future research.

## 2. Literature review

Two streams of literature are related to our study: one is callable products, and the other is revenue management game.

Many forms of callable products have been used in various industries. Some companies use an option named “callback” to recall previously committed advertisement time by paying a predetermined amount. The callable concept is also used by Caterpillar to reduce the inventory risk of its dealers (Sheffi, 2005, pp. 229–231). [Biyalogorsky, Carmon, Fruchter, and Gerstner \(1999\)](#) showed that the use of overselling with opportunistic cancellations can increase expected profits in an airline context. [Biyalogorsky and Gerstner \(2004\)](#) demonstrated that contingent pricing can be used for sellers in response to demand uncertainty. In contingent pricing arrangements, price is contingent on whether the seller succeeds in obtaining a higher price within a specified period. It is shown that contingent pricing is profitable regardless of buyers’ risk attitudes, and that contingent pricing benefits buyers as well as sellers. [Gallego et al. \(2008\)](#) differed from and extended [Biyalogorsky and Gerstner \(2004\)](#) in the following ways. First, [Biyalogorsky and Gerstner \(2004\)](#) considered sales of a single unit of capacity and [Gallego et al. \(2008\)](#) extended the analysis to sales of multiple units. Second, [Biyalogorsky and Gerstner \(2004\)](#) assumed common willingness-to-pay among buyers, whereas [Gallego et al. \(2008\)](#) assumed that demand for callable products is uncertain and depends

on the recall price. [Gallego et al. \(2008\)](#) showed that callable products provide a riskless source of additional revenue to a monopolistic airline. [Biyalogorsky \(2009\)](#) considered a model with strategic consumers who can decide when to show up in the market and investigated whether, in the face of strategic behavior by consumers, it can be profitable for sellers to use contingent pricing to induce the low-high arrival pattern typical in the airline industry. [Elmaghraby, Lippman, Tang, and Yin \(2009\)](#) examined a situation in which the firm offers both callable and non-callable units at different prices at any point in time. They showed that strategic customer behavior can render the customer to be worse off and the retailer to be better off. Therefore, more purchasing options do not necessarily benefit customers. [Aydn, Birbil, and Topaloğlu \(2016\)](#) developed single-leg revenue management models that consider contingent commitment decisions, where commitment option allows passengers to reserve a seat for a fixed duration before making a final purchase decision. We introduce the concept of callable products into a capacity allocation game between two airlines and examine its impact on the revenues and the booking limits of the two airlines.

The second stream of literature related to our study is revenue management game. [Lederer and Nambimadom \(1998\)](#) discussed how the entire airline network determines the routes and frequencies of flights when multiple airlines interact with each other. Using data on U.S. airline departure times from 1975, when fares were regulated, and 1986, when fares were not regulated, [Borenstein and Netz \(1999\)](#) empirically estimated the effect of competition on product differentiation. [Richard \(2003\)](#) analyzed the welfare consequences of airline mergers in terms of ticket price and flight frequency. The above research considers the competition between price, flight frequency and departure time, which is different from seat allocation competition as considered in this paper.

[Netessine and Shumsky \(2005\)](#) was the first published paper that places the seat allocation problem in a competitive framework and examines the seat inventory control problem. The analytical results demonstrated that more seats are protected for high-fare passengers under horizontal competition than when a single airline acts as a monopoly. [Li, Oum, and Anderson \(2007\)](#) showed the existence of an equilibrium booking strategy such that both airlines protect the same number of seats for the high fare and that the total number of seats available for the low fare under competition is smaller than the total number of seats that would be available if the two airlines were to collude. [Li, Zhang, and Zhang \(2008\)](#) extended [Li et al. \(2007\)](#) by incorporating the cost asymmetry of different airlines. While [Netessine and Shumsky \(2005\)](#) took the differentiation approach by assuming separate demand for each fare class offered by an airline, [Li et al. \(2007\)](#) and [Li et al. \(2008\)](#) chose the homogeneous market approach, i.e., two airlines face common market demand and the demand is split between the two airlines. The splitting rules of the demand in [Li et al. \(2007, 2008\)](#) are analogous to Rule 3 (Incremental Random Splitting) in [Lippman and McCardle \(1997\)](#) and generate demand that is independent or perfectly correlated, whereas the demand form in [Netessine and Shumsky \(2005\)](#) is more general as demand of different fare classes and different airlines can be partially correlated. We incorporate the concept of callable products into the framework of [Netessine and Shumsky \(2005\)](#) and examine its impact on the revenues and the booking limits of the two airlines.

Furthermore, [Song and Parlar \(2012\)](#) also studied the capacity allocation game between two airlines, where the demand form is similar to that in [Netessine and Shumsky \(2005\)](#). [Song and Parlar \(2012\)](#) took into account the penalty cost for each reservation of the transfer customers rejected by an airline. They used a nonnested model to approximate the original nested booking limit model and showed the existence of a unique Nash equilibrium in the noncooperative situation. [Zhao, Atkins, \(2002\)](#) made a major

attempt to address the joint pricing and allocation problem when two airlines compete for passengers in one demand class. Lim (2009) examined the practice of overselling in a duopoly context where late-arriving consumers value the good higher than early-arriving ones but the former's arrival is uncertain.

There are some papers studying the airline alliances, including capacity control of the airlines, revenue sharing of the airlines, etc. (e.g., Graf & Kimms, 2011; Kimms & Çetiner, 2012; Hu et al., 2013; Graf & Kimms, 2013). Particularly, Kimms and Graubeger (2016) investigated the problem of two airlines in which they cooperate within an alliance on one hand, but on the other hand they continue to compete for customers within a revenue management setting. It is the first paper to consider simultaneous horizontal and vertical competition within alliances on a network with multiple classes.

### 3. Problem description

Our model is similar to that of Netessine and Shumsky (2005) and the difference is that the airlines in our model could offer callable products. Suppose there are two airlines offering flights between the same origin and the same destination. Use subscripts  $i = 1, 2$  to distinguish the two airlines. Both airlines face two types of customers: low-fare customers and high-fare customers. The low-fare customers accept low-fare tickets only but are willing to book in advance; the high-valuation customers are willing to buy expensive tickets but arrive just before the plane takes off. Therefore, we can assume that the low-fare customers arrive in the first period and the high-fare customers arrive in the second period. Suppose that the capacity of Airline  $i$  is  $C_i$ , and that the prices of the low-fare tickets and high-fare tickets of Airline  $i$  are  $p_{Li}$  and  $p_{Hi}$ , respectively. The initial low-fare demand and the initial high-fare demand of Airline  $i$  are represented by the random variables  $D_{Li}$  and  $D_{Hi}$ , respectively. Assume that the support sets of  $D_{Li}$  and  $D_{Hi}$  are nonnegative and that the cumulative distribution functions of  $D_{Li}$  and  $D_{Hi}$  are differentiable. If either type of customer cannot be satisfied by one airline, they spill to the other airline and can be recaptured by the other airline. We call these spilled customers.

The low-fare tickets are either callable products or noncallable products: if an airline offers callable products, the low-fare tickets are callable products, which means that the low-fare tickets can be recalled at a pre-specified price; if an airline does not offer callable products, the low-fare tickets are noncallable products, which means that the low-fare tickets are regular tickets and cannot be recalled. If an airline offers callable products, when the capacity available for the high-fare customers could not satisfy their demand, the airline can recall some or all of the callable tickets (i.e., low-fare tickets). Suppose the recall price of Airline  $i$  is  $p_i$  ( $p_{Li} < p_i < p_{Hi}$ ). Denote the booking limit of Airline  $i$  as  $B_i$ . Each airline needs to determine the booking limit for the low-fare customers at the beginning of the first period.

Unless particularly specified, we analyze the case where both airlines offer callable products. The expected revenue of Airline  $i$  is:

$$\pi_i = E \left\{ p_{Li} \min(D_{Li}^T, B_i) + p_{Hi} \min(D_{Hi}^T, C_i - \min(D_{Li}^T, B_i)) + (p_{Hi} - p_i) \min((\min(D_{Li}^T, B_i) + D_{Hi}^T - C_i)^+, \min(D_{Li}^T, B_i)) \right\}, \tag{1}$$

where  $D_{Li}^T = D_{Li} + (D_{Lj} - B_j)^+$  is the total low-fare demand for Airline  $i$  and  $D_{Hi}^T = D_{Hi} + (D_{Hj} - C_j)^+$  is the total high-fare demand for Airline  $i$  ( $i, j = 1, 2$  and  $i \neq j$ ). The meanings of the three items in Eq. (1) are as follows: the first item is the revenue from selling the

**Table 1**  
Notation description.

Notation	Meaning
$C_i$	Capacity of Airline $i$
$p_{Hi}$	Price of the high-fare tickets of Airline $i$
$p_{Li}$	Price of the low-fare tickets of Airline $i$
$p_i$	Recall price of Airline $i$
$D_{Hi}$	Initial high-fare demand of Airline $i$
$D_{Li}$	Initial low-fare demand of Airline $i$
$D_{Hi}^T$	Total high-fare demand of Airline $i$
$D_{Li}^T$	Total low-fare demand of Airline $i$
$B_i$	Booking limit of Airline $i$
$\pi_i$	Expected revenue of Airline $i$

low-fare tickets, the second item is the revenue from selling the rest of the tickets to the high-fare customers and the third item is the additional revenue brought by recalling some low-fare tickets. The marginal revenue of one unit of callable product is  $p_{Hi} - p_i$ .

Similar to Netessine and Shumsky (2005), we apply the method described in Rudi (2001, pp. 27–31) to obtain the derivative of Airline  $i$ 's expected revenue:

$$\begin{aligned} \frac{\partial \pi_i}{\partial B_i} &= p_{Li} \Pr(D_{Li}^T > B_i) - p_{Hi} \Pr(D_{Li}^T > B_i, D_{Hi}^T > C_i - B_i) \\ &\quad + (p_{Hi} - p_i) \Pr(D_{Li}^T > B_i, D_{Hi}^T > C_i - B_i) \\ &= \Pr(D_{Li}^T > B_i) (p_{Li} - p_i \Pr(D_{Hi}^T > C_i - B_i | D_{Li}^T > B_i)). \end{aligned} \tag{2}$$

For ease of exposition, Table 1 summarizes the notation we used, where  $i = 1, 2$ .

Note that the booking limit  $B_i$  is the only decision of Airline  $i$ . The prices  $p_{Li}$ ,  $p_{Hi}$  and  $p_i$  are not decision variables and they are assumed to be determined somehow in advance. We consider the effects of different prices with numerical studies in Section 5. It is obvious that, for Airline  $i$  to obtain higher profit, it should be better to determine the prices  $p_{Li}$ ,  $p_{Hi}$  and  $p_i$  and the booking limit  $B_i$  simultaneously. However, due to the complexity of computations and analyses, this problem is rarely considered in the competitive revenue management situation (Zhao & Atkins, (2002). Furthermore, the decisions of pricing and booking limit are at different decision levels in reality, as stated by Petrick, Gönsch, Steinhardt, and Klein (2010), "Revenue management is essentially achieved by the application of two instruments: In a first step, on a rather tactical planning level, price differentiation is performed, leading to a variety of differently priced products defined on the same set of resources. In a second step, on the operational level, the availability of the products is permanently adjusted by means of capacity control, according to the current forecast regarding future demand within the selling horizon". In addition, the focus of the paper is on examining the impact of callable products on the revenues and the booking limits of the airlines. Therefore, we assume that the prices  $p_{Li}$ ,  $p_{Hi}$  and  $p_i$  are exogenously given.

### 4. Model analysis

This section provides a comprehensive model analysis. Section 4.1 presents the conditions under which a unique Nash equilibrium exists. Section 4.2 examines the impact of callable products on the revenues of the airlines when there is no low-fare spill. Section 4.3 compares the booking limits under competition with those under monopoly. Section 4.4 conducts a sensitivity analysis to investigate the impact of price parameters, i.e., ticket prices and recall prices, on the booking limits. Note that only Section 4.2 considers the situation where there is no spill of low-fare customers while Sections 4.1, 4.3 and 4.4 all consider the situation where both low-fare and high-fare customers spill.

4.1. Nash equilibrium conditions

This subsection presents the sufficient conditions for the existence and uniqueness of a pure-strategy Nash equilibrium. We utilize two properties: totally positive of order 2 ( $TP_2$ ) and multivariate totally positive of order 2 ( $MTP_2$ ) (for a thorough discussion of  $TP_2$  and  $MTP_2$ , see Joe, 1997).

**Definition 1.** (Joe, 1997, p. 23). A non-negative function  $b$  on  $A^2$ , where  $A \subset \mathcal{R}$ , is  $TP_2$  if for all  $x_1 < y_1, x_2 < y_2$ , with  $x_1, x_2, y_1, y_2 \in A$ ,

$$b(x_1, x_2)b(y_1, y_2) \geq b(x_1, y_2)b(y_1, x_2). \tag{3}$$

**Definition 2.** (Joe, 1997, p. 24). Random variables  $X$  and  $Y$  are  $TP_2$  if the joint probability density function of  $X$  and  $Y$  is  $TP_2$ .

**Definition 3.** (Joe, 1997, p. 24). Let  $\mathbf{X}$  be a random  $m$ -vector with density  $f$ .  $\mathbf{X}$  of  $f$  is multivariate totally positive of order 2 ( $MTP_2$ ) if

$$f(\mathbf{x} \vee \mathbf{y})f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x})f(\mathbf{y}), \tag{4}$$

for all  $\mathbf{x}, \mathbf{y} \in \mathcal{R}^m$ , where

$$\begin{aligned} \mathbf{x} \vee \mathbf{y} &= (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \dots, \max\{x_m, y_m\}), \\ \mathbf{x} \wedge \mathbf{y} &= (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \dots, \min\{x_m, y_m\}). \end{aligned} \tag{5}$$

The definition of  $TP_2$  indicates that it is more possible for the realizations of the two random variables to be both low or both high, than to be mixed low and high. Many useful bivariate distributions are  $TP_2$ , such as any set of independent random variables, Gamma and  $F$  distributions, the multivariate logistic and the bivariate normal distributions with positive correlation (Kalin & Rinotta, 1980; Netessine & Shumsky, 2005). The  $TP_2$  property can be extended to the  $MTP_2$ .

**Lemma 1.** (Theorem 2.1 in Vives, 2001). Consider a game with  $n$  ( $n \geq 2$ ) players. If the strategy sets are nonempty convex and compact subsets of Euclidean space and the payoff to firm  $i$  is continuous in the actions of all firms and quasiconcave in its own action, there is a Nash equilibrium.

**Proposition 1.** When both airlines offer callable products, if  $D_{Li}$  and  $D_{Hi}^T$  ( $i = 1, 2$ ) are  $TP_2$ , there exists a unique pure-strategy Nash equilibrium. Furthermore, the best response functions of the airlines are decreasing.

Proposition 1 demonstrates that the best response functions are decreasing, i.e., if one airline increases the booking limit, the other airline will decrease the booking limit, which is the same as that in Netessine and Shumsky (2005). Recall that in Netessine and Shumsky (2005) the Nash equilibrium exists but may not be unique. When the low-fare tickets of the airlines are callable, Proposition 1 indicates that the Nash equilibrium is unique, which facilitates the subsequent analysis. Note that Proposition 1 requires  $D_{Li}$  and  $D_{Hi}^T$  to be  $TP_2$  while the realization of  $D_{Hi}^T$  depends on the realizations of  $D_{Hi}$  and  $D_{Hj}$ . Corollary 1 describes conditions on the four underlying demand distributions.

**Corollary 1.** When both airlines offer callable products, if  $(D_{L1}, D_{L2}, D_{H1}, D_{H2})$  are  $MTP_2$  in their density functions, the results in Proposition 1 hold.

In the previous analysis, we assume that, if the low-fare tickets are callable, all the low-fare customers grant the airlines the call option. If only a fraction of the low-fare customers grant the airlines the call option, Proposition 2 gives conditions under which a pure-strategy Nash equilibrium exists.

**Proposition 2.** Suppose only  $\alpha$  ( $0 \leq \alpha \leq 1$ ) proportion of the low-fare customers grant the airlines the call option when the low-fare

tickets are callable. If  $D_{Li}$  and  $D_{Hi}^T$  ( $i = 1, 2$ ) are  $TP_2$  or  $(D_{L1}, D_{L2}, D_{H1}, D_{H2})$  are  $MTP_2$ , a pure-strategy Nash equilibrium exists. In addition, the best-response functions of the airlines are decreasing.

For mathematical tractability, in the following analysis, we still discuss the case where all the low-fare customers grant the airlines the call option if the low-fare tickets are callable. In this situation, the uniqueness of the Nash equilibrium (Proposition 1) facilitates the analysis. If only a fraction of the low-fare customers grant the airlines the call option, we conjecture that the insights of the current paper still hold.

4.2. Impact of callable products on the revenues of the airlines

In this subsection, we consider the case where the low-fare customers do not spill to the other airline while the high-fare customers do spill. This applies to the situation where low-fare demand is so high that the booking limits of the two airlines can always be reached, and therefore low-fare spill is irrelevant (Netessine & Shumsky, 2005). The case where both the low-fare customers and the high-fare customers spill is examined numerically in Section 5.

To investigate the impact of callable products on the revenues of the airlines, we consider three other scenarios: neither of the two airlines offers callable products; only Airline  $i$  ( $i = 1, 2$ ) offers callable products.

(1) If neither of the two airlines offers callable products, the expected revenue of Airline  $i$  is:

$$\pi_i = E[p_{Li} \min(D_{Li}, B_i) + p_{Hi} \min(\tilde{D}_{Hi}^T, C_i - \min(D_{Li}, B_i))], \tag{6}$$

where  $\tilde{D}_{Hi}^T = D_{Hi} + (D_{Hj} - C_j + \min(D_{Lj}, B_j))^+$  is the total high-fare demand for Airline  $i$  ( $i, j = 1, 2$  and  $i \neq j$ ). The first order condition for the maximization of (6) is:

$$\begin{aligned} \frac{\partial \pi_i}{\partial B_i} &= p_{Li} \Pr(D_{Li} > B_i) - p_{Hi} \Pr(D_{Li} > B_i, \tilde{D}_{Hi}^T > C_i - B_i) \\ &= \Pr(D_{Li} > B_i) (p_{Li} - p_{Hi} \Pr(\tilde{D}_{Hi}^T > C_i - B_i | D_{Li} > B_i)) \\ &= 0. \end{aligned} \tag{7}$$

That is,

$$\Pr(\tilde{D}_{Hi}^T > C_i - B_i | D_{Li} > B_i) = \frac{p_{Li}}{p_{Hi}}. \tag{8}$$

(2) If only Airline  $i$  offers callable products, the expected revenues of Airline  $i$  and  $j$  are

$$\begin{aligned} \pi_i &= E[p_{Li} \min(D_{Li}, B_i) + p_{Hi} \min(\tilde{D}_{Hi}^T, C_i - \min(D_{Li}, B_i)) \\ &\quad + (p_{Hi} - p_i) \min((\min(D_{Li}, B_i) + \tilde{D}_{Hi}^T - C_i)^+, \min(D_{Li}, B_i))], \end{aligned} \tag{9}$$

and

$$\pi_j = E[p_{Lj} \min(D_{Lj}, B_j) + p_{Hj} \min(D_{Hj}^T, C_j - \min(D_{Lj}, B_j))], \tag{10}$$

respectively. The first order conditions for the maximization of (9) and (10) are:

$$\begin{aligned} \frac{\partial \pi_i}{\partial B_i} &= p_{Li} \Pr(D_{Li} > B_i) - p_i \Pr(D_{Li} > B_i, \tilde{D}_{Hi}^T > C_i - B_i) \\ &= \Pr(D_{Li} > B_i) (p_{Li} - p_i \Pr(\tilde{D}_{Hi}^T > C_i - B_i | D_{Li} > B_i)) \\ &= 0, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \frac{\partial \pi_j}{\partial B_j} &= p_{Lj} \Pr(D_{Lj} > B_j) - p_{Hj} \Pr(D_{Lj} > B_j, D_{Hj}^T > C_j - B_j) \\ &= \Pr(D_{Lj} > B_j) (p_{Lj} - p_{Hj} \Pr(D_{Hj}^T > C_j - B_j | D_{Lj} > B_j)) \\ &= 0, \end{aligned} \tag{12}$$

respectively. Eqs. (11) and (12) can be simplified as:

$$\Pr(\tilde{D}_{Hi}^T > C_i - B_i | D_{Li} > B_i) = \frac{p_{Li}}{p_i}, \tag{13}$$

and

$$\Pr(D_{Hj}^T > C_j - B_j | D_{Lj} > B_j) = \frac{p_{Lj}}{p_{Hj}}. \tag{14}$$

When  $D_{Li}$  and  $D_{Hi}^T$  are  $TP_2$ , applying a similar analysis as the proof of Proposition 1, it can be shown that there is a unique Nash equilibrium for each game corresponding to the above three scenarios. Let  $\pi_i^{k_1 k_2}$  ( $k_1, k_2 = 0, 1, i = 1, 2$ ) denote the revenue of Airline  $i$  in different scenarios. The superscripts  $k_1$  and  $k_2$  indicate whether the low-fare tickets of the two airlines are callable: 1 is callable while 0 is not. For example,  $\pi_1^{10}$  denotes the revenue of Airline 1 when the low-fare tickets of Airline 1 are callable while those of Airline 2 are not. Similarly,  $B_i^{k_1 k_2}$  is the booking limit of Airline  $i$  in different scenarios.

Recall that Gallego et al. (2008) showed that offering callable products brings a riskless revenue improvement to a monopolistic airline, which is intuitive as an airline offers callable products if and only if it is beneficial to itself. Proposition 3 indicates that, if there is no spill of low-fare customers, offering callable products is a dominant strategy of both airlines and brings Pareto gains to both.

**Proposition 3.** Suppose that  $D_{Li}$  and  $D_{Hi}^T$  ( $i = 1, 2$ ) are  $TP_2$ . If there is no spill of low-fare customers, the following inequalities hold:

- (1)  $\pi_1^{00} < \pi_1^{10}, \pi_1^{01} < \pi_1^{11}$ ;
- (2)  $\pi_2^{00} < \pi_2^{01}, \pi_2^{10} < \pi_2^{11}$ ;
- (3)  $\pi_i^{00} < \pi_i^{11}, i = 1, 2$ .

Parts (1) and (2) of Proposition 3 indicate that offering callable products improves an airline's revenue no matter whether the other airline offers callable products. Thus, offering callable products is a dominant strategy of both airlines. Parts (1) and (2) can be interpreted as follows. If Airline  $j$  does not offer callable products, Airline  $i$ 's ( $i, j = 1, 2$  and  $i \neq j$ ) revenue when Airline  $i$  offers callable products is no lower than that when Airline  $i$  does not offer callable products. If Airline  $j$  offers callable products, the high-fare customers of Airline  $j$  spill to Airline  $i$  only when the realized high-fare demand of Airline  $j$  is higher than the capacity of Airline  $j$ . Given this situation, Airline  $i$ 's revenue when Airline  $i$  offers callable products is no lower than that when Airline  $i$  does not offer callable products either. Thus, offering callable products is a dominant strategy of both airlines.

Part (3) of Proposition 3 implies that offering callable products provides Pareto gains to both airlines, which can be interpreted intuitively. Note that, when one airline does not offer callable products, some of its high-fare customers may spill to the other airline. However, if one airline offers callable products, its high-fare customers spill to the other airline only when the realized high-fare demand is higher than the capacity of the airline. Thus, offering callable products provides Pareto gains to both airlines. In other words, when there is no spill of low-fare customers, Proposition 3 demonstrates that offering callable products is a dominant strategy of both airlines and brings Pareto gains to both. When customers of both fare classes spill, Section 5 numerically shows that offering callable products is no longer a dominant strategy and may harm the revenues of the airlines.

#### 4.3. Comparing the competitors with a monopolist

This subsection compares the behavior of two airlines in competition with that of a monopolist. The monopolist may be two airlines in an alliance to coordinate the revenue management deci-

sions, i.e., booking limits (Graf & Kimms, 2011; Kimms & Çetiner, 2012). Denote the optimal booking limits of the monopolist to be  $B_i^C$  ( $i = 1, 2$ ) and the booking limits in equilibrium to be  $B_i^*$  ( $i = 1, 2$ ), respectively. Assume that  $B_i^*$  and  $B_i^C$  are interior points, i.e.,  $B_i^*, B_i^C \in (0, C_i)$ .

**Proposition 4.** When both airlines offer callable products, if  $D_{Li}$  and  $D_{Hi}^T$  are  $TP_2$ , and the two airlines are symmetric (i.e.,  $p_{L1} = p_{L2}, p_{H1} = p_{H2}, p_1 = p_2, C_1 = C_2, (D_{L1}, D_{H1})$  and  $(D_{L2}, D_{H2})$  are identically distributed), the booking limits under competition are higher than those under monopoly:  $B_i^* \geq B_i^C$ , where  $i = 1, 2$ .

Proposition 4 shows that, when the two airlines are symmetric, the airlines always provide more tickets to low-fare customers in the decentralized situation compared to the centralized situation, which implies that the airlines compete more intensely for low-fare customers. This is different from the result when the two airlines do not offer callable products: if only the high-fare customers spill, the airlines set lower booking limits in the decentralized situation; if only the low-fare customers spill, the airlines set higher booking limits in the decentralized situation (Netessine & Shumsky, 2005). The reason behind the difference is that when the low-fare tickets are callable, the airlines can avoid cannibalization from the low-fare customers by recalling the low-fare tickets, so both airlines pay more attention to the scramble for the low-fare customers.

#### 4.4. Impact of price parameters on the booking limits

The main difference between callable products and the regular products is that callable products can be recalled at a pre-specified recall price. Obviously, recall price has a great impact on the booking limits of the airlines. Although recall price is not taken as a decision variable, we can examine its impact on the booking limits through comparative static analysis.

**Proposition 5.** If  $D_{Li}$  and  $D_{Hi}^T$  ( $i = 1, 2$ ) are  $TP_2$ ,

- (1)  $B_i^*$  decreases with  $p_i$  while increases with  $p_j$  ( $i, j = 1, 2, i \neq j$ );
- (2)  $B_i^*$  increases with  $p_{Li}$  while decreases with  $p_{Lj}$  ( $i, j = 1, 2, i \neq j$ ).

Part (1) of Proposition 5 shows that as the recall price of Airline  $i$  increases, Airline  $i$  lowers its booking limit while Airline  $j$  raises its booking limit. This result is intuitive as the higher the recall price of Airline  $i$ , the higher the cost for Airline  $i$  to recall the low-fare tickets; at the same time, Airline  $j$  has a comparative advantage in recalling the low-fare tickets. This is also consistent with Proposition 1 which states that the best response functions of the airlines are decreasing. The interpretation of Part (2) is similar to that of Part (1). Note that the impact of  $p_{Li}$  on  $B_i$  and  $B_j$  is opposite to that of  $p_i$ , as the airlines weigh the deterministic revenue  $p_{Li}$  against the potential recall cost  $p_i$  when determining the booking limits.

In addition, we find that the booking limits of the airlines are not affected by the prices of the high-fare tickets. For the high-fare customers, the airline could avoid cannibalization from the low-fare customers by recalling the low-fare tickets. Therefore, when setting the booking limits, the airlines focus on the tradeoff between the immediate revenue  $p_{Li}$  and the potential recall cost  $p_i$ .

Proposition 6 examines the case in which the two airlines are symmetric, i.e.,  $p_{L1} = p_{L2} = p_L, p_{H1} = p_{H2} = p_H, p_1 = p_2 = p, C_1 = C_2 = C, (D_{L1}, D_{H1})$  and  $(D_{L2}, D_{H2})$  are identically distributed.

**Proposition 6.** When the two airlines are symmetric, if  $D_{Li}$  and  $D_{Hi}^T$  ( $i = 1, 2$ ) are  $TP_2$ ,  $B_1^*$  is equal to  $B_2^*$  and

- (1) both of them decrease with  $p$ ;
- (2) both of them increase with  $p_L$ .

When the recall price increases, the airlines cost more to recall the low-fare tickets, so they will lower the booking limits. We can consider the following two extreme cases. First, if the recall price is equal to the price of the low-fare tickets, the airlines will set the booking limit to be the capacity; second, if the recall price is equal to the price of the high-fare tickets, the game between the two airlines degenerates to that when the two airlines do not offer callable products. Part (1) of Proposition 6 implies that, when the two airlines are symmetric, the booking limits of the two airlines when they offer callable products are higher than those when they do not offer callable products. When the two airlines are asymmetric, numerical examples in Section 5 shows that the result still holds.

5. Numerical examples

In this section, we run numerical examples to examine, when both the low-fare and high-fare customers spill, whether the two airlines offer callable products and how callable products impact the booking limits and the revenues of the airlines in equilibrium. Referring to the parameter values in Netessine and Shumsky (2005), the parameter values are as follows.

- *Capacity (C<sub>1</sub> and C<sub>2</sub>):* We run two sets of experiments: a symmetric case with C<sub>1</sub> = C<sub>2</sub> = 200, and an asymmetric case with C<sub>1</sub> = 200 and C<sub>2</sub> = 100.
- *Ratio of high fare to low fare (p<sub>H</sub>/p<sub>L</sub>):* We define three scenarios, p<sub>H</sub>/p<sub>L</sub> = [1.3, 2.6, 4], for both the symmetric and asymmetric cases.
- *Recall price (p<sub>1</sub> and p<sub>2</sub>):* The recall prices of the two airlines are assumed to be equal. We set p<sub>1</sub> = p<sub>2</sub> = p<sub>L</sub> + α(p<sub>H</sub> - p<sub>L</sub>), where α = [0.4, 0.6, 0.8].
- *Proportion of demand due to low-fare passengers:* Let μ<sub>Li</sub> and μ<sub>Hi</sub> be the average low-fare demand and the average high-fare demand of Airline i, respectively. For both the symmetric and asymmetric cases, set γ = μ<sub>Li</sub> / (μ<sub>Li</sub> + μ<sub>Hi</sub>) = [0.5, 0.74, 0.9].
- *Total demand and demand faced by each airline:* We consider two scenarios: first, the average total demand is equal to the total airline capacity, i.e., μ<sub>L1</sub> + μ<sub>H1</sub> + μ<sub>L2</sub> + μ<sub>H2</sub> = C<sub>1</sub> + C<sub>2</sub>, referred to as TD = TC case; second, the average total demand is slightly larger than the total airline capacity, where μ<sub>L1</sub> + μ<sub>H1</sub> + μ<sub>L2</sub> + μ<sub>H2</sub> = 1.1(C<sub>1</sub> + C<sub>2</sub>), referred to as TD = 1.1TC case. Describe the allocation of demand between the two airlines in terms of load, where the load for Airline i equals (μ<sub>Li</sub> + μ<sub>Hi</sub>)/C<sub>i</sub>. For ease of exposition, Table 2 presents the loads of the two airlines in different cases.
- *Variability:* Netessine and Shumsky (2005) supposed the coefficient of variation (CV) of the four demand distributions to be the same and CV=[0.2, 0.33, 0.6]. To limit the number of parameters, we assume CV of the four demand distributions to be the same and CV=[0.2, 0.6].
- *Correlation:* Netessine and Shumsky (2005) assumed the correlations among all demand are equal and the correlation ρ = [-0.3, 0.0, 0.3, 0.6]. As stated by Netessine and Shumsky (2005), correlation among airline demand classes is usually small and when correlation is significant, it is more likely to be positive than negative. Thus, to limit the number of parameters, we as-

sume that the correlations among all demand are equal and the correlation ρ = [0.2, 0.5].

- *Probability density:* For each of the scenarios, assume that the demand distribution is a multivariate Normal distribution and is truncated at zero. The negative values of the demand are added to a mass point at zero.

When combined, the above parameters define 2\*3\*3\*3\*6\*2\*2=1296 scenarios. There are 648 scenarios in Netessine and Shumsky (2005). The parameter values in the current paper are the same as those in Netessine and Shumsky (2005) except CV and ρ. In addition, there is no recall price in Netessine and Shumsky (2005). We use a simple gradient algorithm to find the solutions and evaluate the gradients by Monte Carlo integration.

Proposition 3 shows that, when there is no spill of low-fare customers, offering callable products is a dominant strategy and provides Pareto gains to both airlines. When low-fare customers spill, numerical examples demonstrate that whether the two airlines offer callable products and whether offering callable products is beneficial to them mainly depends on the loads and the capacities of the two airlines. For ease of exposition, we refer to the case where the two airlines do not offer callable products as the case without callable products and the case where both airlines could choose whether or not to offer callable products as the case with callable products. We present the numerical results in the symmetric case and the asymmetric case, respectively.

In the symmetric case, we obtain the following three results. (1) When the loads of the two airlines are not equal, and the ratio of high fare to low fare (p<sub>H</sub>/p<sub>L</sub>) is low or the recall price (p<sub>1</sub> and p<sub>2</sub>) is high, the airline with a lower load does not offer callable products while the airline with a higher load offers callable products in equilibrium. Specifically, if the parameter values belong to Table 3, Airline 1 does not offer callable products while Airline 2 offers callable products in equilibrium. Under other parameter values, both airlines offer callable products in equilibrium. This result is intuitive and we could interpret it as follows. Compared to the airline with a higher load, the airline with a lower load is at a disadvantage. The ratio of high fare to low fare being low or the recall price being high indicates that the market is pessimistic to the airlines. This result implies that when the market is pessimistic to the airlines, the airline at a disadvantage does not offer callable products in equilibrium. (2) When the loads of the two airlines are not equal, the revenue of the airline with a higher (resp. lower) load in the case with callable products is higher (resp. lower) than that in the case without callable products. That is, the airline at an advantage always benefits from offering callable products while the airline at a disadvantage is always worse off no matter whether or not it offers callable products in the case with callable products. Specifically, when the loads of Airlines 1 and 2 are different, Airline 2's (resp. Airline 1's) revenue in the case with callable products is higher (resp. lower) than that in the case without callable products. (3) When the loads of the two airlines are equal, both airlines offer callable products in equilibrium and they are better off or worse off by offering callable products simultaneously. If the

Table 2 Loads of the two airlines in different cases.

Cases	l <sub>1</sub>	l <sub>2</sub>
Symmetric, TD=TC	[0.5, 0.75, 1]	[1.5, 1.25, 1]
Symmetric, TD=1.1TC	[0.5, 0.8, 1.1]	[1.7, 1.4, 1.1]
Asymmetric, TD=TC	[0.5, 1, 1.25]	[2, 1, 0.5]
Asymmetric, TD=1.1TC	[0.5, 1.1, 1.4]	[2.3, 1.1, 0.5]

Table 3 Parameter values in the symmetric case under which Airline 1 does not offer callable products while Airline 2 offers callable products.

Cases	l <sub>1</sub>	l <sub>2</sub>	p <sub>H</sub> /p <sub>L</sub>	α
TD=TC	0.5	1.5	130	0.4, 0.6, 0.8
		0.75	260	0.8
TD=1.1TC	0.5	1.7	130	0.4, 0.6, 0.8
		0.8	130	0.8

**Table 4**  
Parameter values in the symmetric case under which both airlines are better off by offering callable products.

Cases	$l_1$	$l_2$	$p_H/p_L$	$\alpha$
TD=TC	1	1	260, 400	0.4
TD=1.1TC	1.1	1.1	260, 400	0.4

**Table 5**  
Parameter values in the asymmetric case under which Airline 1 does not offer callable products while Airline 2 offers callable products.

Cases	$l_1$	$l_2$	$p_H/p_L$	$\alpha$
TD=TC	0.5	2	130 130, 260, 400	0.4, 0.6 0.8
TD=1.1TC	0.5	2.3	130	0.4, 0.6, 0.8

ratio of high fare to low fare ( $p_H/p_L$ ) is high and the recall price ( $p_1$  and  $p_2$ ) is low, both airlines are better off while they are worse off under other parameter values. Specifically, both airlines are better off by offering callable products when the parameter values belong to Table 4. Under other parameter values, both airlines are worse off by offering callable products. Therefore, callable products may bring Pareto gains or Prisoner’s Dilemma to the two airlines.

In the asymmetric case, we obtain the following three results. (1) When the difference between the loads of the two airlines is large, and the ratio of high fare to low fare ( $p_H/p_L$ ) is low or the recall price ( $p_1$  and  $p_2$ ) is high, the airline with a lower load does not offer callable products while the airline with a higher load offers callable products in equilibrium. Specifically, if the parameter values belong to Table 5, Airline 1 does not offer callable products while Airline 2 offers callable products in equilibrium. Under other parameter values, both airlines offer callable products in equilibrium. The interpretation of the result is as follows. The revenues of the airlines depend on the proportion of high-fare demand to capacity (i.e.,  $\mu_{H_i}/C_i$ ) to a large degree. Thus, compared to the other airline, whether an airline is at an advantage depends on the ratio of their proportion of high-fare demand to capacity. Note that  $\mu_{H_i}/C_i = (1 - \gamma)l_i$ , where  $i = 1, 2$ . Obviously, the higher  $l_i$  is, the higher  $\mu_{H_i}/C_i$  is. Therefore, an airline with a higher load is at an advantage no matter whether its capacity is larger than the other airline. This result implies that, when the market is pessimistic, the airline at a disadvantage does not offer callable products in equilibrium. (2) When the loads of the two airlines are not equal, the revenue of the airline with a higher (resp. lower) load in the case with callable products is higher (resp. lower) than that in the case without callable products. Specifically, if the load of Airline  $i$  is higher, Airline  $i$ ’s (resp. Airline  $j$ ’s) revenue in the case with callable products is higher (resp. lower) than that in the case without callable products, where  $i, j = 1, 2$  and  $i \neq j$ . This result is intuitive and is similar to that in the symmetric case. (3) When the loads of the two airlines are equal, both airlines offer callable products in equilibrium. If the ratio of high fare to low fare ( $p_H/p_L$ ) is low or the recall price ( $p_1$  and  $p_2$ ) is high, both airlines are worse off by offering callable products. Specifically, if the parameter values belong to Table 6, both airlines are worse off by offering callable products. Under other parameter values, the airline with a higher capacity is better off while the one with a lower capacity is worse off by offering callable products. The result is intuitive. Note that, when the ratio of high fare to low fare is low or the recall price is high, the market is pessimistic. So both airlines are worse off by offering callable products. Moreover, when the loads of the two airlines are equal, the airline with a higher capacity is at an advantage.

**Table 6**  
Parameter values in the asymmetric case under which both airlines are worse off by offering callable products.

Cases	$l_1$	$l_2$	$p_H/p_L$	$\alpha$
TD=TC	1	1	130 130, 260, 400	0.4, 0.6 0.8
TD=1.1TC	1.1	1.1	130 130, 260, 400	0.6 0.8

In general, when the low-fare customers spill, offering callable products is no longer a dominant strategy and may harm the revenues of the airlines. Numerical examples demonstrate that whether the two airlines offer callable products and whether offering callable products is beneficial to them mainly depends on the loads and the capacities of the two airlines. Especially, when the difference between the loads of the airlines is large, the loads of the airlines play the most important role. We obtain the following two insights: (1) if the ratio of high fare to low fare is low or the recall price is high, the airline with a lower load does not offer callable products in equilibrium; (2) the revenue of the airline with a higher (resp. lower) load is higher (resp. lower) in the case with callable products than that in the case without callable products. When the difference between the loads of the airlines is small, the capacities of the airlines play the most important role. If the difference between the capacities of the airlines is also small, the two airlines are worse off or better off by offering callable products simultaneously. If the difference between the capacities of the airlines is large, the airline with a lower capacity is always worse off by offering callable products while the one with a higher capacity may be worse off or better off which depends on the ratio of high fare to low fare and the recall price.

Furthermore, Proposition 6 implies that, when the two airlines are symmetric, the booking limits of the two airlines when they offer callable products are higher than those when they do not offer callable products. Numerical examples show that, the booking limits of the two airlines in the case with callable products are higher than those in the case without callable products. This result is intuitive. In the case with callable products, one airline may or may not offer callable products in equilibrium. If the airline offers callable products, it can recall some of the low-fare tickets if needed, so it will set a higher booking limit. If the airline does not offer callable products, its opponent will offer callable products and set a higher booking limit. To cope with the competition from its opponent, this airline will set a higher booking limit than that in the case without callable products.

**6. Conclusion**

This paper introduces the concept of callable products into the capacity allocation game between duopolistic airlines. We examine the impact of the introduction of callable products on the revenues and the booking limits of the two airlines. The analytical results demonstrate that, if there is no spill of low-fare customers, offering callable products is a dominant strategy of both airlines and provides Pareto gains to both airlines. If customers of both fare classes spill, numerical examples demonstrate that whether the two airlines offer callable products and whether offering callable products is beneficial to the two airlines mainly depends on the loads and the capacities of them. Moreover, numerical examples demonstrate that the booking limits of the two airlines in the case with callable products are always higher than those in the case without callable products.

There are some limitations in our study. First, to focus on airline competition with regard to booking limits, we take the prices as exogenously given, including the prices of low-fare tickets, the

prices of high-fare tickets and the recall prices. Letting the prices be endogenously determined is a direction for future research. Second, in our model, the low-fare tickets are either callable or non-callable. In reality, airlines can sell both regular low-fare tickets and callable tickets. One might explore the case in which the low-fare customers can choose between regular low-fare tickets and callable tickets. Third, we implicitly assume that the price of regular low-fare tickets and the price of callable tickets are the same. In fact, the prices should be different.

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**Appendix A. Proof of Proposition 1**

**Proof.** Different from the proof of the existence of Nash equilibrium in Netessine and Shumsky (2005), we apply Theorem 2.1 in Vives (2001) (as shown in Lemma 1) to obtain the existence of Nash equilibrium more directly.

We first show the existence of the pure-strategy Nash equilibrium. The strategy space of the game is  $[0, C_1] \times [0, C_2]$  which is a bounded closed convex set. In addition, according to Eq. (2),

$$\begin{aligned} \frac{\partial \pi_i}{\partial B_i} &= Pr(D_{Li}^T > B_i)(p_{Li} - p_i Pr(D_{Hi}^T > C_i - B_i | D_{Li}^T > B_i)) \\ &= Pr(D_{Li}^T > B_i)(p_{Li} - p_i Pr(D_{Hi}^T > C_i - B_i | D_{Li} \\ &> B_i - (D_{Li} - B_j)^+)). \end{aligned} \tag{A.1}$$

As  $D_{Li}$  and  $D_{Hi}^T$  are  $TP_2$ ,  $D_{Li}$  and  $D_{Hi}^T$  are right tail increasing (Proposition 2.3 of Joe, 1997), i.e.,  $Pr(D_{Hi}^T > C_i - B_i | D_{Li} > B_i - (D_{Li} - B_j)^+)$  is increasing in  $B_i$ . It is obvious that  $\partial \pi_i / \partial B_i$  is always positive or first positive and then negative. Therefore,  $\pi_i$  is quasi-concave in  $B_i$ . Apparently,  $\pi_i$  is continuous in both  $B_i$  and  $B_j$ . Referring to Lemma 1, there exists a pure-strategy Nash equilibrium for the game.

In the following, we show that best response functions of the airlines are decreasing. Similar to the proof of Proposition 2 in Netessine and Shumsky (2005), when the best response functions are differentiable, we use the Implicit Function Theorem (IFT) to show that they are non-increasing (Part I). However, Part I does not eliminate the possibility of jumps up. Part II demonstrates that the best response functions do not have jumps up. In Part III, we show the uniqueness of the Nash equilibrium.

Part I: By the IFT,

$$\frac{\partial B_i}{\partial B_j} = - \frac{\partial^2 \pi_i / \partial B_i \partial B_j}{\partial^2 \pi_i / \partial B_i^2} \tag{A.2}$$

At Airline  $i$ 's best response, the inequality  $\partial^2 \pi_i / \partial B_i^2 < 0$  holds. We will show that, when the first order conditions hold, the inequality  $\partial^2 \pi_i / \partial B_i \partial B_j < 0$  holds. Equivalently, we need to show that  $\partial \pi_i / \partial B_i$  monotonically decreases in  $B_j$ .

$$\frac{\partial \pi_i}{\partial B_i} = Pr(D_{Li}^T > B_i)(p_{Li} - p_i Pr(D_{Hi}^T > C_i - B_i | D_{Li}^T > B_i)). \tag{A.3}$$

From Proposition 2.3 of Joe (1997),  $TP_2$  implies that  $D_{Li}$  and  $D_{Hi}^T$  are right tail increasing, so  $Pr(D_{Hi}^T > C_i - B_i | D_{Li} > B_i - (D_{Li} - B_j)^+)$  is increasing in  $B_j$ . Thus the second item in (A.3) is de-

creasing in  $B_j$ , i.e.,  $p_{Li} - p_i Pr(D_{Hi}^T > C_i - B_i | D_{Li}^T > B_i)$  is decreasing in  $B_j$ . If  $B_i^*(B_j^*)$  is some point on player  $i$ 's best response function, then  $\partial \pi_i / \partial B_i |_{(B_i^*, B_j^*)} = 0$ . So  $p_{Li} - p_i Pr(D_{Hi}^T > C_i - B_i | D_{Li}^T > B_i)$  equals to zero at the point  $(B_i^*, B_j^*)$ . Denote  $\epsilon$  as a sufficiently small positive amount, then  $p_{Li} - p_i Pr(D_{Hi}^T > C_i - B_i | D_{Li}^T > B_i)$  is smaller than zero at the point  $(B_i(B_j^* + \epsilon), B_j^* + \epsilon)$ . Thus,  $\partial \pi_i / \partial B_i |_{(B_i(B_j^* + \epsilon), B_j^* + \epsilon)}$  is smaller than zero. Therefore,  $\partial \pi_i / \partial B_i$  decreases with  $B_j$ .

Part II: It needs to eliminate the possibility of jumps up in the best response functions. The proof is similar to Part II in the proof of Proposition 2 in Netessine and Shumsky (2005), so it is omitted.

Parts I and II together know that the best response functions are decreasing.

Part III: We show that the Nash equilibrium is unique. The proof is by contradiction. Suppose there are two different Nash equilibrium solutions  $(B_i^*, B_j^*)$  and  $(B_i^* + \delta_i, B_j^* - \delta_j)$ . As the best responses are decreasing,  $\delta_i$  and  $\delta_j$  are larger than zero or smaller than zero simultaneously. We only consider the case where they are larger than zero. According to the definition of Nash equilibrium and the first order condition for Airline  $i$  in Eq. (2), we have,

$$\begin{aligned} Pr(D_{Hi}^T > C_i - B_i | D_{Li} > B_i - (D_{Li} - B_j)^+) \\ &= Pr(D_{Hi}^T > C_i - B_i - \delta_i | D_{Li} > B_i + \delta_i - (D_{Li} - B_j + \delta_j)^+) \\ &= \frac{p_{Li}}{p_i}. \end{aligned} \tag{A.4}$$

The item after the first equal sign increases with  $\delta_i$  and decreases with  $\delta_j$ . In order to guarantee that the first equality holds,  $\delta_i$  should be smaller than  $\delta_j$ . Doing the same analysis to Airline  $j$ , we obtain that  $\delta_i > \delta_j$ , which makes a contradiction. Thus, there is a unique Nash equilibrium.  $\square$

**Appendix B. Proof of Proposition 3**

**Proof.** We only show that  $\pi_1^{00} \leq \pi_1^{10}$  as the analysis of other parts are similar. Referring to Eqs. (8) and (14), we obtain the following equation:

$$Pr(D_{H2}^T > C_2 - B_2^{10} | D_{L2} > B_2^{10}) = Pr(\tilde{D}_{H2}^T > C_2 - B_2^{00} | D_{L2} > B_2^{00}). \tag{B.1}$$

As  $\tilde{D}_{H2}^T > D_{H2}^T$ , the inequality  $B_2^{10} > B_2^{00}$  holds. Therefore, we have the following inequalities:

$$\begin{aligned} \pi_1^{00} &= \pi_1^{00}(B_1^{00}, B_2^{00}) \leq \pi_1^{00}(B_1^{00}, B_2^{10}) \leq \pi_1^{00}(B_1^{00}(B_2^{10}), B_2^{10}) \\ &\leq \pi_1^{10}(B_1^{00}(B_2^{10}), B_2^{10}) \leq \pi_1^{10}(B_1^{10}, B_2^{10}) = \pi_1^{10}. \end{aligned} \tag{B.2}$$

The first inequality holds as  $\partial \pi_1^{00} / \partial B_2 > 0$ . Note that an incremental increase in the booking limit of one airline results in more high-fare customers for the other airline but has no effect on the low-fare demand of the other airline.

The second inequality holds as in the first scenario  $B_1^{00}(B_2^{10})$  is the best response of Airline 1 to the booking limit  $B_2^{10}$  of Airline 2.

The third inequality holds as callable products can bring riskless revenue improvement.

The fourth inequality holds as in the second scenario  $B_1^{10}$  is the best response of Airline 1 to the booking limit  $B_2^{10}$  of Airline 2.  $\square$

**Appendix C. Proof of Proposition 4**

**Proof.** The objective function of the alliance is the sum of the two airlines' objective functions,  $\pi = \pi_i + \pi_j$ , and the centralized



optimality condition  $\partial(\pi_i + \pi_j)/\partial B_i = 0$  can be written as

$$\begin{aligned} & Pr(D_{Hi}^T > C_i - B_i^C | D_{Li} > B_i^C - (D_{Lj} - B_j^C)^+) \\ &= \frac{p_{Li}}{p_i} + \frac{\partial \pi_j}{\partial B_i} \frac{1}{p_i Pr(D_{Li} > B_i^C - (D_{Lj} - B_j^C)^+)}. \end{aligned} \quad (C.1)$$

Clearly,  $\partial \pi_j / \partial B_i$  is larger than zero as an incremental increase in the booking limit by one airline results in fewer low-fare customers for the other airline but has no effect on the high-fare demand of the other airline.

The decentralized optimality condition  $\partial \pi_i / \partial B_i = 0$  can be written as

$$Pr(D_{Hi}^T > C_i - B_i | D_{Li}^T > B_i) = \frac{p_{Li}}{p_i}. \quad (C.2)$$

Comparing Eqs. (C.2) and (C.1), we find that

$$\begin{aligned} & Pr(D_{Hi}^T > C_i - B_i^* | D_{Li} > B_i^* - (D_{Lj} - B_j^*)^+) > \\ & Pr(D_{Hi}^T > C_i - B_i^C | D_{Li} > B_i^C - (D_{Lj} - B_j^C)^+). \end{aligned} \quad (C.3)$$

According to Proposition 1, the Nash equilibrium exists, so in the symmetric case, there is a symmetric Nash equilibrium  $(B_1^* = B_2^*)$  which is also the unique one as Proposition 1 also demonstrates that the Nash equilibrium is unique. Therefore,  $B_i^* = B_j^* < B_i^C = B_j^C$  or  $B_i^* = B_j^* \geq B_i^C = B_j^C$ . As  $D_{Li}$  and  $D_{Hi}^T$  are  $TP_2$ , only the latter case holds.  $\square$

#### Appendix D. Proof of Proposition 5

**Proof.** The proofs of Part (1) and Part (2) are similar, we only present the proof of Part (1) here.

Denote the Nash equilibrium solution as  $(B_i^*, B_j^*)$  when the recall prices are  $p_i$  and  $p_j$ . If  $p_i$  increases to  $p_i'$ , denote the corresponding Nash equilibrium solution by  $(B_i', B_j')$ .  $(B_i^*, B_j^*)$  and  $(B_i', B_j')$  satisfy the optimality condition (C.2). So,

$$Pr(D_{Hi}^T > C_i - B_i^* | D_{Li} > B_i^* - (D_{Lj} - B_j^*)^+) > \quad (D.1)$$

$$Pr(D_{Hi}^T > C_i - B_i' | D_{Li} > B_i' - (D_{Lj} - B_j')^+),$$

$$Pr(D_{Hj}^T > C_j - B_j^* | D_{Lj} > B_j^* - (D_{Li} - B_i^*)^+) = \quad (D.2)$$

$$Pr(D_{Hj}^T > C_j - B_j' | D_{Lj} > B_j' - (D_{Li} - B_i')^+).$$

Now consider the following four cases:

- $B_i^* < B_i', B_j^* < B_j'$ . Given the  $TP_2$  assumption, the probability item in (D.1) is increasing in both  $B_i$  and  $B_j$ , so this case is impossible to happen.
- $B_i^* < B_i', B_j^* > B_j'$ . As the probability item in (D.1) is increasing in both  $B_i$  and  $B_j$ , the increment of  $B_i$  increases the probability item in (D.1) and the decrement of  $B_j$  decreases the probability item in (D.1). In order to guarantee Inequality (D.1) holds, the inequality  $|B_i^* - B_i'| < |B_j^* - B_j'|$  must hold. However, the analysis of Inequality (D.2) leads to an opposite requirement  $|B_i^* - B_i'| > |B_j^* - B_j'|$ , which is a contradiction.
- $B_i^* > B_i', B_j^* > B_j'$ . Given the  $TP_2$  assumption, the probability item in (D.2) is increasing in both  $B_i$  and  $B_j$ , so this case is impossible to happen.

The only remaining case is that  $B_i^* \geq B_i'$  and  $B_j^* \leq B_j'$ . Therefore,  $B_i^*$  decreases with  $p_i$  while  $B_j^*$  increases with  $p_i$ .  $\square$

#### Appendix E. Proof of Proposition 6

**Proof.** The proofs of Part (1) and Part (2) are similar, we only present the proof of Part (1).

The optimality condition (C.2) requires that, if  $(B_i, B_j)$  is an equilibrium solution, the following equation holds:

$$Pr(D_{Hi}^T > C_i - B_i | D_{Li} > B_i - (D_{Lj} - B_j)^+) = \frac{p_L}{p}. \quad (E.1)$$

According to Proposition 1, the Nash equilibrium exists. Thus, in the symmetric case, there must exist a symmetric Nash equilibrium  $(B_i^*, B_j^*)$  where  $B_i^* = B_j^*$ , which is also the unique one as Proposition 1 also demonstrates that the Nash equilibrium is unique. As  $p$  increases,  $B_i^*$  and  $B_j^*$  increase or decrease simultaneously. If they increase simultaneously as  $p$  increases, the left hand side of Eq. (E.1) increases while the right hand side of (E.1) decreases, which leads to a contradiction. Therefore,  $B_1^*$  and  $B_2^*$  both decrease with  $p$ .  $\square$

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