On a doubly critical Schrödinger system in $\mathbb{R}^4$ with steep potential wells*

Yuanze Wu[a] **& Wenming Zou**[b]

[a] *College of Sciences, China University of Mining and Technology, Xuzhou 221116, P.R. China*

[b] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China*

Abstract: Study the following two-component elliptic system

$$\begin{cases} \Delta u - (\lambda a(x) + a_0)u + u^3 + \beta v^2 u = 0 & \text{in } \mathbb{R}^4, \\ \Delta v - (\lambda b(x) + b_0)v + v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^4, \\ (u, v) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4), \end{cases}$$

where $a_0, b_0 \in \mathbb{R}$ are constants; $\lambda > 0$ and $\beta \in \mathbb{R}$ are parameters and $a(x), b(x) \geq 0$ are potential wells which are not necessarily to be radial symmetric. By using the variational method, we investigate the existence of ground state solutions and general ground state solutions (i.e., possibly semi-trivial) to this system. Indeed, to the best of our knowledge, even the existence of semi-trivial solutions is also unknown in the literature. We observe some concentration behaviors of ground state solutions and general ground state solutions. The phenomenon of phase separations is also excepted. It seems that this is the first result definitely describing the phenomenon of phase separation for critical system in the whole space $\mathbb{R}^4$. Note that both the cubic nonlinearities and the coupled terms of the system are all of critical growth with respect to the Sobolev critical exponent.

Keywords: Elliptic system; Ground state; Steep potential well; Critical Sobolev exponent; Variational method.

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*Supported by NSFC(11371212, 11271386). E-mails: wuyz850306@cumt.edu.cn (Y. Wu); wzou@math.tsinghua.edu.cn (W. Zou)

1 Introduction

We study the following two-component elliptic system

$$\begin{cases} \Delta u - (\lambda a(x) + a_0)u + u^3 + \beta v^2 u = 0, & x \text{ in } \mathbb{R}^4, \\ \Delta v - (\lambda b(x) + b_0)v + v^3 + \beta u^2 v = 0, & x \text{ in } \mathbb{R}^4, \\ (u, v) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4), \end{cases} \qquad (\mathcal{P}_{\lambda,\beta})$$

where $a_0, b_0 \in \mathbb{R}$ are constants and $\lambda > 0$, $\beta \in \mathbb{R}$ are parameters. The potentials $a(x)$ and $b(x)$ satisfy some conditions to be specified later.

It is well known that the solutions of $(\mathcal{P}_{\lambda,\beta})$ are related to the solitary wave solutions to the following two-component system of nonlinear Schrödinger equations

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Psi_1 = \Delta\Psi_1 - \lambda a(x)\Psi_1 + |\Psi_1|^2\Psi_1 + \beta|\Psi_2|^2\Psi_1 = 0, \\ -i\dfrac{\partial}{\partial t}\Psi_2 = \Delta\Psi_2 - \lambda b(x)\Psi_2 + |\Psi_2|^2\Psi_2 + \beta|\Psi_1|^2\Psi_2 = 0, \\ x \text{ in } \mathbb{R}^4, t > 0; \ \ \Psi_j = \Psi_j(t,x) \in \mathbb{C}, \ \ j = 1, 2, \\ \Psi_j(t,x) \to 0 \text{ as } |x| \to +\infty, \ \ t > 0. \end{cases} \qquad (\mathcal{P}^*_{\lambda,\beta})$$

Indeed, set $\Psi_1(t,x) = e^{-ita_0}u(x)$ and $\Psi_2(t,x) = e^{-itb_0}v(x)$, then (Ψ_1, Ψ_2) is called the solitary wave solution of $(\mathcal{P}^*_{\lambda,\beta})$ and (u, v) is a solution of the $(\mathcal{P}_{\lambda,\beta})$ if and only if (Ψ_1, Ψ_2) is a solution of the $(\mathcal{P}^*_{\lambda,\beta})$.

In the literature, the System $(\mathcal{P}^*_{\lambda,\beta})$ defined on an open set Ω (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is called the Gross-Pitaevskii equations (e.g. [27, 45]), which appears in many different physical problems. For example, in the Hartree-Fock theory, the Gross-Pitaevskii equations can be used to describe a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (cf. [22]). The solutions $\Psi_j (j = 1, 2)$ are the corresponding condensate amplitudes and β is the interaction of the states $|1\rangle$ and $|2\rangle$. The interaction is attractive if $\beta > 0$ and repulsive if $\beta < 0$. When the interaction is repulsive, it is expected that the phenomenon of phase separation will happen, that is, the two components of the system tend to separate in different regions as the interaction tends to infinity. The Gross-Pitaevskii equation also arises in nonlinear optics (cf. [1]). Due to the important application in physics, the Gross-Pitaevskii equation $(\mathcal{P}^*_{0,\beta})$ has been studied extensively in the last decades. We refer the readers to [5, 18, 30, 31, 36, 38] and the references therein, where various existence theorems of the solitary wave solutions were established.

When we consider the equation $(\mathcal{P}^*_{\lambda,\beta})$ or $(\mathcal{P}_{\lambda,\beta})$ in $\mathbb{R}^4$, the cubic nonlinearities and the couple terms are all of critical growth, since the Sobolev critical exponent $2^* := 2N/(N-2) = 4$ in $\mathbb{R}^N = \mathbb{R}^4$. By the Pohozaev identity, we can easily conclude that any solution of $(\mathcal{P}_{0,\beta})$ satisfies $\int_{\mathbb{R}^4} a_0u^2 + b_0v^2 dx = 0$ (cf.

[15, 17]). Thus, any solution of $(\mathcal{P}_{0,\beta})$ must be $(0,0)$ in the case of $a_0b_0 > 0$. Due to this reason, to some extent, it seems that $\lambda \neq 0$ is a necessary condition for the existence of non-zero or even non-trivial solutions to $(\mathcal{P}_{\lambda,\beta})$.

Definition 1.1 *We call that $(u,v) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ is a non-zero solution of $(\mathcal{P}_{\lambda,\beta})$ if (u,v) is a solution of $(\mathcal{P}_{\lambda,\beta})$ with $(u,v) \neq (0,0)$; we say $(u,v) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ is a non-trivial solution of $(\mathcal{P}_{\lambda,\beta})$ if (u,v) is a non-zero solution with both $u \neq 0$ and $v \neq 0$.*

To the best of our knowledge, few result has been established for the System $(\mathcal{P}_{\lambda,\beta})$. In this paper, we will study the System $(\mathcal{P}_{\lambda,\beta})$ with $\lambda > 0$ when $a(x), b(x)$ satisfy the following conditions:

($\mathbf{D_1}$) $a(x), b(x) \in C(\mathbb{R}^4)$ and $a(x), b(x) \geq 0$ on $\mathbb{R}^4$.

($\mathbf{D_2}$) There exist $a_\infty, b_\infty \in (0, +\infty)$ such that $\lim_{|x|\to+\infty} a(x) = a_\infty$ and $a(x) \leq a_\infty$ for all $x \in \mathbb{R}^4$ while $\lim_{|x|\to+\infty} b(x) = b_\infty$ and $b(x) \leq b_\infty$ for all $x \in \mathbb{R}^4$.

($\mathbf{D_3}$) $\Omega_a := \text{int } a^{-1}(0)$ and $\Omega_b := \text{int } b^{-1}(0)$ are bounded non-empty domains and have smooth boundaries. Moreover, $\overline{\Omega}_a = a^{-1}(0)$, $\overline{\Omega}_b = b^{-1}(0)$ and $\overline{\Omega}_a \cap \overline{\Omega}_b = \emptyset$.

In the sequel, $\lambda a(x)$ and $\lambda b(x)$ are called the steep potential wells under the conditions (D_1)-(D_3) if the parameter λ is sufficiently large. The depth of the wells is controlled by the parameter λ. An interesting phenomenon for this kind of Schrödinger equations is that, one can expect to find the solutions which are concentrated at the bottom of the wells as the depth goes to infinity. Due to this interesting property, such a topic for the scalar Schrödinger equations was studied extensively in the last decades. We refer the readers to [3, 4, 9, 20, 21, 32, 39, 48] and the references therein. Most of the papers are devoted to the subcritical case. In recent years, the steep potential wells were also introduced to some other elliptic equations and systems, see for example [25, 26, 28, 42, 49] and the references therein. In particular, in [49], the Gross-Pitaevskii equations in $\mathbb{R}^3$ (subcritical case) with steep potential wells were considered and some existence results of the solitary wave solutions were established.

Under the conditions (D_1)-(D_3), the System $(\mathcal{P}_{\lambda,\beta})$ has a variational structure. Indeed, let

$$E_a := \{u \in D^{1,2}(\mathbb{R}^4) \mid \int_{\mathbb{R}^4} a(x)u^2 dx < +\infty\};$$

$$E_b := \{u \in D^{1,2}(\mathbb{R}^4) \mid \int_{\mathbb{R}^4} b(x)u^2 dx < +\infty\}.$$

Then by the condition (D_1), for every $a_0, b_0 \in \mathbb{R}$ and $\lambda > \max\{0, \frac{-a_0}{a_\infty}, \frac{-b_0}{b_\infty}\}$, E_a and E_b are the Hilbert spaces equipped with the following inner products

$$\langle u, v\rangle_{a,\lambda} := \int_{\mathbb{R}^4} \nabla u \nabla v + (\lambda a(x) + a_0)^+ uv dx,$$

$$\langle u, v\rangle_{b,\lambda} := \int_{\mathbb{R}^4} \nabla u \nabla v + (\lambda b(x) + b_0)^+ uvdx,$$

respectively, where $(\cdot)^+ := \max\{\cdot, 0\}$. The corresponding norms are respectively given by

$$\|u\|_{a,\lambda} := \left(\int_{\mathbb{R}^4} |\nabla u|^2 + (\lambda a(x) + a_0)^+ u^2 dx \right)^{\frac{1}{2}}$$

and

$$\|v\|_{b,\lambda} := \left(\int_{\mathbb{R}^4} |\nabla v|^2 + (\lambda b(x) + b_0)^+ v^2 dx \right)^{\frac{1}{2}}.$$

We denote the Hilbert spaces $(E_a, \|\cdot\|_{a,\lambda})$ and $(E_b, \|\cdot\|_{b,\lambda})$ by $E_{a,\lambda}$ and $E_{b,\lambda}$ respectively. Let $E_\lambda := E_{a,\lambda} \times E_{b,\lambda}$ be the Hilbert space with the inner product

$$\langle (u, v), (w, \sigma)\rangle_\lambda := \langle u, w\rangle_{a,\lambda} + \langle v, \sigma\rangle_{b,\lambda}.$$

The corresponding norm is given by $\|(u, v)\|_\lambda := (\|u\|^2_{a,\lambda} + \|v\|^2_{b,\lambda})^{\frac{1}{2}}$. Then by the conditions (D_1)-(D_2) and the Hölder and Sobolev inequalities, for every $\lambda > \max\{0, \frac{-a_0}{a_\infty}, \frac{-b_0}{b_\infty}\}$, there exists $d_\lambda > 0$ such that

$$\|u\|_{L^2(\mathbb{R}^4)} \leq d_\lambda \|u\|_{a,\lambda}, \quad \|v\|_{L^2(\mathbb{R}^4)} \leq d_\lambda \|v\|_{b,\lambda} \tag{1.1}$$

and

$$\|u\|_{L^4(\mathbb{R}^4)} \leq S^{-\frac{1}{2}} \|u\|_{a,\lambda}, \quad \|v\|_{L^4(\mathbb{R}^4)} \leq S^{-\frac{1}{2}} \|v\|_{b,\lambda}, \tag{1.2}$$

for $(u, v) \in E_\lambda$, where $\|\cdot\|_{L^p(\mathbb{R}^4)}$ is the usual norm in $L^p(\mathbb{R}^4)$ for all $p \geq 1$ and S is the best Sobolev embedding constant from $D^{1,2}(\mathbb{R}^4)$ to $L^4(\mathbb{R}^4)$ and given by

$$S := \inf\{\|\nabla u\|^2_{L^2(\mathbb{R}^4)} \mid u \in D^{1,2}(\mathbb{R}^4), \|u\|^2_{L^4(\mathbb{R}^4)} = 1\}.$$

It follows that E_λ is embedded continuously into $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ for $\lambda > \max\{0, \frac{-a_0}{a_\infty}, \frac{-b_0}{b_\infty}\}$. Moreover, by (1.1)–(1.2), the conditions (D_1)–(D_2) and the Hölder inequality, the energy functional $J_{\lambda,\beta}(u, v)$ given by

$$\begin{aligned} &J_{\lambda,\beta}(u, v) \\ &:= \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u|^2 + (\lambda a(x) + a_0)u^2 dx + \frac{1}{2} \int_{\mathbb{R}^4} |\nabla v|^2 + (\lambda b(x) + b_0)v^2 dx \\ &\quad - \frac{1}{4} \int_{\mathbb{R}^4} u^4 dx - \frac{1}{4} \int_{\mathbb{R}^4} v^4 dx - \frac{\beta}{2} \int_{\mathbb{R}^4} u^2 v^2 dx \end{aligned} \tag{1.3}$$

is well defined in E_λ for $\lambda > \max\{0, \frac{-a_0}{a_\infty}, \frac{-b_0}{b_\infty}\}$ and $\beta \in \mathbb{R}$. Furthermore, by a standard argument, we can also show that $J_{\lambda,\beta}(u, v)$ is of C^2 in E_λ and it is the corresponding energy functional to System $(\mathcal{P}_{\lambda,\beta})$. For the sake of convenience, we re-write the energy functional $J_{\lambda,\beta}(u, v)$ by

$$J_{\lambda,\beta}(u, v) = \frac{1}{2}\mathcal{D}_\lambda(u, v) - \frac{1}{4}\mathcal{L}_\beta(u, v),$$

where $\mathcal{D}_\lambda(u,v) := \mathcal{D}_{a,\lambda}(u,u) + \mathcal{D}_{b,\lambda}(v,v)$ with

$$\mathcal{D}_{a,\lambda}(u,v) := \int_{\mathbb{R}^4} (\nabla u \nabla v + (\lambda a(x) + a_0)uv)dx,$$

$$\mathcal{D}_{b,\lambda}(u,v) := \int_{\mathbb{R}^4} (\nabla u \nabla v + (\lambda b(x) + b_0)uv)dx$$

and

$$\mathcal{L}_\beta(u,v) := \|u\|^4_{L^4(\mathbb{R}^4)} + \|v\|^4_{L^4(\mathbb{R}^4)} + 2\beta\|u^2v^2\|_{L^1(\mathbb{R}^4)}.$$

We are interested in finding the ground state solutions of $(\mathcal{P}_{\lambda,\beta})$ for λ sufficiently large.

Definition 1.2 *We say that (u,v) is a ground state solution of $(\mathcal{P}_{\lambda,\beta})$ if (u,v) is a non-trivial solution of $(\mathcal{P}_{\lambda,\beta})$ and the energy of (u,v) given by (1.3) is the least one among all that of the non-trivial solutions to $(\mathcal{P}_{\lambda,\beta})$.*

To the best of our knowledge, the existence of semi-trivial solution to $(\mathcal{P}_{\lambda,\beta})$ is also unknown in the literature. Therefore, we are also concerned with finding the general ground state solutions to $(\mathcal{P}_{\lambda,\beta})$ for λ sufficiently large.

Definition 1.3 *We say $(u,v) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ is a semi-trivial solution of $(\mathcal{P}_{\lambda,\beta})$ if (u,v) is a non-zero solution to $(\mathcal{P}_{\lambda,\beta})$ of the type $(u,0)$ or $(0,v)$; we call (u,v) a general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ if (u,v) is a non-zero solution of $(\mathcal{P}_{\lambda,\beta})$ and its energy is the least one among all that of the non-zero solutions to $(\mathcal{P}_{\lambda,\beta})$.*

Definition 1.4 *Let $\mu_{a,1}$ and $\mu_{b,1}$ denote the first eigenvalues of $(-\Delta, H^1_0(\Omega_a))$ and $(-\Delta, H^1_0(\Omega_b))$, respectively.*

We denote the sets of all eigenvalues of $(-\Delta, H^1_0(\Omega_a))$ and $(-\Delta, H^1_0(\Omega_b))$ by $\sigma(-\Delta, H^1_0(\Omega_a))$ and $\sigma(-\Delta, H^1_0(\Omega_b))$, respectively.

Remark 1.1 *Without loss of generality, we always assume $a_0 \leq b_0$ throughout this paper.*

1.1 The case of $-\mu_{a,1} < a_0$ and $-\mu_{b,1} < b_0$.

Clearly, $J_{\lambda,\beta}(u,v)$ is heavily rely on the properties of $\mathcal{D}_\lambda(u,v)$, a_0 and b_0. Firstly, we note that there exists $\Lambda_0 \geq 0$ such that $\mathcal{D}_\lambda(u,v)$ is positively definite on E_λ for $\lambda > \Lambda_0$ provided that $-\mu_{a,1} < a_0$ and $-\mu_{b,1} < b_0$ (see Lemma 2.3 below for more details). In particular, $\Lambda_0 = 0$ if $a_0 \geq 0$ and $b_0 \geq 0$. Let

$$\mathcal{N}_{\lambda,\beta} := \Big\{(u,v) \in E_\lambda \mid u \neq 0, v \neq 0, \langle D[J_{\lambda,\beta}(u,v)], (u,0)\rangle_{E_\lambda^*,E_\lambda} = \langle D[J_{\lambda,\beta}(u,v)], (0,v)\rangle_{E_\lambda^*,E_\lambda} = 0\Big\}$$

and

$$\mathcal{M}_{\lambda,\beta} := \Big\{(u,v) \in E_\lambda\backslash\{(0,0)\} \mid \langle D[J_{\lambda,\beta}(u,v)], (u,v)\rangle_{E_\lambda^*,E_\lambda} = 0\Big\},$$

where $D[J_{\lambda,\beta}(u,v)]$ is the Frechét derivative of the functional $J_{\lambda,\beta}$ in E_λ at (u,v) and E_λ^* is the dual space of E_λ. It is easy to see that $\mathcal{N}_{\lambda,\beta}$ and $\mathcal{M}_{\lambda,\beta}$ are both nonempty and contains all non-trivial solutions and non-zero solutions of the System $(\mathcal{P}_{\lambda,\beta})$, respectively. Such sets are the so-called Nehari type sets to $(\mathcal{P}_{\lambda,\beta})$ and they are extensively used for finding the ground state solution to nonlinear elliptic systems (cf. [15, 16, 18, 19, 30, 30, 38, 49]). Define

$$m_{\lambda,\beta} := \inf_{(u,v)\in\mathcal{N}_{\lambda,\beta}} J_{\lambda,\beta}(u,v), \qquad m_{\lambda,\beta}^* := \inf_{(u,v)\in\mathcal{M}_{\lambda,\beta}} J_{\lambda,\beta}(u,v). \tag{1.4}$$

Since $\mathcal{D}_\lambda(u,v)$ is positively definite on E_λ, it is also easy to show that $m_{\lambda,\beta}$ and $m_{\lambda,\beta}^*$ are both nonnegative for all $\lambda > \Lambda_0$ and $\beta \in \mathbb{R}$.

Theorem 1.1 *Assume (D_1)-(D_3) and $-\mu_{a,1} < a_0, -\mu_{b,1} < b_0$. If $\lambda > \Lambda_0$, then we have the following conclusions:*

(1) *If $0 \leq a_0 \leq b_0$, then*

$$m_{\lambda,\beta} = \frac{S^2}{2(1+\max\{\beta,0\})}; \quad m_{\lambda,\beta}^* = \frac{S^2}{2(1+\max\{1,\beta\})} \text{ for all } \beta \in \mathbb{R}.$$

Moreover, both $m_{\lambda,\beta}$ and $m_{\lambda,\beta}^*$ can not be attained.

(2) *If $a_0 < 0$, then $m_{\lambda,\beta}^*$ can be attained by a general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ for all $\beta \in \mathbb{R}$. Moreover, there exists $\Lambda_\beta > 0$ such that the general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ must be semi-trivial provided that one of the following conditions holds:*

- *$a_0 < 0 \leq b_0$, $\beta < 1 - \frac{|a_0|}{\mu_{a,1}}$ and $\lambda > \Lambda_\beta$;*
- *$a_0 \leq b_0 < 0$, $\beta < \beta_0$ and $\lambda > \Lambda_\beta$, where*

$$\beta_0 := \min\left\{\frac{1}{2}(1-\frac{|a_0|}{\mu_{a,1}})(1-\frac{|b_0|}{\mu_{b,1}}),\ \frac{1-\frac{|b_0|}{\mu_{b,1}}}{1-\frac{|a_0|}{\mu_{a,1}}},\ \frac{1-\frac{|a_0|}{\mu_{a,1}}}{1-\frac{|b_0|}{\mu_{b,1}}}\right\}.$$

(3) *$m_{\lambda,\beta}$ can be attained by a ground state solution of $(\mathcal{P}_{\lambda,\beta})$ if one of the following additional conditions holds:*

- *$a_0 \leq b_0 < 0$ and $\beta \leq 0$;*
- *$a_0 < 0$ and $\beta > \beta_\lambda$ for some $0 < \beta_\lambda < +\infty$.*

Moreover, if $a_0 < 0$, then $m_{\lambda,\beta} = m^*_{\lambda,\beta}$ for $\beta > \beta_\lambda$.

The next is a by-product of the previous theorem.

Corollary 1.1 *Assume (D_1)-(D_3) and $-\mu_{a,1} < a_0 < 0, -\mu_{b,1} < b_0 < 0$. If $\lambda > \Lambda_0$, then the following equation*

$$-\Delta u + (\lambda a(x) + a_0)u = u^3, \quad u \in H^1(\mathbb{R}^4), \tag{1.5}$$

$$-\Delta v + (\lambda b(x) + b_0)v = v^3, \quad v \in H^1(\mathbb{R}^4), \tag{1.6}$$

have ground state solutions, respectively.

Remark 1.2 *The ground states obtained in Theorem 1.1 and Corollary 1.1 are positive. The Corollary 1.1 can be viewed as the generalization of the celebrated results in [10] obtained by Brézis and Nirenberg, where the equation is defined on the bounded smooth domain. On the other hand, let us recall the following equation which was studied in [6] by Benci and Cerami:*

$$-\Delta u + V(x)u = u^{(N+2)/(N-2)}, \quad u \in H^1(\mathbb{R}^N), \tag{1.7}$$

where $N \geq 3$ and $V(x)$ is a nonnegative function. It was observed when $V(x) \equiv constant \neq 0$, then (1.7) has only trivial solution $u = 0$. Moreover, if $\|V(x)\|_{L^{N/2}}$ is sufficiently small, then (1.7) has at least one solution.

1.2 The case of $a_0 \leq -\mu_{a,1}$ or $b_0 \leq -\mu_{b,1}$

If either $a_0 \leq -\mu_{a,1}$ or $b_0 \leq -\mu_{b,1}$, then there exists $\Lambda_1 > 0$ such that $\mathcal{D}_\lambda(u,v)$ is indefinite on E_λ and has finite augment Morse index for $\lambda > \Lambda_1$ (also see Lemma 2.3 below for more details). In this case, $\mathcal{N}_{\lambda,\beta}$ and $\mathcal{M}_{\lambda,\beta}$ are not the good choice for finding the ground state solution and the general ground state solution of $(\mathcal{P}_{\lambda,\beta})$. For $\lambda > \Lambda_1$, let $\widehat{\mathcal{F}}^{\perp}_{a,\lambda}$ and $\widehat{\mathcal{F}}^{\perp}_{b,\lambda}$ be the negative part of $\mathcal{D}_{a,\lambda}(u,u)$ on $E_{a,\lambda}$ and $\mathcal{D}_{b,\lambda}(v,v)$ on $E_{b,\lambda}$, respectively. Then we can modify $\mathcal{M}_{\lambda,\beta}$ to the following set

$$\begin{aligned}\mathcal{G}_{\lambda,\beta} \quad := \quad & \Big\{(u,v) \in \widetilde{E}_\lambda \mid \langle D[J_{\lambda,\beta}(u,v)], (u,v)\rangle_{E^*_\lambda,E_\lambda} = 0, \\ & \langle D[J_{\lambda,\beta}(u,v)], (w,\sigma)\rangle_{E^*_\lambda,E_\lambda} = 0, \forall (w,\sigma) \in \widehat{\mathcal{F}}^{\perp}_{a,\lambda} \times \widehat{\mathcal{F}}^{\perp}_{b,\lambda}\Big\},\end{aligned} \tag{1.8}$$

where $\widetilde{E}_\lambda := E_\lambda \backslash (\widehat{\mathcal{F}}^{\perp}_{a,\lambda} \times \widehat{\mathcal{F}}^{\perp}_{b,\lambda})$. This kind of set is the so-called Nehari-Pankov type set to $(\mathcal{P}_{\lambda,\beta})$, which was introduced by Pankov in [35] for the scalar

Schrödinger equations with indefinite potentials and was further studied by Szulkin and Weth [40]. For other papers devoted to the indefinite problems, we would like to refer the readers to [7, 8, 24] and the references therein. Define

$$c_{\lambda,\beta} := \inf_{\mathcal{G}_{\lambda,\beta}} J_{\lambda,\beta}. \tag{1.9}$$

Evidently, $c_{\lambda,\beta} \geq 0$ whenever $\beta \geq -1$ since $\mathcal{L}_\beta(u,v)$ is positively definite in this case.

Theorem 1.2 *Assume* (D_1)-(D_3). *Suppose either* $a_0 \leq -\mu_{a,1}$ *with* $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$ *or* $b_0 \leq -\mu_{b,1}$ *with* $-b_0 \notin \sigma(-\Delta, H_0^1(\Omega_b))$. *If* $\lambda > \Lambda_1$, *then* $c_{\lambda,\beta}$ *can be attained by a general ground state solution of* $(\mathcal{P}_{\lambda,\beta})$ *for* $0 \leq \beta < 1$. *Furthermore, if* $a_0 \leq -\mu_{a,1} < 0 \leq b_0$, *then there exists* $\Lambda_\beta^* \geq \Lambda_1$ *such that the general ground state solution of* $(\mathcal{P}_{\lambda,\beta})$ *must be semi-trivial and be of the type* $(u_{\lambda,\beta}, 0)$ *for all* $\lambda \geq \Lambda_\beta^*$. *In particular, where* $u_{\lambda,\beta}$ *is the ground state solution to the equation*

$$-\Delta u + (\lambda a(x) + a_0)u = u^3, \quad u \in H^1(\mathbb{R}^4). \tag{1.10}$$

Remark 1.3

(a) *Theorem 1.2 only gives the existence of the general ground state solution to* $(\mathcal{P}_{\lambda,\beta})$ *for* $0 \leq \beta < 1$ *and* λ *sufficiently large in the case of* $a_0 \leq -\mu_{a,1}$ *with* $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$ *or* $b_0 \leq -\mu_{b,1}$ *with* $-b_0 \notin \sigma(-\Delta, H_0^1(\Omega_b))$. *However, it is still open for us that whether* $(\mathcal{P}_{\lambda,\beta})$ *has the general ground state solution for other* β *in such cases. Indeed, since* $\mathcal{L}_\beta(u,v)$ *is not symmetric in* E_λ *due to the conditions* (D_1)–(D_3) *and even indefinite on* $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ *for* $\beta < -1$, *Lemmas 3.6 and 3.7 which are crucial in the proof of Theorem 1.2 are invalid for* $\beta \in (-\infty, 0) \cup [1, +\infty)$ *in the case of* $a_0 \leq -\mu_{a,1}$ *with* $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$ *or* $b_0 \leq -\mu_{b,1}$ *with* $-b_0 \notin \sigma(-\Delta, H_0^1(\Omega_b))$.

(b) *By Theorem 1.2, it is easy to show that* $(\mathcal{P}_{\lambda,0})$ *has a ground state solution in the case of* $a_0 \leq -\mu_{a,1}$ *with* $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$ *and* $b_0 \leq -\mu_{b,1}$ *with* $-b_0 \notin \sigma(-\Delta, H_0^1(\Omega_b))$. *However, since the dimension of the set for the semi-trivial solutions to* $(\mathcal{P}_{\lambda,\beta})$ *might be infinite, we do not know how to modify the Nehari type set* $\mathcal{N}_{\lambda,\beta}$ *to some Nehari-Pankov type sets as* $\mathcal{G}_{\lambda,\beta}$. *Therefore, it is also open to us that whether* $(\mathcal{P}_{\lambda,\beta})$ *has a ground state solution for* $\beta \neq 0$ *in the case of* $a_0 \leq -\mu_{a,1}$ *with* $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$ *and* $b_0 \leq -\mu_{b,1}$ *with* $-b_0 \notin \sigma(-\Delta, H_0^1(\Omega_b))$.

(c) *To the best of our knowledge, it seems that Theorems 1.2 is the first existence result for* (1.10) *in the indefinite case. By checking the proof of Theorem 1.1 (more precisely, Lemma 4.5), we can also see that* (1.10) *has a ground state solution in some definite case but might not have solutions in the case of* $a_0 \geq 0$.

1.3 The concentration phenomenon as $\lambda \to +\infty$.

Since $a(x), b(x)$ have the potential wells, it is natural to ask whether the ground state solution and the general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ will concentrate at the bottom of $a(x), b(x)$ as $\lambda \to +\infty$. Our results on this aspect can be stated as follows.

Theorem 1.3 *Let $(u_{\lambda,\beta}, v_{\lambda,\beta})$ be the solution of $(\mathcal{P}_{\lambda,\beta})$ obtained by Theorems 1.1 and 1.2. Then we have the following conclusions.*

(1) *If $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is a ground state solution of $(\mathcal{P}_{\lambda,\beta})$ with $\beta \leq 0$ in the case of $a_0 \leq b_0 < 0$, then up to a subsequence $(u_{\lambda,\beta}, v_{\lambda,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ strongly in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ as $\lambda \to +\infty$. Furthermore, $(u_{0,\beta}, v_{0,\beta})$ is also a ground state solution of the system:*

$$\begin{cases} \Delta u - a_0 u + u^3 = 0 & in\ \Omega_a, \\ \Delta v - b_0 v + v^3 = 0 & in\ \Omega_b, \\ (u, v) \in H_0^1(\Omega_a) \times H_0^1(\Omega_b). \end{cases} \tag{1.11}$$

(2) *If $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is a general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ in the case of $a_0 < 0$, then up to a subsequence $(u_{\lambda,\beta}, v_{\lambda,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ strongly in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ as $\lambda \to +\infty$. Furthermore, $(u_{0,\beta}, v_{0,\beta})$ is a semi-trivial general ground state solution of* (1.11).

Remark 1.4 *By checking the proof of Theorem 1.1, we may have $\beta_\lambda \to +\infty$ as $\lambda \to +\infty$ (see Lemmas 3.8 and 4.6 and Proposition 4.4 for more details). Thus, the concentration behaviors described in Theorem 1.3 may not hold in the case of $\beta \geq \beta_\lambda$.*

1.4 Phase separation

Note that the ground state solution and the general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ are also depending on the parameter β, it is natural that we are concerned with the phenomenon of phase separation as $\beta \to -\infty$. Our results on this topic now read as

Theorem 1.4 *Let $(u_{\lambda,\beta}, v_{\lambda,\beta})$ be the ground state solution of $(\mathcal{P}_{\lambda,\beta})$ obtained by Theorem 1.1 with $\beta \leq 0$. Then there exists $\Lambda_2 > 0$ such that $\beta \int_{\mathbb{R}^4} u_{\lambda,\beta}^2 v_{\lambda,\beta}^2 dx \to 0$ as $\beta \to -\infty$ for each $\lambda \geq \Lambda_2$. Furthermore, for every $\beta_n \to -\infty$, up to a subsequence, we also have the following*

(1) *$(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \to (u_{\lambda,\infty}, v_{\lambda,\infty})$ strongly in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ as $n \to \infty$ with $u_{\lambda,\infty} \neq 0$ and $v_{\lambda,\infty} \neq 0$;*

(2) *$u_{\lambda,\infty}$ is the ground state solution of the following equation*

$$-\Delta u + (\lambda a(x) + a_0)u = u^3, \quad u \in H_0^1(\{u_{\lambda,\infty} > 0\})$$

while $v_{\lambda,\infty}$ is the ground state solution to the following equation

$$-\Delta v + (\lambda b(x) + b_0)v = v^3, \quad v \in H_0^1(\{v_{\lambda,\infty} > 0\});$$

(3) *both* $\{u_{\lambda,\infty} > 0\}$ *and* $\{v_{\lambda,\infty} > 0\}$ *are connect domains and* $\{u_{\lambda,\infty} > 0\} = \mathbb{R}^4 \backslash \overline{\{v_{\lambda,\infty} > 0\}}$.

Remark 1.5 *For the Schrödinger system in* $\mathbb{R}^4$ *with critical Sobolev exponent defined in the whole space, Theorem 1.4 seems to be the first result getting the phase separation.*

We point out that such phenomenon for the ground state solution of the Gross-Pitaevskii equations was observed in [18, 33, 47] on a bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$; and [45, 49] on the whole space $\mathbb{R}^2$ or $\mathbb{R}^3$. Such phenomenon for the ground state solution of the elliptic systems with critical Sobolev exponent on a bounded domain in $\mathbb{R}^N (N \geq 4)$ was involved in [15, 16, 19]. In fact, the authors of [15] study the system in $\mathbb{R}^4$ and only get an alternative theorem which can not assert that the phase separation must happen. In [16] (see also [19]), the phase separation is observed when the dimension N of $\mathbb{R}^N$ is ≥ 6 and the system is defined on the bounded domains. For other kinds of elliptic systems with strong competition, the phenomenon of phase separations has also been well studied; we refer the readers to [11, 12, 13] and references therein.

1.5 Concentration behaviors as $\lambda \to +\infty$ and $\beta \to -\infty$

We also study the concentration behaviors of the ground state solution obtained by Theorem 1.1 as $\lambda \to +\infty$ and $\beta \to -\infty$.

Theorem 1.5 *Let* $(u_{\lambda,\beta}, v_{\lambda,\beta})$ *be the ground state solution of* $(\mathcal{P}_{\lambda,\beta})$ *obtained by Theorem 1.1 with* $\beta \leq 0$. *Then for every* $\{(\lambda_n, \beta_n)\}$ *satisfying* $\lambda_n \to +\infty$ *and* $\beta_n \to -\infty$ *as* $n \to \infty$, *we have that* $(u_{\lambda_n,\beta_n}, v_{\lambda_n,\beta_n}) \to (u_{0,0}, v_{0,0})$ *strongly in* $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ *as* $n \to \infty$ *up to a subsequence for some* $(u_{0,0}, v_{0,0}) \in H_0^1(\Omega_a) \times H_0^1(\Omega_b)$. *Furthermore,* $(u_{0,0}, v_{0,0})$ *is also a ground state solution of* (1.11).

The structure of the current paper is organized as follows. In section 2, we study the functionals $\mathcal{D}_\lambda(u, v)$ and $\mathcal{L}_\beta(u, v)$. In section 3, we explore the "manifolds" $\mathcal{N}_{\lambda,\beta}$, $\mathcal{M}_{\lambda,\beta}$ and $\mathcal{G}_{\lambda,\beta}$. The section 4 will be devoted to the existence results. The last section is about the concentration behaviors. Throughout this paper, C and C' will be indiscriminately used to denote generic positive constants and $o_n(1)$ will denote the quantities tending to zero as $n \to \infty$.

2 The functionals $\mathcal{D}_\lambda(u, v)$ and $\mathcal{L}_\beta(u, v)$

In this section, we give some properties of $\mathcal{D}_\lambda(u, v)$ and $\mathcal{L}_\beta(u, v)$. We begin with the study of $\mathcal{L}_\beta(u, v)$. Clearly, $\mathcal{L}_\beta(u, v)$ is positively definite on $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ if $\beta \geq 0$. For $\beta < 0$, let

$$\mathcal{V}_\beta = \{(u, v) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4) \mid \|u\|^4_{L^4(\mathbb{R}^4)} \|v\|^4_{L^4(\mathbb{R}^4)} - \beta^2 \|u^2 v^2\|^2_{L^1(\mathbb{R}^4)} > 0\}, \tag{2.1}$$

then $\mathcal{V}_\beta \neq \emptyset$ and it is easy to see that $\mathcal{L}_\beta(u,v) > 0$ if $(u,v) \in \mathcal{V}_\beta$. In what follows, we will make some observations on the functional $\mathcal{D}_\lambda(u,v) = \mathcal{D}_{a,\lambda}(u,u) + \mathcal{D}_{b,\lambda}(v,v)$, which are inspired by [9] and [21]. We first study the functional $\mathcal{D}_{a,\lambda}(u,u)$. By the condition (D_1), $\int_{\mathbb{R}^4}(\lambda a(x) + a_0)u^2dx \geq 0$ for all $u \in E_\lambda$ with $\lambda > 0$ in the case of $a_0 \geq 0$. It follows that $\mathcal{D}_{a,\lambda}(u,u)$ is positively definite on E_λ with $\lambda > 0$ in the case of $a_0 \geq 0$. When $a_0 < 0$, by the condition (D_3), we have $\Omega_a \subset \mathcal{A}_\lambda$, which is given by

$$\mathcal{A}_\lambda := \{x \in \mathbb{R}^4 \mid \lambda a(x) + a_0 < 0\}. \tag{2.2}$$

Thus, $\mathcal{A}_\lambda \neq \emptyset$ for every $\lambda > 0$. Let

$$\Lambda_{a,0} := \inf\{\lambda > 0 \mid |\mathcal{A}_\lambda| < +\infty\}. \tag{2.3}$$

By conditions (D_1)-(D_2), we can see that $0 < \Lambda_{a,0} = \frac{-a_0}{a_\infty}$. For $\lambda > \Lambda_{a,0}$, we define

$$\mathcal{F}_{a,\lambda} := \{u \in E_{a,\lambda} \mid \mathrm{supp} u \subset \mathbb{R}^4 \backslash \mathcal{A}_\lambda\}.$$

Then by the conditions (D_1)–(D_2), $\mathcal{F}_{a,\lambda}$ is nonempty and $\mathcal{F}_{a,\lambda} \neq E_{a,\lambda}$. Hence, $E_{a,\lambda} = \mathcal{F}_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}^\perp$ and $\mathcal{F}_{a,\lambda}^\perp \neq \emptyset$ for $\lambda > \Lambda_{a,0}$ in the case of $a_0 < 0$, where $\mathcal{F}_{a,\lambda}^\perp$ is the orthogonal complement of $\mathcal{F}_{a,\lambda}$ in $E_{a,\lambda}$. Now, consider the operator $(-\Delta + (\lambda a(x) + a_0)^+)^{-1}(\lambda a(x) + a_0)^-$, where $(\lambda a(x) + a_0)^- = \max\{-(\lambda a(x) + a_0), 0\}$. Clearly, $(-\Delta + (\lambda a(x) + a_0)^+)^{-1}(\lambda a(x) + a_0)^-$ is linear and self-conjugate on $L^2(\mathbb{R}^4)$ for $\lambda > \Lambda_{a,0}$ in the case of $a_0 < 0$. By the definition of $\Lambda_{a,0}$, we can easily show that $(-\Delta + (\lambda a(x) + a_0)^+)^{-1}(\lambda a(x) + a_0)^-$ is also compact on $L^2(\mathbb{R}^4)$ for $\lambda > \Lambda_{a,0}$ in the case of $a_0 < 0$. Thus, by [46, Theorems 4.45 and 4.46], the eigenvalue problem

$$-\Delta u + (\lambda a(x) + a_0)^+ u = \alpha(\lambda a(x) + a_0)^- u \quad \text{on } \mathcal{F}_{a,\lambda}^\perp \tag{2.4}$$

has a sequence of positive eigenvalues $\{\alpha_{a,j}(\lambda)\}$ satisfying

$$0 < \alpha_{a,1}(\lambda) \leq \alpha_{a,2}(\lambda) \leq \cdots \leq \alpha_{a,j}(\lambda) \to +\infty, \text{ as } j \to +\infty.$$

Furthermore, $\{\alpha_{a,j}(\lambda)\}$ can be characterized by

$$\alpha_{a,j}(\lambda) := \inf_{\dim M \geq j, M \subset \mathcal{F}_{a,\lambda}^\perp} \sup_{u \in M \backslash \{0\}} \frac{\int_{\mathbb{R}^4}(|\nabla u|^2 + (\lambda a(x) + a_0)^+ u^2)dx}{\int_{\mathbb{R}^4}(\lambda a(x) + a_0)^- u^2 dx} \tag{2.5}$$

for all $j \in \mathbb{N}$ and the corresponding eigenfunctions $\{e_{a,j}(\lambda)\}$ can be chosen so that $\int_{\mathbb{R}^3}(\lambda a(x) + a_0)^- e_{a,j}^2(\lambda)dx = 1$ for all $j \in \mathbb{N}$ and are a basis of $\mathcal{F}_{a,\lambda}^\perp$.

Lemma 2.1 *Assume (D_1)–(D_3) and $a_0 < 0$. Then $\alpha_{a,j}(\lambda)$ are nondecreasing in $(\Lambda_{a,0}, +\infty)$ for all $j \in \mathbb{N}$ and $\lim_{\lambda \to +\infty} \alpha_{a,j}(\lambda) = \alpha_{a,j}^0$, where $\alpha_{a,j}^0$ are the eigenvalues of the following equation*

$$-\Delta u = \alpha |a_0| u, \quad u \in H_0^1(\Omega_a). \tag{2.6}$$

In particular, $\alpha_{a,1}^0$ is the first eigenvalue of (2.6).

Proof. Let $\lambda_1 \geq \lambda_2 > \Lambda_{a,0}$, then by the definition of $E_{a,\lambda}$, we have $E_{a,\lambda_1} = E_{a,\lambda_2}$ in the sense of sets. It follows from the condition (D_1) that $\mathcal{F}_{a,\lambda_2} \subset \mathcal{F}_{a,\lambda_1}$, which implies $\mathcal{F}^{\perp}_{a,\lambda_1} \subset \mathcal{F}^{\perp}_{a,\lambda_2}$. Thanks to the condition (D_1) and $a_0 < 0$ once more, we have

$$\frac{\int_{\mathbb{R}^4}(|\nabla u|^2 + (\lambda_1 a(x) + a_0)^+ u^2)dx}{\int_{\mathbb{R}^4}(\lambda_1 a(x) + a_0)^- u^2 dx} \geq \frac{\int_{\mathbb{R}^4}(|\nabla u|^2 + (\lambda_2 a(x) + a_0)^+ u^2)dx}{\int_{\mathbb{R}^4}(\lambda_2 a(x) + a_0)^- u^2 dx}$$

for all $u \in \mathcal{F}^{\perp}_{a,\lambda_1}$. Thus, by the definitions of $\alpha_{a,j}(\lambda_1)$ and $\alpha_{a,j}(\lambda_2)$, we can see that $\alpha_{a,j}(\lambda_2) \leq \alpha_{a,j}(\lambda_1)$, that is, $\alpha_{a,j}(\lambda)$ are nondecreasing in $(\Lambda_{a,0}, +\infty)$ for all $j \in \mathbb{N}$. In what follows, we will show that $\lim_{\lambda\to+\infty} \alpha_{a,j}(\lambda) = \alpha^0_{a,j}$, where $\alpha^0_{a,j}$ is an eigenvalue of (2.6). Indeed, by the condition (D_3), for every $j \in \mathbb{N}$, there exists $\{\varphi_m\}_{1\leq m\leq j} \subset C^\infty_0(\Omega_a)$ such that $\text{supp}\varphi_m \cap \text{supp}\varphi_n = \emptyset$ for $m \neq n$. Let $M_0 = \text{span}\{\varphi_1, \cdots, \varphi_j\}$. Then $M_0 \subset \mathcal{F}^{\perp}_{a,\lambda}$ for $\lambda > \Lambda_{a,0}$ due to $a_0 < 0$ and the condition (D_3) once more. It follows from (2.5) that $\alpha_{a,j}(\lambda) \leq \alpha^*_{a,j}$, where

$$\alpha^*_{a,j} := \sup\left\{ \int_{\Omega_a} |\nabla u| dx \mid u \in M_0 \text{ and } \int_{\Omega_a} |a_0| u^2 dx = 1 \right\}.$$

Since $\alpha_{a,j}(\lambda)$ are positive and nondecreasing in $(\Lambda_{a,0}, +\infty)$ for all $j \in \mathbb{N}$, we have

$$\lim_{\lambda\to+\infty} \alpha_{a,j}(\lambda) = \alpha^0_{a,j} \quad \text{with some } \alpha^0_{a,j} > 0 \text{ for all } j \in \mathbb{N}.$$

Meanwhile, by the choice of $\{e_{a,j}(\lambda)\}$, we have

$$\int_{\mathbb{R}^4} (|\nabla e_{a,j}(\lambda)|^2 + (\lambda a(x) + a_0)^+ [e_{a,j}(\lambda)]^2) dx \leq \alpha^*_{a,j}, \tag{2.7}$$

which then implies that $\{e_{a,j}(\lambda)\}$ is bounded in $D^{1,2}(\mathbb{R}^4)$ for $\lambda > \Lambda_{a,0}$. Therefore, up to a subsequence, $e_{a,j}(\lambda) \rightharpoonup e_{a,j}$ weakly in $D^{1,2}(\mathbb{R}^4)$ and $e_{a,j}(\lambda) \to e_{a,j}$ a.e. in $\mathbb{R}^4$ as $\lambda \to +\infty$. By the Fatou lemma and the condition (D_1), we have $\int_{\mathbb{R}^4} a(x) e^2_{a,j} dx = 0$. This together with the condition (D_3), implies $e_{a,j} = 0$ outside Ω_a and $e_{a,j} \in H^1_0(\Omega_a)$. It follows from the condition (D_2), the Sobolev embedding theorem and (2.7) once more that, up to a subsequence, $e_{a,j}(\lambda) \to e_{a,j}$ strongly in $L^2(\mathbb{R}^4)$ as $\lambda \to +\infty$. Now, by the condition (D_3), for every $\psi \in C^\infty_0(\Omega_a) \subset \mathcal{F}^{\perp}_{a,\lambda}$, we can see from a variant of the Lebesgue dominated convergence theorem (cf. [34, Theorem 2.2]) that

$$\begin{aligned}
\int_{\Omega_a} \nabla e_{a,j} \nabla \psi dx &= \lim_{\lambda\to+\infty} \int_{\mathbb{R}^4} \nabla e_{a,j}(\lambda) \nabla \psi dx \\
&= \lim_{\lambda\to+\infty} \alpha_{a,j}(\lambda) \int_{\mathbb{R}^4} (\lambda a(x) + a_0)^- e_{a,j}(\lambda) \psi dx \\
&= \alpha^0_{a,j} \int_{\Omega_a} |a_0| e_{a,j} \psi dx.
\end{aligned}$$

Hence, $(e_{a,j}, \alpha^0_{a,j})$ satisfies (2.6) and $\alpha^0_{a,j}$ are the eigenvalues of (2.6). Note that

$$\alpha_{a,1}(\lambda) = \inf_{u\in\mathcal{F}^{\perp}_{a,\lambda}} \left\{ \int_{\mathbb{R}^4} (|\nabla u|^2 + (\lambda a(x) + a_0)^+ u^2) dx \mid \int_{\mathbb{R}^4} (\lambda a(x) + a_0)^- u^2 dx = 1 \right\}.$$

By the definition of the first eigenvalue to (2.6) and the condition (D_3), we can easily see that $\alpha^0_{a,1}$ is the first eigenvalue to (2.6). $\blacksquare$

Let $\{\alpha_{a,j}\}$ be the eigenvalues of (2.6) and $\{e^*_{a,j}\}$ be the corresponding eigenfunctions. Then it is well known that $\alpha_{a,j} = \frac{\mu_{a,j}}{|a_0|}$, where $\{\mu_{a,j}\}$ is the eigenvalues of the operator $-\Delta$ in $H^1_0(\Omega_a)$. Furthermore, for every $a_0 < 0$, k_a =dim(span$\{e^*_{a,j} \mid \alpha_{a,j} \le 1\}$) is finite. Let

$$\widehat{\mathcal{F}}^{\perp}_{a,\lambda} = \text{span}\{e_{a,j}(\lambda) \mid \alpha_{a,j}(\lambda) \le 1\} \quad \text{and} \quad \widetilde{\mathcal{F}}^{\perp}_{a,\lambda} = \text{span}\{e_{a,j}(\lambda) \mid \alpha_{a,j}(\lambda) > 1\}.$$

Then $\dim(\widehat{\mathcal{F}}^{\perp}_{a,\lambda}) < +\infty$ and $E_{a,\lambda} = \mathcal{F}_{a,\lambda} \oplus \widehat{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \widetilde{\mathcal{F}}^{\perp}_{a,\lambda}$ for all $\lambda > \Lambda_{a,0}$ in the case of $a_0 < 0$. Furthermore, by Lemma 2.1, $\widehat{\mathcal{F}}^{\perp}_{a,\lambda} = \emptyset$ for $\lambda > \Lambda_{a,0}$ sufficiently large, say $\lambda > \overline{\Lambda}_a > \Lambda_{a,0}$, in the case of $-\mu_{a,1} < a_0 < 0$ and $\widehat{\mathcal{F}}^{\perp}_{a,\lambda} \ne \emptyset$ for all $\lambda > \Lambda_{a,0}$ in the case of $a_0 \le -\mu_{a,1}$, where $\mu_{a,1}$ is the first eigenvalue of $(-\Delta, H^1_0(\Omega_a))$.

Lemma 2.2 *Let the conditions (D_1)–(D_3) hold and $a_0 < 0$. Then there exists $\Lambda^*_a \ge \overline{\Lambda}_a$ such that dim$(\widehat{\mathcal{F}}^{\perp}_{a,\lambda})$ is independent of $\lambda \ge \Lambda^*_a$ and dim$(\widehat{\mathcal{F}}^{\perp}_{a,\lambda}) \le k_a$ for all $\lambda \ge \Lambda^*_a$.*

Proof. In the proof of Lemma 2.1, we obtain that $e_{a,j}(\lambda) \rightharpoonup e_{a,j}$ weakly in $D^{1,2}(\mathbb{R}^4)$ and $e_{a,j}(\lambda) \to e_{a,j} \in H^1_0(\Omega_a)$ strongly in $L^2(\mathbb{R}^4)$ as $\lambda \to +\infty$ up to a subsequence and $\lim_{\lambda\to+\infty} \alpha_{a,j}(\lambda) = \alpha^0_{a,j}$, where $(e_{a,j}(\lambda), \alpha_{a,j}(\lambda))$ and $(e_{a,j}, \alpha^0_{a,j})$ satisfy (2.4) and (2.6), respectively. Since $(\lambda a(x) + a_0)^- \le |a_0|$ due to the condition (D_1), by a variant of the Lebesgue dominated convergence theorem (cf. [34, Theorem 2.2]), we have

$$\lim_{\lambda\to+\infty} \alpha_{a,j}(\lambda) \int_{\mathbb{R}^4} (\lambda a(x) + a_0)^-[e_{a,j}(\lambda)]^2 dx = \alpha^0_{a,j} \int_{\mathbb{R}^4} |a_0| e^2_{a,j} dx.$$

This together with the Fatou's lemma and the conditions (D_1)–(D_3), implies $e_{a,j}(\alpha) \to e_{a,j}$ strongly in $D^{1,2}(\mathbb{R}^4)$ as $\lambda \to +\infty$ up to a subsequence. Now, suppose there exist $j \ne i$ such that $\alpha^0_{a,j} = \alpha^0_{a,i} = \alpha_{a,k}$ for some $k \in \mathbb{N}$. Then one of the following two cases must happen:

(1) $e_{a,j} = e_{a,i}$;

(2) $e_{a,j} \ne e_{a,i}$.

If case (1) happen, then by $a_0 < 0$, Lemma 2.1 and the definition of $e_{a,j}(\lambda)$ and $e_{a,i}(\lambda)$, we have

$$\begin{aligned}
&2\alpha_{a,k} \\
&= \lim_{\lambda\to+\infty}(\alpha_{a,j}(\lambda)+\alpha_{a,i}(\lambda)) \\
&= \lim_{\lambda\to+\infty}\bigg(\int_{\mathbb{R}^4}(|\nabla e_{a,j}(\lambda)|^2+(\lambda a(x)+a_0)^+[e_{a,j}(\lambda)]^2)dx \\
&\quad+\int_{\mathbb{R}^4}(|\nabla e_{a,i}(\lambda)|^2+(\lambda a(x)+a_0)^+[e_{a,i}(\lambda)]^2)dx\bigg) \\
&= \lim_{\lambda\to+\infty}\int_{\mathbb{R}^4}(|\nabla(e_{a,j}(\lambda)-e_{a,i}(\lambda))|^2+(\lambda a(x)+a_0)^+[e_{a,j}(\lambda)-e_{a,i}(\lambda)]^2)dx \\
&= 0,
\end{aligned}$$

which is impossible. Therefore, we must have the case (2). Without loss of generality, we may assume that $\int_{\Omega_a}\nabla e_{a,j}\nabla e_{a,i}dx = 0$ in this case. Let

$$j^*_{a,0} = \inf\{j \in \mathbb{N} \mid \alpha^0_{a,j} > 1\}. \tag{2.8}$$

Then by Lemma 2.1, there exists $\Lambda^*_a > \overline{\Lambda}_a$ such that $\dim(\widehat{\mathcal{F}}^{\perp}_{a,\lambda}) = j^*_{a,0} - 1$ is independent of $\lambda \geq \Lambda^*_a$ and less than or equal to k_a for $\lambda \geq \Lambda^*_a$. ∎

Remark 2.1 *Clearly, the functional $\mathcal{D}_{b,\lambda}(v,v)$ is also positive definite on $E_{b,\lambda}$ for $\lambda > 0$ in the case of $b_0 \geq 0$. In the case of $b_0 < 0$, we can similarly define $\mathcal{B}_\lambda$, $\Lambda_{b,0}$, $\mathcal{F}_{b,\lambda}$, $\alpha_{b,j}(\lambda)$, $\widehat{\mathcal{F}}^{\perp}_{b,\lambda}$, $\widetilde{\mathcal{F}}^{\perp}_{b,\lambda}$, k_b and $j^*_{b,0}$. Then dim$(\widehat{\mathcal{F}}^{\perp}_{b,\lambda}) < +\infty$ and $E_{b,\lambda} = \mathcal{F}_{b,\lambda} \oplus \widehat{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \widetilde{\mathcal{F}}^{\perp}_{b,\lambda}$ for all $\lambda > \Lambda_{b,0}$. Furthermore, by a similar argument as used in Lemma 2.1, we have $\widehat{\mathcal{F}}^{\perp}_{b,\lambda} = \emptyset$ for $\lambda > \Lambda_{b,0}$ sufficiently large, say $\lambda > \overline{\Lambda}_b > \Lambda_{b,0}$, in the case of $-\mu_{b,1} < b_0 < 0$ and $\widehat{\mathcal{F}}^{\perp}_{b,\lambda} \neq \emptyset$ for all $\lambda > \Lambda_{b,0}$ in the case of $b_0 \leq -\mu_{b,1}$, where $\mu_{b,1}$ is the first eigenvalue of $(-\Delta, H^1_0(\Omega_b))$. By a similar argument as used in Lemma 2.2, we also can see that there exists $\Lambda^*_b > \overline{\Lambda}_b$ such that dim$(\widehat{\mathcal{F}}^{\perp}_{b,\lambda})$ is independent of $\lambda \geq \Lambda^*_b$ and dim$(\widehat{\mathcal{F}}^{\perp}_{b,\lambda}) = j^*_{b,0} - 1 \leq k_b$ for all $\lambda \geq \Lambda^*_b$.*

Now, we have the following decomposition of E_λ:

(1) $E_\lambda = \mathcal{F}_{a,\lambda} \times \mathcal{F}_{b,\lambda}$ for $\lambda > 0$ in the case of $b_0 \geq a_0 \geq 0$.

(2) $E_\lambda = (\widetilde{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}) \times \mathcal{F}_{b,\lambda}$ for $\lambda > \overline{\Lambda}_a$ in the case of $-\mu_{a,1} < a_0 < 0 \leq b_0$.

(3) $E_\lambda = (\widetilde{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}) \times (\widetilde{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \mathcal{F}_{b,\lambda})$ for $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$ in the case of $-\mu_{a,1} < a_0 < 0$ and $-\mu_{b,1} < b_0 < 0$.

(4) $E_\lambda = (\widehat{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \widetilde{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}) \times \mathcal{F}_{b,\lambda}$ for $\lambda > \Lambda_{a,0}$ in the case of $a_0 \leq -\mu_{a,1} < 0 \leq b_0$.

(5) $E_\lambda = (\widehat{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \widetilde{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}) \times (\widetilde{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \mathcal{F}_{b,\lambda})$ for $\lambda > \max\{\Lambda_{a,0}, \overline{\Lambda}_b\}$ in the case of $a_0 \leq -\mu_{a,1}$ and $-\mu_{b,1} < b_0 < 0$.

(6) $E_\lambda = (\widetilde{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}) \times (\widehat{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \widetilde{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \mathcal{F}_{b,\lambda})$ for $\lambda > \max\{\Lambda_{b,0}, \overline{\Lambda}_a\}$ in the case of $-\mu_{a,1} < a_0 \leq b_0 \leq -\mu_{b,1}$.

(7) $E_\lambda = (\widehat{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \widetilde{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}) \times (\widehat{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \widetilde{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \mathcal{F}_{b,\lambda})$ for $\lambda > \max\{\Lambda_{a,0}, \Lambda_{b,0}\}$ in the case of $a_0 \leq -\mu_{a,1}$, $b_0 \leq -\mu_{b,1}$.

Moreover, we have the following estimates.

Lemma 2.3 *Let the conditions* (D_1)*–*(D_3) *hold and* $a_0, b_0 \in \mathbb{R}$*. Then*

(i) $\mathcal{D}_{a,\lambda}(u,u) = \|u\|^2_{a,\lambda}$ *on* $\mathcal{F}_{a,\lambda}$ *and* $\mathcal{D}_{b,\lambda}(v,v) = \|v\|^2_{b,\lambda}$ *on* $\mathcal{F}_{b,\lambda}$ *for all* $\lambda > 0$.

(ii) $\mathcal{D}_{a,\lambda}(u,u) \geq (1 - \frac{1}{\alpha_{a,j_{a,\lambda}}(\lambda)})\|u\|^2_{a,\lambda}$ *on* $\widetilde{\mathcal{F}}^{\perp}_{a,\lambda}$ *and*

$$\mathcal{D}_{b,\lambda}(v,v) \geq (1 - \frac{1}{\alpha_{b,j_{b,\lambda}}(\lambda)})\|v\|^2_{b,\lambda}$$

on $\widetilde{\mathcal{F}}^{\perp}_{b,\lambda}$ *for all* $\lambda > \max\{\Lambda_{a,0}, \Lambda_{b,0}\}$*, where* $j_{a,\lambda}$ $=$*dim*$(\widehat{\mathcal{F}}^{\perp}_{a,\lambda}) + 1$ *and* $j_{b,\lambda}$ $=$*dim*$(\widehat{\mathcal{F}}^{\perp}_{b,\lambda}) + 1$.

(iii) $\mathcal{D}_{a,\lambda}(u,u) \leq 0$ *on* $\widehat{\mathcal{F}}^{\perp}_{a,\lambda}$ *and* $\mathcal{D}_{b,\lambda}(v,v) \leq 0$ *on* $\widehat{\mathcal{F}}^{\perp}_{b,\lambda}$ *for* $\lambda > \max\{\Lambda_{a,0}, \Lambda_{b,0}\}$.

Proof. The conclusions follow immediately from the definitions of $\widehat{\mathcal{F}}^{\perp}_{a,\lambda}$, $\widetilde{\mathcal{F}}^{\perp}_{a,\lambda}$, $\mathcal{F}_{a,\lambda}$ and $\widehat{\mathcal{F}}^{\perp}_{b,\lambda}$, $\widetilde{\mathcal{F}}^{\perp}_{b,\lambda}$, $\mathcal{F}_{b,\lambda}$. ▮

By Lemma 2.3, we can see that the functional $\mathcal{D}_\lambda(u,v)$ is positively definite on E_λ in the cases of (1)–(3) and indefinite on E_λ in the cases of (4)–(7). For the sake of convenience, we always denote

$$E_\lambda = (\widetilde{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}) \times (\widetilde{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \mathcal{F}_{b,\lambda})$$

and

$$E_\lambda = (\widehat{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \widetilde{\mathcal{F}}^{\perp}_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}) \times (\widehat{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \widetilde{\mathcal{F}}^{\perp}_{b,\lambda} \oplus \mathcal{F}_{b,\lambda})$$

in the definite case and the indefinite case, respectively.

3 The sets $\mathcal{N}_{\lambda,\beta}$, $\mathcal{M}_{\lambda,\beta}$ and $\mathcal{G}_{\lambda,\beta}$

In this section, we will drive some properties of the sets $\mathcal{N}_{\lambda,\beta}$, $\mathcal{M}_{\lambda,\beta}$ and $\mathcal{G}_{\lambda,\beta}$. We start by the observations on $\mathcal{M}_{\lambda,\beta}$. It is well known that $\mathcal{M}_{\lambda,\beta}$ is closely linked to the so-called fibering maps of $J_{\lambda,\beta}(u,v)$, which are the functions defined on $\mathbb{R}^+$ and given by $\overline{T}_{\lambda,\beta,u,v}(t) = J_{\lambda,\beta}(tu,tv)$ for each $(u,v) \in E_\lambda \backslash \{(0,0)\}$. Clearly, $\overline{T}_{\lambda,\beta,u,v}(t) \in C^2(\mathbb{R}^+)$. Moreover, $\overline{T}'_{\lambda,\beta,u,v}(t) = 0$ is equivalent to $(tu,tv) \in \mathcal{M}_{\lambda,\beta}$. In particular, $\overline{T}'_{\lambda,\beta,u,v}(1) = 0$ if and only if $(u,v) \in \mathcal{M}_{\lambda,\beta}$.

Lemma 3.1 *Assume* (D_1)-(D_3) *hold and* $\mathcal{D}_\lambda(u,v)$ *is positively definite on* E_λ. *Then for every* $(u,v) \in E_\lambda\backslash\{(0,0)\}$ *with* $\mathcal{L}_\beta(u,v) > 0$, *there exists a unique* $t_{u,v} = \left(\frac{\mathcal{D}_\lambda(u,v)}{\mathcal{L}_\beta(u,v)}\right)^{\frac{1}{2}}$ *such that* $\overline{T}'_{\lambda,\beta,u,v}(t_{u,v}) = 0$, $(t_{u,v}u, t_{u,v}v) \in \mathcal{M}_{\lambda,\beta}$ *and* $\overline{T}_{\lambda,\beta,u,v}(t_{u,v}) = \max_{t\geq 0} \overline{T}_{\lambda,\beta,u,v}(t)$. *Furthermore, for every* $(u,v) \in E_\lambda\backslash\{(0,0)\}$ *with* $\mathcal{L}_\beta(u,v) \leq 0$, *we have* $\mathcal{X}_{u,v} \cap \mathcal{M}_{\lambda,\beta} = \emptyset$, *where* $\mathcal{X}_{u,v} = \{(tu,tv) \mid t \in \mathbb{R}^+\}$.

Proof. The proof is very standard, so we omit it here. ∎

Due to Lemma 3.1, we can see that

$$m^*_{\lambda,\beta} = \inf_{E_\lambda\backslash\{(0,0)\}} \frac{\mathcal{D}_\lambda(u,v)^2}{4\mathcal{L}_\beta(u,v)}. \tag{3.1}$$

Lemma 3.2 *Let* (D_1)-(D_3) *hold and* $\mathcal{D}_\lambda(u,v)$ *be positively definite on* E_λ. *If* (u,v) *is the minimizer of* $J_{\lambda,\beta}(u,v)$ *on* $\mathcal{M}_{\lambda,\beta}$, *then we have* $D[J_{\lambda,\beta}(u,v)] = 0$ *in* E^*_λ.

Proof. The proof is standard. Since $J_{\lambda,\beta}(u,v)$ is C^2 in E_λ, by the method of Lagrange multipliers, there exists $\nu \in \mathbb{R}$ such that

$$D[J_{\lambda,\beta}(u,v)] - \nu D[\Psi_{\lambda,\beta}(u,v)] = 0 \quad \text{in} \quad E^*_\lambda,$$

where $\Psi_{\lambda,\beta}(u,v) = \langle D[J_{\lambda,\beta}(u,v)], (u,v)\rangle_{E^*_\lambda,E_\lambda}$. Multiplying this equation with (u,v) and noting that $(u,v) \in \mathcal{M}_{\lambda,\beta}$, we have

$$-\nu\langle D[\Psi_{\lambda,\beta}(u,v)], (u,v)\rangle_{E^*_\lambda,E_\lambda} = 2\nu\mathcal{D}_\lambda(u,v) = 0.$$

Since $\mathcal{D}_\lambda(u,v)$ is positively definite on E_λ, we must have $\nu = 0$. It follows that $D[J_{\lambda,\beta}(u,v)] = 0$ in E^*_λ, which completes the proof. ∎

We next look at the set $\mathcal{N}_{\lambda,\beta}$. From the point of the fibering maps, $\mathcal{N}_{\lambda,\beta}$ is closely linked to the functions defined on $\mathbb{R}^+ \times \mathbb{R}^+$ and given by $T_{\lambda,\beta,u,v}(t,s) = J_{\lambda,\beta}(tu,sv)$ for each $(u,v) \in (E_{a,\lambda}\backslash\{0\}) \times (E_{b,\lambda}\backslash\{0\})$. $T_{\lambda,\beta,u,v}(t,s) \in C^2(\mathbb{R}^+ \times \mathbb{R}^+)$ and

$$\frac{\partial T_{\lambda,\beta,u,v}}{\partial t}(t,s) = \frac{\partial T_{\lambda,\beta,u,v}}{\partial s}(t,s) = 0$$

is equivalent to $(tu,sv) \in \mathcal{N}_{\lambda,\beta}$. In particular, $\frac{\partial T_{\lambda,\beta,u,v}}{\partial t}(1,1) = \frac{\partial T_{\lambda,\beta,u,v}}{\partial s}(1,1) = 0$ if and only if $(u,v) \in \mathcal{N}_{\lambda,\beta}$.

Lemma 3.3 *Assume* (D_1)-(D_3) *hold and* $\beta \leq 0$. *If* $\mathcal{D}_{a,\lambda}(u,u)$ *and* $\mathcal{D}_{b,\lambda}(v,v)$ *are respectively definite on* $E_{a,\lambda}$ *and* $E_{b,\lambda}$, *then we have the following.*

(1) *If* $(u,v) \in \mathcal{V}_{\lambda,\beta}$, *then there exists a unique* $(t_{\lambda,\beta}(u,v), s_{\lambda,\beta}(u,v)) \in \mathbb{R}^+\times\mathbb{R}^+$ *such that*

$$(t_{\lambda,\beta}(u,v)u, s_{\lambda,\beta}(u,v)v) \in \mathcal{N}_{\lambda,\beta},$$

where $\mathcal{V}_{\lambda,\beta} = E_\lambda \cap \mathcal{V}_\beta$ *and* $\mathcal{V}_\beta$ *is given by* (2.1) *and* $t_{\lambda,\beta}(u,v)$ *and* $s_{\lambda,\beta}(u,v)$ *are respectively given by*

$$t_{\lambda,\beta}(u,v) = \left(\frac{\|v\|^4_{L^4(\mathbb{R}^4)}\mathcal{D}_{a,\lambda}(u,u) - \beta\|u^2v^2\|_{L^1(\mathbb{R}^4)}\mathcal{D}_{b,\lambda}(v,v)}{\|u\|^4_{L^4(\mathbb{R}^4)}\|v\|^4_{L^4(\mathbb{R}^4)} - \beta^2\|u^2v^2\|^2_{L^1(\mathbb{R}^4)}} \right)^{\frac{1}{2}} \tag{3.2}$$

and

$$s_{\lambda,\beta}(u,v) = \left(\frac{\|u\|^4_{L^4(\mathbb{R}^4)}\mathcal{D}_{b,\lambda}(v,v) - \beta\|u^2v^2\|_{L^1(\mathbb{R}^4)}\mathcal{D}_{a,\lambda}(u,u)}{\|u\|^4_{L^4(\mathbb{R}^4)}\|v\|^4_{L^4(\mathbb{R}^4)} - \beta^2\|u^2v^2\|^2_{L^1(\mathbb{R}^4)}} \right)^{\frac{1}{2}}. \tag{3.3}$$

Moreover, $T_{\lambda,\beta,u,v}(t_{\lambda,\beta}(u,v), s_{\lambda,\beta}(u,v)) = \max_{t\geq 0, s\geq 0} T_{\lambda,\beta,u,v}(t,s)$*. In particular, we have*

$$T_{\lambda,\beta,u,v}(1,1) = \max_{t\geq 0, s\geq 0} T_{\lambda,\beta,u,v}(t,s) \tag{3.4}$$

for all $(u,v) \in \mathcal{N}_{\lambda,\beta}$*.*

(2) *If* $(u,v) \in E_\lambda \backslash \mathcal{V}_{\lambda,\beta}$*, then* $\mathcal{X}_{u,v} \cap \mathcal{N}_{\lambda,\beta} = \emptyset$*, where* $\mathcal{X}_{u,v} = \{(tu, sv) \mid (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+\}$*.*

Proof. Since $\mathcal{D}_{a,\lambda}(u,u)$ and $\mathcal{D}_{b,\lambda}(v,v)$ are respectively positively definite on $E_{a,\lambda}$ and $E_{b,\lambda}$, the proof is similar to that of [49, Lemma 3.1] and only some trivial modifications are needed, so we omit the details here. $\blacksquare$

The relation between $\mathcal{N}_{\lambda,\beta}$ and $T_{\lambda,\beta,u,v}(t,s)$ for $\beta > 0$ is quite different from the case of $\beta \leq 0$. In the case of $0 < \beta < 1$, we have from the Hölder inequality that $\mathcal{V}_{\lambda,\beta} = E_\lambda \backslash \{0\}$. However, the properties described in Lemma 3.3 may not hold for all $(u,v) \in \mathcal{V}_{\lambda,\beta} = E_\lambda \backslash \{0\}$ except (3.4).

Lemma 3.4 *Assume* (D_1)*-*(D_3) *hold and* $\beta \in (0,1)$*. If* $\mathcal{D}_{a,\lambda}(u,u)$ *and* $\mathcal{D}_{b,\lambda}(v,v)$ *are positively definite on* $E_{a,\lambda}$ *and* $E_{b,\lambda}$ *respectively, then* (3.4) *holds for every* $(u,v) \in \mathcal{N}_{\lambda,\beta}$*.*

Proof. Suppose $(u,v) \in \mathcal{N}_{\lambda,\beta}$ and consider the following two-component systems of algebraic equations

$$\begin{cases} \mathcal{D}_{a,\lambda}(u,u) - \|u\|^4_{L^4(\mathbb{R}^4)}t - \beta\|u^2v^2\|_{L^1(\mathbb{R}^4)}s = 0, \\ \mathcal{D}_{b,\lambda}(v,v) - \|v\|^4_{L^4(\mathbb{R}^4)}s - \beta\|u^2v^2\|_{L^1(\mathbb{R}^4)}t = 0. \end{cases} \tag{3.5}$$

Since $(u,v) \in \mathcal{N}_{\lambda,\beta}$, we can see that (3.5) has a unique solution $(1,1)$. It follows that $(1,1)$ is the unique critical point of $T_{\lambda,\beta,u,v}(t,s)$ in $\mathbb{R}^+ \times \mathbb{R}^+$. By the fact that $\beta \in (0,1)$, a direct calculation and the Hölder inequality, we have

$$\frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial t^2}(1,1) = -2\|u\|^4_{L^4(\mathbb{R}^4)} < 0$$

and

$$\begin{vmatrix} \frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial t^2}(1,1) & \frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial t \partial s}(1,1) \\ \frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial s \partial t}(1,1) & \frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial s^2}(1,1) \end{vmatrix}$$
$$= 4(\|u\|^4_{L^4(\mathbb{R}^4)}\|v\|^4_{L^4(\mathbb{R}^4)} - \beta^2\|u^2v^2\|^2_{L^1(\mathbb{R}^4)}) > 0.$$

Note that $\mathcal{D}_{a,\lambda}(u,u)$ and $\mathcal{D}_{b,\lambda}(v,v)$ are positively definite on $E_{a,\lambda}$ and $E_{b,\lambda}$ respectively and $\beta \in (0,1)$, by Lemma 2.3, (1.2) and a standard argument, we can obtain that $T_{\lambda,\beta,u,v}(t,s) > 0$ for $|(t,s)|$ sufficiently small and $T_{\lambda,\beta,u,v}(t,s) \to -\infty$ as $|(t,s)| \to +\infty$. These imply that $(1,1)$ is the global maximum point of $T_{\lambda,\beta,u,v}(t,s)$ in $\mathbb{R}^+ \times \mathbb{R}^+$, i.e., (3.4) holds. ∎

Remark 3.1 *The relation between $\mathcal{N}_{\lambda,\beta}$ and $T_{\lambda,\beta,u,v}(t,s)$ for $\beta \geq 1$ is much more complicated than that of $\beta < 1$ and even* (3.4) *does not hold for some $(u,v) \in \mathcal{N}_{\lambda,\beta}$ in this case.*

Lemma 3.5 *Suppose that* (D_1)-(D_3) *hold and $\mathcal{D}_\lambda(u,v)$ is positively definite on E_λ. If (u,v) is the minimizer of $J_{\lambda,\beta}(u,v)$ on $\mathcal{N}_{\lambda,\beta}$ with $\beta < 1$, then $D[J_{\lambda,\beta}(u,v)] = 0$ in E^*_λ.*

Proof. The proof is also standard. We only give the proof for the case of $-\mu_{a,1} < a_0 < 0$ and $-\mu_{b,1} < b_0 < 0$, since other cases are more simple and can be proved in a similar way due to Lemma 2.3. Since $J_{\lambda,\beta}(u,v)$ is C^2 in E_λ, by the method of Lagrange multipliers, there exists $\nu_1, \nu_2 \in \mathbb{R}$ such that

$$D[J_{\lambda,\beta}(u,v)] - \nu_1 D[\Psi^{*,1}_{\lambda,\beta}(u,v)] - \nu_2 D[\Psi^{*,2}_{\lambda,\beta}(u,v)] = 0 \text{ in } E^*_\lambda,$$

where

$$\Psi^{*,1}_{\lambda,\beta}(u,v) = \langle D[J_{\lambda,\beta}(u,v)], (u,0)\rangle_{E^*_\lambda,E_\lambda}$$

and

$$\Psi^{*,2}_{\lambda,\beta}(u,v) = \langle D[J_{\lambda,\beta}(u,v)], (0,v)\rangle_{E^*_\lambda,E_\lambda}.$$

Multiplying this equation with $(u,0)$ and $(0,v)$ respectively and noting that $(u,v) \in \mathcal{N}_{\lambda,\beta}$, we have

$$\begin{cases} 2\nu_1\|u\|^4_{L^4(\mathbb{R}^4)} + 2\nu_2\beta\|u^2v^2\|_{L^1(\mathbb{R}^4)} = 0, \\ 2\nu_2\|v\|^4_{L^4(\mathbb{R}^4)} + 2\nu_1\beta\|u^2v^2\|_{L^1(\mathbb{R}^4)} = 0. \end{cases}$$

It follows that either $\nu_1 = \nu_2 = 0$ or $\|u\|^4_{L^4(\mathbb{R}^4)}\|v\|^4_{L^4(\mathbb{R}^4)} - \beta^2\|u^2v^2\|^2_{L^1(\mathbb{R}^4)} = 0$. By the Hölder inequality and Lemma 3.3, we can see that $\mathcal{N}_{\lambda,\beta} \subset \mathcal{V}_{\lambda,\beta}$ with $\beta < 1$. Therefore, we must have $\nu_1 = \nu_2 = 0$, which implies $D[J_{\lambda,\beta}(u,v)] = 0$ in E^*_λ. ∎

Next we consider the set $\mathcal{G}_{\lambda,\beta}$. Since $\mathcal{G}_{\lambda,\beta}$ is modified from $\mathcal{M}_{\lambda,\beta}$, the firbering maps $\overline{T}_{\lambda,\beta,u,v}(t)$ also need to be modified. For every $(u,v) \in E_\lambda \backslash \{(0,0)\}$, we define

$$G_{\lambda,\beta,u,v}(w,\sigma,t) : \widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp} \times \mathbb{R}^+ \to \mathbb{R}$$

by $G_{\lambda,\beta,u,v}(w,\sigma,t) = J_{\lambda,\beta}(w+t\widetilde{u}, \sigma+t\widetilde{v})$, then $G(w,\sigma,t)$ is C^2 in $\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp} \times \mathbb{R}^+$, where $\widetilde{u}$ and $\widetilde{v}$ are the projections of u and v on $\widetilde{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathcal{F}_{a,\lambda}$ and $\widetilde{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathcal{F}_{b,\lambda}$. In what follows, we will borrow some ideas from [40] to observe the set $\mathcal{G}_{\lambda,\beta}$ by $G_{\lambda,\beta,u,v}(w,\sigma,t)$.

Lemma 3.6 *Assume (D_1)-(D_3) hold and either $a_0 \le -\mu_{a,1}$ or $b_0 \le -\mu_{b,1}$. If $0 \le \beta < 1$ and $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$, then for every $(u,v) \in \mathcal{G}_{\lambda,\beta}$, $G_{\lambda,\beta,u,v}(\widehat{u},\widehat{v},1) \ge G_{\lambda,\beta,u,v}(w,\sigma,t)$ for all $t \in (0,+\infty)$ and $(w,\sigma) \in \widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp}$, where $(\widehat{u},\widehat{v})$ is the projection of (u,v) in $\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp}$. Furthermore, the equality holds if and only if $t = 1$ and $(w,\sigma) = (\widehat{u},\widehat{v})$.*

Proof. We only give the proof for the case of $a_0 \le -\mu_{a,1}$ and $b_0 \le -\mu_{b,1}$, since the proofs of other cases are similar and more simple due to Lemma 2.3. Suppose $(u,v) \in \mathcal{G}_{\lambda,\beta}$, $t \in (0,+\infty)$ and $(w,\sigma) \in \widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp}$. Then we have

$$\begin{aligned}
&J_{\lambda,\beta}(u,v) - J_{\lambda,\beta}(tu+w, tv+\sigma)\\
&= \frac{1}{2}(\mathcal{D}_\lambda(u,v) - \mathcal{D}_\lambda(tu+w, tv+\sigma)) - \frac{1}{4}\int_{\mathbb{R}^4} u^4 + v^4 + 2\beta u^2 v^2 dx\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^4} (tu+w)^4 + (tv+\sigma)^4 + 2\beta(tu+w)^2(tv+\sigma)^2 dx\\
&= \frac{1}{2}((1-t^2)\mathcal{D}_{a,\lambda}(u,u) - 2t\mathcal{D}_{a,\lambda}(u,w) - \mathcal{D}_{a,\lambda}(w,w))\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^4} (tu+w)^4 - u^4 dx\\
&\quad + \frac{1}{2}((1-t^2)\mathcal{D}_{b,\lambda}(v,v) - 2t\mathcal{D}_{b,\lambda}(v,\sigma) - \mathcal{D}_{b,\lambda}(\sigma,\sigma))\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^4} (tv+\sigma)^4 - v^4 dx\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^4} 2\beta((tu+w)^2(tv+\sigma)^2 - u^2v^2) dx.
\end{aligned}$$

It follows from the definition of $\mathcal{G}_{\lambda,\beta}$, $(w,\sigma)\in\widehat{\mathcal{F}}_{a,\lambda}^{\perp}\times\widehat{\mathcal{F}}_{b,\lambda}^{\perp}$ and Lemma 2.3 that

$$\begin{aligned}
&J_{\lambda,\beta}(u,v)-J_{\lambda,\beta}(tu+w,tv+\sigma)\\
&=-\frac{1}{2}\mathcal{D}_{a,\lambda}(w,w)+\frac{1}{4}\int_{\mathbb{R}^4}(tu+w)^4-u^4+2(1-t^2)u^4-4tu^3wdx\\
&\quad-\frac{1}{2}\mathcal{D}_{b,\lambda}(\sigma,\sigma)+\frac{1}{4}\int_{\mathbb{R}^4}(tv+\sigma)^4-v^4+2(1-t^2)v^4-4tv^3\sigma dx\\
&\quad+\frac{\beta}{2}\int_{\mathbb{R}^4}(tu+w)^2(tv+\sigma)^2-u^2v^2+(2-2t^2)u^2v^2dx\\
&\quad-\beta\int_{\mathbb{R}^4}tv^2uw+tu^2v\sigma dx\\
&\geq\frac{1}{4}\int_{\mathbb{R}^4}(tu+w)^4+u^4-2u^2(tu+w)^2+2u^2w^2dx\\
&\quad+\frac{1}{4}\int_{\mathbb{R}^4}(tv+\sigma)^4+v^4-2v^2(tv+\sigma)^2+2v^2\sigma^2dx\\
&\quad+\frac{\beta}{2}\int_{\mathbb{R}^4}((tu+w)^2-u^2)((tv+\sigma)^2-v^2)+v^2w^2+u^2\sigma^2dx\\
&=\frac{1}{4}\int_{\mathbb{R}^4}\Big(((tu+w)^2-u^2)^2+((tv+\sigma)^2-v^2)^2+\\
&\quad 2\beta((tu+w)^2-u^2)((tv+\sigma)^2-v^2)\Big)dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^4}u^2w^2+v^2\sigma^2+\beta v^2w^2+\beta u^2\sigma^2dx.
\end{aligned}$$

Since $\beta\in[0,1)$, we have

$$J_{\lambda,\beta}(u,v)-J_{\lambda,\beta}(tu+w,tv+\sigma)\geq\frac{1}{4}\int_{\mathbb{R}^4}(|(tu+w)^2-u^2|-|(tv+\sigma)^2-v^2|)^2dx\geq 0$$

and the equalities hold if and only if $t=1$ and $(w,\sigma)=(0,0)$. ∎

By Lemma 3.6, we have the following important observation for $\mathcal{G}_{\lambda,\beta}$.

Lemma 3.7 *Assume (D_1)-(D_3) hold and either $a_0\leq-\mu_{a,1}$ or $b_0\leq-\mu_{b,1}$. If $0\leq\beta<1$ and $\lambda>\max\{\overline{\Lambda}_a,\overline{\Lambda}_b\}$, then for every $(u,v)\in\widetilde{E}_\lambda$, there exists a unique $(w_\lambda^0,\sigma_\lambda^0,t_\lambda^0)\in\widehat{\mathcal{F}}_{a,\lambda}^{\perp}\times\widehat{\mathcal{F}}_{b,\lambda}^{\perp}\times\mathbb{R}^+$ such that $(u_{\lambda,\beta}^0,v_{\lambda,\beta}^0)=(w_\lambda^0+t_\lambda^0\widetilde{u},\sigma_\lambda^0+t_\lambda^0\widetilde{v})\in\mathcal{G}_{\lambda,\beta}$, where $\widetilde{u}$ and $\widetilde{v}$ are the projections of u and v on $\widetilde{\mathcal{F}}_{a,\lambda}^{\perp}\oplus\mathcal{F}_{a,\lambda}$ and $\widetilde{\mathcal{F}}_{b,\lambda}^{\perp}\oplus\mathcal{F}_{b,\lambda}$. Furthermore, we also have*

$$G_{\lambda,\beta,u,v}(w_\lambda^0,\sigma_\lambda^0,t_\lambda^0)=\max_{\widehat{\mathcal{F}}_{a,\lambda}^{\perp}\times\widehat{\mathcal{F}}_{b,\lambda}^{\perp}\times\mathbb{R}^+}G_{\lambda,\beta,u,v}(w,\sigma,t).\tag{3.6}$$

Proof. We only give the proof for the case of $a_0\leq-\mu_{a,1}$ and $b_0\leq-\mu_{b,1}$, since the proofs of other cases are similar and more simple due to Lemma 2.3.

Clearly, $(\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathbb{R}^+ u) \times (\widehat{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathbb{R}^+ v) = (\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{u}) \times (\widehat{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{v})$ for every $(u, v) \in E_\lambda$ with $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$. By the definitions of $\widehat{\mathcal{F}}_{a,\lambda}^{\perp}$ and $\widehat{\mathcal{F}}_{b,\lambda}^{\perp}$, we can also see that $\dim((\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{u}) \times (\widehat{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{v})) < +\infty$ for every $(u, v) \in E_\lambda$ with $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$. On the other hand, since $(u, v) \in \widetilde{E}_\lambda$ with $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$, we must have $\widetilde{u} \neq 0$ or $\widetilde{v} \neq 0$. Since $0 \leq \beta < 1$, by Lemma 2.3 and (1.2), we have that

$$
\begin{aligned}
& J_{\lambda,\beta}(t\widetilde{u}, t\widetilde{v}) \\
= \;& \frac{t^2}{2}(\mathcal{D}_{a,\lambda}(\widetilde{u}, \widetilde{u}) + \mathcal{D}_{b,\lambda}(\widetilde{v}, \widetilde{v})) \\
& -\frac{t^4}{4}(\|\widetilde{u}\|_{L^4(\mathbb{R}^4)}^4 + \|\widetilde{v}\|_{L^4(\mathbb{R}^4)}^4 + 2\beta\|\widetilde{u}^2\widetilde{v}^2\|_{L^1(\mathbb{R}^4)}^1) \\
\geq \;& \frac{t^2 d_\lambda^*}{2}(\|\widetilde{u}\|_{a,\lambda}^2 + \|\widetilde{v}\|_{b,\lambda}^2) - \frac{t^4 S^{-2}}{4}(\|\widetilde{u}\|_{a,\lambda}^2 + \|\widetilde{v}\|_{b,\lambda}^2)^2 \\
= \;& \frac{t^2}{4}(\|\widetilde{u}\|_{a,\lambda}^2 + \|\widetilde{v}\|_{b,\lambda}^2)(2d_\lambda^* - t^2 S^{-2}(\|\widetilde{u}\|_{a,\lambda}^2 + \|\widetilde{v}\|_{b,\lambda}^2)) \\
> \;& 0
\end{aligned} \tag{3.7}
$$

for $t > 0$ sufficiently small, where

$$
d_\lambda^* = \min\left\{1 - \frac{1}{\alpha_{a,j_{a,\lambda}}(\lambda)}, 1 - \frac{1}{\alpha_{a,j_{b,\lambda}}(\lambda)}\right\} > 0 \tag{3.8}
$$

and $j_{a,\lambda}$ and $j_{b,\lambda}$ are given by Lemma 2.3. Note that $0 \leq \beta < 1$, then for every $(w, \sigma) \in (\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{u}) \times (\widehat{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{v})$ with $\|w\|_{a,\lambda}^2 + \|\sigma\|_{b,\lambda}^2 = 1$, we have from the Hölder inequality that $\mathcal{L}_\beta(w, \sigma) > 0$, which then implies

$$
J_{\lambda,\beta}(Rw, R\sigma) \;\; \leq \;\; \frac{R^2}{2} - \frac{R^4}{4}(\|w\|_{L^4(\mathbb{R}^4)}^4 + \|\sigma\|_{L^4(\mathbb{R}^4)}^4 + 2\beta\|w^2\sigma^2\|_{L^1(\mathbb{R}^4)}) \to -\infty
$$

as $R \to +\infty$. Since $\dim((\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{u}) \times (\widehat{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{v})) < +\infty$, there exists $R_\lambda > 0$ such that

$$
J_{\lambda,\beta}(R_\lambda w, R_\lambda \sigma) \leq -1 \tag{3.9}
$$

for all $(w, \sigma) \in (\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{u}) \times (\widehat{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{v})$ with $\|w\|_{a,\lambda}^2 + \|\sigma\|_{b,\lambda}^2 = 1$. Since $G_{\lambda,\beta,u,v}(w, \sigma, t)$ is of C^2 in $\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp} \times \mathbb{R}^+$, noting

$$
\dim((\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{u}) \times (\widehat{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathbb{R}^+ \widetilde{v})) < +\infty,
$$

there exists $(w_\lambda^0, \sigma_\lambda^0, t_\lambda^0) \in \widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp} \times \mathbb{R}^+$ such that (3.6) holds. It follows that $(w_\lambda^0, \sigma_\lambda^0, t_\lambda^0)$ is a critical point of $G_{\lambda,\beta,u,v}(w, \sigma, t)$ in $\widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp} \times \mathbb{R}^+$. Therefore, $(u_{\lambda,\beta}^0, v_{\lambda,\beta}^0) = (w_\lambda^0 + t_\lambda^0 \widetilde{u}, \sigma_\lambda^0 + t_\lambda^0 \widetilde{v}) \in \mathcal{G}_{\lambda,\beta}$. Note that $(u_{\lambda,\beta}^0, v_{\lambda,\beta}^0) \in \mathcal{G}_{\lambda,\beta}$ and $(w_\lambda^0, \sigma_\lambda^0, t_\lambda^0)$ satisfy (3.6), by Lemma 3.6, $(w_\lambda^0, \sigma_\lambda^0, t_\lambda^0)$ must be unique, which completes the proof. ∎

Remark 3.2 *By Lemma 2.2 and Remark 2.1, we have* $j_{a,\lambda} = j^*_{a,0} + 1 \leq k_a + 1$ *for* $\lambda > \Lambda^*_a$ *and* $j_{b,\lambda} = j^*_{b,0} + 1 \leq k_b + 1$ *for* $\lambda > \Lambda^*_b$*. It follows that* d^*_λ *given by* (3.8) *is independent of* $\lambda > \Lambda^*_a$*. Furthermore, if* $(u, v) \in E_a \times E_b$ *satisfies that* $\|(u, v)\|_\lambda \leq C$ *and* $C > 0$ *is independent of* λ*, then* R_λ *given by* (3.9) *is also independent of* λ *sufficient large, say* $\lambda \geq \Lambda^*_0 \geq \max\{\Lambda^*_a, \Lambda^*_b\}$.

In what follows, we will give some estimates of $J_{\lambda,\beta}(u, v)$ on the sets $\mathcal{N}_{\lambda,\beta}$, $\mathcal{M}_{\lambda,\beta}$ and $\mathcal{G}_{\lambda,\beta}$. More precisely, we will give some estimates of $m_{\lambda,\beta}$, $m^*_{\lambda,\beta}$ and $c_{\lambda,\beta}$. We begin with the estimates of the upper boundary to $m_{\lambda,\beta}$ and $m^*_{\lambda,\beta}$. Let $I_{\Omega_a}(u)$ and $I_{\Omega_b}(v)$ be two functionals respectively defined on $H^1_0(\Omega_a)$ and $H^1_0(\Omega_b)$, which are given by

$$I_{\Omega_a}(u) := \frac{1}{2}\int_{\Omega_a} |\nabla u|^2 + a_0 u^2 dx - \frac{1}{4}\int_{\Omega_a} u^4 dx,$$

$$I_{\Omega_b}(v) := \frac{1}{2}\int_{\Omega_b} |\nabla v|^2 + b_0 v^2 dx - \frac{1}{4}\int_{\Omega_b} v^4 dx.$$

Then it is well known that $I_{\Omega_a}(u)$ and $I_{\Omega_b}(v)$ are of C^2 in $H^1_0(\Omega_a)$ and $H^1_0(\Omega_b)$, respectively. Define

$$\mathcal{N}_a := \{u \in H^1_0(\Omega_a)\backslash\{0\} \mid I'_{\Omega_a}(u)u = 0\},$$

$$\mathcal{N}_b := \{v \in H^1_0(\Omega_b)\backslash\{0\} \mid I'_{\Omega_b}(v)v = 0\}.$$

Then it is easy to show that $\mathcal{N}_a$ and $\mathcal{N}_b$ are all nonempty. Let

$$m_a := \inf_{\mathcal{N}_a} I_a(u), \quad m_b = \inf_{\mathcal{N}_b} I_b(v).$$

Then it is well known that $m_a = \frac{1}{4}S^2$ in the case of $a_0 \geq 0$ and $m_a < \frac{1}{4}S^2$ in the case of $-\mu_{a,1} < a_0 < 0$ while $m_b = \frac{1}{4}S^2$ in the case of $b_0 \geq 0$ and $m_b < \frac{1}{4}S^2$ in the case of $-\mu_{b,1} < b_0 < 0$ due to the condition (D_3) (cf. [37]).

Lemma 3.8 *Let* (D_1)*-*(D_3) *hold and* $\mathcal{D}_\lambda(u, v)$ *be positively definite in* E_λ*. Then* $m_a + m_b \geq m_{\lambda,\beta}$ *and* $\min\{m_a, m_b\} \geq m^*_{\lambda,\beta}$ *for all* $\beta \in \mathbb{R}$*.*

Proof. Without loss of generality, we assume $m_a \leq m_b$. Since $\mathcal{N}_a \times \mathcal{N}_b \subset \mathcal{N}_{\lambda,\beta}$ and $\mathcal{N}_a \times \{0\} \subset \mathcal{M}_{\lambda,\beta}$ by the condition (D_3), the conclusion follows immediately from a similar argument as used in [49, Lemma 3.2]. ∎

We next give some estimates of the lower bound of $m_{\lambda,\beta}$ and $m^*_{\lambda,\beta}$. Let

$$I_{a,\lambda}(u) = \frac{1}{2}\mathcal{D}_{a,\lambda}(u, u) - \frac{1}{4}\|u\|^4_{L^4(\mathbb{R}^4)} \quad \text{and} \quad I_{b,\lambda}(v) = \frac{1}{2}\mathcal{D}_{b,\lambda}(v, v) - \frac{1}{4}\|v\|^4_{L^4(\mathbb{R}^4)}.$$

Then by (1.1)–(1.2), $I_{a,\lambda}(u)$ is well defined on $E_{a,\lambda}$ and $I_{b,\lambda}(v)$ is well defined on $E_{b,\lambda}$ respectively for $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$. Moreover, by a standard argument, we can see that $I_{a,\lambda}(u)$ and $I_{b,\lambda}(v)$ are of C^2 in $E_{a,\lambda}$ and $E_{b,\lambda}$, respectively. Denote

$$\mathcal{N}_{a,\lambda} = \{u \in E_{a,\lambda}\backslash\{0\} \mid I'_{a,\lambda}(u)u = 0\}, \tag{3.10}$$

$$\mathcal{N}_{b,\lambda} = \{u \in E_{b,\lambda}\backslash\{0\} \mid I'_{b,\lambda}(u)u = 0\}. \tag{3.11}$$

Then $\mathcal{N}_{a,\lambda}$ and $\mathcal{N}_{b,\lambda}$ are nonempty if $\mathcal{D}_{a,\lambda}(u,u)$ and $\mathcal{D}_{b,\lambda}(v,v)$ are positively definite in $E_{a,\lambda}$ and $E_{b,\lambda}$ respectively.

Lemma 3.9 *Assume that (D_1)-(D_3) hold and $\mathcal{D}_{a,\lambda}(u,u)$ and $\mathcal{D}_{b,\lambda}(v,v)$ are positively definite on $E_{a,\lambda}$ and $E_{b,\lambda}$ respectively, then for $\beta \leq 0$, we have $m_{\lambda,\beta} \geq m_{a,\lambda} + m_{b,\lambda}$ and $m^*_{\lambda,\beta} \geq \min\{m_{a,\lambda}, m_{b,\lambda}\}$, where $m_{a,\lambda} = \inf_{\mathcal{N}_{a,\lambda}} I_{a,\lambda}(u)$ and $m_{b,\lambda} = \inf_{\mathcal{N}_{b,\lambda}} I_{b,\lambda}(v)$.*

Proof. Since $\mathcal{D}_{a,\lambda}(u,u)$ and $\mathcal{D}_{b,\lambda}(v,v)$ are positively definite on $E_{a,\lambda}$ and $E_{b,\lambda}$ respectively and Lemma 3.3 holds, the proof of $m_{\lambda,\beta} \geq m_{a,\lambda} + m_{b,\lambda}$ is similar to [49, Lemma 3.2]. For the proof of $m^*_{\lambda,\beta} \geq \min\{m_{a,\lambda}, m_{b,\lambda}\}$, note that by $\beta \leq 0$, we must have $\min\{\mathcal{D}_{a,\lambda}(u,u) - \|u\|^4_{L^4(\mathbb{R}^4)}, \mathcal{D}_{b,\lambda}(v,v) - \|v\|^4_{L^4(\mathbb{R}^4)}\} \leq 0$ for all $(u,v) \in \mathcal{M}_{\lambda,\beta}$. It follows that there exists $t \in (0,1]$ such that either $tu \in \mathcal{N}_{a,\lambda}$ or $tv \in \mathcal{N}_{b,\lambda}$, which together with Lemma 3.1, implies $m^*_{\lambda,\beta} \geq \min\{m_{a,\lambda}, m_{b,\lambda}\}$. It completes the proof. $\blacksquare$

When $\mathcal{D}_\lambda(u,v)$ is positively indefinite in E_λ, the situation is somewhat different. In this case, due to Lemma 2.3, we have that either $a_0 \leq -\mu_{a,1}$ or $b_0 \leq -\mu_{b,1}$ if $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$.

Lemma 3.10 *Assume that (D_1)-(D_3) hold and that either $a_0 \leq -\mu_{a,1}$ or $b_0 \leq -\mu_{b,1}$. If $0 \leq \beta < 1$ and $\lambda \geq \Lambda^*_0$, then we have*

$$c_{\lambda,\beta} \geq \alpha_0 > 0,$$

where $\alpha_0 > 0$ is a constant independent of $\beta \in (-1,1)$ and $\lambda \geq \Lambda^*_0$.

Proof. As in the proof of Lemma 3.6, we only give the proof for the case of $a_0 \leq -\mu_{a,1}$ and $b_0 \leq -\mu_{b,1}$, since the proofs of other cases are similar and more simple due to Lemma 2.3. Let $(u,v) \in \mathcal{G}_{\lambda,\beta}$. Then $u = \widehat{u} + \widetilde{u}$ and $v = \widehat{v} + \widetilde{v}$ with $\widetilde{u} \neq 0$ or $\widetilde{v} \neq 0$, where $\widehat{u}$, $\widetilde{u}$, $\widehat{v}$ and $\widetilde{v}$ are the projections of u and v on $\widehat{\mathcal{F}}^\perp_{a,\lambda}$, $\widetilde{\mathcal{F}}^\perp_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}$, $\widehat{\mathcal{F}}^\perp_{b,\lambda}$ and $\widetilde{\mathcal{F}}^\perp_{b,\lambda} \oplus \mathcal{F}_{b,\lambda}$, respecitvely. By Lemma 3.6 and a similar argument as used in (3.7), we can see that

$$J_{\lambda,\beta}(u,v) \quad \geq \quad \frac{t^2}{4}(\|\widetilde{u}\|^2_{a,\lambda} + \|\widetilde{v}\|^2_{b,\lambda})(2d^*_\lambda - t^2 S^{-2}(\|\widetilde{u}\|^2_{a,\lambda} + \|\widetilde{v}\|^2_{b,\lambda}))$$

for all $t \geq 0$, where d^*_λ is given by (3.8). Since $\widetilde{u} \neq 0$ or $\widetilde{v} \neq 0$, there exists $t_\lambda \in (0,+\infty)$ such that $t^2_\lambda(\|\widetilde{u}\|^2_{a,\lambda} + \|\widetilde{v}\|^2_{b,\lambda}) = d^*_\lambda S^2$. It follows that $J_{\lambda,\beta}(u,v) \geq \frac{(d^*_\lambda)^2 S^2}{4}$. Note that $(u,v) \in \mathcal{G}_{\lambda,\beta}$ is arbitrary, we must have $c_{\lambda,\beta} \geq \frac{(d^*_\lambda)^2 S^2}{4} := \alpha_\lambda > 0$ for $0 \leq \beta < 1$ and $\lambda > \Lambda^*_0$. It remains to show that $\alpha_\lambda \geq \alpha_0 > 0$ for some α_0 independent of $0 \leq \beta < 1$ and $\lambda \geq \Lambda^*_0$. Indeed, by Lemma 2.2 and Remark 2.1, $j_{a,\lambda} = j^*_{a,0} + 1 \leq k_a + 1$ for $\lambda > \Lambda^*_a$ and $j_{b,\lambda} = j^*_{b,0} + 1 \leq k_b + 1$ for $\lambda > \Lambda^*_b$. Then by Lemma 2.1, we have

$$d^*_\lambda \geq C^*_0 := \min\left\{1 - \frac{1}{\alpha_{a,j^*_{a,0}+1}(\Lambda^*_0)}, 1 - \frac{1}{\alpha_{a,j^*_{b,0}+1}(\Lambda^*_0)}\right\} > 0.$$

We close the proof by taking $\alpha_0 = \frac{S^2(C_0^*)^2}{4}$. ∎

If we also have $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$, then by the condition (D_3) and the results of [14, 41], $I_a(u)$ has a least energy critical point u_a in $H_0^1(\Omega_a)$ with the energy value $0 < I_a(u_a) < \frac{1}{4}S^2$. Similarly, if $-b_0 \notin \sigma(-\Delta, H_0^1(\Omega_b))$, then $I_b(v)$ has a least energy critical point v_b in $H_0^1(\Omega_b)$ with the energy value $0 < I_b(v_b) < \frac{1}{4}S^2$ due to the condition (D_3) and the results of [14, 41].

Lemma 3.11 *Let (D_1)-(D_3) hold. Further, assume that either $a_0 \leq -\mu_{a,1}$ with $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$ or $b_0 \leq -\mu_{b,1}$ with $-b_0 \notin \sigma(-\Delta, H_0^1(\Omega_b))$, then we have that*

$$\frac{1}{4}S^2 > \limsup_{\lambda\to+\infty} c_{\lambda,\beta} \quad \textit{for all } 0 \leq \beta < 1.$$

Proof. We only give the proof for the case of $a_0 \leq -\mu_{a,1}$ with $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$, since another case can be proved in a similar way. Since u_a is a nonzero critical point of $I_a(u)$ with the energy value $0 < I_a(u_a) < \frac{1}{4}S^2$, we can see that $\int_{\Omega_a}(|\nabla u_a|^2 + a_0 u_a^2)dx > 0$. We claim that there exists $\Lambda_0 \geq \Lambda_0^*$ such that $u_a \in E_{a,\lambda}\backslash\widehat{\mathcal{F}}_{a,\lambda}^{\perp}$ for $\lambda > \Lambda_0$. Indeed, by the condition (D_3) once more, we can see that $u_a \in E_{a,\lambda}$ for $\lambda > \Lambda_{a,0}$. If there exists $\{\lambda_n\}$ with $\lambda_n \to +\infty$ as $n \to \infty$ such that $u_a \in \widehat{\mathcal{F}}_{a,\lambda_n}^{\perp}$, then by Lemmas 2.1 and 2.2 and the definition of $\widehat{\mathcal{F}}_{a,\lambda_n}^{\perp}$, we must have $u_a = \sum_{i=1}^{j_{a,0}^*} d_i e_{a,i}$, where $j_{a,0}^*$ is given by (2.8) and $e_{a,i}$ satisfy (2.6) with $\alpha_{a,i} \leq 1$ for all $i = 1, 2, \cdots j_{a,0}^*$. This implies $\int_{\Omega_a}(|\nabla u_a|^2 + a_0 u_a^2)dx \leq 0$ and it is impossible. Now, by Lemma 3.7, there exists a unique $(w_\lambda^0, \sigma_\lambda^0, t_\lambda^0) \in \widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp} \times \mathbb{R}^+$ such that $(u_{\lambda,\beta}^0, v_{\lambda,\beta}^0) = (w_\lambda^0 + t_\lambda^0(u_a - \widehat{u}_{a,\lambda}), \sigma_\lambda^0) \in \mathcal{G}_{\lambda,\beta}$ for $\lambda \geq \Lambda_0$, where $\widehat{u}_{a,\lambda}$ and $\widehat{v}_{b,\lambda}$ are the projections of u_a and v_b in $\widehat{\mathcal{F}}_{a,\lambda}^{\perp}$ and $\widehat{\mathcal{F}}_{b,\lambda}^{\perp}$ respectively. It follows from Remark 3.2 that $(w_\lambda^0, \sigma_\lambda^0, t_\lambda^0) \to (w_0, \sigma_0, t_0)$ strongly in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4) \times \mathbb{R}^+$ as $\lambda \to +\infty$. Moreover, we also have that $(\widehat{u}_{a,\lambda}, \widehat{v}_{b,\lambda}) \to (\widehat{u}_a, \widehat{v}_b)$, where $\widehat{u}_a$ and $\widehat{v}_b$ are the projections of u_a and v_b in $\text{span}\{e_{a,j}^* \mid \alpha_{a,j} \leq 1\}$ and $\text{span}\{e_{b,j}^* \mid \alpha_{b,j} \leq 1\}$ respectively. Due to the condition (D_3), we must have that $(w_0, \sigma_0) \in H_0^1(\Omega_a) \times H_0^1(\Omega_b)$, which together with the condition (D_3) and $u_a \in H_0^1(\Omega_a)$, implies

$$\begin{aligned} &\limsup_{\lambda\to+\infty} J_{\lambda,\beta}(w_\lambda^0 + t_\lambda^0(u_a - \widehat{u}_{a,\lambda}), \sigma_\lambda^0) \\ &\leq \limsup_{\lambda\to+\infty} I_{a,\lambda}(w_\lambda^0 + t_\lambda^0(u_a - \widehat{u}_{a,\lambda})) \leq I_a(t_0(u_a - \widehat{u}_a) + w_0). \end{aligned} \tag{3.12}$$

By a similar argument as used in the proof of Lemma 3.6 (see also [40, Proposition 2.3]), we have that $I_a(t_0(u_a - \widehat{u}_a) + w_0) < I_a(u_a)$. Thus, $\frac{1}{4}S^2 > I_a(u_a) \geq \limsup_{\lambda\to+\infty} c_{\lambda,\beta}$ for $0 \leq \beta < 1$, which completes the proof. ∎

Next we prepare some estimates which are useful in the following sections.

Lemma 3.12 *Assume (D_1)-(D_3) and $\beta \leq 0$. If $\mathcal{D}_\lambda(u, v)$ is positively definite in E_λ, then there exists $d_{\lambda,\beta} > 0$ such that $\|u\|_{L^4(\mathbb{R}^4)}^4 \|v\|_{L^4(\mathbb{R}^4)}^4 - \beta^2 \|u^2v^2\|_{L^1(\mathbb{R}^4)}^2 > d_{\lambda,\beta}$ for all $(u, v) \in \mathcal{N}_{\lambda,\beta}$.*

Proof. Due to Lemmas 2.3 and 3.3, the conclusion can be obtained by a similar argument as used in the proof of [49, Lemma 3.3] and only some trivial modifications needed, so we omit the details here. ∎

Lemma 3.13 *Assume that* (D_1)-(D_3) *hold and* $\mathcal{D}_\lambda(u,v)$ *is positively definite in* E_λ *in the case of* $a_0 < 0 \le b_0$. *If*

$$0 < \beta < 1 - \frac{1}{\alpha_{a,1}(\lambda)}, \tag{3.13}$$

then we have

$$\|u\|^2_{L^4(\mathbb{R}^4)} \ge \frac{(1 - \frac{1}{\alpha_{a,1}(\lambda)}) - \beta}{1 - \beta(1 - \frac{1}{\alpha_{a,1}(\lambda)})} S > 0 \tag{3.14}$$

and

$$\|v\|^2_{L^4(\mathbb{R}^4)} \ge S > 0 \tag{3.15}$$

for all $(u,v) \in \mathcal{N}_{\lambda,\beta}$ *with* $J_{\lambda,\beta}(u,v) \le \frac{1}{4}S^2$.

Proof. Since $\mathcal{D}_\lambda(u,v)$ is positively definite in E_λ, without loss of generality, we may assume that $E_\lambda = (\widetilde{\mathcal{F}}^\perp_{a,\lambda} \oplus \mathcal{F}_{a,\lambda}) \times (\widetilde{\mathcal{F}}^\perp_{b,\lambda} \oplus \mathcal{F}_{b,\lambda})$. Suppose $(u,v) \in \mathcal{N}_{\lambda,\beta}$ with $J_{\lambda,\beta}(u,v) \le \frac{1}{4}S^2$, then by Lemma 2.3, (1.2) and the Hölder inequality, we can see that

$$(1 - \tfrac{1}{\alpha_{a,1}(\lambda)})S \le \|u\|^2_{L^4(\mathbb{R}^4)} + \beta\|v\|^2_{L^4(\mathbb{R}^4)}, \tag{3.16}$$

$$S \le \|v\|^2_{L^4(\mathbb{R}^4)} + \beta\|u\|^2_{L^4(\mathbb{R}^4)}. \tag{3.17}$$

Since $J_{\lambda,\beta}(u,v) \le \frac{1}{2}S^2$, we also have from Lemma 2.3 and (1.2) that

$$(1 - \frac{1}{\alpha_{a,1}(\lambda)})\|u\|^2_{L^4(\mathbb{R}^4)} + \|v\|^2_{L^4(\mathbb{R}^4)} \le S. \tag{3.18}$$

We can obtain (3.14) by (3.16) and (3.18) while (3.15) can be obtained by (3.17) and (3.18) due to (3.13), which completes the proof. ∎

Lemma 3.14 *Assume that* (D_1)-(D_3) *hold and that* $\mathcal{D}_\lambda(u,v)$ *is positively definite in* E_λ *in the case of* $a_0 \le b_0 < 0$. *If*

$$0 < \beta < \min\left\{\frac{1}{2}(1 - \frac{1}{\alpha_{a,1}(\lambda)})(1 - \frac{1}{\alpha_{b,1}(\lambda)}), \frac{1 - \frac{1}{\alpha_{b,1}(\lambda)}}{1 - \frac{1}{\alpha_{a,1}(\lambda)}}, \frac{1 - \frac{1}{\alpha_{a,1}(\lambda)}}{1 - \frac{1}{\alpha_{b,1}(\lambda)}}\right\}, \tag{3.19}$$

then we have

$$\|u\|^2_{L^4(\mathbb{R}^4)} \ge \frac{(1 - \frac{1}{\alpha_{a,1}(\lambda)})(1 - \frac{1}{\alpha_{b,1}(\lambda)}) - 2\beta}{(1 - \frac{1}{\alpha_{b,1}(\lambda)}) - \beta(1 - \frac{1}{\alpha_{a,1}(\lambda)})} S > 0 \tag{3.20}$$

and

$$\|v\|_{L^4(\mathbb{R}^4)}^2 \geq \frac{(1-\frac{1}{\alpha_{b,1}(\lambda)})(1-\frac{1}{\alpha_{a,1}(\lambda)})-2\beta}{(1-\frac{1}{\alpha_{a,1}(\lambda)})-\beta(1-\frac{1}{\alpha_{b,1}(\lambda)})}S > 0 \tag{3.21}$$

for all $(u,v) \in \mathcal{N}_{\lambda,\beta}$ *with* $J_{\lambda,\beta}(u,v) \leq \frac{1}{2}S^2$.

Proof. Since $\mathcal{D}_\lambda(u,v)$ is positively definite in E_λ, without loss of generality, we may assume that $E_\lambda = (\widetilde{\mathcal{F}}_{a,\lambda}^\perp \oplus \mathcal{F}_{a,\lambda}) \times (\widetilde{\mathcal{F}}_{b,\lambda}^\perp \oplus \mathcal{F}_{b,\lambda})$. Suppose $(u,v) \in \mathcal{N}_{\lambda,\beta}$ with $J_{\lambda,\beta}(u,v) \leq \frac{1}{2}S^2$, then by Lemma 2.3, (1.2) and the Hölder inequality, we can see that

$$(1-\frac{1}{\alpha_{a,1}(\lambda)})S \leq \|u\|_{L^4(\mathbb{R}^4)}^2 + \beta\|v\|_{L^4(\mathbb{R}^4)}^2, \tag{3.22}$$

$$(1-\frac{1}{\alpha_{b,1}(\lambda)})S \leq \|v\|_{L^4(\mathbb{R}^4)}^2 + \beta\|u\|_{L^4(\mathbb{R}^4)}^2. \tag{3.23}$$

Since $J_{\lambda,\beta}(u,v) \leq \frac{1}{2}S^2$, we also have from Lemma 2.3 and (1.2) that

$$(1-\frac{1}{\alpha_{a,1}(\lambda)})\|u\|_{L^4(\mathbb{R}^4)}^2 + (1-\frac{1}{\alpha_{b,1}(\lambda)})\|v\|_{L^4(\mathbb{R}^4)}^2 \leq 2S. \tag{3.24}$$

We can obtain (3.20) by (3.22) and (3.24) while (3.21) can be obtained by (3.23) and (3.24) due to (3.19), which completes the proof. ∎

4 The existence results

Note that we have assumed $b_0 \geq a_0$, without loss of generality, one of the following four cases must happen:

(*i*) $b_0 \geq a_0 \geq 0$;

(*ii*) $-\mu_{a,1} < a_0 < 0 \leq b_0$;

(*iii*) $-\mu_{a,1} < a_0 < 0$, $-\mu_{b,1} < b_0 < 0$ and $b_0 \geq a_0$;

(*iv*) $a_0 \leq -\mu_{a,1}$ or $b_0 \leq -\mu_{b,1}$ and $a_0 \leq b_0$.

Let us first consider the case of $b_0 \geq a_0 \geq 0$. In this case, $\mathcal{D}_\lambda(u,v) = \|(u,v)\|_\lambda$ for all $\lambda > 0$ due to Lemma 2.3. Let

$$\mathcal{E}_\beta(u,v) := \frac{1}{2}\|\nabla u\|_{L^2(\mathbb{R}^4)}^2 + \frac{1}{2}\|\nabla v\|_{L^2(\mathbb{R}^4)}^2 - \frac{1}{4}\mathcal{L}_\beta(u,v). \tag{4.1}$$

Then $\mathcal{E}_\beta(u,v)$ is a C^2 functional on $D^{1,2}(\mathbb{R}^4) \times D^{1,2}(\mathbb{R}^4)$. Denote $D^{1,2}(\mathbb{R}^4) \times D^{1,2}(\mathbb{R}^4)$ by $\mathcal{D}$ and define

$$\mathcal{M}_\beta^* := \{(u,v) \in \mathcal{D}\backslash\{(0,0)\} \mid \langle D[\mathcal{E}_\beta(u,v)], (u,v)\rangle_{\mathcal{D}^*,\mathcal{D}} = 0\}$$

and

$$\mathcal{N}_\beta^* := \Big\{(u,v)\in\mathcal{D} \mid u\neq 0, v\neq 0, \langle D[\mathcal{E}_\beta(u,v)],(u,0)\rangle_{\mathcal{D}^*,\mathcal{D}} = \langle D[\mathcal{E}(u,v)],(0,v)\rangle_{\mathcal{D}^*,\mathcal{D}} = 0\Big\}, \tag{4.2}$$

where $D[\mathcal{E}_\beta(u,v)]$ is the Frechét derivative of the functional $\mathcal{E}_\beta$ in $\mathcal{D}$ at (u,v) and $\mathcal{D}^*$ is the dual space of $\mathcal{D}$. The accurate value of m_β^{**} is firstly obtained by [17, Theorem 3.1] (see also [15, Theorem 1.5]), namely,

Lemma 4.1 *(See [17, Theorem 3.1]) Let $\beta>0$. Then $m_\beta^{**}=\frac{1}{2(1+\beta)}S^2$.*

Lemma 4.2 *Let $\beta>1$. Then $m_\beta^0=m_\beta^{**}=\frac{1}{2(1+\beta)}S^2$, where*

$$m_\beta^0=\inf_{\mathcal{M}_\beta^*}\mathcal{E}_\beta(u,v),\quad m_\beta^{**}=\inf_{\mathcal{N}_\beta^*}\mathcal{E}_\beta(u,v).$$

Proof. The idea of this proof comes from [17]. Clearly, $\mathcal{N}_\beta^*\subset\mathcal{M}_\beta^*$. Thus, it is easy to see that $m_\beta^0\leq m_\beta^{**}$. By Lemma 4.1,

$$m_\beta^{**}=\frac{1}{2(1+\beta)}S^2\quad\text{for }\beta>1. \tag{4.3}$$

Thus, $m_\beta^0\leq m_\beta^{**}=\frac{1}{2(1+\beta)}S^2$ for $\beta>1$. On the other hand, by a standard argument, we also have

$$\begin{aligned} m_\beta^0 &= \inf_{(u,v)\in\mathcal{D}\setminus\{(0,0)\}}\max_{t\geq 0}\mathcal{E}_\beta(tu,tv)\\ &= \inf_{(u,v)\in\mathcal{D}\setminus\{(0,0)\}}\frac{(\|\nabla u\|_{L^2(\mathbb{R}^4)}^2+\|\nabla v\|_{L^2(\mathbb{R}^4)}^2)^2}{4\mathcal{L}_\beta(u,v)}. \end{aligned} \tag{4.4}$$

It follows from the Hölder and Sobolev inequalities that $m_\beta^0\geq\frac{S^2}{4(1+\beta)}$. Next, we will show that $m_\beta^0\geq m_\beta^{**}$. Let $\{(u_n,v_n)\}\subset\mathcal{M}_\beta^*$ be a minimizing sequence of $\mathcal{E}_\beta(u,v)$. Then it is easy to show that $\{(u_n,v_n)\}$ is bounded in $\mathcal{D}$. Without loss of generality, we assume $(u_n,v_n)\rightharpoonup(u_0,v_0)$ weakly in $\mathcal{D}$ and $(u_n,v_n)\to(u_0,v_0)$ a.e. in $\mathbb{R}^4\times\mathbb{R}^4$ as $n\to\infty$. Denote $w_n=u_n-u_0$ and $\sigma_n=v_n-v_0$. Then by the Sobolev inequality, the Brezís-Lieb lemma and [17, Lemma 2.3], we have

$$m_\beta^0=\mathcal{E}_\beta(u_0,v_0)+\mathcal{E}_\beta(w_n,\sigma_n)+o_n(1) \tag{4.5}$$

and

$$0=\langle D[\mathcal{E}_\beta(u_0,v_0)],(u_0,v_0)\rangle_{\mathcal{D}^*,\mathcal{D}}+\langle D[\mathcal{E}_\beta(w_n,\sigma_n)],(w_n,\sigma_n)\rangle_{\mathcal{D}^*,\mathcal{D}}+o_n(1). \tag{4.6}$$

Case 1: $(u_0,v_0)\neq(0,0)$. In this case, we can see from (4.4) that

$$\frac{(\|\nabla u_0\|_{L^2(\mathbb{R}^4)}^2+\|\nabla v_0\|_{L^2(\mathbb{R}^4)}^2)^2}{\mathcal{L}_\beta(u_0,v_0)}\geq 4m_\beta^0=\|\nabla u_n\|_{L^2(\mathbb{R}^4)}^2+\|\nabla v_n\|_{L^2(\mathbb{R}^4)}^2+o_n(1).$$

It follows that

$$\|\nabla u_0\|^2_{L^2(\mathbb{R}^4)} + \|\nabla v_0\|^2_{L^2(\mathbb{R}^4)} \geq \|u_0\|^4_{L^4(\mathbb{R}^4)} + 2\beta\|u_0^2 v_0^2\|_{L^1(\mathbb{R}^4)} + \|v_0\|^4_{L^4(\mathbb{R}^4)},$$

which together with (4.1) and (4.6), implies that

$$\mathcal{E}_\beta(u_0, v_0) > 0 \text{ and } \langle D[\mathcal{E}_\beta(w_n, \sigma_n)], (w_n, \sigma_n)\rangle_{\mathcal{D}^*, \mathcal{D}} \leq o_n(1). \qquad (4.7)$$

If $\|\nabla w_n\|_{L^2(\mathbb{R}^4)} + \|\nabla \sigma_n\|_{L^2(\mathbb{R}^4)} \geq C + o_n(1)$, then by (4.7), there exists $0 < t_n \leq 1 + o_n(1)$ such that $(t_n w_n, t_n v_n) \in \mathcal{M}^*_\beta$ for n large enough. Since $\beta > 1$, by Lemma 3.1 and similar arguments as used in (4.5), we can see that

$$\begin{aligned} m^0_\beta &\geq \mathcal{E}_\beta(t_n u_n, t_n v_n) + o_n(1) \\ &= \mathcal{E}_\beta(t_n u_0, t_n v_0) + \mathcal{E}_\beta(t_n w_n, t_n \sigma_n) + o_n(1) \\ &\geq \mathcal{E}_\beta(t_n u_0, t_n v_0) + m^0_\beta + o_n(1). \end{aligned}$$

It follows that $t_n \to 0$ as $n \to \infty$, which is impossible due to $\|\nabla w_n\|_{L^2(\mathbb{R}^4)} + \|\nabla \sigma_n\|_{L^2(\mathbb{R}^4)} \geq C + o_n(1)$ and $(t_n w_n, t_n v_n) \in \mathcal{M}^*_\beta$ for n large enough. Therefore, we must have $\|\nabla w_n\|_{L^2(\mathbb{R}^4)} + \|\nabla \sigma_n\|_{L^2(\mathbb{R}^4)} \to 0$ as $n \to \infty$ up to a subsequence. It follows from the Sobolev inequality and (4.5)–(4.7) that $\mathcal{E}_\beta(u_0, v_0) = m^0_\beta$ and $(u_0, v_0) \in \mathcal{M}^*_\beta$. If $u_0 = 0$ or $v_0 = 0$, then by the Sobolev inequality and (4.4), we can see that $m^0_\beta \geq \frac{1}{4}S^2$. It contradicts to $m^0_\beta \leq m^{**}_\beta$ and (4.3), since $\beta > 1$. Hence, both $u_0 \neq 0$ and $v_0 \neq 0$. Since $\mathcal{E}_\beta(u, v)$ is C^2, by a similar argument as used in the proof of Lemma 3.2, we have $D[\mathcal{E}_\beta(u_0, v_0)] = 0$ in $\mathcal{D}^*$. Hence, $(u_0, v_0) \in \mathcal{N}^*_\beta$ and $m^0_\beta \geq m^{**}_\beta$.

Case 2: $(u_0, v_0) = (0, 0)$.

In this case, $(w_n, \sigma_n) = (u_n, v_n)$. By (4.5) and $m^0_\beta \geq \frac{S^2}{4(1+\beta)}$, we must have $\|\nabla w_n\|_{L^2(\mathbb{R}^4)} + \|\nabla \sigma_n\|_{L^2(\mathbb{R}^4)} \geq C + o_n(1)$. If $w_n \to 0$ or $\sigma_n \to 0$ strongly in $D^{1,2}(\mathbb{R}^4)$ as $n \to \infty$, then we can see from the Sobolev inequality and (4.4) that $m^0_\beta \geq \frac{1}{4}S^2$, which is impossible since $m^0_\beta \leq m^{**}_\beta$, (4.3) holds and $\beta > 1$. Therefore, we must have both $w_n \not\to 0$ and $\sigma_n \not\to 0$ strongly in $D^{1,2}(\mathbb{R}^4)$ as $n \to \infty$. Now, by a similar argument as used in [17, Lemma 2.5], we can get a contradiction. ∎

Due to Lemma 4.2, we can give a precise description on $m_{\lambda,\beta}$ and $m^*_{\lambda,\beta}$ in the case of $b_0 \geq a_0 \geq 0$.

Lemma 4.3 *Assume that* (D_1)-(D_3) *hold. If* $b_0 \geq a_0 \geq 0$*, then*

$$m_{\lambda,\beta} = \frac{1}{2(1 + \max\{\beta, 0\})} S^2, \quad m^*_{\lambda,\beta} = \frac{1}{2(1 + \max\{1, \beta\})} S^2$$

for all $\beta \in \mathbb{R}$ *and* $\lambda > 0$.

Proof. For the sake of clarity, the proof will be performed through the following five steps.

Step 1. We prove that $m^*_{\lambda,\beta} = \frac{1}{4}S^2$ and $m_{\lambda,\beta} = \frac{1}{2}S^2$ for $\lambda > 0$ and $\beta \leq 0$.

Indeed, thanks to the Sobolev inequality and the condition (D_1), we have $m_{a,\lambda} \geq \frac{1}{4}S^2$ and $m_{b,\lambda} \geq \frac{1}{4}S^2$ in the case of $b_0 \geq a_0 \geq 0$. It follows from Lemmas 3.8 and 3.9 that $m^*_{\lambda,\beta} = \frac{1}{4}S^2$ and $m_{\lambda,\beta} = \frac{1}{2}S^2$ for $\lambda > 0$ and $\beta \leq 0$.

Step 2. We prove that $m^*_{\lambda,\beta} = \frac{1}{4}S^2$ for $\lambda > 0$ and $0 < \beta \leq 1$.

Indeed, by Lemma 3.8, we can see that $m^*_{\lambda,\beta} \leq \frac{1}{4}S^2$ for $\lambda > 0$ and $0 < \beta \leq 1$. It remains to show that $m^*_{\lambda,\beta} \geq \frac{1}{4}S^2$ for all $\lambda > 0$ and $0 < \beta \leq 1$. Suppose the contrary, we have $m^*_{\lambda',\beta'} < \frac{1}{4}S^2$ for some $\lambda' > 0$ and $0 < \beta' \leq 1$. By the definition of $m^*_{\lambda',\beta'}$, there exists $(u_\delta, v_\delta) \in \mathcal{M}_{\lambda',\beta'}$ satisfying $J_{\lambda',\beta'}(u_\delta, v_\delta) \leq m_{\lambda',\beta'} + \delta$ for some $\delta \in (0, \frac{1}{4}S^2 - m_{\lambda',\beta'})$. Since $(u_\delta, v_\delta) \in \mathcal{M}_{\lambda',\beta'}$ with $\lambda' > 0$, by the Sobolev inequality, the condition (D_1) and $b_0 \geq a_0 \geq 0$, we have

$$\mathcal{L}_{\beta'}(u_\delta, v_\delta) \leq 4m_{\lambda',\beta'} + 4\delta < S^2, \tag{4.8}$$

and

$$S\|u_\delta\|^2_{L^4(\mathbb{R}^4)} + S\|v_\delta\|^2_{L^4(\mathbb{R}^4)} \leq \mathcal{D}_{\lambda'}(u_\delta, v_\delta) = \mathcal{L}_{\beta'}(u_\delta, v_\delta). \tag{4.9}$$

Combining (4.8)-(4.9), we can obtain that $\|u_\delta\|^2_{L^4(\mathbb{R}^4)} + \|v_\delta\|^2_{L^4(\mathbb{R}^4)} < S$. On the other hand, thanks to the Hölder inequality, $0 < \beta' \leq 1$ and (4.9), we can see that $\|u_\delta\|^2_{L^4(\mathbb{R}^4)} + \|v_\delta\|^2_{L^4(\mathbb{R}^4)} \geq S$, which is a contradiction.

Step 3. We prove that $m_{\lambda,\beta} = \frac{1}{2(1+\beta)}S^2$ for $\lambda > 0$ and $\beta \in (0, 1)$.

Indeed, consider the following family of functions:

$$\psi^*_\varepsilon(x) = \frac{2\sqrt{2}\varepsilon}{\varepsilon^2 + |x|^2}, \quad \varepsilon > 0.$$

Then $\psi_\varepsilon(x) = \psi^*_\varepsilon(x)\eta(x) \in H^1_0(B_R)$, where $\eta \in C^\infty_0(B_R)$ with $\eta \equiv 1$ on $B_{\frac{R}{2}}$. Furthermore, it is well known that $\|\psi_\varepsilon\|^4_{L^4(\mathbb{R}^4)} = S^2 + O(\varepsilon^4)$, $\|\nabla\psi_\varepsilon\|^2_{L^2(\mathbb{R}^4)} = S^2 + O(\varepsilon^2)$ and $\|\psi_\varepsilon\|^2_{L^2(\mathbb{R}^4)} = o(\varepsilon)$ (cf. [37]). It follows from the condition (D_1) that

$$\|\psi_\varepsilon\|^4_{L^4(\mathbb{R}^4)}\mathcal{D}_{a,\lambda}(\psi_\varepsilon, \psi_\varepsilon) - \beta\|(\psi_\varepsilon)^4\|_{L^1(\mathbb{R}^4)}\mathcal{D}_{b,\lambda}(\psi_\varepsilon, \psi_\varepsilon) = S^4(1 - \beta + o(\varepsilon)) \tag{4.10}$$

and

$$\|\psi_\varepsilon\|^4_{L^4(\mathbb{R}^4)}\mathcal{D}_{b,\lambda}(\psi_\varepsilon, \psi_\varepsilon) - \beta\|(\psi_\varepsilon)^4\|_{L^1(\mathbb{R}^4)}\mathcal{D}_{a,\lambda}(\psi_\varepsilon, \psi_\varepsilon) = S^4(1 - \beta + o(\varepsilon)). \tag{4.11}$$

Since

$$\|\psi_\varepsilon\|^8_{L^4(\mathbb{R}^4)} - \beta^2\|(\psi_\varepsilon)^4\|^2_{L^1(\mathbb{R}^4)} = (1 - \beta^2)\|\psi_\varepsilon\|^8_{L^4(\mathbb{R}^4)}, \tag{4.12}$$

by (4.10)–(4.11), we can see that the proof of Lemma 3.3 still works for ε sufficiently small in the case of $\beta \neq 1$. Thus, there exist $t_{\lambda,\beta}(\psi_\varepsilon, \psi_\varepsilon)$ and $s_{\lambda,\beta}(\psi_\varepsilon, \psi_\varepsilon)$ respectively given by (3.2) and (3.3) such that

$$(t_{\lambda,\beta}(\psi_\varepsilon, \psi_\varepsilon)\psi_\varepsilon, s_{\lambda,\beta}(\psi_\varepsilon, \psi_\varepsilon)\psi_\varepsilon) \in \mathcal{N}_{\lambda,\beta},$$

which then implies

$$m_{\lambda,\beta} \le J_{\lambda,\beta}(t_{\lambda,\beta}(\psi_\varepsilon,\psi_\varepsilon)\psi_\varepsilon, s_{\lambda,\beta}(\psi_\varepsilon,\psi_\varepsilon)\psi_\varepsilon) = \frac{1}{2(1+\beta)}S^2 + o(\varepsilon).$$

It follows that $m_{\lambda,\beta} \le \frac{1}{2(1+\beta)}S^2$. It remains to show that $m_{\lambda,\beta} \ge \frac{1}{2(1+\beta)}S^2$. Indeed, let $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\beta}$ be a minimizing sequence of $J_{\lambda,\beta}(u,v)$. Since $m_{\lambda,\beta} \le \frac{1}{2(1+\beta)}S^2$, we have

$$\mathcal{L}_\beta(u_n, v_n) \le 4m_{\lambda,\beta} + o_n(1) \le \frac{2}{1+\beta}S^2 + o_n(1). \tag{4.13}$$

Note that the condition (D_1) holds, $\lambda > 0$ and $b_0 \ge a_0 \ge 0$, by the Sobolev inequality and the fact that $(u_n, v_n) \in \mathcal{N}_{\lambda,\beta}$, we have

$$S\|u_n\|^2_{L^4(\mathbb{R}^4)} \le \mathcal{D}_{a,\lambda}(u_n, u_n) = \|u_n\|^4_{L^4(\mathbb{R}^4)} + \beta\|u_n^2 v_n^2\|_{L^1(\mathbb{R}^4)} \tag{4.14}$$

and

$$S\|v_n\|^2_{L^4(\mathbb{R}^4)} \le \mathcal{D}_{b,\lambda}(v_n, v_n) = \|v_n\|^4_{L^4(\mathbb{R}^4)} + \beta\|u_n^2 v_n^2\|_{L^1(\mathbb{R}^4)}. \tag{4.15}$$

Thanks to (4.13)–(4.15), we can see that

$$\|u_n\|^2_{L^4(\mathbb{R}^4)} + \|v_n\|^2_{L^4(\mathbb{R}^4)} \le \frac{2}{1+\beta}S + o_n(1),$$

which together with (4.14) and (4.15) and the Hölder inequality, implies

$$\|u_n\|^2_{L^4(\mathbb{R}^4)} \ge \frac{1}{1+\beta}S + o_n(1) \quad \text{and} \quad \|v_n\|^2_{L^4(\mathbb{R}^4)} \ge \frac{1}{1+\beta}S + o_n(1). \tag{4.16}$$

Since $(u_n, v_n) \in \mathcal{N}_{\lambda,\beta}$ and $\beta > 0$, we must have from (4.14)–(4.16) that $m_{\lambda,\beta} \ge \frac{1}{2(1+\beta)}S^2 + o_n(1)$. The conclusion follows from letting $n \to \infty$.

Step 4. We prove that $m_{\lambda,1} = \frac{1}{4}S^2$ for $\lambda > 0$.

Indeed, for every $\lambda > 0$, we consider the following two-component systems of algebraic equations

$$\begin{cases} \mathcal{D}_{a,\lambda}(\psi_\varepsilon,\psi_\varepsilon) - \|\psi_\varepsilon\|^4_{L^4(\mathbb{R}^4)}t - \|\psi_\varepsilon^4\|_{L^1(\mathbb{R}^4)}s = 0, \\ \mathcal{D}_{b,\lambda}(\psi_\varepsilon,\psi_\varepsilon) - \|\psi_\varepsilon\|^4_{L^4(\mathbb{R}^4)}s - \|\psi_\varepsilon^4\|_{L^1(\mathbb{R}^4)}t = 0, \end{cases} \tag{4.17}$$

where ψ_ε is given in Step 3. Since $\|\psi_\varepsilon\|^4_{L^4(\mathbb{R}^4)} = S^2 + O(\varepsilon^4)$, $\|\nabla\psi_\varepsilon\|^2_{L^2(\mathbb{R}^4)} = S^2 + O(\varepsilon^2)$ and $\|\psi_\varepsilon\|^2_{L^2(\mathbb{R}^4)} = o(\varepsilon)$, by the condition (D_1), we can see that (4.17) can be solved in $\mathbb{R}^+\times\mathbb{R}^+$ for ε sufficiently small and the solutions $(t_\varepsilon, s_\varepsilon)$ satisfies $t_\varepsilon + s_\varepsilon = 1 + o(\varepsilon)$. Thus, we can choose $t_\varepsilon > 0$ and $s_\varepsilon > 0$ for ε sufficiently small such that $(\sqrt{t_\varepsilon}\psi_\varepsilon, \sqrt{s_\varepsilon}\psi_\varepsilon) \in \mathcal{N}_{\lambda,1}$. It follows that

$$m_{\lambda,1} \le J_{\lambda,1}(\sqrt{t_\varepsilon}\psi_\varepsilon, \sqrt{s_\varepsilon}\psi_\varepsilon) = \frac{1}{4}S^2 + o(\varepsilon). \tag{4.18}$$

Letting $\varepsilon \to 0^+$ in (4.18), we have $m_{\lambda,1} \leq \frac{1}{4}S^2$ for all $\lambda > 0$. Since $m_{\lambda,1} \geq m^*_{\lambda,1}$ for all $\lambda > 0$, by the conclusion of Step 2, we can see that $m_{\lambda,1} \geq \frac{1}{4}S^2$ for all $\lambda > 0$.

Step 5. We prove that $m^*_{\lambda,\beta} = m_{\lambda,\beta} = \frac{1}{2(1+\beta)}S^2$ for $\lambda > 0$ and $\beta > 1$.

Indeed, since $\mathcal{N}_{\lambda,\beta} \subset \mathcal{M}_{\lambda,\beta}$, we can see that $m^*_{\lambda,\beta} \leq m_{\lambda,\beta}$. Note that (4.10)–(4.12) still hold for $\lambda > 0$ and $\beta > 1$, thus, we also have $m_{\lambda,\beta} \leq \frac{1}{2(1+\beta)}S^2$ for $\lambda > 0$ and $\beta > 1$ by similar arguments as used in Step 3. In what follows, we will show that $m^*_{\lambda,\beta} \geq \frac{1}{2(1+\beta)}S^2$ for $\lambda > 0$ and $\beta > 1$. Indeed, for every $\delta > 0$, we can take $(u_\delta, v_\delta) \in \mathcal{M}_{\lambda,\beta}$ such that $J_{\lambda,\beta}(u_\delta, v_\delta) \leq m^*_{\lambda,\beta} + \delta$. By a standard argument, there exists $t_\delta > 0$ such that $(t_\delta u_\delta, t_\delta v_\delta) \in \mathcal{M}^*_\beta$. It follows from the condition (D_1), $\lambda > 0$, $b_0 \geq a_0 \geq 0$ and Lemmas 3.1 and 4.2 that

$$\delta + m^*_{\lambda,\beta} \geq J_{\lambda,\beta}(u_\delta, v_\delta) \geq J_{\lambda,\beta}(t_\delta u_\delta, t_\delta v_\delta) \geq \mathcal{E}_\beta(t_\delta u_\delta, t_\delta v_\delta) \geq m^0_\beta = \frac{S^2}{2(1+\beta)}.$$

The conclusion follows by letting $\delta \to 0^+$. $\blacksquare$

With Lemma 4.3 in hands, we can obtain the following

Proposition 4.1 *Let (D_1)-(D_3) hold. If $b_0 \geq a_0 \geq 0$, then $m_{\lambda,\beta}$ and $m^*_{\lambda,\beta}$ can not be attained for all $\beta \in \mathbb{R}$ and $\lambda > 0$.*

Proof. For the sake of clarity, the proof will be performed through the following four steps.

Step 1. We prove that $m_{\lambda,\beta}$ and $m^*_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta \leq 0$.

Firstly, we assume that there exists $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{N}_{\lambda,\beta}$ such that

$$J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = m_{\lambda,\beta} \text{ for } \lambda > 0 \text{ and } \beta \leq 0.$$

Then by Lemma 4.3, we must have

$$\mathcal{D}_\lambda(u_{\lambda,\beta}, v_{\lambda,\beta}) = 2S^2. \tag{4.19}$$

On the other hand, since $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{N}_{\lambda,\beta}$ with $\lambda > 0$ and $\beta \leq 0$, by the Sobolev inequality, the condition (D_1) and $b_0 \geq a_0 \geq 0$, we can see that

$$\|\nabla u_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)} \geq S^2 \text{ and } \|\nabla v_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)} \geq S^2 \tag{4.20}$$

and

$$\|u_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} \geq S \text{ and } \|v_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} \geq S. \tag{4.21}$$

By (4.19), (4.20), (D_1) and recall that $b_0 \geq a_0 \geq 0$, we have that

$$\|\nabla u_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)} = \|\nabla v_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)} = S^2$$

and

$$\int_{\mathbb{R}^4} a(x) u^2_{\lambda,\beta} dx = \int_{\mathbb{R}^4} b(x) v^2_{\lambda,\beta} dx = 0.$$

Thanks to the condition (D_3), $u_{\lambda,\beta} \in H_0^1(\Omega_a)$ and $v_{\lambda,\beta} \in H_0^1(\Omega_b)$ and it follows from (4.21) that

$$\frac{\|\nabla u_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)}}{\|u_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)}} = \inf_{u \in H_0^1(\Omega_a)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^4)}}{\|u\|^2_{L^4(\mathbb{R}^4)}},$$

$$\frac{\|\nabla v_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)}}{\|v_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)}} = \inf_{v \in H_0^1(\Omega_b)\setminus\{0\}} \frac{\|\nabla v\|^2_{L^2(\mathbb{R}^4)}}{\|v\|^2_{L^4(\mathbb{R}^4)}},$$

which contradicts to the Talenti's results in [44], since Ω_a and Ω_b are both bounded domains. Thus, $m_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta \leq 0$.

We next prove that $m^*_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta \leq 0$. In fact, assume that there exists $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{M}_{\lambda,\beta}$ such that $J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = m^*_{\lambda,\beta}$ for $\lambda > 0$ and $\beta \leq 0$. Then by Lemma 4.3 again, we must have

$$\mathcal{D}_\lambda(u_{\lambda,\beta}, v_{\lambda,\beta}) = S^2. \tag{4.22}$$

On the other hand, since $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{M}_{\lambda,\beta}$ with $\lambda > 0$ and $\beta \leq 0$, by the Sobolev inequality, the condition (D_1) and $b_0 \geq a_0 \geq 0$, we can see that

$$\|\nabla u_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)} + \|\nabla v_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)} \geq S^2 \tag{4.23}$$

and

$$\|u_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} + \|v_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} \geq S. \tag{4.24}$$

(4.23) together with (4.22), the condition (D_1) and $b_0 \geq a_0 \geq 0$ once more, implies that

$$\|\nabla u_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)} + \|\nabla v_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)} = S^2$$

and

$$\int_{\mathbb{R}^4} a(x) u^2_{\lambda,\beta} dx = \int_{\mathbb{R}^4} b(x) v^2_{\lambda,\beta} dx = 0.$$

Thanks to the condition (D_3), $u_{\lambda,\beta} \in H_0^1(\Omega_a)$ and $v_{\lambda,\beta} \in H_0^1(\Omega_b)$ and it follows from (4.24) and the condition (D_3) again that

$$\frac{\|\nabla w_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)}}{\|w_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)}} = \inf_{w \in H_0^1(\Omega_a \cup \Omega_b)\setminus\{0\}} \frac{\|\nabla w\|^2_{L^2(\mathbb{R}^4)}}{\|w\|^2_{L^4(\mathbb{R}^4)}},$$

where $w_{\lambda,\beta} = u_{\lambda,\beta} + v_{\lambda,\beta}$. It contradicts to the results in [44], since Ω_a and Ω_b are both bounded domains. Thus, $m^*_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta \leq 0$.

Step 2. We prove that $m_{\lambda,\beta}$ and $m^*_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta \in (0, 1)$.

We first prove that $m_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta \in (0, 1)$. Indeed, suppose that there exists $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{N}_{\lambda,\beta}$ such that $J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = m_{\lambda,\beta}$

for $\lambda > 0$ and $\beta \in (0,1)$. Then by similar arguments as used in Step 3 of Lemma 4.3, we can show that

$$\|u_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} = \frac{1}{1+\beta}S, \quad \|v_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} = \frac{1}{1+\beta}S \tag{4.25}$$

and

$$\|u^2_{\lambda,\beta}v^2_{\lambda,\beta}\|_{L^1(\mathbb{R}^4)} = \frac{S^2}{(1+\beta)^2}.$$

It follows from $\beta \in (0,1)$ that

$$\|u_{\lambda,\beta}\|^4_{L^4(\mathbb{R}^4)}\|v_{\lambda,\beta}\|^4_{L^4(\mathbb{R}^4)} - \beta^2\|u^2_{\lambda,\beta}v^2_{\lambda,\beta}\|^2_{L^1(\mathbb{R}^4)} = \frac{1-\beta^2}{(1+\beta)^2}S^2 > 0.$$

On the other hand, by Lemma 4.3, $\mathcal{D}_\lambda(u_{\lambda,\beta}, v_{\lambda,\beta}) = \frac{2}{1+\beta}S^2$. It follows from the condition (D_1) and $b_0 \geq a_0 \geq 0$ that

$$\lambda\int_{\mathbb{R}^4} a(x)u^2_{\lambda,\beta}dx \leq \frac{2}{1+\beta}S^2 \quad \text{and} \quad \lambda\int_{\mathbb{R}^4} b(x)v^2_{\lambda,\beta}dx \leq \frac{2}{1+\beta}S^2.$$

Hence, there exists $0 < \lambda' < \lambda$ such that

$$\begin{aligned}\frac{1-\beta^2}{4(1+\beta)^2}S^2 \geq (\lambda-\lambda')\Big|&\|v_{\lambda,\beta}\|^4_{L^4(\mathbb{R}^4)}\int_{\mathbb{R}^4} a(x)u^2_{\lambda,\beta}dx \\ &+\beta\|u^2_{\lambda,\beta}v^2_{\lambda,\beta}\|_{L^1(\mathbb{R}^4)}\int_{\mathbb{R}^4} b(x)v^2_{\lambda,\beta}dx\Big| > 0\end{aligned}$$

and

$$\begin{aligned}\frac{1-\beta^2}{4(1+\beta)^2}S^2 \geq (\lambda-\lambda')\Big|&\|u_{\lambda,\beta}\|^4_{L^4(\mathbb{R}^4)}\int_{\mathbb{R}^4} b(x)v^2_{\lambda,\beta}dx \\ &+\beta\|u^2_{\lambda,\beta}v^2_{\lambda,\beta}\|_{L^1(\mathbb{R}^4)}\int_{\mathbb{R}^4} a(x)u^2_{\lambda,\beta}dx\Big| > 0,\end{aligned}$$

which implies

$$(t_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}), s_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})) \in [\frac{\sqrt{3}}{2}, 1] \times [\frac{\sqrt{3}}{2}, 1],$$

where $t_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})$ and $s_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})$ are given by (3.2) and (3.3), respectively. Therefore, by a similar argument as used in Lemma 3.3, we must have that

$$(t_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})u_{\lambda,\beta}, s_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})v_{\lambda,\beta}) \in \mathcal{N}_{\lambda',\beta}.$$

Since $\beta \in (0,1)$, by Lemmas 3.4 and 4.3, we have from $0 < \lambda' < \lambda$ and the condition (D_1) that

$$J_{\lambda',\beta}(t_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})u_{\lambda,\beta}, s_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})v_{\lambda,\beta}) = m_{\lambda',\beta}.$$

That is, $(t_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})u_{\lambda,\beta}, s_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})v_{\lambda,\beta})$ is the minimizer of $J_{\lambda',\beta}(u, v)$ on $\mathcal{N}_{\lambda',\beta}$. By similar arguments as used in (4.25), we can also obtain that

$$\|t_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})u_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} = \frac{1}{1+\beta}S, \quad \|s_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})v_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} = \frac{1}{1+\beta}S,$$

which together with (4.25), implies $t_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = 1$ and $s_{\lambda',\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = 1$. Note that $(u_{\lambda,\beta}, v_{\lambda,\beta})$ are the minimizers for both $J_{\lambda,\beta}(u, v)$ on $\mathcal{N}_{\lambda,\beta}$ and $J_{\lambda',\beta}(u, v)$ on $\mathcal{N}_{\lambda',\beta}$. By $\lambda > \lambda' > 0$, $\beta \in (0, 1)$, condition (D_1) and Lemma 4.3, we can see that

$$\int_{\mathbb{R}^4} a(x)u^2_{\lambda,\beta}dx = \int_{\mathbb{R}^4} b(x)v^2_{\lambda,\beta}dx = 0.$$

Thanks to the condition (D_3) and $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{N}_{\lambda,\beta}$, we have $u_{\lambda,\beta} \in \mathcal{N}_a$ and $v_{\lambda,\beta} \in \mathcal{N}_b$. Since $b_0 \geq a_0 \geq 0$, we must have $\|u_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} \geq S$ and $\|v_{\lambda,\beta}\|^2_{L^4(\mathbb{R}^4)} \geq S$, which contradicts to (4.25). Since $m^*_{\lambda,\beta} = \frac{1}{4}S^2$ for all $\lambda > 0$ and $\beta \leq 1$, we can prove that $m^*_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta \in (0, 1)$ by a similar argument as used in Step. 1.

Step 3. We prove that $m_{\lambda,1}$ and $m^*_{\lambda,1}$ can not be attained for $\lambda > 0$.

We first prove that $m_{\lambda,1}$ can not be attained for $\lambda > 0$. Indeed, suppose that there exists $(u_{\lambda,1}, v_{\lambda,1}) \in \mathcal{N}_{\lambda,1}$ such that $J_{\lambda,1}(u_{\lambda,1}, v_{\lambda,1}) = m_{\lambda,1}$. By Lemma 4.3 and the Hölder inequality, we can see that $\|u_{\lambda,1}\|^2_{L^4(\mathbb{R}^4)} + \|v_{\lambda,1}\|^2_{L^4(\mathbb{R}^4)} \geq S$. On the other hand, thanks to a similar argument as used in (4.8)-(4.9), we have $\|u_{\lambda,1}\|^2_{L^4(\mathbb{R}^4)} + \|v_{\lambda,1}\|^2_{L^4(\mathbb{R}^4)} \leq S$ due to Lemma 4.3. Thus, we must have $\|u_{\lambda,1}\|^2_{L^4(\mathbb{R}^4)} + \|v_{\lambda,1}\|^2_{L^4(\mathbb{R}^4)} = S$. Since $(u_{\lambda,1}, v_{\lambda,1}) \in \mathcal{N}_{\lambda,1}$, by similar arguments as used in (4.22)–(4.24) and the Hölder inequality, we can see that

$$\int_{\mathbb{R}^4} a(x)u^2_{\lambda,1}dx = \int_{\mathbb{R}^4} b(x)v^2_{\lambda,1}dx = 0, \quad S\|u_{\lambda,1}\|^2_{L^4(\mathbb{R}^4)} = \|\nabla u_{\lambda,1}\|^2_{L^2(\mathbb{R}^4)}$$

and

$$S\|v_{\lambda,1}\|^2_{L^4(\mathbb{R}^4)} = \|\nabla v_{\lambda,1}\|^2_{L^2(\mathbb{R}^4)}.$$

By the condition (D_3), $u_{\lambda,\beta} \in H^1_0(\Omega_a)$ and $v_{\lambda,\beta} \in H^1_0(\Omega_b)$, which contradicts to the Talenti's results of [44], since Ω_a and Ω_b are both bounded domains. Thus, $m_{\lambda,1}$ can not be attained for $\lambda > 0$. Since $m^*_{\lambda,1} = \frac{1}{4}S^2$ for all $\lambda > 0$, we can prove that $m^*_{\lambda,1}$ can not be attained for $\lambda > 0$ by a similar argument as used in Step 1.

Step 4. We prove that $m_{\lambda,\beta}$ and $m^*_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta > 1$.

We first prove that $m_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta > 1$. Indeed, suppose that there exists $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{N}_{\lambda,\beta}$ such that $J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = m_{\lambda,\beta}$. Without loss of generality, we may assume $u_{\lambda,\beta} \geq 0$ and $v_{\lambda,\beta} \geq 0$. Clearly, $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{M}_{\lambda,\beta}$. By a standard argument, we can see that there exists $t_{\lambda,\beta} > 0$ such that $(t_{\lambda,\beta}u_{\lambda,\beta}, t_{\lambda,\beta}v_{\lambda,\beta}) \in \mathcal{M}^*_\beta$. Since the condition (D_1) holds and $b_0 \geq a_0 \geq 0$, by Lemmas 3.1 and 4.3, we have that

$$\frac{1}{2(1+\beta)}S^2 = J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) \geq J_{\lambda,\beta}(t_{\lambda,\beta}u_{\lambda,\beta}, t_{\lambda,\beta}v_{\lambda,\beta}) \geq \mathcal{E}_\beta(t_{\lambda,\beta}u_{\lambda,\beta}, t_{\lambda,\beta}v_{\lambda,\beta}),$$

which together with Lemma 4.2, implies $\mathcal{E}_\beta(t_{\lambda,\beta}u_{\lambda,\beta}, t_{\lambda,\beta}v_{\lambda,\beta}) = m^0_\beta$. Use a similar argument as that in the proof of Lemma 4.2, we have

$$D[\mathcal{E}_\beta(t_{\lambda,\beta}u_{\lambda,\beta}, t_{\lambda,\beta}v_{\lambda,\beta})] = 0 \text{ in } \mathcal{D}^*.$$

Therefore, by the maximum principle, we can see that $u_{\lambda,\beta} > 0$ and $v_{\lambda,\beta} > 0$ on $\mathbb{R}^4$. Due to [17, Theorem 3.1] and $\beta > 1$, we must have $t_{\lambda,\beta}u_{\lambda,\beta} = t_{\lambda,\beta}v_{\lambda,\beta} = U_{\lambda,\beta}$, where $U_{\lambda,\beta}$ is given in [17, Theorem 3.1] and satisfies

$$\|U_{\lambda,\beta}\|^4_{L^4(\mathbb{R}^4)} = \frac{S^2}{(1+\beta)^2}; \quad \|\nabla U_{\lambda,\beta}\|^2_{L^2(\mathbb{R}^4)} = \frac{S^2}{1+\beta}.$$

It follows from Lemma 4.3 and $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{N}_{\lambda,\beta}$ that

$$\frac{S^2}{2(1+\beta)} = J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = \frac{S^2}{2t^4_{\lambda,\beta}(1+\beta)},$$

which then implies $t_{\lambda,\beta} = 1$. By Lemma 4.3 and $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{N}_{\lambda,\beta}$ once more, we must have

$$\begin{aligned}
\frac{S^2}{2(1+\beta)} &= J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) \\
&= \frac{1}{4}\mathcal{D}_\lambda(u_{\lambda,\beta}, v_{\lambda,\beta}) \\
&= \frac{S^2}{2(1+\beta)} + \int_{\mathbb{R}^4} (\lambda a(x) + a_0)u^2_{\lambda,\beta} + (\lambda b(x) + b_0)v^2_{\lambda,\beta}dx.
\end{aligned}$$

It is impossible since $u_{\lambda,\beta} > 0$, $v_{\lambda,\beta} > 0$ on $\mathbb{R}^4$, $b_0 \geq a_0 \geq 0$ and the conditions (D_1)–(D_3) hold. Note that $m^*_{\lambda,\beta} = m_{\lambda,\beta}$ for all $\lambda > 0$ in the case of $\beta > 1$, we can also show that $m^*_{\lambda,\beta}$ can not be attained for $\lambda > 0$ and $\beta > 1$ by a similar argument as above, which completes the proof. ∎

We next consider the case of $-\mu_{a,1} < a_0 < 0 \leq b_0$. Due to Lemma 2.3, we always assume $\lambda > \overline{\Lambda}_a$ in this case. Let us consider the Nehari type set $\mathcal{M}_{\lambda,\beta}$ in what follows. Since $\mathcal{D}_{a,\lambda}(u, u)$ and $\mathcal{D}_{b,\lambda}(v, v)$ are both definite on $E_{a,\lambda}$ and $E_{b,\lambda}$ respectively for $\lambda > \overline{\Lambda}_a$ in the case of $-\mu_{a,1} < a_0 < 0 \leq b_0$, we can see that Lemma 3.1 holds for all $E_\lambda\backslash\{(0, 0)\}$ for $\lambda > \overline{\Lambda}_a$ and $\beta \geq 0$. Furthermore, we also have the following.

Lemma 4.4 *Assume (D_1)-(D_3) and $-\mu_{a,1} < a_0 < 0 \leq b_0$. If $\beta \geq 0$ and $\lambda > \overline{\Lambda}_a$, then* $m^*_{\lambda,\beta} \geq \frac{S^2(\alpha_{a,1}(\lambda)-1)^2}{4\max\{1,\beta\}[\alpha_{a,1}(\lambda)]^2} > 0$.

Proof. Since $-\mu_{a,1} < a_0 < 0 \leq b_0$ for $\lambda > \overline{\Lambda}_a$, we have $E_\lambda = (\widetilde{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathcal{F}_{a,\lambda}) \times \mathcal{F}_{b,\lambda}$. Thanks to Lemmas 2.1 and 2.3 and $-\mu_{a,1} < a_0 < 0$, for every $\varepsilon > 0$, we have from the Sobolev inequality that

$$\varepsilon + m_{\lambda,\beta}^* \geq \frac{S(\alpha_{a,1}(\lambda) - 1)}{4\alpha_{a,1}(\lambda)}(\|u_\varepsilon\|_{L^4(\mathbb{R}^4)}^2 + \|v_\varepsilon\|_{L^4(\mathbb{R}^4)}^2) \tag{4.26}$$

for some $(u_\varepsilon, v_\varepsilon) \in \mathcal{M}_{\lambda,\beta}$ with $\lambda > \overline{\Lambda}_a$ and $\beta \geq 0$. On the other hand, since $(u_\varepsilon, v_\varepsilon) \in \mathcal{M}_{\lambda,\beta}$ with $\lambda > \overline{\Lambda}_a$ and $\beta \geq 0$, we can see from the Hölder and Sobolev inequalities that

$$\|u_\varepsilon\|_{L^4(\mathbb{R}^4)}^2 + \|v_\varepsilon\|_{L^4(\mathbb{R}^4)}^2 \geq \frac{S(\alpha_{a,1}(\lambda) - 1)}{\alpha_{a,1}(\lambda)\max\{1, \beta\}}. \tag{4.27}$$

Combining (4.26) and (4.27), we can obtain that $m_{\lambda,\beta}^* \geq \frac{S^2(\alpha_{a,1}(\lambda)-1)^2}{4\max\{1,\beta\}[\alpha_{a,1}(\lambda)]^2} > 0$ for $\lambda > \overline{\Lambda}_a$ and $\beta \geq 0$ by letting $\varepsilon \to 0^+$. $\blacksquare$

Proposition 4.2 *Let the conditions* (D_1)*–*(D_3) *hold and* $-\mu_{a,1} < a_0 < 0 \leq b_0$*. If* $\beta \geq 0$ *and* $\lambda > \overline{\Lambda}_a$*, then* $J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = m_{\lambda,\beta}^*$ *and* $D[J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})] = 0$ *in* E_λ^* *for some* $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in \mathcal{M}_{\lambda,\beta}$*.*

Proof. Let $\{(u_n, v_n)\} \subset \mathcal{M}_{\lambda,\beta}$ be a minimizing sequence of $J_{\lambda,\beta}(u, v)$. Since $\lambda > \overline{\Lambda}_a$, by Lemma 2.3, we can see that $\{(u_n, v_n)\}$ is bounded both in E_λ and $\mathcal{D}$. Without loss of generality, we may assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly both in E_λ and $\mathcal{D}$ and $(u_n, v_n) \to (u_0, v_0)$ a.e. in $\mathbb{R}^4 \times \mathbb{R}^4$ as $n \to \infty$. Clearly, one of the following two cases must happen:

(*i*) $(u_0, v_0) = (0, 0)$;

(*ii*) $(u_0, v_0) \neq (0, 0)$.

If the case (*i*) happen, then by the Sobolev embedding theorem and the condition (D_2), we have that

$$\int_{\mathbb{R}^4} (\lambda a(x) + a_0)^- u_n^2 dx = o_n(1). \tag{4.28}$$

It follows from $\{(u_n, v_n)\} \subset \mathcal{M}_{\lambda,\beta}$ and (1.2) that

$$\begin{aligned} S(\|u_n\|_{L^4(\mathbb{R}^4)}^2 + \|v_n\|_{L^4(\mathbb{R}^4)}^2) &\leq \|u_n\|_{a,\lambda}^2 + \|v_n\|_{b,\lambda}^2 + o_n(1) \\ &= \mathcal{D}_\lambda(u_n, v_n) + o_n(1) \\ &= \mathcal{L}_\beta(u_n, v_n) + o_n(1). \end{aligned} \tag{4.29}$$

If $\beta \leq 1$, then we can see from (4.29) that $\|u_n\|_{L^4(\mathbb{R}^4)}^2 + \|v_n\|_{L^4(\mathbb{R}^4)}^2 \geq S + o_n(1)$. On the other hand, by (1.2) and Lemma 3.8, we have that

$$\|u_n\|_{L^4(\mathbb{R}^4)}^2 + \|v_n\|_{L^4(\mathbb{R}^4)}^2 \leq 4\min\{m_a, m_b\}S^{-1}.$$

Note that $\min\{m_a, m_b\} < \frac{1}{4}S^2$ in the case of $-\mu_{a,1} < a_0 < 0 \le b_0$, we get a contradiction. Thus, we must have $\beta > 1$. By a similar argument as used in Step 5 to Lemma 4.3, we can see that $m^*_{\lambda,\beta} \le \frac{1}{2(1+\beta)}S^2$ in the case of $-\mu_{a,1} < a_0 < 0 \le b_0$ for $\beta > 1$. By (4.28) and Lemma 4.4, it is easy to see that there exist $0 < t_n \le 1 + o_n(1)$ such that $(t_n u_n, t_n v_n) \in \mathcal{M}^*_\beta$. Hence, by Lemma 4.2 and the fact that $\{(u_n, v_n)\} \subset \mathcal{M}_{\lambda,\beta}$ is a minimizing sequence of $J_{\lambda,\beta}(u, v)$, we can see that

$$\frac{1}{2(1+\beta)}S^2 \le \frac{1}{4}\mathcal{L}_\beta(t_n u_n, t_n v_n) \le \frac{1}{4}\mathcal{L}_\beta(u_n, v_n) \le \frac{1}{2(1+\beta)}S^2 + o_n(1).$$

It follows that $t_n \to 1$ as $n \to \infty$, $m^*_{\lambda,\beta} = \frac{1}{2(1+\beta)}S^2$ and $\{(t_n u_n, t_n v_n)\} \subset \mathcal{M}^*_\beta$ is a minimizing sequence of $\mathcal{E}_\beta(u, v)$. Due to a similar argument as used in Case 2 of Lemma 4.2, we can get a contradiction. Thus, we must have the case (ii). In this case, by (3.1) and the Fatou lemma, we can see that

$$\frac{\mathcal{D}_\lambda(u_0, v_0)^2}{4\mathcal{L}_\beta(u_0, v_0)} \ge m^*_{\lambda,\beta} = \frac{1}{4}\mathcal{L}_\beta(u_n, v_n) + o_n(1) \ge \frac{1}{4}\mathcal{L}_\beta(u_0, v_0) + o_n(1).$$

It follows that

$$\langle D[J_{\lambda,\beta}(u_0, v_0)], (u_0, v_0)\rangle_{E^*_\lambda, E_\lambda} \ge 0. \tag{4.30}$$

Let $(w_n, \sigma_n) = (u_n - u_0, v_n - v_0)$. Then by the Brezís-Lieb lemma and [17, Lemma 2.3], the Sobolev embedding theorem and (D_2), we get

$$\langle D[J_{\lambda,\beta}(w_n, \sigma_n)], (w_n, \sigma_n)\rangle_{E^*_\lambda, E_\lambda} + \langle D[J_{\lambda,\beta}(u_0, v_0)], (u_0, v_0)\rangle_{E^*_\lambda, E_\lambda} = o_n(1),$$

which together with (4.30), implies

$$\langle D[J_{\lambda,\beta}(w_n, \sigma_n)], (w_n, \sigma_n)\rangle_{E^*_\lambda, E_\lambda} \le o_n(1). \tag{4.31}$$

Due to (3.1) and (4.30)–(4.31), we can use a similar argument as used in the case (i) of Lemma 4.2 to show that $(w_n, \sigma_n) \to (0, 0)$ strongly in $L^4(\mathbb{R}^4) \times L^4(\mathbb{R}^4)$ as $n \to \infty$ up to a subsequence. By (4.31) and Lemma 2.3, $(w_n, \sigma_n) \to (0, 0)$ strongly in E_λ as $n \to \infty$ up to a sequence. Hence, $J_{\lambda,\beta}(u_0, v_0) = m^*_{\lambda,\beta}$. Thanks to Lemma 3.2, we have that $D[J_{\lambda,\beta}(u_0, v_0)] = 0$ in E^*_λ. ∎

By Proposition 4.2, we can see that $(\mathcal{P}_{\lambda,\beta})$ has a general ground state solution $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in E_\lambda$ for all $\beta \ge 0$ and $\lambda > \overline{\Lambda}_a$. Furthermore, we have the following

Lemma 4.5 *Let $(u_{\lambda,0}, v_{\lambda,0})$ be the general ground state solution of $(\mathcal{P}_{\lambda,0})$ obtained by Proposition 4.2. Then $(u_{\lambda,0}, v_{\lambda,0})$ is a semi-trivial solution of $(\mathcal{P}_{\lambda,0})$ and of the type $(u_{\lambda,0}, 0)$. Furthermore, $u_{\lambda,0}$ is a least energy critical point of $I_{a,\lambda}(u)$.*

Proof. Suppose $v_{\lambda,0} \ne 0$. Since $(u_{\lambda,0}, v_{\lambda,0})$ is a non-zero solution of $(\mathcal{P}_{\lambda,\beta})$, by the condition (D_1), $\lambda > 0$ and $b_0 \ge 0$, we can see from the Sobolev inequality that $\|v_{\lambda,0}\|^4_{L^4(\mathbb{R}^4)} \geq S^2$. Note that the condition (D_3) holds, it is well known that $\|v_{\lambda,0}\|^4_{L^4(\mathbb{R}^4)} > S^2$. Hence,

$$J_{\lambda,\beta}(u_{\lambda,0}, v_{\lambda,0}) \geq \frac{1}{4}(\|u_{\lambda,0}\|^4_{L^4(\mathbb{R}^4)} + \|v_{\lambda,0}\|^4_{L^4(\mathbb{R}^4)}) > \frac{1}{4}S^2,$$

which contradicts to Lemma 3.8. Hence, $(u_{\lambda,0}, v_{\lambda,0})$ is a semi-trivial solution of $(\mathcal{P}_{\lambda,0})$ and of the type $(u_{\lambda,0}, 0)$. It follows from $\mathcal{N}_{a,\lambda} \times \{0\} \subset \mathcal{M}_{\lambda,0}$ for all $\lambda > \overline{\Lambda}_a$ that $u_{\lambda,0}$ is also a least energy critical point of $I_{a,\lambda}(u)$, where $\mathcal{N}_{a,\lambda}$ is given by (3.10). $\blacksquare$

By Lemma 2.1, we have $\lim\limits_{\lambda\to+\infty} \alpha_{a,1}(\lambda) = \dfrac{\mu_{a,1}}{|a_0|} < 1$ in the case of $-\mu_{a,1} < a_0 < 0 \leq b_0$. It follows that for $0 < \beta < 1 - \frac{\mu_{a,1}}{|a_0|}$, there exists $\Lambda_1^* > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$ such that $0 < \beta < 1 - \frac{1}{\alpha_{a,1}(\lambda)}$ for $\lambda \geq \Lambda_1^*$.

Lemma 4.6 *Let $(u_{\lambda,\beta}, v_{\lambda,\beta})$ be the general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ obtained by Proposition 4.2. Then we have*

(1) *$(u_{\lambda,0}, 0)$ is a general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ for all $\beta < 0$.*

(2) *For every $\beta \in (0, 1 - \frac{\mu_{a,1}}{|a_0|})$, there exists $\Lambda_\beta > \Lambda_1^*$ such that $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is a semi-trivial solution of $(\mathcal{P}_{\lambda,\beta})$ and of the type $(u_{\lambda,\beta}, 0)$ with $\lambda > \Lambda_\beta$.*

(3) *There exists $\beta_\lambda > 0$ such that $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is a non-trivial solution of $(\mathcal{P}_{\lambda,\beta})$ for $\beta > \beta_\lambda$.*

Proof. (1) Clearly, $(u_{\lambda,0}, 0)$ is a solution of $(\mathcal{P}_{\lambda,\beta})$ for all $\beta < 0$. It follows that $m^*_{\lambda,0} \geq m^*_{\lambda,\beta}$ for all $\beta < 0$. On the other hand, since $\beta < 0$, for all $(u, v) \in \mathcal{M}_{\lambda,\beta}$, there exists $0 < t \leq 1$ such that $(tu, tv) \in \mathcal{M}_{\lambda,0}$. By Lemma 3.1, we have

$$J_{\lambda,\beta}(u, v) \geq J_{\lambda,\beta}(tu, tv) \geq J_{\lambda,0}(tu, tv) \geq m^*_{\lambda,0},$$

which implies $m^*_{\lambda,\beta} \geq m^*_{\lambda,0}$ for all $\beta < 0$. Therefore, $(u_{\lambda,0}, 0)$ is a general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ for all $\beta < 0$.

(2) Suppose the contrary, there exists $\{\lambda_n\}$ with $\lambda_n \to +\infty$ as $n \to \infty$ and $\beta \in (0, 1 - \frac{\mu_{a,1}}{|a_0|})$ such that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \in \mathcal{N}_{\lambda_n,\beta}$ and

$$J_{\lambda_n,\beta}(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) = m^*_{\lambda_n,\beta}.$$

Thanks to Lemma 3.8, we have that $\{(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})\}$ is bounded in

$$\mathcal{D} = D^{1,2}(\mathbb{R}^4) \times D^{1,2}(\mathbb{R}^4).$$

Without loss of generality, we assume $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \rightharpoonup (u_{0,\beta}, v_{0,\beta})$ weakly in $\mathcal{D}$ and $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ a.e. in $\mathbb{R}^4 \times \mathbb{R}^4$ as $n \to \infty$. Note that $\lambda_n \to +\infty$ as $n \to \infty$, we can see from the condition (D_1) and Lemma 3.8 once more that

$$\int_{\mathbb{R}^4} a(x)u^2_{\lambda_n,\beta} + b(x)v^2_{\lambda_n,\beta}dx \to 0 \quad \text{as } n \to \infty. \tag{4.32}$$

By (D_3) and the Fatou's lemma, we see that $(u_{0,\beta}, v_{0,\beta}) \in H_0^1(\Omega_a) \times H_0^1(\Omega_b)$ with $u_{0,\beta} = 0$ outside Ω_a and $v_{0,\beta} = 0$ outside Ω_b. It follows from the Sobolev embedding theorem, the condition (D_2) and (4.32) once more that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ strongly in $L^2(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ as $n \to \infty$. Since $H_0^1(\Omega_a) \times H_0^1(\Omega_b) \subset E_\lambda$, by the condition (D_3), it is easy to see that $I_a'(u_{0,\beta}) = 0$ and $I_b'(v_{0,\beta}) = 0$ in $H^{-1}(\Omega_a)$ and $H^{-1}(\Omega_b)$, respectively. Thus, we have from Lemma 3.8 that

$$\begin{aligned}
&\min\{m_a, m_b\} && (4.33)\\
&\geq \limsup_{n\to\infty} m^*_{\lambda_n,\beta}\\
&\geq \liminf_{n\to\infty} \frac{1}{4}\mathcal{D}_{\lambda_n}(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})\\
&\geq \frac{1}{4}(\|\nabla u_{0,\beta}\|^2_{L^2(\mathbb{R}^4)} + a_0\|u_{0,\beta}\|^2_{L^2(\mathbb{R}^4)} && (4.34)\\
&\quad +\|\nabla v_{0,\beta}\|^2_{L^2(\mathbb{R}^4)} + b_0\|v_{0,\beta}\|^2_{L^2(\mathbb{R}^4)}). && (4.35)
\end{aligned}$$

Since Ω_b is bounded and $-\mu_{a,1} < a_0 < 0 \leq b_0$, it is well known that $I_b(v_{0,\beta}) > m_b = \frac{1}{4}S^2$ if $v_{0,\beta} \neq 0$, which together with (4.35) and the condition (D_3) once more, implies that $v_{0,\beta} = 0$. Note that $u_{\lambda_n,\beta} \to u_{0,\beta}$ strongly in $L^2(\mathbb{R}^4)$ as $n \to \infty$, by Lemma 4.3, we can see that $u_{0,\beta}$ is a least energy critical point of $I_a(u)$. Thanks to (4.35) and $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ strongly in $L^2(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ as $n \to \infty$ once more, we can see that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, 0)$ strongly in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ as $n \to \infty$. Since $0 < \beta < 1 - \frac{1}{\alpha_{a,1}(\lambda_n)}$ for n sufficiently large, we can see from Lemma 3.13 that

$$\|v_{\lambda_n,\beta}\|^2_{L^4(\mathbb{R}^4)} \geq S + o_n(1),$$

which is a contradiction.

(3) By (3.1) and the condition (D_2), we can see that

$$\begin{aligned}
m^*_{\lambda,\beta} &\leq \frac{(\mathcal{D}_{a,\lambda}(u_{\lambda,0}, u_{\lambda,0}) + \mathcal{D}_{b,\lambda}(u_{\lambda,0}, u_{\lambda,0}))^2}{8(1+\beta)\|u_{\lambda,0}\|^4_{L^4(\mathbb{R}^4)}}\\
&\leq \frac{2m_{a,\lambda}}{1+\beta} + \frac{(\lambda(a_\infty + b_\infty) + b_0 - a_0)\|u_{\lambda,0}\|^2_{L^2(\mathbb{R}^4)}}{2(1+\beta)}\\
&\quad + \frac{(\lambda(a_\infty + b_\infty) + b_0 - a_0)^2\|u_{\lambda,0}\|^4_{L^2(\mathbb{R}^4)}}{8(1+\beta)\|u_{\lambda,0}\|^4_{L^4(\mathbb{R}^4)}},
\end{aligned}$$

where $m_{a,\lambda} = \inf_{\mathcal{N}_{a,\lambda}} I_{a,\lambda}(u)$. Thus, $m^*_{\lambda,\beta} \to 0$ as $\beta \to +\infty$. By $-\mu_{a,1} < a_0 < 0$ and Lemma 2.3, we have $m_{a,\lambda} > 0$ for all $\lambda > \overline{\Lambda}_a$. Thus, there exists $\beta_\lambda \in (0, +\infty)$ such that $m^*_{\lambda,\beta} < m_{a,\lambda}$ for $\beta > \beta_\lambda$. Since $-\mu_{a,1} < a_0 < 0$, it is easy to show that $m_{a,\lambda} \leq \frac{1}{4}S^2$. If $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is of the type $(0, v_{\lambda,\beta})$ for some $\beta > \beta_\lambda$, then by a similar argument as used in the proof of Lemma 4.5, we can see that $m_{\lambda,\beta} > \frac{1}{4}S^2$, which is impossible. If $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is of the type $(u_{\lambda,\beta}, 0)$ for some $\beta > \beta_\lambda$, then also by a similar argument as used in the proof of Lemma 4.5, we can obtain that $m^*_{\lambda,\beta} \geq m_{a,\lambda}$, which is also a contradiction. Therefore, $(u_{\lambda,\beta}, v_{\lambda,\beta})$ must be a non-trivial solution of $(\mathcal{P}_{\lambda,\beta})$ for $\beta > \beta_\lambda$. ∎

Next, we consider the case of $-\mu_{a,1} < a_0 < 0$, $-\mu_{b,1} < b_0 < 0$ and $b_0 \geq a_0$. Due to Lemma 2.3, we always assume $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$ in this case.

Lemma 4.7 *Assume (D_1)-(D_3) and $-\mu_{a,1} < a_0 < 0$ and $-\mu_{b,1} < b_0 < 0$. Then $\lim_{\lambda\to+\infty} m_{a,\lambda} = m_a$ and $\lim_{\lambda\to+\infty} m_{b,\lambda} = m_b$, where m_a and m_b are given by Lemma 3.8 and $m_{a,\lambda}$ and $m_{b,\lambda}$ are given by Lemma 3.9.*

Proof. We only give the proof of $\lim_{\lambda\to+\infty} m_{a,\lambda} = m_a$. Due to the condition (D_1), it is easy to show that $m_{a,\lambda}$ is nondecreasing by λ. Thus, combine with (D_3), it implies $\lim_{\lambda\to+\infty} m_{a,\lambda} \leq m_a$. By Lemma 4.5, $m_{a,\lambda}$ can be attained by some $u_{\lambda,0} \in E_\lambda$ for $\lambda > \overline{\Lambda}_a$. Now, thanks to a similar argument as used in the proof of (2) to Lemma 4.6, for every $\lambda_n \to +\infty$ as $n \to \infty$, we can see that $u_{\lambda_n,0} \rightharpoonup u_{0,0}$ weakly in $D^{1,2}(\mathbb{R}^4)$, and $u_{\lambda_n,0} \to u_{0,0}$ a.e. in $\mathbb{R}^4$ and $u_{\lambda_n,0} \to u_{0,0}$ strongly in $L^2(\mathbb{R}^4)$ as $n \to \infty$ up to a subsequence. It $u_{0,0} = 0$, then by the condition (D_2) and (4.32), we have

$$S\|u_{\lambda_n,0}\|^2_{L^4(\mathbb{R}^4)} \leq \mathcal{D}_{a,\lambda}(u_{\lambda_n,0}, u_{\lambda_n,0}) + o_n(1) = \|u_{\lambda_n,0}\|^4_{L^4(\mathbb{R}^4)} + o_n(1).$$

It follows that $\|u_{\lambda_n,0}\|^4_{L^4(\mathbb{R}^4)} \geq S^2 + o_n(1)$. Thus, we must have that $m_{a,\lambda_n} \geq \frac{1}{4}S^2 + o_n(1)$. It is impossible since $m_a < \frac{1}{4}S^2$ due to $a_0 < 0$. Thus, we must have $u_{0,0} \neq 0$. By a similar argument as used in the proof of (2) to Lemma 4.6, we have that $I'_a(u_{0,0}) = 0$ in $H^{-1}(\Omega_a)$. Therefore, $\lim_{\lambda\to+\infty} m_{a,\lambda} = m_a$. ∎

Thanks to Lemma 2.3, we can see that every minimizing sequence of $J_{\lambda,\beta}(u,v)$ on $\mathcal{N}_{\lambda,\beta}$ is bounded in E_λ in this case. Using Lemma 3.12, the implicit function theorem and the Ekeland principle in a standard way (cf. [15]), we can obtain a $(PS)_{m_{\lambda,\beta}}$ sequence of $J_{\lambda,\beta}(u,v)$ in $\mathcal{N}_{\lambda,\beta}$ for $\beta \leq 0$, denoted by $\{(u_n, v_n)\}$.

Proposition 4.3 *Let the conditions (D_1)–(D_3) hold and $\beta \leq 0$. Then $m_{\lambda,\beta}$ can be attained by a ground state solution of $(\mathcal{P}_{\lambda,\beta})$ for $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$.*

Proof. Since $\{(u_n, v_n)\}$ is bounded in E_λ, without loss of generality, we may assume $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E_λ as $n \to \infty$. It follows that $D[J_{\lambda,\beta}(u_0, v_0)] = 0$ in E^*_λ.

Case 1: $u_0 = 0$ and $v_0 = 0$. In this case, by a similar argument as used in the proof of Proposition 4.2, we can obtain that $\|u_n\|^2_{L^4(\mathbb{R}^4)} \geq S + o_n(1)$ and $\|v_n\|^2_{L^4(\mathbb{R}^4)} \geq S + o_n(1)$. It follows that $m_{\lambda,\beta} \geq \frac{1}{2}S^2$, which contradicts to Lemma 3.8.

Case 2: $u_0 = 0$ and $v_0 \neq 0$. Let $\sigma_n = v_n - v_0$. Then $(u_n, \sigma_n) \rightharpoonup (0, 0)$ weakly in E_λ as $n \to \infty$. It follows from the Sobolev inequality, the Brezís-Lieb lemma and [17, Lemma 2.3] that

$$J_{\lambda,\beta}(u_n, v_n) = J_{\lambda,\beta}(u_n, \sigma_n) + I_{b,\lambda}(v_0) + o_n(1), \tag{4.36}$$

$$\langle D[J_{\lambda,\beta}(u_n,\sigma_n)],(u_n,0)\rangle_{E_\lambda^*,E_\lambda} = \langle D[J_{\lambda,\beta}(u_n,v_n)],(u_n,0)\rangle_{E_\lambda^*,E_\lambda} + o_n(1), \quad (4.37)$$

$$\langle D[J_{\lambda,\beta}(u_n,\sigma_n)],(0,\sigma_n)\rangle_{E_\lambda^*,E_\lambda}$$
$$= \langle D[J_{\lambda,\beta}(u_n,v_n)],(0,v_n)\rangle_{E_\lambda^*,E_\lambda} + I'_{b,\lambda}(v_0)v_0 + o_n(1). \quad (4.38)$$

Since $u_n^2 v_n \rightharpoonup 0$ in $L^{\frac{4}{3}}(\mathbb{R}^4)$, by $D[J_{\lambda,\beta}(u_n,v_n)] = o_n(1)$ strongly in E_λ^* as $n \to \infty$, we must have $I'_{b,\lambda}(v_0)v_0 = 0$. It follows that $I_{b,\lambda}(v_0) \geq m_{b,\lambda}$. If $\|\sigma_n\|_{L^4(\mathbb{R}^4)} \geq C + o_n(1)$, then by a similar argument as used in the proof of Proposition 4.2, we can see from (4.37) and (4.38) that $\|u_n\|^2_{L^4(\mathbb{R}^4)} \geq S + o_n(1)$ and $\|\sigma_n\|^2_{L^4(\mathbb{R}^4)} \geq S + o_n(1)$, which together with (4.36), implies $m_{\lambda,\beta} \geq \frac{1}{2}S^2$. It contradicts to Lemma 3.8. Thus, we must have $\sigma_n \to 0$ strongly in $L^4(\mathbb{R}^4)$ as $n \to \infty$. Since $u_0 = 0$, we still have $\|u_n\|^2_{L^4(\mathbb{R}^4)} \geq S + o_n(1)$. Now, by (4.36) once more, we can see that $m_{\lambda,\beta} \geq \frac{1}{4}S^2 + m_{b,\lambda}$. Thanks to Lemmas 3.8 and 4.7, it is impossible for λ sufficient large. Without loss of generality, we assume $m_{\lambda,\beta} \geq \frac{1}{4}S^2 + m_{b,\lambda}$ can not hold for $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$.

Case 3: $u_0 \neq 0$ and $v_0 = 0$. We can exclude this case by a similar argument as used in the Case 2.

Combine with the above 3 cases, now we must have $u_0 \neq 0$ and $v_0 \neq 0$. It follows that $(u_0, v_0) \in \mathcal{N}_{\lambda,\beta}$, which together with the Fatou's lemma, implies that $J_{\lambda,\beta}(u_0,v_0) = m_{\lambda,\beta}$. Thus, (u_0,v_0) is a ground state solution of $(\mathcal{P}_{\lambda,\beta})$ for $\lambda > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$ and $\beta \leq 0$. ∎

We next consider the general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ in the case of $-\mu_{a,1} < a_0 < 0$, $-\mu_{b,1} < b_0 < 0$ and $b_0 \geq a_0$. By Lemma 2.1, we have $\lim_{\lambda\to+\infty} \alpha_{a,1}(\lambda) = \frac{\mu_{a,1}}{|a_0|} < 1$ and $\lim_{\lambda\to+\infty} \alpha_{b,1}(\lambda) = \frac{\mu_{b,1}}{|b_0|} < 1$ in this case. Let

$$\beta_0 := \min\left\{\frac{1}{2}(1 - \frac{|a_0|}{\mu_{a,1}})(1 - \frac{|b_0|}{\mu_{b,1}}), \frac{1 - \frac{|b_0|}{\mu_{b,1}}}{1 - \frac{|a_0|}{\mu_{a,1}}}, \frac{1 - \frac{|a_0|}{\mu_{a,1}}}{1 - \frac{|b_0|}{\mu_{b,1}}}\right\}.$$

Then it is easy to see that $\beta_0 \leq 1$. It follows that for $0 < \beta < \beta_0$, there exists $\Lambda_2^* > \max\{\overline{\Lambda}_a, \overline{\Lambda}_b\}$ such that

$$0 < \beta < \min\left\{\frac{1}{2}(1 - \frac{1}{\alpha_{a,1}(\lambda)})(1 - \frac{1}{\alpha_{b,1}(\lambda)}), \frac{1 - \frac{1}{\alpha_{b,1}(\lambda)}}{1 - \frac{1}{\alpha_{a,1}(\lambda)}}, \frac{1 - \frac{1}{\alpha_{a,1}(\lambda)}}{1 - \frac{1}{\alpha_{b,1}(\lambda)}}\right\}$$

as long as $\lambda \geq \Lambda_2^*$. By checking the proofs of Proposition 4.2 and Lemma 4.6, we can see that they still work for $\lambda \geq \Lambda_2^*$ and $\beta < \beta_0$ since Lemma 3.14 holds. Thus, we can obtain the following.

Proposition 4.4 *Let the conditions (D_1)–(D_3) hold and $\beta \geq 0$. If $-\mu_{a,1} < a_0 < 0$, $-\mu_{b,1} < b_0 < 0$ and $\lambda \geq \Lambda_2^*$, then $m^*_{\lambda,\beta}$ can be attained by a general ground state solution of $(\mathcal{P}_{\lambda,\beta})$. Moreover, we have the following.*

(1) *The general ground state solution of* $(\mathcal{P}_{\lambda,0})$ *is also a general ground state solution of* $(\mathcal{P}_{\lambda,\beta})$ *for* $\beta < 0$.

(2) *For every* $\beta \in (0, \beta_0)$, *there exists* $\Lambda_\beta > \Lambda_1^*$ *such that* $(u_{\lambda,\beta}, v_{\lambda,\beta})$ *is a semi-trivial solution of* $(\mathcal{P}_{\lambda,\beta})$ *and of the type* $(u_{\lambda,\beta}, 0)$ *with* $\lambda > \Lambda_\beta$.

(3) *There exists* $\beta_\lambda > 0$ *such that* $m^*_{\lambda,\beta}$ *can be attained by a ground state solution of* $(\mathcal{P}_{\lambda,\beta})$ *with* $\beta \geq \beta_\lambda$.

Now, we are ready to give the proof of Theorem 1.1.
Proof of Theorem 1.1: In fact, it is a straightforward consequence of Lemmas 4.3 and 4.5-4.6 and Propositions 4.1-4.4. $\blacksquare$

In the following part of this section, we will consider the case of $a_0 \leq -\mu_{a,1}$ or $b_0 \leq -\mu_{b,1}$. Let $\mathcal{G}_{\lambda,\beta}$ be the Nehari-Pankov type set related to $\mathcal{M}_{\lambda,\beta}$, which is given by (1.8). Then it is easy to see that $\mathcal{G}_{\lambda,\beta}$ contains all non-zero solutions of $(\mathcal{P}_{\lambda,\beta})$. In what follows, we will borrow some ideas from [40] to show that $c_{\lambda,\beta} = \inf_{\mathcal{G}_{\lambda,\beta}} J_{\lambda,\beta}(u,v)$ can be attained by some non-zero solutions of $(\mathcal{P}_{\lambda,\beta})$ for λ sufficiently large and $0 \leq \beta < 1$. Denote the map $(u,v) \to (u^0_{\lambda,\beta}, v^0_{\lambda,\beta})$ by $(\check{m}^0_{\lambda,\beta}(u,v), \hat{m}^0_{\lambda,\beta}(u,v))$, where $(u^0_{\lambda,\beta}, v^0_{\lambda,\beta})$ is given by Lemma 3.7. Then we can obtain the following

Lemma 4.8 *Assume* (D_1)-(D_3) *and* $a_0 \leq -\mu_{a,1}$ *or* $b_0 \leq -\mu_{b,1}$. *If* $0 \leq \beta < 1$ *and* $\lambda \geq \Lambda_0^*$, *then the map* $(u,v) \to (\check{m}^0_{\lambda,\beta}(u,v), \hat{m}^0_{\lambda,\beta}(u,v))$ *is continuous on* $\widetilde{E}_\lambda$, *where* Λ_0^* *is given by Lemma 3.10.*

Proof. We only give the proof for the case $a_0 \leq -\mu_{a,1}$ and $b_0 \leq -\mu_{b,1}$, since other cases are more simple and can be proved in a similar way due to Lemma 2.3. Let $(u_n, v_n) \to (u,v)$ strongly in $\widetilde{E}_\lambda$ as $n \to \infty$. By Lemma 3.7, we can see that

$$(\check{m}^0_{\lambda,\beta}(u_n,v_n), \hat{m}^0_{\lambda,\beta}(u_n,v_n)) = (w^0_n + t^0_n\widetilde{u}_n, \sigma^0_n + t^0_n\widetilde{v}_n)$$

and

$$(\check{m}^0_{\lambda,\beta}(u,v), \hat{m}^0_{\lambda,\beta}(u,v)) = (w^0 + t^0\widetilde{u}, \sigma^0 + t^0\widetilde{v}).$$

Since $0 \leq \beta < 1$ and $\dim(\widehat{\mathcal{F}}^\perp_{a,\lambda} \oplus \mathbb{R}^+\widetilde{u}) \times (\widehat{\mathcal{F}}^\perp_{b,\lambda} \oplus \mathbb{R}^+\widetilde{v}) < +\infty$ due to Lemma 2.2, there exists $R_\lambda > 0$ such that $J_{\lambda,\beta}(R_\lambda w, R_\lambda \sigma) \leq -1$ for all $(w,\sigma) \in (\widehat{\mathcal{F}}^\perp_{a,\lambda} \oplus \mathbb{R}^+\widetilde{u}) \times (\widehat{\mathcal{F}}^\perp_{b,\lambda} \oplus \mathbb{R}^+\widetilde{v})$ with $\|w\|^2_{a,\lambda} + \|\sigma\|^2_{b,\lambda} = 1$. Since $(u_n, v_n) \to (u,v)$ strongly in $\widetilde{E}_\lambda$ as $n \to \infty$, we have $J_{\lambda,\beta}(R_\lambda w, R_\lambda \sigma) \leq 0$ for all $(w,\sigma) \in (\widehat{\mathcal{F}}^\perp_{a,\lambda} \oplus \mathbb{R}^+\widetilde{u}_n) \times (\widehat{\mathcal{F}}^\perp_{b,\lambda} \oplus \mathbb{R}^+\widetilde{v}_n)$ with $\|w\|^2_{a,\lambda} + \|\sigma\|^2_{b,\lambda} = 1$ and n sufficiently large. It follows from (3.6) that $\{(w^0_n + t^0_n\widetilde{u}_n, \sigma^0_n + t^0_n\widetilde{v}_n)\}$ is bounded in $\widetilde{E}_\lambda$. Without loss of generality, we may assume that $(w^0_n + t^0_n\widetilde{u}_n, \sigma^0_n + t^0_n\widetilde{v}_n) \rightharpoonup (w^0_0 + t^0_0\widetilde{u}, \sigma^0_0 + t^0_0\widetilde{v})$ weakly in $\widetilde{E}_\lambda$ and $(w^0_n + t^0_n\widetilde{u}_n, \sigma^0_n + t^0_n\widetilde{v}_n) \to (w^0_0 + t^0_0\widetilde{u}, \sigma^0_0 + t^0_0\widetilde{v})$ a.e. in $\mathbb{R}^4 \times \mathbb{R}^4$ as $n \to \infty$.

Since $\dim \widehat{\mathcal{F}}_{a,\lambda}^{\perp} \times \widehat{\mathcal{F}}_{b,\lambda}^{\perp} < +\infty$ and $(u_n, v_n) \to (u, v)$ strongly in $\widetilde{E}_\lambda$ as $n \to \infty$, we have $(w_n^0 + t_n^0 \widetilde{u}_n, \sigma_n^0 + t_n^0 \widetilde{v}_n) \to (w_0^0 + t_0^0 \widetilde{u}, \sigma_0^0 + t_0^0 \widetilde{v})$ strongly in $\widetilde{E}_\lambda$ as $n \to \infty$. Now, by (3.6), we can see that

$$\begin{aligned} J_{\lambda,\beta}(w_n^0 + t_n^0 \widetilde{u}_n, \sigma_n^0 + t_n^0 \widetilde{v}_n) &\geq J_{\lambda,\beta}(w^0 + t^0 \widetilde{u}_n, \sigma^0 + t^0 \widetilde{v}_n) \\ &= J_{\lambda,\beta}(w^0 + t^0 \widetilde{u}, \sigma^0 + t^0 \widetilde{v}) + o_n(1). \end{aligned}$$

and

$$\begin{aligned} J_{\lambda,\beta}(w^0 + t^0 \widetilde{u}, \sigma^0 + t^0 \widetilde{v}) &\geq J_{\lambda,\beta}(w_0^0 + t_0^0 \widetilde{u}, \sigma_0^0 + t_0^0 \widetilde{v}) \\ &= J_{\lambda,\beta}(w_n^0 + t_n^0 \widetilde{u}_n, \sigma_n^0 + t_n^0 \widetilde{v}_n) + o_n(1). \end{aligned}$$

Note that (w^0, σ^0, t^0) is the unique one satisfying (3.6) for (u, v), we must have $w_0^0 = w^0$, $\sigma_0^0 = \sigma^0$ and $t_0^0 = t^0$. Since $(u_n, v_n) \to (u, v)$ strongly in $\widetilde{E}_\lambda$ as $n \to \infty$, we can see that the map $(\check{m}_{\lambda,\beta}^0(u, v), \hat{m}_{\lambda,\beta}^0(u, v))$ is continuous on $\widetilde{E}_\lambda$ for $0 \leq \beta < 1$ and $\lambda \geq \Lambda_0^*$. $\blacksquare$

Let

$$\mathbb{B}_1^+ := \{(u, v) \in \widetilde{E}_\lambda \mid \|u\|_{a,\lambda}^2 + \|v\|_{b,\lambda}^2 = 1\}$$

and consider the following functional

$$\Phi_{\lambda,\beta}(u, v) := J_{\lambda,\beta}(\check{m}_{\lambda,\beta}^0(u, v), \hat{m}_{\lambda,\beta}^0(u, v)).$$

Then by Lemmas 3.7 and 4.8, $\Phi_{\lambda,\beta}(u, v)$ is well defined and continuous on $\mathbb{B}_1^+$ for $0 \leq \beta < 1$ and $\lambda \geq \Lambda_0^*$. Furthermore, we also have the following

Lemma 4.9 *Assume that* (D_1)-(D_3) *hold and that either* $a_0 \leq -\mu_{a,1}$ *or* $b_0 \leq -\mu_{b,1}$. *If* $0 \leq \beta < 1$ *and* $\lambda \geq \Lambda_0^*$, *then* $\Phi_{\lambda,\beta}(u, v)$ *is* C^1 *on* $\mathbb{B}_1^+$. *Moreover,*

$$\langle D[\Phi_{\lambda,\beta}(u, v)], (w, \sigma)\rangle_{E_\lambda^*, E_\lambda} = \langle D[J_{\lambda,\beta}(\check{m}_{\lambda,\beta}^0(u, v), \hat{m}_{\lambda,\beta}^0(u, v))], (t_\lambda^0 \widetilde{w}, t_\lambda^0 \widetilde{\sigma})\rangle_{E_\lambda^*, E_\lambda} \tag{4.39}$$

for all (u, v) *and* $(w, \sigma) \in \mathbb{B}_1^+$, *where* $t_\lambda^0 \in \mathbb{R}^+$ *is given by* (3.6) *and only depends on* (u, v); $\widetilde{w}$ *and* $\widetilde{\sigma}$ *are the projections of* w *and* σ *on* $\widetilde{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathcal{F}_{a,\lambda}$ *and* $\widetilde{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathcal{F}_{b,\lambda}$.

Proof. We just give the proof for the case $a_0 \leq -\mu_{a,1}$ and $b_0 \leq -\mu_{b,1}$. Let (w, σ) and $(u, v) \in \mathbb{B}_1^+$. Then $w = \widehat{w} + \widetilde{w}$, $u = \widehat{u} + \widetilde{u}$ and $\sigma = \widehat{\sigma} + \widetilde{\sigma}$, $v = \widehat{v} + \widetilde{v}$ with $\widetilde{u} \neq 0$ or $\widetilde{v} \neq 0$ and $\widetilde{w} \neq 0$ or $\widetilde{\sigma} \neq 0$, where $\widehat{w}$, $\widehat{u}$ and $\widehat{\sigma}$, $\widehat{v}$ are the projections of w, u and σ, v on $\widehat{\mathcal{F}}_{a,\lambda}^{\perp}$ and $\widehat{\mathcal{F}}_{b,\lambda}^{\perp}$ while $\widetilde{w}$, $\widetilde{u}$ and $\widetilde{\sigma}$, $\widetilde{v}$ are the projections of w, u and σ, v on $\widetilde{\mathcal{F}}_{a,\lambda}^{\perp} \oplus \mathcal{F}_{a,\lambda}$ and $\widetilde{\mathcal{F}}_{b,\lambda}^{\perp} \oplus \mathcal{F}_{b,\lambda}$. By the implicit function theorem, it is easy to see that there exist $\delta > 0$ and a C^1 function $t(l)$ on $(-\delta, \delta)$ satisfying $t(l) \in [\frac{1}{2}, \frac{3}{2}]$, $t(0) = 1$ and $t'(0) = -(\langle u, w\rangle_{a,\lambda} + \langle v, \sigma\rangle_{b,\lambda})$ such that $(w_l, \sigma_l) \in \mathbb{B}_1^+$ for $l \in (-\delta, \delta)$, where $(w_l, \sigma_l) = (t(l)u + lw, t(l)v + l\sigma)$. Now, by the mean value theorem and Lemma 3.7, we can see that

$$\begin{aligned} &\Phi_{\lambda,\beta}(u, v) - \Phi_{\lambda,\beta}(w_l, \sigma_l) \\ &\leq J_{\lambda,\beta}(w_\lambda^0 + t_\lambda^0 \widetilde{u}, \sigma_\lambda^0 + t_\lambda^0 \widetilde{v}) \\ &\quad - J_{\lambda,\beta}(w_\lambda^0 + t_\lambda^0 (t(l)\widetilde{u} + l\widetilde{w}), \sigma_\lambda^0 + t_\lambda^0 (t(l)\widetilde{v} + l\widetilde{\sigma})) \qquad (4.40) \\ &= -\langle D[J_{\lambda,\beta}(\rho_1(l), \rho_2(l))], (t_\lambda^0((t(l) - 1)\widetilde{u} + l\widetilde{w}), t_\lambda^0((t(l) - 1)\widetilde{v} + l\widetilde{\sigma}))\rangle_{E_\lambda^*, E_\lambda} \end{aligned}$$

and

$$
\begin{aligned}
&\Phi_{\lambda,\beta}(u,v)-\Phi_{\lambda,\beta}(w_l,\sigma_l)\\
&\geq J_{\lambda,\beta}(w^0_{\lambda,l}+t^0_{\lambda,l}\widetilde{u},\sigma^0_{\lambda,l}+t^0_{\lambda,l}\widetilde{v})\\
&\quad -J_{\lambda,\beta}(w^0_{\lambda,l}+t^0_{\lambda,l}(t(l)\widetilde{u}+l\widetilde{w}),\sigma^0_{\lambda,l}+t^0_{\lambda,l}(t(l)\widetilde{v}+l\widetilde{\sigma})) \qquad (4.41)\\
&=-\langle D[J_{\lambda,\beta}(\rho^*_1(l),\rho^*_2(l))],(t^0_{\lambda,l}((t(l)-1)\widetilde{u}+l\widetilde{w}),t^0_{\lambda,l}((t(l)-1)\widetilde{v}+l\widetilde{\sigma}))\rangle_{E^*_\lambda,E_\lambda},
\end{aligned}
$$

where

$$
\begin{aligned}
\rho_1(l)&=w^0_\lambda+t^0_\lambda(t(l)\rho^1_l+(1-\rho^1_l))\widetilde{u}+l\rho^1_l t^0_\lambda\widetilde{w},\\
\rho_2(l)&=\sigma^0_\lambda+t^0_\lambda(t(l)\rho^2_l+(1-\rho^2_l))\widetilde{v}+l\rho^2_l t^0_\lambda\widetilde{\sigma},\\
\rho^*_1(l)&=w^0_{\lambda,l}+t^0_{\lambda,l}(t(l)\rho^{1,*}_l+(1-\rho^{1,*}_l))\widetilde{u}+l\rho^{1,*}_l t^0_{\lambda,l}\widetilde{w}
\end{aligned}
$$

and

$$
\rho^*_2(l)=\sigma^0_{\lambda,l}+t^0_{\lambda,l}(t(l)\rho^{2,*}_l+(1-\rho^{2,*}_l))\widetilde{v}+l\rho^{2,*}_l t^0_{\lambda,l}\widetilde{\sigma}
$$

with $\rho^1_l,\rho^2_l,\rho^{1,*}_l,\rho^{2,*}_l\in(0,1)$. Since $(w_l,\sigma_l)\to(u,v)$ as $l\to 0$, by Lemmas 3.7 and 4.8 and (4.40)–(4.41), we have from $(\check{m}^0_{\lambda,\beta}(u,v),\hat{m}^0_{\lambda,\beta}(u,v)))\in\mathcal{G}_{\lambda,\beta}$ that

$$
\begin{aligned}
&\lim_{l\to 0^+}\frac{\Phi_{\lambda,\beta}(w_l,\sigma_l)-\Phi_{\lambda,\beta}(u,v)}{l}\\
=&\ \langle D[J_{\lambda,\beta}(\check{m}^0_{\lambda,\beta}(u,v),\hat{m}^0_{\lambda,\beta}(u,v))],(t^0_\lambda(t'(0)\widetilde{u}+\widetilde{w}),t^0_\lambda(t'(0)\widetilde{v}+\widetilde{\sigma}))\rangle_{E^*_\lambda,E_\lambda}\\
=&\ \langle D[J_{\lambda,\beta}(\check{m}^0_{\lambda,\beta}(u,v),\hat{m}^0_{\lambda,\beta}(u,v))],(t^0_\lambda\widetilde{w},t^0_\lambda\widetilde{\sigma})\rangle_{E^*_\lambda,E_\lambda},
\end{aligned}
$$

where $t^0_\lambda\in\mathbb{R}^+$ is given by (3.6) and only relies on (u,v). It follows that $\Phi_{\lambda,\beta}(u,v)$ is of C^1 on $\mathbb{B}^+_1$ and (4.39) holds. ∎

Thanks to Lemma 4.9, we can obtain the following

Proposition 4.5 *Let (D_1)-(D_3) hold. Assume that either $a_0\leq-\mu_{a,1}$ with $-a_0\notin\sigma(-\Delta,H^1_0(\Omega_a))$ or $b_0\leq-\mu_{b,1}$ with $-b_0\notin\sigma(-\Delta,H^1_0(\Omega_b))$. If $0\leq\beta<1$ and $\lambda\geq\Lambda^*_0$, then $c_{\lambda,\beta}$ can be attained by a non-zero solution of $(\mathcal{P}_{\lambda,\beta})$.*

Proof. We only consider the case $a_0\leq-\mu_{a,1}$ with $-a_0\notin\sigma(-\Delta,H^1_0(\Omega_a))$ and $b_0\leq-\mu_{b,1}$ with $-b_0\notin\sigma(-\Delta,H^1_0(\Omega_b))$, the other cases are more simple in view of Lemma 2.3. Let

$$
\widetilde{c}_{\lambda,\beta}=\inf_{\mathbb{B}^+_1}\Phi_{\lambda,\beta}(u,v).
$$

Then by Lemmas 3.6 and 3.7, it is easy to see that $\widetilde{c}_{\lambda,\beta}=c_{\lambda,\beta}$. Since $\Phi_{\lambda,\beta}$ is C^1 due to Lemma 4.9, we can apply the implicit function theorem and the Ekeland principle in a standard way (cf. [15]) to show that there exists $\{(u_n,v_n)\}\subset\mathbb{B}^+_1$ such that $\Phi_{\lambda,\beta}(u_n,v_n)=\widetilde{c}_{\lambda,\beta}+o_n(1)$ and $\langle D[\Phi_{\lambda,\beta}(u_n,v_n)],(w,\sigma)\rangle_{E^*_\lambda,E_\lambda}=o_n(1)$ for all $(w,\sigma)\in\mathbb{B}^+_1$. For the sake of clarity, the remaining proof will be performed by several steps.

Step 1. We prove that $\{\check{m}^0_{\lambda,\beta}(u_n,v_n),\hat{m}^0_{\lambda,\beta}(u_n,v_n)\}$ is bounded in E_λ.

We denote $(\check{m}^0_{\lambda,\beta}(u_n,v_n),\hat{m}^0_{\lambda,\beta}(u_n,v_n))$ by (ϕ_n,ψ_n) for the sake of convenience. By the definition of $\Phi_{\lambda,\beta}(u,v)$, it is easy to see that $(\phi_n,\psi_n)\in\mathcal{G}_{\lambda,\beta}$ and $J_{\lambda,\beta}(\phi_n,\psi_n)=c_{\lambda,\beta}+o_n(1)$. It follows that

$$c_{\lambda,\beta}+o_n(1)=J_{\lambda,\beta}(\phi_n,\psi_n)=\frac{1}{4}\mathcal{L}_\beta(\phi_n,\psi_n). \tag{4.42}$$

If $\|\phi_n\|_{L^4(\mathbb{R}^4)}+\|\psi_n\|_{L^4(\mathbb{R}^4)}\to+\infty$ as $n\to\infty$ up to a subsequence, then by (4.42), $0\leq\beta<1$ and the Hölder inequality, we can see that

$$o_n(1)=\frac{c_{\lambda,\beta}+o_n(1)}{\|\phi_n\|^4_{L^4(\mathbb{R}^4)}+\|\psi_n\|^4_{L^4(\mathbb{R}^4)}}\geq\frac{1}{4}(1-\beta)>0,$$

which is a contradiction. Thus, $\{(\phi_n,\psi_n)\}$ is bounded in $L^4(\mathbb{R}^4)\times L^4(\mathbb{R}^4)$. Since $\lambda\geq\Lambda^*_0>\max\{\Lambda_{a,0},\Lambda_{b,0}\}$, by the definitions of $\Lambda_{a,0}$ and $\Lambda_{b,0}$ given by (2.3) and Remark 2.1, we can obtain that

$$\int_{\mathbb{R}^4}(\lambda a(x)+a_0)^-\phi_n^2dx\leq|a_0||\mathcal{A}_\lambda|^{\frac{1}{2}}\|\phi_n\|^2_{L^4(\mathbb{R}^4)}, \tag{4.43}$$

$$\int_{\mathbb{R}^4}(\lambda b(x)+b_0)^-v\psi_n^2dx\leq|b_0||\mathcal{B}_\lambda|^{\frac{1}{2}}\|\psi_n\|^2_{L^4(\mathbb{R}^4)}, \tag{4.44}$$

where $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$ are given by (2.2) and Remark 2.1. This implies

$$\begin{aligned}&c_{\lambda,\beta}+o_n(1)\\ =\;&J_{\lambda,\beta}(\phi_n,\psi_n)\\ =\;&\frac{1}{4}(\mathcal{D}_{a,\lambda}(\phi_n,\phi_n)+\mathcal{D}_{b,\lambda}(\psi_n,\psi_n))\\ \geq\;&\frac{1}{4}(\|\phi_n\|^2_{a,\lambda}+\|\psi_n\|^2_{b,\lambda})-\frac{1}{4}(|a_0||\mathcal{A}_\lambda|^{\frac{1}{2}}\|\phi_n\|^2_{L^4(\mathbb{R}^4)}+|b_0||\mathcal{B}_\lambda|^{\frac{1}{2}}\|\psi_n\|^2_{L^4(\mathbb{R}^4)}).\end{aligned}$$

Hence $\{(\phi_n,\psi_n)\}$ is bounded in E_λ.

Step 2. We prove that $\{\phi_n,\psi_n\}$ is a $(PS)_{c_{\lambda,\beta}}$ sequence of $J_{\lambda,\beta}(u,v)$. Indeed, by the definition of $\Psi_{\lambda,\beta}(u,v)$, it is easy to see that

$$J_{\lambda,\beta}(\phi_n,\psi_n)=c_{\lambda,\beta}+o_n(1).$$

It remains to show that $D[J_{\lambda,\beta}(\phi_n,\psi_n)]=o_n(1)$ strongly in E^*_λ. Let $(\varphi,\eta)\in E_\lambda$. Without loss of generality, we may assume $(\varphi,\eta)\neq(0,0)$ and $\|\varphi\|^2_{a,\lambda}+\|\eta\|^2_{b,\lambda}=1$. If $(\varphi,\eta)\in\widehat{\mathcal{F}}^\perp_{a,\lambda}\times\widehat{\mathcal{F}}^\perp_{b,\lambda}$, then due to $(\phi_n,\psi_n)\in\mathcal{G}_{\lambda,\beta}$, we have

$$\langle D[J_{\lambda,\beta}(\phi_n,\psi_n)],(\varphi,\eta)\rangle_{E^*_\lambda,E_\lambda}=0\quad\text{for all }n\in\mathbb{N}.$$

Otherwise, by Lemma 4.9, we can see that

$$\langle D[J_{\lambda,\beta}(\phi_n,\psi_n)],(t^0_{\lambda,n}\widetilde{\varphi},t^0_{\lambda,n}\widetilde{\eta})\rangle_{E^*_\lambda,E_\lambda}=o_n(1),$$

where $\widetilde{\varphi}$ and $\widetilde{\eta}$ are the projections of φ and η on $\widetilde{\mathcal{F}}_{a,\lambda}^{\perp}\oplus\mathcal{F}_{a,\lambda}$ and $\widetilde{\mathcal{F}}_{b,\lambda}^{\perp}\oplus\mathcal{F}_{b,\lambda}$ and $t_{\lambda,n}^0\in\mathbb{R}^+$ is given by (3.6) and only depends on (ϕ_n,ψ_n). If $t_{\lambda,n}^0\to 0$ as $n\to\infty$, then by Lemma 2.3, (3.6) and Step 1, we must have $J_{\lambda,\beta}(\phi_n,\psi_n)\leq 0$, which implies $c_{\lambda,\beta}\leq 0$. It is impossible since Lemma 3.10 holds for $\lambda\geq\Lambda_0^*$. Therefore, by $(\phi_n,\psi_n)\in\mathcal{G}_{\lambda,\beta}$, we have

$$\langle D[J_{\lambda,\beta}(\phi_n,\psi_n)],(\varphi,\eta)\rangle_{E_\lambda^*,E_\lambda}=o_n(1)\quad\text{for all }(\varphi,\eta)\in E_\lambda.$$

Step 3. We prove that there exists $(u_{\lambda,\beta},v_{\lambda,\beta})\in\mathcal{G}_{\lambda,\beta}$ such that

$$D[J_{\lambda,\beta}(u_{\lambda,\beta},v_{\lambda,\beta})]=0$$

in E_λ^* and $J_{\lambda,\beta}(u_{\lambda,\beta},v_{\lambda,\beta})=c_{\lambda,\beta}$.

Without loss of generality, we may assume that $(\phi_n,\psi_n)\rightharpoonup(u_{\lambda,\beta},v_{\lambda,\beta})$ weakly in E_λ and $(\phi_n,\psi_n)\to(u_{\lambda,\beta},v_{\lambda,\beta})$ a.e. in $\mathbb{R}^4\times\mathbb{R}^4$ as $n\to\infty$. Clearly, $D[J_{\lambda,\beta}(u_{\lambda,\beta},v_{\lambda,\beta})]=0$ in E_λ^*. In this situation, two cases may occur:

(1) $(u_{\lambda,\beta},v_{\lambda,\beta})=(0,0)$;

(2) $(u_{\lambda,\beta},v_{\lambda,\beta})\neq(0,0)$.

If case (1) happens, then $(\phi_n,\psi_n)\rightharpoonup(0,0)$ weakly in E_λ as $n\to\infty$. Since $|\mathcal{A}_\lambda|<+\infty$ and $|\mathcal{B}_\lambda|<+\infty$, by the condition (D_1), we can see that

$$\int_{\mathbb{R}^4}(\lambda a(x)+a_0)\phi_n^2dx\geq\int_{\mathbb{R}^4}(\lambda a(x)+a_0)^+\phi_n^2dx+o_n(1)\geq o_n(1)$$

and

$$\int_{\mathbb{R}^4}(\lambda b(x)+b_0)\psi_n^2dx\geq\int_{\mathbb{R}^4}(\lambda b(x)+b_0)^+\psi_n^2dx+o_n(1)\geq o_n(1),$$

where $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$ are given by (2.2) and Remark 2.1. By using the Sobolev inequality and $(\phi_n,\psi_n)\in\mathcal{G}_{\lambda,\beta}$, we have that

$$S(\|\phi_n\|_{L^4(\mathbb{R}^4)}^2+\|\psi_n\|_{L^4(\mathbb{R}^4)}^2)\leq\mathcal{L}_\beta(\phi_n,\psi_n)+o_n(1).$$

Note that $0\leq\beta<1$ and $J_{\lambda,\beta}(\phi_n,\psi_n)=c_{\lambda,\beta}+o_n(1)$, by Lemma 3.10, we have $S+o_n(1)\leq\|\phi_n\|_{L^4(\mathbb{R}^4)}^2+\|\psi_n\|_{L^4(\mathbb{R}^4)}^2$, which then implies

$$\|\phi_n\|_{L^4(\mathbb{R}^4)}^4+\|\psi_n\|_{L^4(\mathbb{R}^4)}^4+2\beta\|\phi_n^2\psi_n^2\|_{L^1(\mathbb{R}^4)}\geq S^2+o_n(1).$$

Now, by Lemma 3.11, we can see that

$$\begin{aligned}\frac{1}{4}S^2>c_{\lambda,\beta}+o_n(1)&=J_{\lambda,\beta}(\phi_n,\psi_n)\\&=\frac{1}{4}(\|\phi_n\|_{L^4(\mathbb{R}^4)}^4+\|\psi_n\|_{L^4(\mathbb{R}^4)}^4+2\beta\|\phi_n^2\psi_n^2\|_{L^1(\mathbb{R}^4)})\\&\geq\frac{1}{4}S^2+o_n(1),\end{aligned}$$

it is impossible. Therefore, we must have the case (2). In this case we can easily see that $J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) \geq c_{\lambda,\beta}$. On the other hand, since $0 \leq \beta < 1$, we must have

$$|\phi_n|^4 + |\psi_n|^4 + 2\beta|\phi_n|^2|\psi_n|^2 \geq 0 \quad \text{all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}^4.$$

It follows from the Fatou's lemma that

$$\begin{aligned} &\liminf_{n\to\infty} \int_{\mathbb{R}^4} (|\phi_n|^4 + |\psi_n|^4 + 2\beta|\phi_n|^2|\psi_n|^2)dx \\ &\geq \int_{\mathbb{R}^4} (|u_{\lambda,\beta}|^4 + |v_{\lambda,\beta}|^4 + 2\beta|u_{\lambda,\beta}|^2|v_{\lambda,\beta}|^2)dx, \end{aligned}$$

hence

$$\begin{aligned} c_{\lambda,\beta} &= \lim_{n\to\infty} J_{\lambda,\beta}(\phi_n, \psi_n) - \frac{1}{2}\langle D[J_{\lambda,\beta}(\phi_n, \psi_n)], (\phi_n, \psi_n)\rangle_{E_\lambda^*, E_\lambda} \\ &= \frac{1}{4}\lim_{n\to\infty} \int_{\mathbb{R}^4} (|\phi_n|^4 + |\psi_n|^4 + 2\beta|\phi_n|^2|\psi_n|^2)dx \\ &\geq \frac{1}{4}\int_{\mathbb{R}^4} (|u_{\lambda,\beta}|^4 + |v_{\lambda,\beta}|^4 + 2\beta|u_{\lambda,\beta}|^2|v_{\lambda,\beta}|^2)dx \\ &= J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) - \frac{1}{2}\langle D[J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})], (u_{\lambda,\beta}, v_{\lambda,\beta})\rangle_{E_\lambda^*, E_\lambda} \\ &= J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}). \end{aligned}$$

Therefore, $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is a general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ for $\lambda \geq \Lambda_0^*$ and $0 \leq \beta < 1$. ∎

For the general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ obtained by Proposition 4.5, we also have the following properties.

Proposition 4.6 *Let $(u_{\lambda,\beta}, v_{\lambda,\beta})$ be the general ground state solution of $(\mathcal{P}_{\lambda,\beta})$ obtained by Proposition 4.5. Then*

(1) *$(u_{\lambda,0}, v_{\lambda,0})$ is a semi-trivial solution of $(\mathcal{P}_{\lambda,0})$ and must be of the type $(u_{\lambda,0}, 0)$ in the case of $a_0 \leq -\mu_{a,1}$ with $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$ and $b_0 \geq 0$. Furthermore, $(u_{\lambda,0}, 0)$ is a least energy critical point of $I_{a,\lambda}(u)$.*

(2) *For every $0 \leq \beta < 1$, there exists $\Lambda_\beta^* \geq \Lambda_0^*$ such that $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is a semi-trivial solution of $(\mathcal{P}_{\lambda,\beta})$ and must be of the type $(u_{\lambda,\beta}, 0)$ for $\lambda > \Lambda_\beta^*$ in the case of $a_0 \leq -\mu_{a,1}$ with $-a_0 \notin \sigma(-\Delta, H_0^1(\Omega_a))$ and $b_0 \geq 0$.*

Proof. (1) Since Lemma 3.11 holds, by a similar argument as used in the proof of Lemma 4.5, we can get the conclusion.

(2) Suppose the contrary, $(u_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0})$ is non-trivial for some $\beta_0 \in [0, 1)$ and $\{\lambda_n\}$ with $\lambda_n \to +\infty$ as $n \to \infty$. Since Lemma 3.11 holds and $\beta_0 \in [0, 1)$, by a similar argument as used in Step 1 of the proof to Proposition 4.5, we can see that $\{(u_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0})\}$ is bounded in $L^4(\mathbb{R}^4) \times L^4(\mathbb{R}^4)$. Without loss of generality, we assume $(u_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0}) \rightharpoonup (u_{0,\beta_0}, v_{0,\beta_0})$ weakly in $L^4(\mathbb{R}^4) \times L^4(\mathbb{R}^4)$

and $(u_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0}) \to (u_{0,\beta_0}, v_{0,\beta_0})$ a.e. in $\mathbb{R}^4 \times \mathbb{R}^4$ as $n \to \infty$. Since $|\mathcal{A}_{\lambda_n}| < +\infty$ and is nonincreasing for $n \in \mathbb{N}$ due to $\lambda_n \to +\infty$, by similar arguments as used in (4.32) and (4.43), we can obtain that

$$\int_{\mathbb{R}^4} (a(x) + \frac{a_0}{\lambda_n})^+ u^2_{\lambda_n,\beta_0} + b(x) v^2_{\lambda_n,\beta_0} dx \to 0 \quad \text{as } n \to \infty, \tag{4.45}$$

where $\mathcal{A}_{\lambda_n}$ is given by (2.2). By the Fatou lemma, we have from (4.45) that $(u_{0,\beta_0}, v_{0,\beta_0}) \in H^1_0(\Omega_a) \times H^1_0(\Omega_b)$ with $u_{0,\beta_0} = 0$ outside Ω_a and $v_{0,\beta_0} = 0$ outside Ω_b. It follows from the Sobolev embedding theorem, the condition (D_2) and (4.45) once more that $(u_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0}) \to (u_{0,\beta_0}, v_{0,\beta_0})$ strongly in $L^2(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ as $n \to \infty$. Since $H^1_0(\Omega_a) \times H^1_0(\Omega_b) \subset E_\lambda$ and $(u_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0})$ is the general ground state solution of $(\mathcal{P}_{\lambda_n,\beta_0})$ for all $n \in \mathbb{N}$, by the condition (D_3), it is easy to see that $I'_a(u_{0,\beta_0}) = 0$ and $I'_b(v_{0,\beta_0}) = 0$ in $H^{-1}(\Omega_a)$ and $H^{-1}(\Omega_b)$, respectively. Thanks to Lemma 3.11 and a similar argument as used in (4.35), we must have $v_{0,\beta_0} = 0$. Let $w_{\lambda_n,\beta_0} = u_{\lambda_n,\beta_0} - u_{0,\beta_0}$. Then $w_{\lambda_n,\beta_0} \to 0$ strongly in $L^2(\mathbb{R}^4)$ and $w_{\lambda_n,\beta_0} \rightharpoonup 0$ weakly in $L^4(\mathbb{R}^4)$ as $n \to \infty$. Furthermore, by the condition (D_3), the Brezís-Lieb lemma and [17, Lemma 2.3], we also have

$$J_{\lambda_n,\beta_0}(u_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0}) = J_{\lambda_n,\beta_0}(w_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0}) + I_a(u_{0,\beta_0}) + o_n(1). \tag{4.46}$$

Now, similar to (4.35), we can get that $u_{0,\beta_0} \neq 0$. Due to $-a_0 \notin \sigma(-\Delta, H^1_0(\Omega_a))$ and the result of [14], u_{0,β_0} is a least energy critical point of $I_a(u)$. This together with (4.46), implies $(w_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0}) \to (0,0)$ strongly in $L^4(\mathbb{R}^4) \times L^4(\mathbb{R}^4)$ as $n \to \infty$. It follows from the condition (D_1) and the fact that $(u_{\lambda_n,\beta_0}, v_{\lambda_n,\beta_0})$ is a non-trivial solution of $(\mathcal{P}_{\lambda_n,\beta_0})$ that $\|u_{0,\beta_0}\|^2_{L^4(\mathbb{R}^4)} \geq \frac{S}{\beta_0}$, which is impossible. It remains to show that $u_{\lambda,\beta} \neq 0$ for $0 \leq \beta < 1$ and $\lambda > \Lambda^*_\beta$. Indeed, if $u_{\lambda,\beta} = 0$, then by $b_0 \geq 0$ and the condition (D_1), we can see that $c_{\lambda,\beta} \geq \frac{1}{4}S^2$, which contradicts to Lemma 3.6 once more. ∎

We close this section by

Proof of Theorem 1.2: This theorem follows immediately from Propositions 4.5–4.6. ∎

5 Concentration behaviors

This section is devoted to the concentration behaviors of the ground state solutions and the general ground state solutions to $(\mathcal{P}_{\lambda,\beta})$ obtained by Theorems 1.1 and 1.2. We first study such a property as $\lambda \to +\infty$.

Proof of Theorem 1.3: Let $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})$ be the solution obtained by Theorems 1.1 and 1.2 with $\lambda_n \to +\infty$ as $n \to \infty$. Then by a similar argument as used in the proof of (3) to Proposition 4.6, we can see that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, v_{0,\beta}) \in H^1_0(\Omega_a) \times H^1_0(\Omega_b)$ strongly in $(L^4(\mathbb{R}^4) \times L^4(\mathbb{R}^4)) \cap (L^2(\mathbb{R}^4) \times L^2(\mathbb{R}^4))$ as $n \to \infty$ up to a subsequence if $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})$ is the ground state solution with $\beta \leq 0$ in the case of $a_0 \leq b_0 < 0$ or $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})$ is the general ground state solution in the case of $a_0 < 0$. Furthermore, $(u_{0,\beta}, v_{0,\beta})$ is a solution of the equations (1.11). It follows from $D[J_{\lambda_n,\beta}(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})] = 0$ in $E^*_{\lambda_n}$ that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ in $D^{1,2}(\mathbb{R}^4) \times D^{1,2}(\mathbb{R}^4)$ as $n \to \infty$ up to a subsequence. Thus, $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ as $n \to \infty$ up to a subsequence. If $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})$ is the ground state solution with $\beta \leq 0$ in the case of $a_0 \leq b_0 < 0$, then by Lemmas 3.12 and 3.13, $(u_{0,\beta}, v_{0,\beta})$ is nontrivial. Thanks to lemma 3.8, $(u_{0,\beta}, v_{0,\beta})$ must be the ground state solution of (1.11). If $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})$ is the general ground state solution in the case of $a_0 < 0$, then by Lemma 3.8 once more, we have that either $(u_{0,\beta}, v_{0,\beta})$ is semi-trivial or $(u_{0,\beta}, v_{0,\beta}) = (0, 0)$. Note that Lemma 3.1 holds, thus, it is easy to show that $m_{\lambda_n,\beta}$ is nondecreasing for n and $m_{\lambda_n,\beta} > 0$ for all $n \in \mathbb{N}$. Therefore, we must have $(u_{0,\beta}, v_{0,\beta})$ is semi-trivial. Due to lemma 3.8, we can also see that $(u_{0,\beta}, v_{0,\beta})$ is a general ground state solution to (1.11), which completes the proof. ∎

We next study the concentration behaviors when $\beta \to -\infty$.

Proof of Theorem 1.4: Let $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})$ be the ground state solution of $(\mathcal{P}_{\lambda,\beta_n})$ obtained by Theorem 1.1 with $\beta_n \to -\infty$ as $n \to \infty$. By Lemma 3.8, we have $\|(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\|^2_\lambda \leq 4(m_a + m_b) < S^2$. It follows from (1.1) that $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\}$ is bounded in E_λ and $\mathcal{H}$. Without loss of generality, we assume $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \rightharpoonup (u_{\lambda,\infty}, v_{\lambda,\infty})$ weakly in E_λ and $\mathcal{H}$ and $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \to (u_{\lambda,\infty}, v_{\lambda,\infty})$ a.e. in $\mathbb{R}^4 \times \mathbb{R}^4$ as $n \to \infty$. For the sake of clarity, the following proof will be further performed by several steps.

Step 1. We prove that $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \to (u_{\lambda,\infty}, v_{\lambda,\infty})$ strongly in $\mathcal{H}$ as $n \to \infty$ with $u_{\lambda,\infty} \neq 0$ and $v_{\lambda,\infty} \neq 0$. Indeed, one of the following two cases must happen:

(1) $u_{\lambda,\infty} = 0$ or $v_{\lambda,\infty} = 0$;

(2) $u_{\lambda,\infty} \neq 0$ and $v_{\lambda,\infty} \neq 0$.

If the Case (1) happens, then without loss of generality, we may assume $u_{\lambda,\infty} = 0$. By a similar argument as used in Proposition 4.2, we can see that $\|u_{\lambda,\beta_n}\|^4_{L^4(\mathbb{R}^4)} \geq S^2 + o_n(1)$. On the other hand, since $\beta_n < 0$, there exists $0 < s_n \leq 1 + o_n(1)$ such that $s_n v_{\lambda,\beta_n} \in \mathcal{N}_{b,\lambda}$. Now, by Lemma 3.3, we have

$$J_{\lambda,\beta_n}(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \geq J_{\lambda,\beta_n}(u_{\lambda,\beta_n}, s_n v_{\lambda,\beta_n}) \geq \frac{1}{4}S^2 + m_{b,\lambda} + o_n(1). \tag{5.1}$$

Thanks to Lemma 4.7, $m_{b,\lambda} \to m_b$ as $\lambda \to \infty$. Thus, there exists $\Lambda_3 > 0$ such that (5.1) is impossible for $\lambda > \Lambda_3$ due to Lemma 3.8. Therefore, we must have the Case (2). In this case, it is easily see from the Fatou's lemma that $\int_{\mathbb{R}^4} u^2_{\lambda,\infty} v^2_{\lambda,\infty} = 0$. On the other hand, since $\beta_n < 0$, multiplying $(\mathcal{P}_{\lambda,\beta_n})$ with $(u_{\lambda,\infty}, v_{\lambda,\infty})$, we obtain that

$$\mathcal{D}_{a,\lambda}(u_{\lambda,\infty}, u_{\lambda,\infty}) \leq \|u_{\lambda,\infty}\|^4_{L^4(\mathbb{R}^4)} \text{ and } \mathcal{D}_{b,\lambda}(v_{\lambda,\infty}, v_{\lambda,\infty}) \leq \|v_{\lambda,\infty}\|^4_{L^4(\mathbb{R}^4)}.$$

Thanks to Lemma 3.3, there exist $0 < t_n^* \leq$ and $0 < s_n^* \leq 1$ such that $(t_n^* u_{\lambda,\infty}, s_n^* v_{\lambda,\infty}) \in \mathcal{N}_{\lambda,\beta_n}$ for all n, which, together with the Sobolev embedding theorem, implies

$$\mathcal{D}_\lambda(t_n^* u_{\lambda,\infty}, s_n^* v_{\lambda,\infty}) \geq \mathcal{D}_\lambda(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \geq \mathcal{D}_\lambda(u_{\lambda,\infty}, v_{\lambda,\infty}) + o_n(1).$$

It follows from Lemma 2.3 and (1.1) that $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \to (u_{\lambda,\infty}, v_{\lambda,\infty})$ strongly in $\mathcal{H}$ as $n \to \infty$.

Step 2. We prove that $u_{\lambda,\infty} \in H_0^1(\{u_{\lambda,\infty} > 0\})$ and $v_{\lambda,\infty} \in H_0^1(\{v_{\lambda,\infty} > 0\})$.

Indeed, since the conditions (D_1)–(D_3) hold, u_{λ,β_n} and v_{λ,β_n} are both positive in $\mathbb{R}^4$ by the maximum principle and Lemma 3.5. It follows that $u_{\lambda,\infty}$ and $v_{\lambda,\infty}$ are both nonnegative in $\mathbb{R}^4$. Thanks to Step 1 and [3, Propositions 3.8 and 3.9], $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\}$ is also bounded in $L^\infty(\mathbb{R}^4) \times L^\infty(\mathbb{R}^4)$. By [43, Theorem 1.7], $\{(\nabla u_{\lambda,\beta_n}, \nabla v_{\lambda,\beta_n})\}$ is bounded in $L^\infty(\mathbb{R}^4) \times L^\infty(\mathbb{R}^4)$. By the interior H^2–regularity (see for example Theorem 1 in p.309 of [23]), we can see that $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\} \in H^2_{loc}(\mathbb{R}^4) \times H^2_{loc}(\mathbb{R}^4)$. It follows from the L^p-regularity (see for example Theorem B.2 in [37]) and the fact that $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\}$ is bounded in $L^\infty(\mathbb{R}^4) \times L^\infty(\mathbb{R}^4)$ that $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\} \in W^{2,p}_{loc}(\mathbb{R}^4) \times W^{2,p}_{loc}(\mathbb{R}^4)$ for all $p \geq 2$. Thanks to the Sobolev embedding theorem, we have $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\} \in C^1(\mathbb{R}^4) \times C^1(\mathbb{R}^4)$. It follows from Ascoli–Arzelá theorem that $u_{\lambda,\beta_n} \to u_{\lambda,\infty}$ and $v_{\lambda,\beta_n} \to v_{\lambda,\infty}$ strongly in $C_{loc}(\mathbb{R}^4)$ as $n \to \infty$ with $u_{\lambda,\infty}, v_{\lambda,\infty} \in C(\mathbb{R}^4)$. Now, for $x, y \in \mathbb{R}^4$, we can see from the fact that $\{\nabla u_{\lambda,\beta_n}\}$ is bounded in $L^\infty(\mathbb{R}^4)$ that

$$\begin{aligned}
& |u_{\lambda,\infty}(x) - u_{\lambda,\infty}(y)| \\
\leq\ & |u_{\lambda,\infty}(x) - u_{\lambda,\beta_n}(x)| + |u_{\lambda,\beta_n}(x) - u_{\lambda,\beta_n}(y)| + |u_{\lambda,\infty}(y) - u_{\lambda,\beta_n}(y)| \\
\leq\ & \|\nabla u_{\lambda,\beta_n}\|_{L^\infty(\mathbb{R}^4)} |x - y| + o_n(1) \\
\leq\ & C|x - y| + o_n(1).
\end{aligned}$$

Let $n \to \infty$ in both side of the above inequality, then we obtain that $u_{\lambda,\infty}$ is local Lipschitz in $\mathbb{R}^4$. Similarly, $v_{\lambda,\infty}$ is also local Lipschitz in $\mathbb{R}^4$. Thus,$H_0^1(\{u_{\lambda,\infty} > 0\})$ and $H_0^1(\{v_{\lambda,\infty} > 0\})$ are both well defined. Furthermore, due to the extension theorem, we also have $u_{\lambda,\infty} \in H_0^1(\{u_{\lambda,\infty} > 0\})$ and $v_{\lambda,\infty} \in H_0^1(\{v_{\lambda,\infty} > 0\})$. Now, we can use similar arguments as used in the proof of [15, Theorem 1.4] (see also [49, Theorem 1.3]) to show the conclusions. ∎

We close this section by

Proof of Theorem 1.5: By Lemmas 3.8 and 3.9, we have $m_{\lambda,\beta} \in [m_{a,\lambda} + m_{b,\lambda}, m_a + m_b]$ for all $\lambda > \Lambda_0$ and $\beta \leq 0$ in the case of $a_0 \leq b_0 < 0$. Furthermore, by Lemma 4.7, we have $m_{a,\lambda} + m_{b,\lambda} \to m_a + m_b$ as $\lambda \to +\infty$. Thus, for every $\{(\lambda_n, \beta_n)\}$ satisfying $\lambda_n \to +\infty$ and $\beta_n \to -\infty$ as $n \to \infty$, by a similar argument as used in Theorem 1.3, we can see that the ground state solution of $(\mathcal{P}_{\lambda_n,\beta_n})$ obtained in Theorem 1.1 in the case of $a_0 \leq b_0 < 0$ has the same concentration behaviors as described in Theorem 1.3. ∎

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