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Inventory management for dual sales channels with inventory-level-dependent demand

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This paper studies the inventory management problem of dual channels operated by one vendor. Demands of dual channels are inventory-level-dependent. We propose a multi-period stochastic dynamic programming model which shows that under mild conditions, the myopic inventory policy is optimal for the infinite horizon problem. To investigate the importance of capturing demand dependency on inventory levels, we consider a heuristic where the vendor ignores demand dependency on inventory levels, and compare the optimal inventory levels with those recommended by the heuristic. Through numerical examples, we show that the vendor may order less for dual channels than those recommended by the heuristic, and the difference between the inventory levels in the two cases can be so large that the demand dependency on inventory levels cannot be neglected. In the end, we numerically examine the impact of different ways to treat unmet demand and obtain some managerial insights.

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Introduction

Nowadays, customers can buy products through multiple channels, for example, physical stores, online stores, catalogue/mail order, interactive television and telesales. Although different channels have their own advantages and disadvantages, physical stores and online stores are the most common channels used by consumers. In the physical stores, customers can feel and touch products physically, but they incur travel costs to go to the stores. On the Internet, customers have easy access to online stores with convenience to obtain information about products, but they cannot immediately acquire the products. Therefore, different channels satisfy different preferences and needs of customers. While some customers prefer a particular channel, there are also customers who choose channels according to prices and service levels (eg, fill rates, sales efforts).

To satisfy the different needs of customers, many vendors establish online stores as a supplement to their physical stores, such as Gome and Suning, the two largest household appliances sales companies in China. On the other hand, some pure Internet retailers, such as iParty.com, a company that sells party supplies online (http://www.iparty.com), have expanded their businesses by opening physical stores. Multi-channel retailing can provide a better service to customers, thus leads to increased customer loyalty (Wallace *et al*, 2004). Meanwhile, it also faces some challenges in operations along with opportunities, as ‘a company’s activities in one channel influence a customer’s decision on whether and how to use another channel’ (Müller-Lankenau *et al*, 2004). Thus, the vendors should take into account the interactions between dual channels when integrating physical and online channels. In this paper, we study the inventory management problem of dual sales channels operated by one vendor.

In traditional literature on inventory management, researchers usually assume that demand is exogenous and is not influenced by inventory level. However, Wolfe (1968), Achabal *et al* (1990) and Koschat (2008) provide empirical evidence that displayed inventory increases demand. Balakrishnan *et al* (2004, 2008) suggest that inventory can stimulate demand for a variety of reasons, such as ‘increasing product visibility, kindling latent demand, signalling a popular product, or providing consumers an assurance of future availability’. Moreover, Dana and Petruzzi (2001) suggest that inventory can increase sales because consumers’ utilities increase when a product’s fill rate increases.

Although there are many papers on ‘inventory-level-dependent demand’ (ILDD) (eg, Datta and Pal, 1990; Balakrishnan *et al*, 2004), the inventory management problem of dual channels has not been considered. Therefore, in this paper, we study the inventory management problem of dual channels when the demands of dual sales depend on the inventory levels of both channels (we refer to this type of demand also as ILDD). We investigate how demand dependency on inventory levels (DDIL) affects the optimal inventory policy and the optimal inventory levels of dual channels.

In this paper, we first formulate a multi-period stochastic dynamic programming model and show that under mild

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conditions, the myopic policy is optimal for the infinite horizon problem. Then we investigate the sensitivity of the optimal inventory levels and service levels with respect to the different parameters in the demand functions. To analyse the importance of capturing DDIL, we consider a heuristic where the vendor ignores DDIL, and compare the optimal inventory levels with those recommended by the heuristic. Through numerical examples, we show that sometimes the vendor may order less for dual channels than those recommended by the heuristic, and the difference between the inventory levels in the two cases can be so large that the effect of the DDIL cannot be neglected. In the end, we numerically examine how different ways of treating unmet demand affect the optimal inventory levels and obtain some managerial insights. All proofs are presented in the Appendix.

Literature review

The most relevant literature is the research on inventory management with ILDD. A number of marketing papers have explained and provided empirical evidence showing that a store's sales depend on the amount of stock displayed on its shelves. For example, by taking a large sample, Curhan (1972) finds an average value of 0.212 for space elasticity (using a multiplicative power function). The work of Desmet and Renaudin (1998) indicates that the direct shelf-space elasticities are significant, varying between 0.15 and 0.8 for numerous product categories. Koschat (2008) presents empirical evidence in the magazine industry that demand can indeed vary with inventory, and further quantifies the magnitude of these inventory effects which are twofold. An inventory decrease for one brand can, first, result in a decrease of demand for the brand and, second, in an increase of demand for a competing brand. For a comprehensive review, Urban (2005a) reviews approximately 60 publications on inventory control models with ILDD and classifies the operations management literature in this area into two distinct streams—models in which the demand rate is a function of: (1) the initial inventory level in the cycle (the first type of model) and (2) the instantaneous inventory level (the second type of model).

Our paper is closely related to the first type of model. Gerchak and Wang (1994) consider a periodic-review inventory system without fixed cost where the demand in each period depends randomly, in a multiplicative form, on the starting inventory level. They investigate both the lost sales and backlogging models and show that the optimal policies are myopic. Balakrishnan *et al* (2008) first characterize the profit maximization policy for a single period model with a general inventory-dependent demand distribution and given price, and then extend it to the joint inventory-pricing decisions case with a multiplicative demand model. Chen *et al* (2012) extend the first part of Balakrishnan *et al* (2008) to a general multi-period setting. For cases with and without fixed order costs, Chen *et al* (2012) show that the optimal inventory replenishment policies are of the base-stock type and of the (s, S) type, respectively. Urban (2005b) considers a similar model, in which demands in each period are serially correlated and dependent on the initial net inventory level in a general additive form. He develops an adaptive, base stock policy and illustrates the performance of the policy through extensive numerical examples. Dana and Petruzzi (2001) consider a single-period newsvendor model where the uncertain demand depends on both price and inventory level. They assume that the expected utility maximizing consumers choose between visiting the firm and consuming an exogenous outside option. Given that the inventory level affects the availability of the product, it is shown that the firm holds more inventory, provides a higher fill rate, and earns higher profits when it internalizes the effect of its inventory on demand. In our study, we assume that the demands of dual channels are stochastic and are linear functions of the inventory levels. Compared with the demand forms in the above papers, our demand form seems too simple. However, all the above papers consider the inventory management problem of one channel, while we first investigate the inventory management of dual channels with ILDD. In addition, the explicit form of the demands in our model makes both the modelling and analysis of the periodic-review model much easier, and permits the derivation of explicit, implementable solutions, rather than merely characterization of the nature of optimal policies. Through numerical examples, we show that the vendor does not always order more inventories than those when the vendor ignores the effect of the DDIL, and sometimes he orders less for both channels, which leads to lower service levels. This result is obviously different from that in Dana and Petruzzi (2001). The second type of model assumes that the demand is a function of the instantaneous inventory level such that the demand rate changes as the inventory is depleted. Relevant papers include Baker and Urban (1988), Khmelnitsky and Gerchak (2002), Alfares (2007), etc.

Owing to the complexity of the models, most of the research has focused on the deterministic problem. The only models published to date that incorporate stochastic ILDD are the newsvendor and finite-horizon periodic-review models of Gerchak and Wang (1994) and Chen *et al* (2012), the continuous-time model of Benkherouf *et al* (2001), the newsvendor problem of Dana and Petruzzi (2001) and the newsvendor model with general demand of Balakrishnan *et al* (2008). Note that the demands in our models are also stochastic and are dependent on the inventory levels of dual channels.

Another relevant stream of research is the shelf-space allocation models, which state that as the shelf space increases, the sales also increase. However, most of the literature assumes that the shelf space allocated and the stock level are equal to each other. Martńez-de-Albéniz and Roels (2011) present an up-to-date review. Recently, Baron *et al* (2011) analyse the problem of joint shelf-space allocation and inventory decisions for multiple items competing over shelf space with demand that depends on both shelf space allocated and instantaneous inventory level.

The third stream of literature related to our study is the inventory management problems in the multi-channel system.

Seigfert *et al* (2006) develop inventory models for both a dedicated and an integrated supply chain. In an integrated supply chain, excess stock at retail stores could be used to fill some online orders. They also propose a contract to coordinate the supply chain. Wu and Chiang (2011) discuss a system in which a wholesaler sells fashion products through two channels with asymmetric sales horizons. The demands in the system are exogenous random variables. The wholesaler can improve profitability by employing joint procurement and inventory reallocation as a recourse action in response to the dynamics of sales. Liang *et al* (2011) compare site-to-store and store-to-site strategies for dual-channel integration. The site-to-store (or store-to-site) strategy can fill unmet orders in the physical channel (or online channel) from the inventory in the online channel (or physical channel). In these three papers, unmet demand in one channel can be satisfied by the remaining inventory of the other channel while in our paper there is no inventory transhipment between dual channels. Geng and Mallik (2007) consider the problem that a manufacturer distributes his product to the end-consumers both through the independent retailer and his own direct channel. If one channel is out of stock, a fraction of the unsatisfied customers visit the other channel, which induces inventory competition between the channels. Different from the competition form in Geng and Mallik (2007), dual channels in our model compete through inventory level, and unmet demands are either lost or backlogged and will not switch to the other channel. All the above papers consider the inventory management problem of one period and do not take into account demand dependency on inventory levels. We study the periodic-review inventory systems in which the demands of dual channels are dependent on the inventory levels of both channels.

Model development and optimal inventory policy

The inventory system considered in this paper relates to a vendor selling goods to customers through dual channels, a physical store and an online store. The system is operated for multiple periods. Customer demands at both stores are stochastic. Unmet demand in the physical store is lost, whereas that in the online store is backlogged. Here, customers backlogged will pay at the time they are satisfied. Selling prices in dual channels are exogenously determined. The vendor makes replenishment decisions at the beginning of each period, with zero lead time for replenishment. The replenishment decision of the vendor is to determine the inventory levels of dual channels. We assume that no set-up cost is incurred for replenishment and that there is no inventory transhipment between dual channels. The objective of the vendor is to maximize the total profit of the entire system. Using the subscript 1 to denote the physical store and 2 for the online store, we introduce the following notation ($i = 1, 2$):

r_i: selling price
h_i: unit inventory-holding cost
l_1: unit lost-sales penalty cost of the physical store
l_2: unit backorder penalty cost of the online store
c_i: unit variable order cost
γ: discount factor ($0<\gamma<1$)
M_i: capacity of the warehouse
T: number of periods during the decision horizon

Let, in a period, y_1 and y_2 be the inventory levels of dual channels after replenishment, respectively. Referring to the idea in the literature (eg, Gerchak and Wang, 1994; Urban, 2005b; Chen *et al*, 2012), assume that the demand in one channel increases with its inventory level. At the same time, it is also assumed that the demand in one channel decreases with the inventory level of the other channel, which is reasonable as dual channels will compete for the customers who choose channels according to the different prices and service levels (eg, fill rates, sales efforts) of dual channels. We adopt the following demand structure:

$$D_1(y_1, y_2) = a_1 y_1 - b_1 y_2 + \varepsilon_1, \tag{1}$$

$$D_2(y_1, y_2) = -a_2 y_1 + b_2 y_2 + \varepsilon_2. \tag{2}$$

In the above formulas, $\varepsilon_1(\in [v_1, u_1])$ and $\varepsilon_2(\in [v_2, u_2])$ are independent random variables with distribution functions $F_1(\cdot)$ and $F_2(\cdot)$, respectively, representing demands of loyal customers to individual channels. Other than the loyal customers, more demand is attracted to a particular channel with a linear relationship to its own inventory level (ie, $a_1 y_1$ and $b_2 y_2$), and also a fraction of the demand is drawn to the other channel with a linear relationship to the other inventory level (ie, $-b_1 y_2$ and $-a_2 y_1$). Hence, parameters a_1 and b_2 reflect the demand attraction effect of the inventories and parameters b_1 and a_2 reflect the competition effect of the inventories. The explicit and stochastic functional forms in Equations (1) and (2) make both the modelling and analysis of the periodic-review model much easier, and permit the derivation of explicit, implementable solutions, rather than merely characterization of the nature of optimal policies. To avoid trivial cases, suppose that a_1, a_2, b_1 and b_2 all fall into the interval $(0, 1)$, and that the inequalities $b_1 \leqslant a_1$, $a_2 \leqslant b_2$, $a_1 + b_1 \leqslant 1$ and $a_2 + b_2 \leqslant 1$ hold. Moreover, to guarantee non-negative $D_1(y_1, y_2)$ and $D_2(y_1, y_2)$, assume that b_1 is smaller than v_1/M_2 and a_2 is smaller than v_2/M_1.

Denote $x(t) = (x_1(t), x_2(t))$ as the state of period t, which corresponds to the initial inventories of dual channels in period t, and $y(t) = (y_1(t), y_2(t))$ as the inventory levels after replenishment. As the unmet demand in the physical store is lost, while in the online store it is backlogged, $x_1(t)$ is nonnegative whereas $x_2(t)$ can be either negative or nonnegative. As no set-up cost is incurred for replenishment, the vendor can replenish inventory in every period without more cost caused from frequent set-up. Thus, unmet demand must be satisfied in the next period to avoid more penalty costs. Therefore, under the optimal inventory policy, the online store should replenish the inventory up to a nonnegative level at the beginning of each period, that is, $y_2(t) \geqslant (x_2(t))^+$.

Finite horizon problem

Suppose the system is operated for T periods. In period t ($1 \leqslant t \leqslant T$), on observed state (x_1, x_2), if the vendor replenishes the inventories of dual channels to y_1 and y_2, respectively, the corresponding revenues of dual channels are:

$$r_1 E \min[y_1, D_1(y_1, y_2)], \tag{3}$$

and

$$\begin{aligned} & r_2 E(\min[y_2, D_2(y_1, y_2)] + \gamma [x_2(t+1)]^-) \\ & \quad = r_2 E[D_2(y_1, y_2)] - (1-\gamma) r_2 E[D_2(y_1, y_2) - y_2]^+, \end{aligned} \tag{4}$$

where z^+ is the positive part of z, and z^- is the negative part of z. The revenue of the physical store in Equation (3) is obvious. The revenue of the online store, as shown at the left-hand side of Equation (4), includes two parts: the revenue from the demand immediately satisfied in the current period and the revenue from the unmet demand that is satisfied in the next period. (Note that we discount the revenue from unmet demand received in the next period to the current period.)

The resultant profits of dual channels are as follows:

$$\begin{aligned} & r_1 E \min[y_1, D_1(y_1, y_2)] - h_1 E[y_1 - D_1(y_1, y_2)]^+ \\ & \quad - l_1 E[D_1(y_1, y_2) - y_1]^+ - c_1(y_1 - x_2), \end{aligned} \tag{5}$$

and

$$\begin{aligned} & r_2 E[D_2(y_1, y_2)] - (1-\gamma) r_2 E[D_2(y_1, y_2) - y_2]^+ \\ & \quad - h_2 E[y_2 - D_2(y_1, y_2)]^+ - l_2 E[D_1(y_1, y_2) - y_1]^+ \\ & \quad - c_2(y_2 - x_2). \end{aligned} \tag{6}$$

In the next period, the state transfers to $([y_1 - D_1(y_1, y_2)]^+, y_2 - D_2(y_1, y_2))$. We denote

$$\begin{aligned} C(y_1, y_2) = & -c_1 y_1 - c_2 y_2 + r_1 E \min[y_1, D_1(y_1, y_2)] \\ & - h_1 E[y_1 - D_1(y_1, y_2)]^+ - l_1 E[D_1(y_1, y_2) - y_1]^+ \\ & + r_2 E[D_2(y_1, y_2)] - ((1-\gamma) r_2 + l_2) E[D_2(y_1, y_2) - y_2]^+ \\ & - h_2 E[y_2 - D_2(y_1, y_2)]^+. \end{aligned}$$

Denote $V_t(x_1, x_2)$ as the expected profit-onward from period t by the optimal inventory policy for a given state (x_1, x_2) at period t. Assume

$$V_{T+1}(x_1, x_2) = c_1 x_1 + c_2 x_2, \tag{7}$$

for the end period.

The optimality equations are expressed as follows:

$$V_t(x_1, x_2) = c_1 x_1 + c_2 x_2 + \max_{x_1 \leqslant y_1 \leqslant M_1, x_2^+ \leqslant y_2 \leqslant M_2} \{H_t(y_1, y_2)\}, \quad 1 \leqslant t \leqslant T, \tag{8}$$

$$\begin{aligned} H_t(y_1, y_2) = & C(y_1, y_2) + \gamma E V_{t+1}([y_1 - D_1(y_1, y_2)]^+, \\ & y_2 - D_2(y_1, y_2)), \quad 1 \leqslant t \leqslant T. \end{aligned} \tag{9}$$

Lemma 1 (Topkis, 1968) *If $f(u, \omega)$ is a convex real-valued function on the convex set A, then $g(u) = \inf_{\omega:(u,\omega) \in A} f(u, \omega)$ is convex on $\{u$: there exists $(u, \omega) \in A\}$.*

Theorem 1 (1) *$H_t(y_1, y_2)$ is concave in y_1 and y_2 for all $1 \leqslant t \leqslant T$.*
(2) *$V_t(x_1, x_2)$ is concave and decreasing in x_1 and x_2 for all $1 \leqslant t \leqslant T+1$.*

As the replenishment decisions of dual channels are correlated, there is no simple structure of the optimal inventory policy in the finite horizon problem. Nevertheless, Theorem 1 can be useful in calculating the optimal policy.

To facilitate discussion of the infinite horizon problem, we rewrite optimality equations (7) to (9). By referring to the method in chapter 9.4.2 of Zipkin (2000), we define

$$V_t^+(x_1, x_2) = -c_1 x_1 - c_2 x_2 + V_t(x_1, x_2), \quad 1 \leqslant t \leqslant T+1, \tag{10}$$

and

$$\begin{aligned} C^+(y_1, y_2) = & -(r_1 + l_1) E[D_1(y_1, y_2) - y_1]^+ \\ & - (h_1 - \gamma c_1) E[y_1 - D_1(y_1, y_2)]^+ + r_1 E[D_1(y_1, y_2)] \\ & - ((1-\gamma) r_2 + l_2) E[D_2(y_1, y_2) - y_2]^+ \\ & - h_2 E[y_2 - D_2(y_1, y_2)]^+ + (r_2 - \gamma c_2) E[D_2(y_1, y_2)] \\ & - c_1 y_1 - c_2(1-\gamma) y_2. \end{aligned}$$

The optimality equations (7)-(9) are transformed to the following recursion:

$$V_{T+1}^+(x_1, x_2) = 0, \tag{11}$$

$$\begin{aligned} H_t(y_1, y_2) = & C^+(y_1, y_2) + \gamma E\{V_{t+1}^+([y_1 - D_1(y_1, y_2)]^+, \\ & y_2 - D_2(y_1, y_2))\}, \quad 1 \leqslant t \leqslant T, \end{aligned} \tag{12}$$

$$\begin{aligned} V_t^+(x_1, x_2) = \max\{ & H_t(y_1, y_2) : x_1 \leqslant y_1 \leqslant M_1, \\ & x_2^+ \leqslant y_2 \leqslant M_2\}, \quad 1 \leqslant t \leqslant T. \end{aligned} \tag{13}$$

$C^+(y_1, y_2)$ in Equation (12) is the single period system profit after transformation and is concave in y_1 and y_2, as shown in Lemma 2.

Lemma 2 *$C^+(y_1, y_2)$ is a concave function in y_1 and y_2.*

Infinite horizon problem

For the infinite horizon system, we consider a stationary inventory policy, denoted as (π_1, π_2). Here, π_1 specifies the replenishment rule of the physical store, by which the order-up-to level is determined for a given state, and π_2 has a similar meaning. In this subsection, we continue to use notation $C(y_1, y_2)$ and $C^+(y_1, y_2)$ previously defined.

Let $V(x_1, x_2 | \pi_1, \pi_2)$ denote the total expected discounted profit of dual channels over the infinite horizon for a given initial state

$(x_1(1)=x_1, x_2(1)=x_2)$ when using policy (π_1, π_2). Then,

$$V(x_1,x_2 \mid \pi_1,\pi_2) = E\Bigg\{\sum_{t=1}^{\infty}\gamma^t[c_1x_1(t)+c_2x_2(t) + C(y_1(t),y_2(t))] \mid x_1(1) = x_1, x_2(1) = x_2, x_1(t)\leqslant y_1(t) \leqslant M_1, (x_2(t))^+ \leqslant y_2(t)\leqslant M_2\Bigg\}. \quad (14)$$

Substituting $x_1(t+1)=[y_1-D_1(y_1,y_2)]^+$ and $x_2(t+1)=y_2-D_2(y_1,y_2)$ into (14), we obtain

$$V(x_1,x_2 \mid \pi_1,\pi_2) = c_1x_1+c_2x_2 + E\left[\sum_{t=1}^{\infty}\gamma^t C^+(y_1(t),y_2(t)) \mid x_1(t)\leqslant y_1(t) \leqslant M_1, (x_2(t))^+ \leqslant y_2(t)\leqslant M_2\right].$$

Lemma 3 *If* $1-a_1-b_2+a_1b_2-a_2b_1\neq 0$ *and* $F_1(\cdot)$, $F_2(\cdot)$ *are both strictly increasing distribution functions, the maximizer of* $C^+(y_1, y_2)$ *in domain* $\Omega=\{0\leqslant y_1\leqslant M_1, 0\leqslant y_2\leqslant M_2\}$ *is unique.*

Suppose that $1-a_1-b_2+a_1b_2-a_2b_1\neq 0$ and that $F_1(\cdot)$, $F_2(\cdot)$ are both strictly increasing distribution functions (under these conditions, the optimal inventory policy will have a simple structure). Let $y^*=(y_1^*, y_2^*)$ be the maximizer of $C^+(y_1, y_2)$ in domain Ω.

Define the following functions:

$$z_1(x) = \arg\max_{0\leqslant y_1\leqslant M_1} C^+(y_1,x), \text{ for } y_2^*< x \leqslant M_2,$$

$$z_2(x) = \arg\max_{0\leqslant y_2\leqslant M_2} C^+(x,y_2), \text{ for } y_1^*< x \leqslant M_1.$$

Lemma 4 *If* $1-a_1-b_2+a_1b_2-a_2b_1\neq 0$ *and* $F_1(\cdot)$, $F_2(\cdot)$ *are both strictly increasing distribution functions, then* $z_1(x_2)$ *and* $z_2(x_1)$ *are non-increasing functions.*

Denote the following regions:

$$\Phi = \{(x_1,x_2) : 0\leqslant x_1\leqslant M_1, x_2\leqslant M_2\},$$

$$\Phi_1 = \{(x_1,x_2) : x\leqslant y^*\}\cap\Phi,$$

$$\Phi_2 = \{(x_1,x_2) : y_1^*<x_1\leqslant M_1 \text{ and } x_2<z_2(x_1)\}\cap\Phi,$$

$$\Phi_3 = \{(x_1,x_2) : x_1<z_1(x_2) \text{ and } y_2^*<x_2\leqslant M_2\}\cap\Phi,$$

$$\Phi_4 = \Phi-\Phi_1-\Phi_2-\Phi_3.$$

Define the following inventory policy:

$$\bar{y}(x_1,x_2) = \begin{cases} y^*, & (x_1,x_2)\in\Phi_1,\\ (x_1,z_2(x_1)), & (x_1,x_2)\in\Phi_2,\\ (z_1(x_2),x_2), & (x_1,x_2)\in\Phi_3,\\ (x_1,x_2), & (x_1,x_2)\in\Phi_4.\end{cases}$$

Theorem 2 gives conditions under which the myopic inventory policy (maximizing the profit in the current period) is optimal for the infinite horizon problem.

Theorem 2 *If* $1-a_1-b_2+a_1b_2-a_2b_1\neq 0$ *and* $F_1(\cdot)$, $F_2(\cdot)$ *are both strictly increasing distribution functions, then* $\bar{y}(x_1,x_2)$ *is the myopic policy and is optimal for the infinite horizon problem.*

The rest of the paper focuses on the infinite horizon problem.

Properties

To obtain analytical formulas of the optimal inventory levels of dual channels, assume the initial inventories of dual channels in the first period to be both zero hereafter. The following corollary is straightforward from Theorem 2.

Corollary 1 *Suppose that* $1-a_1-b_2+a_1b_2-a_2b_1\neq 0$ *and that* $F_1(\cdot)$, $F_2(\cdot)$ *are both strictly increasing distribution functions. If initial inventories of dual channels in the first period are both zero, the optimal inventory policy is to order inventory up to* y^* *in every period.*

In the following, we assume that $1-a_1-b_2+a_1b_2-a_2b_1\neq 0$ and that $F_1(\cdot)$, $F_2(\cdot)$ are both strictly increasing distribution functions. The infinite horizon problem can be transformed to a single-period problem as follows:

$$\begin{aligned} \text{maximize} \quad & C^+(y_1,y_2)\\ \text{s.t.} \quad & 0\leqslant y_1\leqslant M_1\\ & 0\leqslant y_2\leqslant M_2. \end{aligned} \quad (15)$$

Let (y_1', y_2') denote one of the maximizers of $C^+(y_1, y_2)$. As $C^+(y_1, y_2)$ is concave, $\partial C^+(y_1, y_2)/\partial y_1$ and $\partial C^+(y_1, y_2)/\partial y_2$ are both equal to zero at the point (y_1', y_2'). Thus, we solve $F_1((1-a_1)y_1'+b_1y_2')$ and $F_2(a_2y_1'+(1-b_2)y_2')$:

$$F_1((1-a_1)y_1'+b_1y_2') = \frac{l_1}{k_1}+\frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1}, \quad (16)$$

$$F_2(a_2y_1'+(1-b_2)y_2') = \frac{l_2-\gamma(r_2-c_2)}{k_2} + \frac{(r_2-c_2)(1-a_1)-(r_1-c_1)b_1}{[(1-a_1)(1-b_2)-a_2b_1]k_2}, \quad (17)$$

where $k_1=r_1+l_1+h_1-\gamma c_1$ and $k_2=(1-\gamma)r_2+l_2+h_2$. Denote $A=(r_1-c_1)(1-b_2)-(r_2-c_2)a_2$ and $B=(r_2-c_2)(1-a_1)-(r_1-c_1)b_1$.

In practice, $1-b_2$ should be much larger than a_2, and $1-a_1$ should be much larger than b_1. Therefore, provided that the profit margin of the physical store (ie, r_1-c_1) is not much different from that of the online store (ie, r_2-c_2), A and B are larger than zero. Thus, we focus our attention on the cases where $A \geqslant 0$ and $B \geqslant 0$. Then, the right-hand sides of Equations (16) and (17) are larger than 0. On the other hand, if the right-hand side of Equations (16) or (17) is larger than 1, y_1' or y_2' will be infinite, resulting in trivial cases. Therefore, we only consider the cases in which the right-hand sides of Equations (16) and (17) are between 0 and 1. From Equations (16) and (17), we obtain

$$y_1' = \frac{\left[\begin{array}{l}(1-b_2)F_1^{-1}\left(\dfrac{l_1}{k_1}+\dfrac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1}\right)\\ \quad -b_1F_2^{-1}\left(\dfrac{l_2-\gamma(r_2-c_2)}{k_2}+\dfrac{(r_2-c_2)(1-a_1)-(r_1-c_1)b_1}{[(1-a_1)(1-b_2)-a_2b_1]k_2}\right)\end{array}\right]}{(1-a_1)(1-b_2)-a_2b_1}, \tag{18}$$

$$y_2' = \frac{\left[\begin{array}{l}(1-a_1)F_2^{-1}\left(\dfrac{l_2-\gamma(r_2-c_2)}{k_2}+\dfrac{(r_2-c_2)(1-a_1)-(r_1-c_1)b_1}{[(1-a_1)(1-b_2)-a_2b_1]k_2}\right)\\ \quad -a_2F_1^{-1}\left(\dfrac{l_1}{k_1}+\dfrac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1}\right)\end{array}\right]}{(1-a_1)(1-b_2)-a_2b_1}. \tag{19}$$

Note that $1-b_2$ is much larger than b_1, and $1-a_1$ is much larger than a_2. Therefore, provided that the cost factors (ie, r_1, c_1, l_1 and h_1) and scale of the loyal customers (ie, u_1 and v_1) of the physical store are not much different from those of the online store, y_1' and y_2' are nonnegative. Hereafter, we focus on cases where (y_1', y_2') satisfies constraints $0 \leqslant y_1' \leqslant M_1$ and $0 \leqslant y_2' \leqslant M_2$. Then, $y_1^* = y_1'$, $y_2^* = y_2'$.

Sensitivity analysis

In this subsection, we conduct sensitivity analysis to the optimal inventory levels and optimal service levels with respect to the parameters reflecting the demand attraction effect of the inventories (ie, a_1, b_2). The sensitivity with respect to the parameters reflecting the competition effect of the inventories (ie, b_1, a_2) is studied through numerical examples in the next section.

Proposition 1 (1) *y_1^* is increasing in a_1, and y_2^* is decreasing in a_1.*
(2) *y_1^* is decreasing in b_2, and y_2^* is increasing in b_2.*

The proposition is intuitive. As the increment of a_1 indicates that the demand attraction effect of the inventory in the physical store becomes stronger, we should raise the inventory level of the physical store. To reduce the competition from the online store, we should lower the inventory level of the online store. The interpretation of Part (2) is similar.

The optimal service levels (here, the service level denotes the probability of an arbitrarily arriving customer being served from stock on hand, ie, Prob{number of demands in a period $\leqslant$ inventory-on-hand at the beginning of the period}) of dual channels are as follows:

$$s_1^* = \frac{l_1}{k_1} + \frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1},$$

$$s_2^* = \frac{l_2-\gamma(r_2-c_2)}{k_2} - \frac{(r_1-c_1)b_1-(r_2-c_2)(1-a_1)}{[(1-a_1)(1-b_2)-a_2b_1]k_2}.$$

Proposition 2 (1) *s_1^* increases in a_1 and b_1, while s_2^* decreases in a_1 and b_1.*
(2) *s_1^* decreases in a_2 and b_2, while s_2^* increases in a_2 and b_2.*

We only interpret the former half of Part (1) as the interpretations of other parts are similar. It is known from Proposition 1 that the optimal inventory level of the physical store increases in a_1. From Equation (1), the demand of the physical store also increases with a_1. 's_1^* increases in a_1' in Part (1) indicates that for the physical store, as the demand attraction effect of the inventory in the physical store becomes stronger, the increment rate of the optimal inventory level should be higher than that of the demand, leading to a higher service level. From Equation (2), the demand of the physical store decreases in b_1. If the optimal inventory level of the physical store increases in b_1, the optimal service level of the physical store will increase in b_1; if the optimal inventory level of the physical store decreases in b_1, 's_1^* increases in b_1' in Part (1) implies that for the physical store, as the competition effect of the inventory in the online store becomes stronger, the decrement rate of the optimal inventory level should be lower than that of the demand, leading to a higher service level.

Ignoring demand dependency on inventory levels

To analyse the importance of capturing demand dependency on inventory levels, we consider a heuristic as follows: the vendor ignores demand dependency on inventory levels and regards demands in dual channels as ε_1 and ε_2, respectively, to make replenishment decisions. The inventory problem of the system is simplified to two independent newsvendor problems. The inventory levels recommended by the heuristic are $y_1'' = F_1^{-1}((r_1-c_1+l_1)/k_1)$ and $y_2'' = F_2^{-1}(((1-\gamma)(r_2-c_2)+l_2)/k_2)$, respectively, with corresponding service levels denoted as s_1'' $(s_1'' = P(y_1'' > D_1(y_1'', y_2'')))$ and s_2'' $(s_1'' = P(y_2'' > D_2(y_1'', y_2'')))$. The following proposition compares y_1'' and y_2'' with the optimal inventory levels of dual channels (ie, y_1^* and y_2^*).

Denote

$$C = (r_1-c_1)[(1-b_2)a_1+a_2b_1]-(r_2-c_2)a_2,$$

$$D = (r_2-c_2)[(1-a_1)b_2+a_2b_1]-(r_1-c_1)b_1,$$

$$e_1 = a_1-a_2-(a_1b_2-a_2b_1),$$

$$e_2 = b_2-b_1-(a_1b_2-a_2b_1).$$

Proposition 3 (1) *When* $C \geqslant 0$ *and* $D \geqslant 0$, *it follows that* $y_2^* \geqslant y_2''$ *if* $y_1^* \leqslant y_1''$, *and* $y_1^* \geqslant y_1''$ *if* $y_2^* \leqslant y_2''$.
(2) *When* $C \geqslant 0$, $D \geqslant 0$, $e_1 \geqslant 0$ *and* $e_2 \geqslant 0$, *it follows that* $y_1^* + y_2^* \geqslant y_1'' + y_2''$.

To understand intuitively the meanings of C and D, we interpret Equations (16) and (17) in a way similar to that in the newsvendor problem. Equation (16) can be written as:

$$F_1\left(y_1^* - \left(a_1 y_1^* - b_1 y_2^*\right)\right) = \frac{l_1}{k_1} + \frac{(r_1 - c_1)(1 - b_2) - (r_2 - c_2)a_2}{((1 - a_1)(1 - b_2) - a_2 b_1)k_1}$$

$$= \frac{r_1 - c_1 + l_1 + \dfrac{C}{(1 - a_1)(1 - b_2) - a_2 b_1}}{k_1}.$$

Demand in the physical store is composed of two parts: the deterministic part $a_1 y_1^* - b_1 y_2^*$ and the stochastic part ε_1. Therefore, $y_1^* - (a_1 y_1^* - b_1 y_2^*)$ can be viewed as the inventory used to satisfy the stochastic demand from loyal customers. $k_1 = r_1 + l_1 + h_1 - \gamma c_1$ is the total cost caused by either overstock or understock of the physical store. $r_1 - c_1 + l_1 + C/[(1 - a_1)(1 - b_2) - a_2 b_1]$ can be interpreted as the cost caused by understock of the physical store, which includes two parts where $r_1 - c_1 + l_1$ is the direct cost caused by understock and $C/[(1 - a_1)(1 - b_2) - a_2 b_1]$ is the indirect cost to the whole system due to the effect of the DDIL. Thus, $C \geqslant 0$ indicates that the understock of the physical store is detrimental to the whole system, which implies that raising the inventory level of the physical store solely is beneficial to the system. D has a similar meaning to that of C. Part (1) shows that if raising the inventory level of an individual channel solely is beneficial to the system, the result that one channel orders less than that recommended by the heuristic can only appear under the consideration of reducing its competition with the other channel. Thus, the other channel should order more than that recommended by the heuristic.

The conditions $e_1 \geqslant 0$ and $e_2 \geqslant 0$ can be interpreted as that the demand attraction effect of the inventories is more significant than the competition effect. Part (2) of Proposition 3 indicates that when the demand attraction effect of the inventories is more significant than the competition effect, and increasing the inventory levels of an individual channel solely is beneficial to the system, the optimal total inventory level of the system should be higher than that recommended by the heuristic.

Numerical analysis

There are a few questions we wish to investigate through numerical analysis. First, how do the optimal inventory levels of dual channels change with the parameters in the demand functions that reflect the competition effect of the inventories? Second, is there any importance in capturing demand dependency on inventory levels? Third, what is the impact of different ways to treat unmet demand? In the next subsections, we describe the numerical experiments designed to answer these questions and their results. For all the numerical examples, we let $\gamma = 0.9$, $M_1 = 1000$, $M_2 = 1000$.

Sensitivity of optimal inventory levels to different parameters

We examine the sensitivity of the optimal inventory levels of dual channels (ie, y_1^* and y_2^*) with respect to the parameters reflecting the demand attraction effect of the inventories (ie, a_1 and b_2) in Proposition 1. In this subsection, we investigate the sensitivity of the optimal inventory levels with respect to the parameters reflecting the competition effect of the inventories (ie, b_1 and a_2). As the results of b_1 and a_2 are similar, we only present the results of b_1 herein.

We first run a numerical example for a given set of values of the parameters. The values of the parameters are as follows: $a_1 = 0.2$, $a_2 = 0.05$, $b_2 = 0.1$, $r_1 = 36$, $r_2 = 32$, $c_1 = 20$, $c_2 = 16$, $h_1 = 2$, $h_2 = 1.5$, $l_1 = 36$ and $l_2 = 24$. ε_1 is uniformly distributed in the interval [150, 500], and ε_2 is uniformly distributed in the interval [150, 400]. b_1 varies from 0.01 to 0.1. $a_1 > b_2$ states that the demand attraction effect of the inventory in the physical store is more significant than that of the inventory in the online store. This is due to the fact that the inventory of the physical store is visible to the customers while the inventory of the online store is usually invisible. $a_1 > b_1$ and $b_2 > a_2$ indicate that the influence of the other channel's inventory is weaker than that of the channel's inventory. ε_1 being stochastically larger than ε_2 indicates that the physical store has a stochastically higher demand (which is not required for the result, ie, the results in this subsection also hold if ε_2 is stochastically larger than ε_1). Table 1 presents the optimal inventory levels of dual channels. We observe that y_1^* and y_2^* both decrease in b_1. As the competition between dual channels intensifies as b_1 increases, both channels lower the inventory levels to alleviate competition.

To explore whether the decreasing trends of y_1^* and y_2^* with respect to b_1 still hold in more general situations, we choose the values of the parameters to be random numbers uniformly distributed in the intervals given in Table 2. Note that the inequalities $a_1 > b_2$, $a_1 > b_1$ and $b_2 > a_2$ still hold and that the retail store has a stochastic higher demand than the online store. We run 10 000 numerical examples, and restrict our analysis to 9542 of them which satisfy the conditions $\{A \geqslant 0, B \geqslant 0\}$ and guarantee the formulas (18) and (19) to be meaningful. To eliminate the impact of varying values of the parameters, we

Table 1 Optimal inventory levels of dual channels under different values of b_1

b_1	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
y_1^*	353.74	351.64	349.60	347.62	345.69	343.82	342.00	340.24	338.54	336.90
y_2^*	178.34	176.11	173.88	171.64	169.39	167.14	164.88	162.62	160.35	158.07

Table 2 The value intervals of the parameters

parameters	a_i	b_i	r_i	c_i	h_i	l_i	v_i	u_i
$i=1$	[0.1, 0.2]	[0.01, 0.1]	[30, 36]	[16, 20]	[1.5, 2]	[30, 36]	[100, 150]	[400, 500]
$i=2$	[0.01, 0.05]	[0.05, 0.1]	[26, 32]	[12, 16]	[1, 1.5]	[18, 24]	[100, 150]	[300, 400]

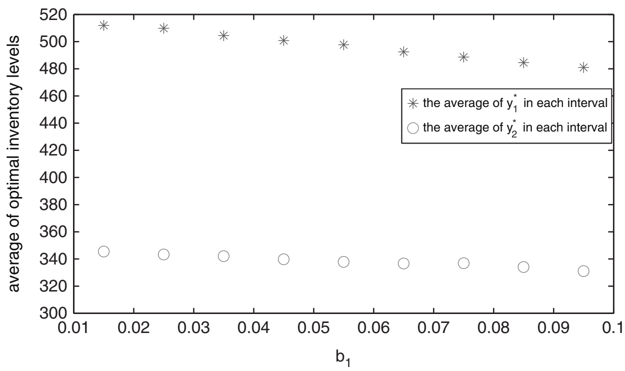

Figure 1 The average of the optimal inventory levels of dual channels in the nine intervals.

divide the values of b_1 into nine intervals $[0.01i, 0.01(i+1)]$ $(i=1, \ldots, 9)$ and average the optimal inventory level of each channel in each interval respectively. Figure 1 plots the average of y_i* $(i=1, 2)$ in the nine intervals, and it is observed that as b_1 increases, y_1^* and y_2^* exhibit an overall trend of decreasing, which is consistent with the results presented in Table 1.

Impact of demand dependency on inventory levels

This subsection analyses the importance of capturing demand dependency on inventory levels. We run numerical examples to compare the optimal inventory levels and the optimal service levels with those recommended by the heuristic. To make the results more general, we consider the following four parameter settings. In each setting, the parameters are random numbers uniformly distributed in the given intervals,

the first setting	all the parameter intervals are the same as those in Table 2
the second setting	$r_2 \sim [30, 36]$, other parameters intervals are the same as those in Table 2
the third setting	$u_2 \sim [600, 700]$, other parameters intervals are the same as those in Table 2
the fourth setting	$r_2 \sim [30, 36]$ and $u_2 \sim [600, 700]$, other parameters intervals are the same as those in Table 2

In the second setting, the retail price in the online store is stochastically equal to that in the physical store. In the third setting, the demand of the online store is stochastically higher than that of the physical store. In the fourth setting, the retail prices in the two stores are stochastically equal to each other, and the demand of the online store is stochastically higher.

We run 10 000 numerical examples for each setting. Let m be the total number of numerical examples that satisfy the conditions $\{A \geqslant 0, B \geqslant 0\}$ and guarantee the formulas of y_1^* and y_2^* (Equations (18) and (19)) to be meaningful. We restrict our analysis to the m examples. Denote k_1, k_2, k_3, k_4 as the number of the numerical examples that satisfy the conditions $\{y_1^* \geqslant y_1'', y_2^* \geqslant y_2''\}$, $\{y_1^* > y_1'', y_2^* < y_2''\}$, $\{y_1^* < y_1'', y_2^* > y_2''\}$ and $\{y_1^* < y_1'', y_2^* < y_2''\}$ respectively, and m_1, m_2, m_3, m_4 as the number of the numerical examples that satisfy the conditions $\{s_1^* \geqslant s_1'', s_2^* \geqslant s_2''\}$, $\{s_1^* > s_1'', s_2^* < s_2''\}$, $\{s_1^* < s_1'', s_2^* > s_2''\}$ and $\{s_1^* < s_1'', s_2^* < s_2''\}$, respectively. In addition, we introduce the following notation:

$\rho_1^{\max}$ (resp. $\rho_1^{\min}$)	the maximum (resp. minimum) value of $(y_1^* - y_1'')/y_1^*$
$\rho_2^{\max}$ (resp. $\rho_2^{\min}$)	the maximum (resp. minimum) value of $(y_2^* - y_2'')/y_2^*$
$\rho_3^{\max}$ (resp. $\rho_3^{\min}$)	the maximum (resp. minimum) value of $(y_1^* + y_2^* - y_1'' - y_2'')/(y_1^* + y_2^*)$
$\delta_1^{\max}$ (resp. $\delta_1^{\min}$)	the maximum (resp. minimum) value of $s_1^* - s_1''$
$\delta_2^{\max}$ (resp. $\delta_2^{\min}$)	the maximum (resp. minimum) value of $s_2^* - s_2''$

Table 3 presents the results of the numerical examples. We have the following two main observations. First, k_4 (resp. m_4) is larger than zero in the third and fourth settings, which means that the optimal inventory (resp. service) levels of dual channels may be less (resp. lower) than those recommended by the heuristic. That is, the vendor does not always order more inventories than those when the vendor ignores the effect of the DDIL, and sometimes he orders less for both channels, which leads to lower service levels. This result is obviously different from that in Dana and Petruzzi (2001) which shows that the vendor holds more inventory, provides a higher fill rate, and earns higher profits when it internalizes the effect of its inventory on demand. Note that the vendor sells products only through one channel in Dana and Petruzzi (2001), while there are dual channels in our model. Thus, the vendor in our paper may order less for both channels to mitigate competition between the two channels. Compared with m, k_4 (resp.m_4) is small in all the four settings, which indicates that in most of the cases the optimal inventory (resp. service) level of at least one channel is higher than that recommended by the heuristic. Second, the differences between the optimal inventory levels and those recommended by the heuristic can be significantly large. For example, for the physical store, it can reach up to 24%; for the online store, it can be as large as 28%; and for the total inventory, it can reach approximately 18%. Furthermore, there are also considerable differences between the optimal service levels and those recommended by the heuristic. For larger a_i or b_i $(i=1,2)$, numerical examples can show that the differences can be even larger. Therefore, the effect of the DDIL cannot be neglected.

Impact of different ways to treat unmet demand

In the previous sections, we assume that unmet demand in the physical store is lost, while in the online store it is backlogged (called the first case). We now investigate how different ways of treating unmet demand affect the optimal inventory levels of dual channels. We consider two other cases. One is that unmet demands in both channels are lost (the second case, also called the lost-sales case) and the other one is that unmet demands are backlogged in both channels (the third case, also called the backorder case). Assuming that the optimal inventory levels corresponding to these two cases are (y_1^{**}, y_2^{**}) and (y_1^{***}, y_2^{***}), respectively, and using the same method as that for the first case, it is easy to obtain the formulas of y_1^{**} and y_1^{***} $(i=1,2)$, which are omitted here.

Table 3 Comparison between the optimal inventory (service) levels and those recommended by the heuristic

Four settings	*First*	*Second*	*Third*	*Fourth*
M	9553	9686	9519	9681
k_1	8135	8793	8983	9250
k_2	1418	893	242	83
k_3	0	0	273	345
k_4	0	0	21	3
m_1	8672	9176	9129	9399
m_2	881	510	201	64
m_3	0	0	168	215
m_4	0	0	21	3
ρ_1^{max}	0.2392	0.2336	0.2236	0.2243
ρ_2^{max}	0.1289	0.1547	0.2602	0.2803
ρ_3^{max}	0.1824	0.1823	0.1726	0.1753
ρ_1^{min}	0.0234	0.0176	−0.0538	−0.0663
ρ_2^{min}	−0.0448	−0.0435	−0.0429	−0.0332
ρ_3^{min}	0.0099	0.0158	−0.0192	−0.0087
δ_1^{max}	0.3372	0.3392	0.3224	0.3252
δ_2^{max}	0.1924	0.2205	0.1762	0.1902
δ_1^{min}	0.0390	0.0286	−0.0457	−0.0559
δ_2^{min}	−0.0824	−0.0714	−0.0325	−0.0246

Applying the same parameter values as those in the four settings of the previous subsection, we run 10 000 numerical examples for each setting to compare the optimal inventory levels in the three cases. In the numerical examples, the unit backorder penalty cost of the physical store and the unit lost-sales penalty cost of the online store are assumed to be random numbers both uniformly distributed in the interval [24, 30]. Denote n as the total number of the numerical examples that satisfy the conditions $\{A \geqslant 0, B \geqslant 0\}$ and guarantee the formulas of the optimal inventory levels in the three cases (for the first case, they are Equations (18) and (19)) to be meaningful. We restrict our analysis to the n examples. Introduce the following notation:

ϕ_1^{max} (resp. ϕ_1^{min}) the maximum (resp. minimum) value of $(y_1^* - y_1^{**})/y_1^*$

ϕ_2^{max} (resp. ϕ_2^{min}) the maximum (resp. minimum) value of $(y_1^* - y_1^{***})/y_1^*$

ϕ_3^{max} (resp. ϕ_3^{min}) the maximum (resp. minimum) value of $(y_2^{**} - y_2^*)/y_2^*$

ϕ_4^{max} (resp. ϕ_4^{min}) the maximum (resp. minimum) value of $(y_2^{***} - y_2^*)/y_2^*$

Table 4 presents the results of the numerical examples. Notice that ϕ_i^{max} and ϕ_i^{min} $(i=1,2,3,4)$ are all non-negative. Thus, the inequalities $y_1^* \geqslant \max(y_1^{**}, y_1^{***})$ and $y_2^* \leqslant \min(y_2^{**}, y_2^{***})$ hold. That is, among the three cases, the optimal inventory level of the physical store is the highest and the optimal inventory level of the online store is the lowest both in the first case. Therefore, in the first case, the difference between the optimal inventory levels of dual channels is the largest. Since, in the first case, unmet demand in the physical store is lost, while in the online store it can be satisfied in the next period, the vendor orders more inventory for the physical store, and less for the online store to reduce its competition with the physical store. As the unit lost-sales penalty costs of dual channels in the lost-sales case are always higher than the unit backorder penalty costs in the backorder case, the vendor may have incentive to order more inventories in the lost-sales case. However, as higher inventories intensify competition between dual channels, there is no clear order relationship between the optimal inventory levels in the lost-sales and the backorder cases. Further, as the differences among the optimal inventory levels in the three cases could be as large as 10%, the vendor should carefully choose the ways to treat unmet demands in dual channels and make the optimal replenishment decisions correspondingly.

Table 4 Comparison between the optimal inventory levels in the three different cases

n	ϕ_1^{max}	ϕ_2^{max}	ϕ_3^{max}	ϕ_4^{max}	ϕ_1^{min}	ϕ_2^{min}	ϕ_3^{min}	ϕ_4^{min}
9555	0.0064	0.0387	0.0794	0.0033	0.0000	0.0000	0.0046	0.0000
9674	0.0063	0.0448	0.0696	0.0032	0.0000	0.0000	0.0017	0.0000
9520	0.0141	0.0378	0.0914	0.0015	0.0001	0.0000	0.0060	0.0000
9676	0.0143	0.0445	0.0982	0.0018	0.0000	0.0000	0.0014	0.0000

Conclusion

In this paper, we study the inventory management problem of dual sales channels operated by one vendor. Demands of dual channels are inventory-level-dependent, increasing with the inventory level of its own channel and decreasing with the inventory level of the other channel. We show that, under mild conditions, the myopic inventory policy is optimal for the infinite horizon problem. Then we analyse how the different parameters in the demand functions affect the optimal inventory levels and service levels of dual channels. To investigate the importance of capturing the DDIL, we consider a heuristic where the vendor ignores the effect of the DDIL, and compare the optimal inventory levels with those recommended by the heuristic. Through numerical examples, we show that the vendor may order less for dual channels than those when he ignores the effect of the DDIL and that the difference between the inventory levels in the two cases can be so large that the effect of the DDIL cannot be neglected. For a more in-depth investigation, we examine numerically how different ways of treating unmet demand affect the optimal inventory levels, and we obtain some managerial insights. One of the interesting observations is that, when unmet demand in the physical store is lost and that in the online store is backlogged, the difference between the optimal inventory levels of dual channels is at its largest.

There are several directions for future research. First, demands of dual channels are inventory-level-dependent in our paper, with a linear relationship to the inventory levels of both channels. More general ILDD models can be considered in multi-channel inventory studies. Second, in this paper, the vendor's inventory decisions only affect demands in the current period. However, the inventory decision in one period may affect demands thereafter. Azadivar *et al* (2010) consider the discrete multi-period dynamic inventory control problem where customers follow a simple satisfaction-based demand process, where their probability of demand depends on whether their demand was satisfied last time. Therefore, as one of the future directions, one can establish models in which the demand depends on the firm's history of inventory levels or on the firm's history of service levels. Third, empirical studies for the ILDD in the dual channels context can be an interesting future research direction. Finally, a system with set-up cost is also worthy of future study.

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Appendix

Proof of Theorem 1. Applying Lemma 1, the theorem can be proven by induction on t. □

Proof of Lemma 2. Let $k_1=r_1+l_1+h_1-\gamma c_1$, $k_2=(1-\gamma)r_2+l_2+h_2$. Denote f_1 and f_2 as the probability density functions of ε_1 and ε_2, respectively. Then,

$$\frac{\partial^2}{\partial y_1^2}C^+(y_1,y_2) = -(1-a_1)^2k_1f_1((1-a_1)y_1+b_1y_2) - a_2^2k_2f_2(a_2y_1+(1-b_2)y_2)\leqslant 0,$$

$$\frac{\partial^2}{\partial y_2^2}C^+(y_1,y_2) = -b_1^2k_1f_1((1-a_1)y_1+b_1y_2) - (1-b_2)^2k_2f_2(a_2y_1+(1-b_2)y_2)\leqslant 0,$$

$$\frac{\partial^2}{\partial y_1\partial y_2}C^+(y_1,y_2) = -(1-a_1)b_1k_1f_1((1-a_1)y_1+b_1y_2) - a_2(1-b_2)k_2f_2(a_2y_1+(1-b_2)y_2)\leqslant 0.$$

Therefore,

$$\frac{\partial^2}{\partial y_1^2}C^+(y_1,y_2)\frac{\partial^2}{\partial y_2^2}C^+(y_1,y_2)-\left(\frac{\partial^2}{\partial y_1\partial y_2}C^+(y_1,y_2)\right)^2 = (1-a_1-b_2+a_1b_2-a_2b_1)^2k_1k_2f_1((1-a_1)y_1 + b_1y_2)f_2(a_2y_1+(1-b_2)y_2))\geqslant 0.$$

Thus, $C^+(y_1,y_2)$ is concave in y_1 and y_2. □

Proof of Lemma 3. We have shown in the proof of Lemma 2 that

$$\frac{\partial^2}{\partial y_1^2}C^+(y_1,y_2)\frac{\partial^2}{\partial y_2^2}C^+(y_1,y_2)-\left(\frac{\partial^2}{\partial y_1\partial y_2}C^+(y_1,y_2)\right)^2 = (1-a_1-b_2+a_1b_2-a_2b_1)^2k_1k_2f_1((1-a_1)y_1 + b_1y_2)f_2(a_2y_1+(1-b_2)y_2)\geqslant 0.$$

When the inventories of dual channels are replenished to y_1 and y_2, respectively, $(1-a_1)y_1+b_1y_2=y_1-(a_1y_1-b_1y_2)$ can be viewed as the inventory used to satisfy the demand of loyal customers in the physical store. So inequalities $v_1\leqslant(1-a_1)y_1+b_1y_2\leqslant u_1$ and $v_2\leqslant a_2y_1+(1-b_2)y_2\leqslant u_2$ hold under the optimal policy. Thus, we can restrict our decision space to $\Omega_1=\{(y_1,y_2){:}v_1\leqslant(1-a_1)y_1+b_1y_2\leqslant u_1,$ $v_2\leqslant a_2y_1+(1-b_2)y_2\leqslant u_2, 0\leqslant y_1\leqslant M_1, 0\leqslant y_2\leqslant M_2\}$. Since F_1 and F_2 are strictly increasing, f_1 is larger than zero on the interval $[v_1,u_1]$ and f_2 is larger than zero on the interval $[v_2,u_2]$. Therefore, as long as $1-a_1-b_2+a_1b_2-a_2b_1\neq 0$, $C^+(y_1,y_2)$ is strictly concave in domain Ω_1. Thus, the maximizer of $C^+(y_1,y_2)$ is unique in domain Ω_1. Therefore, the maximizer of $C^+(y_1,y_2)$ is unique in domain Ω. □

Proof of Lemma 4. For simplicity, denote $C_i^+(y_1,y_2)=(\partial/\partial y_i)C^+(y_1,y_2)$ and $C_{ij}^+(y_1,y_2)=(\partial^2/\partial y_i\partial y_j)C^+(y_1,y_2)$, where $i,j=1,2$. According to the definition of $z_2(x_1)$, we have

$$C_2^+(x_1,z_2(x_1))=0. \qquad \text{(A.1)}$$

Taking derivative of the two sides of Equation (A.1) with respect to x_1, we have: $C_{21}^+(x_1,z_2(x_1))+z_2'(x_1)C_{22}^+(x_1,z_2(x_1))=0$. Since $C^+(y_1,y_2)$ is concave, $C_{22}^+(x_1,z_2(x_1))\leqslant 0$ for arbitrary x_1. We have shown in the proof of Lemma 2 that $C_{21}^+(x_1,z_2(x_1))\leqslant 0$. Thus, $z_2'(x_1)\leqslant 0$. Similarly, $z_1'(x_2)\leqslant 0$. □

Proof of Theorem 2 By Lemmas 3 and 4, it is not difficult to show that $\bar{y}(x_1,x_2)$ is the myopic policy. Next, we show that the myopic policy is optimal for the infinite horizon problem. The total expected discounted profit of dual channels over the infinite horizon for a given initial state $(x_1(1)=x_1, x_2(1)=x_2)$ when using the policy (π_1,π_2) is

$$V(x_1,x_2\mid\pi_1,\pi_2)=c_1x_1+c_2x_2 + E\left[\sum_{t=1}^{\infty}\gamma^tC^+(y_1(t),y_2(t))\mid x_1(t) \leqslant y_1(t)\leqslant M_1,(x_2(t))^+\leqslant y_2(t)\leqslant M_2\right].$$

For each t and every possible $(x_1(t),x_2(t))$, the policy $\bar{y}(x_1,x_2)$ set $y(t)=(y_1(t),y_2(t))$ to maximize $C^+(y_1(t),$

$y_2(t)$), subject to $x_1(t) \leqslant y_1(t) \leqslant M_1$, $(x_2(t))^+ \leqslant y_2(t) \leqslant M_2$. Moreover, if $(x_1(t), x_2(t)) \in \Phi_1$ and we continue to apply the policy $\bar{y}(x_1, x_2)$, we shall have $y(t+1) = y^*$, $y(t+2) = y^*$, etc. On the other hand, if $x_1(t) > y_1^*$ or $x_2(t) > y_2^*$, the policy $\bar{y}(x_1, x_2)$ can still be shown to be optimal. We only analyse the case where $(x_1(t), x_2(t)) \in \Phi_2$ (The analyses of other two cases where $(x_1(t), x_2(t)) \in \Phi_i$ ($i = 3$, 4) are similar.) If $(x_1(t), x_2(t)) \in \Phi_2$, then $y_1^* < x_1(t) \leqslant M_1$ and $x_2 < z_2(x_1)$. As $z_2(.)$ is a non-increasing function, $z_2(x_1)$ is lower than y_2^*. In the next period, if $x_1(t+1) \leqslant y_1^*$, then the inventory is replenished to (y_1^*, y_2^*). If $x_1(t+1) > y_1^*$, as $x_1(t+1) \leqslant x_1(t)$, the inequalities $z_2(x_1(t+1)) \geqslant z_2(x_1(t))$ and $z_2(x_1(t+1)) \leqslant y_2^*$ hold. Therefore, whatever values of the demand happen to occur, $x_1(t+1)$ will be as small as possible, and the feasible range for $y_1(t+1)$ as large as possible. The same is true for subsequent times. In sum, this policy maximizes $C^+(y_1(t), y_2(t))$ for all t and demands sequences. It certainly maximizes the expected discounted sum of these profits, and so is optimal. □

Proof of Proposition 3 (1) Suppose $y_2^* \leqslant y_2''$, we prove $y_1^* \geqslant y_1''$. Denote

$$E_1 = \frac{l_1}{k_1} + \frac{(r_1 - c_1)(1 - b_2) - (r_2 - c_2)a_2}{[(1 - a_1)(1 - b_2) - a_2 b_1]k_1},$$

$$E_2 = \frac{l_2 - \gamma(r_2 - c_2)}{k_2} - \frac{(r_1 - c_1)b_1 - (r_2 - c_2)(1 - a_1)}{[(1 - a_1)(1 - b_2) - a_2 b_1]k_2}.$$

Then

$$y_1^* = \frac{(1 - b_2)F_1^{-1}(E_1) - b_1 F_2^{-1}(E_2)}{(1 - a_1)(1 - b_2) - a_2 b_1},$$

$$y_2^* = \frac{(1 - a_1)F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1)}{(1 - a_1)(1 - b_2) - a_2 b_1},$$

$$y_2^* - y_2'' = \frac{\begin{bmatrix} (1 - a_1)F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1) \\ -[(1 - a_1)(1 - b_2) - a_2 b_1]F_2^{-1}\left(\frac{(1-\gamma)(r_2 - c_2) + l_2}{k_2}\right) \end{bmatrix}}{(1 - a_1)(1 - b_2) - a_2 b_1}.$$

Since $(1 - a_1)(1 - b_2) - a_2 b_1 \geqslant 0$, we obtain

$$(1 - a_1)F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1) - [(1 - a_1)(1 - b_2) - a_2 b_1]F_2^{-1}\left(\frac{(1-\gamma)(r_2 - c_2) + l_2}{k_2}\right) \leqslant 0.$$

As $D \geqslant 0$, the inequality $E_2 - ((1-\gamma)(r_2 - c_2) + l_2)/(k_2) \geqslant 0$ holds. Thus, $F_2^{-1}(E_2) - F_2^{-1}((1-\gamma)(r_2 - c_2) + l_2)/(k_2) \geqslant 0$.

Therefore,

$$[(1 - a_1)b_2 + a_2 b_1]F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1) \leqslant (1 - a_1)F_2^{-1}(E_2) - [(1 - a_1)(1 - b_2) - a_2 b_1]F_2^{-1}\left(\frac{(1-\gamma)(r_2 - c_2) + l_2}{k_2}\right) - a_2 F_1^{-1}(E_1) \leqslant 0. \quad (21)$$

Now, we prove that $y_1^* - y_1'' \geqslant 0$.

$$y_1^* - y_1'' = \frac{\begin{matrix} (1 - b_2)F_1^{-1}(E_1) - b_1 F_2^{-1}(E_2) \\ -[(1 - a_1)(1 - b_2) - a_2 b_1]F_1^{-1}\left(\frac{r_1 - c_1 + l_1}{k_1}\right) \end{matrix}}{(1 - a_1)(1 - b_2) - a_2 b_1}.$$

Since $C \geqslant 0$, the inequality $F_1^{-1}(E_1) \geqslant F_1^{-1}((r_1 - c_1 + l_1)/k_1)$ holds. Thus,

$$y_1^* - y_1'' \geqslant \frac{[(1 - b_2)a_1 + a_2 b_1]F_1^{-1}(E_1) - b_1 F_2^{-1}(E_2)}{(1 - a_1)(1 - b_2) - a_2 b_1}, \quad (22)$$

in which, $[(1 - b_2)a_1 + a_2 b_1]F_1^{-1}(E_1) - b_1 F_1^{-1}(E_2)$ can be rewritten as:

$$-\frac{b_1\left[((1 - a_1)b_2 + a_2 b_1)F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1)\right]}{(1 - a_1)b_2 + a_2 b_1} + \frac{[(1 - a_1)b_2 + a_2 b_1][(1 - b_2)a_1 + a_2 b_1] - a_2 b_1}{(1 - a_1)b_2 + a_2 b_1} F_1^{-1}(E_1).$$

According to Inequality (21), the first item is larger than zero. We only need to show that the second item is larger than zero. It suffices to show that $[(1 - a_1)b_2 + a_2 b_1][(1 - b_2)a_1 + a_2 b_1] - a_2 b_1$ is larger than zero. $[(1 - a_1)b_2 + a_2 b_1][(1 - b_2)a_1 + a_2 b_1] - a_2 b_1$ can be easily factored into $[(1 - a_1)(1 - b_2) - a_2 b_1](a_1 b_2 - a_2 b_1)$. Since the inequalities $(1 - a_1)(1 - b_2) - a_2 b_1 \geqslant 0$, $1 - a_1 \geqslant b_1$ and $1 - b_2 \geqslant a_2$ always hold, $[(1 - a_1)b_2 + a_2 b_1][(1 - b_2)a_1 + a_2 b_1] - a_2 b_1$ is larger than zero. Till now, we have proved that $y_1^* \geqslant y_1''$. Similarly, when $y_1^* \leqslant y_1''$, the inequality $y_2^* \geqslant y_2''$ holds.
(2) If $C \geqslant 0$ and $D \geqslant 0$, we have

$$y_1^* + y_2^* - y_1'' - y_2'' = \frac{[(1 - b_2)a_1 + a_2 b_1]F_1^{-1}(E_1) - b_1 F_2^{-1}(E_2)}{(1 - a_1)(1 - b_2) - a_2 b_1} + \frac{[(1 - a_1)b_2 + a_2 b_1]F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1)}{(1 - a_1)(1 - b_2) - a_2 b_1}$$

$$= \frac{\begin{bmatrix} (a_1 - a_2 + a_2 b_1 - a_1 b_2)F_1^{-1}(E_1) \\ + (b_2 - b_1 + a_2 b_1 - a_1 b_2)F_2^{-1}(E_2) \end{bmatrix}}{(1 - a_1)(1 - b_2) - a_2 b_1} = \frac{e_1 F_1^{-1}(E_1) + e_2 F_2^{-1}(E_2)}{(1 - a_1)(1 - b_2) - a_2 b_1}.$$

Thus, if $e_1 \geqslant 0$, $e_2 \geqslant 0$, $C \geqslant 0$, $D \geqslant 0$, the inequality $y_1^* + y_2^* - y_1'' - y_2'' \geqslant 0$ holds. This completes the proof. □

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