CONTINUATION METHODS FOR COMPUTING Z-/H-EIGENPAIRS OF NONNEGATIVE TENSORS

YUEH-CHENG KUO*, WEN-WEI LIN†, AND CHING-SUNG LIU‡

Abstract. In this paper, a homotopy continuation method for the computation of nonnegative Z-/H-eigenpairs of a nonnegative tensor is presented. We show that the homotopy continuation method is guaranteed to compute a nonnegative eigenpair. Additionally, using degree analysis, we show that the number of positive Z-eigenpairs of an irreducible nonnegative tensor is odd. A novel homotopy continuation method is proposed to compute an odd number of positive Z-eigenpairs, and some numerical results are presented.

Key words. continuation method, nonnegative tensor, Z-eigenpair, H-eigenpair, tensor eigenvalue problem

AMS subject classifications. 65F15, 65F50

1. Introduction.

An mth-order tensor $\mathcal{A} \in \mathbb{F}^{n_1 \times n_2 \times \cdots \times n_m}$ is a multidimensional or m-way array, where $\mathbb{F}$ is a field. A first-order tensor is a vector, a second-order tensor is a matrix, and tensors of order three or higher are called higher-order tensors. When $n := n_1 = n_2 = \cdots = n_m$, $\mathcal{A}$ is called an mth-order, n-dimensional tensor. We denote the set of all mth-order, n-dimensional tensors on the field $\mathbb{F}$ by $\mathbb{F}^{[m,n]}$. For a tensor $\mathcal{A} \in \mathbb{F}^{[m,n]}$, the tensor eigenvalues and eigenvectors have been considered in many literatures [6, 21, 28, 29, 30], there are two particularly interesting definitions called Z-eigenvalues and H-eigenvalues (see the definition later on). Tensor eigenproblems have found applications in automatic control [1, 2, 3], magnetic resonance imaging [32, 31], spectral hypergraph theory [14, 16], higher order Markov chains [5, 10, 20], etc.

Unlike the matrix eigenvalue problem, computing eigenvalues of a general higher-order tensor is NP-hard [11]. Recently, Chen, Han and Zhou [8] proposed a homotopy continuation method for finding all eigenpairs of a general tensor. For the tensors of a certain type, such as symmetric or nonnegative tensors, there are several algorithms for computing one or some eigenpairs (including Z-eigenpair and H-eigenpair) .

For the computation of Z-eigenpairs, Kolda and Mayo [18] proposed a shifted symmetric higher-order power method (SS-HOPM) for real symmetric tensors. Gleich, Lim, and Yu [10] proposed, a always-stochastic Newton iteration for finding nonnegative Z-eigenpair of nonnegative tensors arising in a multilinear PageRank problem.

For the computation of H-eigenpairs, Ng, Qi, and Zhou [25] proposed a power-type method, NQZ algorithm, for the largest H-eigenvalue of weakly primitive nonnegative tensors. Some modeled versions of the power-type method have been proposed in [24, 34, 35]. Recently, Liu, Guo and Lin [22, 23] proposed a Newton-Noda iteration (NNI) for finding the largest H-eigenvalue of weakly irreducible nonnegative tensors.

For a high-order nonnegative tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, $\mathcal{A}$ has nonnegative Z-eigenpairs and H-eigenpairs, but they are not unique (see [4, 6, 7, 6]). In many applications [5, 10, 14, 16, 20], computing the nonnegative Z-/H-eigenpairs is an important subject.

*Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan; yckuo@nuk.edu.tw

†Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan wwlin@math.nctu.edu.tw

‡Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan; chingsungliu@nuk.edu.tw

Therefore, a central concern is how to avoid computing all the eigenvalues to find a few nonnegative Z-eigenpairs and H-eigenpairs. SS-HOPM [18, 10] and NQZ [25] can be used to compute a nonnegative Z-eigenpair and H-eigenpair, respectively, but the convergence may be quite slow. The always-stochastic Newton's method [10] is a fast-converging algorithm when the starting point is sufficiently close to a solution. However, it's interestingly enough that the authors [10] also provided a nonnegative tensor with a unique nonnegative Z-eigenpair, and the always-stochastic Newton's method fails to find it. Based on the reasons above mentioned, this motivates us to develop a continuation method to ensure the global convergence for nonnegative Z-eigenpairs. The main contributions of this article are highlighted in the following items.

- For *nonnegative Z-eigenpairs*: we construct a linear homotopy $H_{\rm Z}(\mathbf{x}, \lambda, t) = \mathbf{0}$, $t \in [0, 1]$, where $H_{\rm Z}(\mathbf{x}, \lambda, 0) = \mathbf{0}$ has only one positive solution, $(\mathbf{x}_0, \lambda_0)$, and all real solutions of $H_{\rm Z}(\mathbf{x}, \lambda, 1) = \mathbf{0}$ are Z-eigenpairs of $\mathcal{A}$.
  1. We show that the solution curve of $H_{\rm Z}(\mathbf{x}, \lambda, t) = \mathbf{0}$ with initial $(\mathbf{x}_0, \lambda_0, 0)$ is smooth. Furthermore, we also show that the solution curve will reach a nonnegative solution of $H_{\rm Z}(\mathbf{x}, \lambda, 1) = \mathbf{0}$ if all nonnegative solutions of $H_{\rm Z}(\mathbf{x}, \lambda, 1) = \mathbf{0}$ are isolated (see Theorems 3.2 and 3.5). Hence, in this case, homotopy continuation method is guaranteed to compute the nonnegative Z-eigenpair of $\mathcal{A}$.
  2. If $\mathcal{A}$ is irreducible and all nonnegative solutions of $H_{\rm Z}(\mathbf{x}, \lambda, 1) = \mathbf{0}$ are isolated, then we show that the number of positive Z-eigenpairs of $\mathcal{A}$, counting multiplicities, is $2k + 1$ for some integer $k \geqslant 0$ (see Corollary 3.8).
  3. We propose a novel homotopy continuation method to compute an odd number of positive Z-eigenpairs for an irreducible nonnegative tenor $\mathcal{A}$ (see the flowchart in Figure 4.1).
- For *nonnegative H-eigenpairs*: we construct a linear homotopy $H_{\rm H}(\mathbf{x}, \lambda, t) = \mathbf{0}$, $t \in [0, 1]$, where $H_{\rm H}(\mathbf{x}, \lambda, 0) = \mathbf{0}$ has only one positive solution, $(\mathbf{x}_0, \lambda_0)$, and all real solutions of $H_{\rm H}(\mathbf{x}, \lambda, 1) = \mathbf{0}$ are H-eigenpairs of $\mathcal{A}$. We show that the solution curve of $H_{\rm H}(\mathbf{x}, \lambda, t) = \mathbf{0}$ with initial $(\mathbf{x}_0, \lambda_0, 0)$ can be parameterized by $t \in [0, 1)$. If the nonnegative solutions of $H_{\rm H}(\mathbf{x}, \lambda, 1) = \mathbf{0}$ are isolated, then the solution curve will reach a nonnegative solution of $H_{\rm H}(\mathbf{x}, \lambda, 1) = \mathbf{0}$ (see Theorem 3.11), and hence, homotopy continuation method is guaranteed to compute the nonnegative H-eigenpair of $\mathcal{A}$. Note that if $\mathcal{A}$ is weakly irreducible, then $H_{\rm H}(\mathbf{x}, \lambda, 1) = \mathbf{0}$ has only one positive isolated solution (see Theorem 3.12).

This paper is organized as follows. The notations and preliminary results are in Section 2. In Section 3, we develop homotopy continuation methods to compute the nonnegative Z-eigenpairs and H-eigenpair of a nonnegative tensor $\mathcal{A}$ and show that the continuation methods are guaranteed to compute the nonnegative eigenpairs. In Section 4, we propose a novel homotopy continuation method to compute an odd number of positive Z-eigenpairs for an irreducible nonnegative tenor. Some numerical results are presented in Section 5. Conclusion of this paper is given in Section 6.

2. Preliminaries. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ be the complex field or the real field. An mth-order rank-1 tensor $\mathcal{A} = [\mathcal{A}_{i_1,i_2,\cdots,i_m}] \in \mathbb{F}^{n_1\times n_2\times\cdots\times n_m}$ is defined as the outer product of m nonzero vectors $\mathbf{u}_k \in \mathbb{F}^{n_k}$ for $k = 1, \cdots, m$, denoted by $\mathbf{u}_1 \circ \mathbf{u}_2 \circ \ldots \circ \mathbf{u}_m$. That is,

$$\mathcal{A}_{i_1,i_2,\cdots,i_m} = u_{1,i_1} u_{2,i_2} \cdots u_{m,i_m},$$

where u_{k,i_k} is the i_k-th component of vector $\mathbf{u}_k$. The *k-mode product* of a tensor $\mathcal{A} \in \mathbb{F}^{n_1 \times n_2 \times \cdots \times n_m}$ with a vector $\mathbf{x} = (x_1, \cdots, x_{n_k})^\top \in \mathbb{F}^{n_k}$ is denoted by $\mathcal{A} \times_k \mathbf{x}$ and is $(m-1)$th-order tensor with size $n_1 \times \cdots \times n_{k-1} \times n_{k+1} \times \cdots \times n_m$. Elementwise, we have

$$(\mathcal{A} \times_k \mathbf{x})_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} = \sum_{i_k=1}^{n_k} \mathcal{A}_{i_1,i_2,\cdots,i_m} x_{i_k}.$$

For a tensor $\mathcal{A} \in \mathbb{F}^{[m,n]}$ and a vector $\mathbf{x} = (x_1, \cdots, x_n)^\top \in \mathbb{F}^n$, we denote $\mathcal{A}\mathbf{x}^{m-1} = \mathcal{A} \times_2 \mathbf{x} \times_3 \cdots \times_m \mathbf{x}$ and $\mathbf{x}^{[\ell]} = (x_1^\ell, \cdots, x_n^\ell)$, where ℓ is a positive real number.

Let $\mathbb{R}_{\geqslant 0}^{[m,n]}$ ($\mathbb{R}_{>0}^{[m,n]}$) denote the set of all real nonnegative (positive) mth-order n-dimensional tensors. We use calligraphic letters to denote tensors, capital letters to denote matrices and lowercase (bold) letters to denote scalars (vectors). For a tensor $\mathcal{A}$, $\mathcal{A} \geqslant 0$ ($\mathcal{A} > 0$) denotes a nonnegative (positive) tensor with nonnegative (positive) entries. A real square nonsingular M-matrix B can be written as $sI - A$ with $A \geq 0$ if $s > \rho(A)$, and a singular M-matrix if $s = \rho(A)$, where $\rho(\cdot)$ is the spectral radius. We use the 2-norm for vectors and matrices, and all vectors are n-vectors and all matrices are $n \times n$, unless specified otherwise.

2.1. Tensor eigenvalues and eigenvectors. The following definition of Z-eigenvalues and H-eigenvalues was introduced by Qi in [28, 29].

Definition 2.1. *Suppose that $\mathcal{A}$ is an mth-order n-dimensional tensor.*

(i) $\lambda \in \mathbb{R}$ is called a Z-eigenvalue of $\mathcal{A}$ with the corresponding Z-eigenvector $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ (or $(\lambda, \mathbf{x})$ is a Z-eigenpair) if $(\lambda, \mathbf{x})$ satisfies

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}, \ \textit{with } \|\mathbf{x}\| = 1. \tag{2.1}$$

(ii) $\lambda \in \mathbb{R}$ is called a H-eigenvalue of $\mathcal{A}$ with the corresponding H-eigenvector $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ (or $(\lambda, \mathbf{x})$ is a H-eigenpair) if $(\lambda, \mathbf{x})$ satisfies

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}. \tag{2.2}$$

In [8, 27], the authors proved that if the tensor $\mathcal{A}$ is generic, then the number of isolated solutions of (2.1) and of (2.2) with $\|\mathbf{x}\| = 1$ are exactly $\frac{(m-1)^n - 1}{m-2}$ and $n(m-1)^{n-1}$, respectively.

The Perron-Frobenius theorems for (weakly) irreducible nonnegative tensor have been widely investigated. The definition of (weakly) irreducible tensor was introduced in [17].

Definition 2.2. *Suppose that $\mathcal{A}$ is an mth-order n-dimensional tensor.*

(i) $\mathcal{A}$ is called reducible if there exists a nonempty proper subset $S \subset \{1, 2, \cdots, n\}$ such that

$$\mathcal{A}_{i_1,i_2,\cdots,i_m} = 0, \ \forall i_1 \in S, \ \forall i_2, \ldots, i_m \notin S.$$

If $\mathcal{A}$ is not reducible, then $\mathcal{A}$ is called irreducible.

(ii) $\mathcal{A}$ is called weakly irreducible if for every nonempty subset $S \subset \{1, 2, \cdots, n\}$ there exist $i_1 \in S$ and $i_2, \cdots, i_m$ with at least one $i_q \notin S$, $q = 2, \cdots, m$ such that $\mathcal{A}_{i_1,i_2,\cdots,i_m} \neq 0$.

Note that when $m = 2$, the definitions of an irreducible tensor and a weakly irreducible tensor are the same as the definition of an irreducible matrix. From the definitions, it is easily seen that if $\mathcal{A}$ is irreducible then $\mathcal{A}$ is weakly irreducible.

The existence of nonnegative Z-eigenpair (see [4, 6]) or H-eigenpair (see [7, 6]) of a nonnegative tensor $\mathcal{A}$ have been investigated. They satisfy the following properties.

- **Z-eigenpair**: Let $\mathcal{A} \in \mathbb{R}_{\geqslant 0}^{[n,m]}$ and $\mathcal{Z}(\mathcal{A})$ be the set of all Z-eigenvalues of $\mathcal{A}$. Then $\mathcal{A}$ has Z-eigenpair $(\lambda_0, \mathbf{x}_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}^n$, i.e., $\mathcal{Z}(\mathcal{A}) \neq \emptyset$. In fact, the set $\mathcal{Z}(\mathcal{A})$ is not necessarily a finite set in general (see Example 3.6 in [4]). The set $\mathcal{Z}(\mathcal{A})$ and the Z-eigenpair $(\lambda_0, \mathbf{x}_0)$ of $\mathcal{A}$ satisfy the following statements:
  1. The set $\mathcal{Z}(\mathcal{A})$ is bounded. It follows from Proposition 3.3 of [4] that
  $$\varrho(\mathcal{A}) \equiv \sup\{|\lambda| \mid \lambda \in \mathcal{Z}(\mathcal{A})\} \leqslant \max_{1 \leqslant i \leqslant n} \sqrt{n} \sum_{i_2, \cdots, i_m = 1}^{n} \mathcal{A}_{i, i_2, \cdots, i_m}. \tag{2.3}$$
  2. If $\mathcal{A}$ is irreducible, then $\lambda_0 > 0$ and $\mathbf{x}_0 > 0$.
  3. If $\mathcal{A}$ is weakly symmetric,[1] then the cardinality of $\mathcal{Z}(\mathcal{A})$ is finite,
  $$\varrho(\mathcal{A}) \in \mathcal{Z}(\mathcal{A}) \text{ and } \varrho(\mathcal{A}) = \max_{\mathbf{x} \in \mathbb{R}^n, \ \|\mathbf{x}\| = 1} \mathcal{A}\mathbf{x}^m.$$
- **H-eigenpair**: If $\mathcal{A} \in \mathbb{R}_{\geqslant 0}^{[n,m]}$ then $\mathcal{A}$ has H-eigenpairs $(\lambda_0, \mathbf{x}_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}^n$. Suppose that $\mathcal{A}$ is weakly irreducible then the eigenpair $(\lambda_0, \mathbf{x}_0)$ satisfies the following statements:
  1. $\lambda_0 > 0$ and $\mathbf{x}_0 > 0$.
  2. If λ is an eigenvalue with nonnegative eigenvector, then $\lambda = \lambda_0$. Moreover, the nonnegative eigenvector is unique up to a multiplicative constant.
  3. If λ is an eigenvalue then $|\lambda| \leqslant \lambda_0$.

The following lemma is straightforward.

Lemma 2.1. *Let $\mathcal{A} \in \mathbb{R}_{>0}^{[m,n]}$. Then $\mathcal{A}$ has no Z-eigenpair (or H-eigenpair) on $\partial(\mathbb{R}_{\geqslant 0}^{n+1})$, where $\partial(\mathbb{R}_{\geqslant 0}^{n+1})$ is the boundary of $\mathbb{R}_{\geqslant 0}^{n+1}$.*

Proof. Assume that $(\lambda, \mathbf{x}) \in \partial(\mathbb{R}_{\geqslant 0}^{n+1})$ is a Z-eigenpair (or H-eigenpair) of $\mathcal{A} \in \mathbb{R}_{>0}^{[m,n]}$. Suppose that $\mathbf{x} \in \partial(\mathbb{R}_{\geqslant 0}^{n})$, then there exists $i \in \{1, 2, \ldots, n\}$ such that $x_i = 0$, where x_i is ith component of $\mathbf{x}$. Then the ith component of the vector $\mathcal{A}\mathbf{x}^{m-1}$ is zero, i.e., $\sum_{i_2, \cdots, i_m = 1}^{n} \mathcal{A}_{i, i_2, \cdots, i_m} x_{i_2} \cdots x_{i_m} = 0$. This is a contradiction because $\mathcal{A}_{i, i_2, \cdots, i_m} > 0$ and $\mathbf{x} \in \mathbb{R}_{\geqslant 0}$ is a nonzero vector. Hence, $\mathbf{x} > 0$. Since $\mathcal{A} > 0$ and $\mathbf{x} > 0$, we have $\lambda > 0$. Hence, $(\lambda, \mathbf{x}) \notin \partial(\mathbb{R}_{\geqslant 0}^{n+1})$. □

Next, we describe all Z-eigenpairs and H-eigenpairs of a rank-1 nonnegative tensor.

Lemma 2.2. *Let $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_m \in \mathbb{R}_{\geqslant 0}^{[m,n]}$, where $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}_{\geqslant 0}^{n}$ are nonzero vectors. Then*

(i) Z-eigenpairs: Let $\mathbf{x}_0 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$ and $\lambda_0 = \|\mathbf{x}_1\| \prod_{k=2}^{m} \left(\mathbf{x}_k^\top \mathbf{x}_0\right)$. Then $(\lambda_0, \mathbf{x}_0)$, $((-1)^m \lambda_0, -\mathbf{x}_0)$ and $(0, \mathbf{w})$ with $\mathbf{w} \in \bigcup_{k=2}^{m} \operatorname{span}\{\mathbf{x}_k\}^\perp$, $\|\mathbf{w}\| = 1$ are Z-eigenpairs of $\mathcal{A}_0$. In addition, if $\mathbf{x}_1$ is a positive vector, then the eigenvalue $\lambda_0 > 0$ with positive eigenvector $\mathbf{x}_0$. Furthermore, if $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}_{>0}^{n}$, then $\mathbf{x}_0 \in \mathbb{R}_{>0}^{n}$ is the unique nonnegative eigenvector of $\mathcal{A}_0$.

(ii) H-eigenpairs: Let $\lambda_0 = \prod_{k=2}^{m} \left(\mathbf{x}_k^\top \mathbf{x}_1^{[1/(m-1)]}\right)$ and $\mathbf{x}_0 = \frac{\mathbf{x}_1^{[1/(m-1)]}}{\|\mathbf{x}_1^{[1/(m-1)]}\|}$. Then $(\lambda_0, \mathbf{x}_0)$ and $(0, \mathbf{w})$ with $\mathbf{w} \in \bigcup_{k=2}^{m} \operatorname{span}\{\mathbf{x}_k\}^\perp$, $\mathbf{w} \neq \mathbf{0}$ are H-eigenpairs of $\mathcal{A}_0$. In addition, if $\mathbf{x}_1$ is a positive vector, then the eigenvalue $\lambda_0 > 0$ with positive eigenvector $\mathbf{x}_0$. Furthermore, if $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}_{>0}^{n}$, then $c\mathbf{x}_0 \in \mathbb{R}_{>0}^{n}$ with $c > 0$ is the unique nonnegative eigenvector of $\mathcal{A}_0$.

Proof. (i) Suppose that $(0, \mathbf{w})$ is a Z-eigenpair of tensor $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_m$. Then
$$\mathcal{A}_0 \mathbf{w}^{m-1} = (\mathbf{x}_2^\top \mathbf{w}) \cdots (\mathbf{x}_m^\top \mathbf{w}) \mathbf{x}_1 = \mathbf{0}.$$

[1] A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called weakly symmetric if it satisfies $\frac{d}{d\mathbf{x}} \mathcal{A}\mathbf{x}^m = m\mathcal{A}\mathbf{x}^{m-1}$.

Since $\mathbf{x}_1 \neq \mathbf{0}$, we obtain that $\prod_{k=2}^{m}(\mathbf{x}_k^\top \mathbf{w}) = 0$ and hence, there exists a $k \in \{2, \ldots, m\}$ such that $\mathbf{x}_k^\top \mathbf{w} = 0$. So, the Z-eigenvector $\mathbf{w} \in \bigcup_{k=2}^{m} \operatorname{span}\{\mathbf{x}_k\}^\perp$ and $\|\mathbf{w}\| = 1$. Suppose that $(\lambda, \mathbf{w})$ with $\lambda \neq 0$ is a Z-eigenpair of tensor $\mathcal{A}_0$. Then

$$\lambda \mathbf{w} = \mathcal{A}_0 \mathbf{w}^{m-1} = \left(\prod_{k=2}^{m} \left(\mathbf{x}_k^\top \mathbf{w} \right) \right) \mathbf{x}_1.$$

Since the Z-eigenvector is a unit vector, we obtain that $\mathbf{w} = \mathbf{x}_0$ or $\mathbf{w} = -\mathbf{x}_0$ is a Z-eigenvector corresponding to Z-eigenvalue $\lambda = \lambda_0$ or $\lambda = (-1)^m \lambda_0$, respectively, where $\mathbf{x}_0 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$ and $\lambda_0 = \|\mathbf{x}_1\| \prod_{k=2}^{m} \left(\mathbf{x}_k^\top \mathbf{x}_0 \right)$. If $\mathbf{x}_1 > 0$, then it is easily seen that $\lambda_0 > 0$. Furthermore, if $\mathbf{x}_1, \cdots, \mathbf{x}_m \in \mathbb{R}^n_{>0}$, then $\left(\bigcup_{k=2}^{m} \operatorname{span}\{\mathbf{x}_k\}^\perp \right) \bigcap \mathbb{R}^n_{\geqslant 0} = \{\mathbf{0}\}$ and hence $\mathbf{x}_0$ is the unique nonnegative eigenvector of $\mathcal{A}_0$.

(*ii*) Similarly, suppose that $(0, \mathbf{w})$ is a H-eigenpair of tensor $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_m$. Then we obtain the H-eigenvector $\mathbf{w} \in \bigcup_{k=2}^{m} \operatorname{span}\{\mathbf{x}_k\}^\perp$ and $\mathbf{w} \neq \mathbf{0}$. Suppose that $(\lambda, \mathbf{w})$ with $\lambda \neq 0$ is a H-eigenpair of tensor $\mathcal{A}_0$. Then

$$\lambda \mathbf{w}^{[m-1]} = \mathcal{A}_0 \mathbf{w}^{m-1} = \left(\prod_{k=2}^{m} (\mathbf{x}_k^\top \mathbf{w}) \right) \mathbf{x}_1.$$

Since $\lambda \neq 0$, we have $\mathbf{w} = c\mathbf{x}_0 = c \frac{\mathbf{x}_1^{[1/(m-1)]}}{\|\mathbf{x}_1^{[1/(m-1)]}\|}$ and $\lambda = \lambda_0 \equiv \prod_{k=2}^{m} \left(\mathbf{x}_k^\top \mathbf{x}_1^{[1/(m-1)]} \right)$, where $c \neq 0$. The eigenvalue $\lambda \geqslant 0$, because $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n_{\geqslant 0}$. Since $\lambda \neq 0$, we obtain that $\mathbf{x}_k^\top \mathbf{x}_1^{[1/(m-1)]} \neq 0$ for each $k \in \{2, \cdots, m\}$. If $\mathbf{x}_1 > 0$, then $\lambda_0 = \prod_{k=2}^{m} \left(\mathbf{x}_k^\top \mathbf{x}_1^{[1/(m-1)]} \right) > 0$ because $\mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbb{R}^n_{\geqslant 0}$ are nonzero. Furthermore, if $\mathbf{x}_1, \cdots, \mathbf{x}_m \in \mathbb{R}^n_{>0}$, then $\left(\bigcup_{k=2}^{m} \operatorname{span}\{\mathbf{x}_k\}^\perp \right) \bigcap \mathbb{R}^n_{\geqslant 0} = \{\mathbf{0}\}$. So, $c\mathbf{x}_0$ with $c > 0$ is the unique nonnegative eigenvector of $\mathcal{A}_0$. This completes the proof. □

2.2. The basic theorems of continuation methods. In the following, we will introduce some preliminary theorems which are useful in study of continuation methods.

DEFINITION 2.3. *Let $H : \mathbb{R}^n \to \mathbb{R}^k$ be a continuously differentiable function (denoted by $H \in C^1(\mathbb{R}^n)$). A point $\mathbf{p} \in \mathbb{R}^k$ is called regular value if* $\operatorname{rank}(\mathscr{D}_{\mathbf{x}} H(\mathbf{x}_*)) = \min\{n, k\}$ *for all $\mathbf{x}_* \in H^{-1}(\mathbf{p}) \subseteq \mathbb{R}^n$, where $\mathscr{D}_{\mathbf{x}} H(\mathbf{x})$ denotes the partial derivatives of $H(\mathbf{x})$.*

Now, we state the Parameterized Sard's Theorem. The proof can be found in [9].

THEOREM 2.3 (Parameterized Sard's Theorem [9]). *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^q$ be open sets, and $P : U \times V \to \mathbb{R}^k$ be a smooth map. If $\mathbf{0} \in \mathbb{R}^k$ is a regular value of P, then for almost all $\mathbf{c} \in V$, $\mathbf{0}$ is a regular value of $H(\cdot) \equiv P(\cdot, \mathbf{c})$.*

Suppose that $\mathbf{p} \in \mathbb{R}^n$ is a regular value of a continuously differentiable function $F : \mathbb{R}^n \to \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ is open bounded and $\mathbf{p} \notin F(\partial\Omega)$. Then the set $F^{-1}(\mathbf{p}) \cap \overline{\Omega}$ is a finite set (see Lemma 3.5 in [19]). The "degree" of $F : \overline{\Omega} \to \mathbb{R}^n$ at a point $\mathbf{p} \in \mathbb{R}^n$ plays an important role in investigating the solution of $F(\mathbf{x}) = \mathbf{p}$. The definition of the degree for F is as follows.

DEFINITION 2.4 (See [19]). *Let $F : \overline{\Omega} \to \mathbb{R}^n$ be a continuously differentiable function, where $\Omega \subseteq \mathbb{R}^n$ is an open bounded subset. Let $\mathbf{p} \in \mathbb{R}^n$ be a regular value of F and $\mathbf{p} \notin F(\partial\Omega)$. The the degree of F on Ω for $\mathbf{p}$ is defined as:*

$$\deg(F, \Omega, \mathbf{p}) = \sum_{\mathbf{x} \in F^{-1}(\mathbf{p}) \cap \overline{\Omega}} \operatorname{Sgn}[\det \mathscr{D}_{\mathbf{x}} F(\mathbf{x})].$$

Remark 2.4. *The conditions of the definition for the degree of F on Ω for $\mathbf{p}$ can be relaxed. It only requires the function F satisfies (i) $F \in C^1(\overline{\Omega})$ and (ii) $\mathbf{p} \notin F(\partial\Omega)$ (see Definition 3.19 in [19]). That is, the condition, $\mathbf{p}$ is a regular value of F, can be omitted in the definition. The main idea is that if $\mathbf{p}$ is not a regular value then the degree can be defined as $\deg(F,\Omega,\mathbf{p}) \equiv \deg(F,\Omega,\mathbf{q})$, where $\mathbf{q}$ is a regular value of F and $\|\mathbf{q}-\mathbf{p}\| < \inf_{\mathbf{x}\in\partial\Omega}\|F(\mathbf{x})-\mathbf{p}\|$.*

Theorem 2.5 (Homotopy Invariance of Degree, see [19]). *Let $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ bounded open set and $\mathbf{p} \in \mathbb{R}^n$ satisfy:*

(i) $H \in C^2(\overline{\Omega}\times[0,1])$ and $H(\mathbf{u},t) \neq \mathbf{p}$ on $\partial\Omega \times [0,1]$,

(ii) $F^0(\mathbf{u}) \equiv H(\mathbf{u},0)$, $F(\mathbf{u}) \equiv H(\mathbf{u},1)$,

(iii) $\mathbf{p}$ is a regular value for H on $\overline{\Omega}\times[0,1]$ and for F^0 and F on $\overline{\Omega}$.

Then $\deg(H(\cdot,t),\Omega,\mathbf{p})$ is independent of $t \in [0,1]$. In particular, $\deg(F^0,\Omega,\mathbf{p}) = \deg(F,\Omega,\mathbf{p})$.

Remark 2.6. *Without the restriction (i) in Theorem 2.5, the theorem is true for the weakly definition of degree (see Remark 2.4).*

2.3. Linear homotopies. Given a tensor $\mathcal{A} \in \mathbb{R}_{\geqslant 0}^{[m,n]}$, let

$$\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_m \in \mathbb{R}_{>0}^{[m,n]}, \tag{2.4}$$

be a rank-1 tensor, where $\mathbf{x}_1,\ldots,\mathbf{x}_m \in \mathbb{R}_{>0}^n$ are generic. We define some systems of polynomial equations:

$$F_{\mathrm{Z}}^0(\mathbf{x},\lambda) = \begin{pmatrix} \mathcal{A}_0\mathbf{x}^{m-1} - \lambda\mathbf{x} \\ \mathbf{x}^\top\mathbf{x} - 1 \end{pmatrix} = \mathbf{0}, \quad F_{\mathrm{Z}}(\mathbf{x},\lambda) = \begin{pmatrix} \mathcal{A}\mathbf{x}^{m-1} - \lambda\mathbf{x} \\ \mathbf{x}^\top\mathbf{x} - 1 \end{pmatrix} = \mathbf{0}, \tag{2.5}$$

and

$$F_{\mathrm{H}}^0(\mathbf{x},\lambda) = \begin{pmatrix} \mathcal{A}_0\mathbf{x}^{m-1} - \lambda\mathbf{x}^{[m-1]} \\ \mathbf{x}^\top\mathbf{x} - 1 \end{pmatrix} = \mathbf{0}, \ F_{\mathrm{H}}(\mathbf{x},\lambda) = \begin{pmatrix} \mathcal{A}\mathbf{x}^{m-1} - \lambda\mathbf{x}^{[m-1]} \\ \mathbf{x}^\top\mathbf{x} - 1 \end{pmatrix} = \mathbf{0}. \tag{2.6}$$

From Definition 2.1, if the H-eigenvector of $\mathcal{A}$ is a unit vector, then the Z-eigenpair and H-eigenpair should satisfy $F_{\mathrm{Z}}(\mathbf{x},\lambda) = \mathbf{0}$ and $F_{\mathrm{H}}(\mathbf{x},\lambda) = \mathbf{0}$, respectively. Let

$$\mathcal{A}(t) = (1-t)\mathcal{A}_0 + t\mathcal{A}. \tag{2.7}$$

Now, we consider two linear homotopies

$$H_{\mathrm{Z}}(\mathbf{x},\lambda,t) = \begin{pmatrix} \mathcal{A}(t)\mathbf{x}^{m-1} - \lambda\mathbf{x} \\ \mathbf{x}^\top\mathbf{x} - 1 \end{pmatrix} = \mathbf{0}, \quad \text{for } t \in [0,1] \tag{2.8}$$

and

$$H_{\mathrm{H}}(\mathbf{x},\lambda,t) = \begin{pmatrix} \mathcal{A}(t)\mathbf{x}^{m-1} - \lambda\mathbf{x}^{[m-1]} \\ \mathbf{x}^\top\mathbf{x} - 1 \end{pmatrix} = \mathbf{0}, \quad \text{for } t \in [0,1]. \tag{2.9}$$

It is easily seen that $H_{\mathrm{Z}}(\mathbf{x},\lambda,0) = F_{\mathrm{Z}}^0(\mathbf{x},\lambda)$, $H_{\mathrm{Z}}(\mathbf{x},\lambda,1) = F_{\mathrm{Z}}(\mathbf{x},\lambda)$, $H_{\mathrm{H}}(\mathbf{x},\lambda,0) = F_{\mathrm{H}}^0(\mathbf{x},\lambda)$ and $H_{\mathrm{H}}(\mathbf{x},\lambda,1) = F_{\mathrm{H}}(\mathbf{x},\lambda)$.

We then have the following results.

Theorem 2.7. *For any $t \in [0,1)$, $H_{\mathrm{Z}}(\mathbf{x},\lambda,t) \neq \mathbf{0}$ and $H_{\mathrm{H}}(\mathbf{x},\lambda,t) \neq \mathbf{0}$ on the boundary of $\mathbb{R}_{\geqslant 0}^{n+1}$.*

Proof. For any $t \in [0,1)$, $\mathcal{A}(t) = (1-t)\mathcal{A}_0 + t\mathcal{A} > 0$ because $\mathcal{A}_0 > 0$ and $\mathcal{A} \geqslant 0$. It follows from Lemma 2.1 that $H_{\mathrm{Z}}(\mathbf{x},\lambda,t) \neq \mathbf{0}$ and $H_{\mathrm{H}}(\mathbf{x},\lambda,t) \neq \mathbf{0}$ on $\partial(\mathbb{R}_{\geqslant 0}^{n+1})$. ∎

3. Homotopy Continuation Methods. In this section, we propose homotopy continuation methods for computing nonnegative Z-/H-eigenpairs of a real nonnegative tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$. The following lemma is useful in our later analysis.

LEMMA 3.1. *Suppose that $A \in \mathbb{R}^{n\times n}$ is an irreducible singular M-matrix and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{>0}$. Then the matrix* $\left[\begin{array}{c|c} A & \mathbf{x} \\ \hline \mathbf{y}^\top & 0 \end{array}\right] \in \mathbb{R}^{(n+1)\times(n+1)}$ *is invertible.*

Proof. Suppose that there exists a nonzero vector $\mathbf{z} = (\mathbf{z}_1^\top, z_2)^\top \in \mathbb{R}^{n+1}$ such that

$$\left[\begin{array}{c|c} A & \mathbf{x} \\ \hline \mathbf{y}^\top & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{z}_1 \\ z_2 \end{array}\right] = \mathbf{0}. \tag{3.1}$$

Then $A\mathbf{z}_1 = -z_2\mathbf{x}$. Since A is an irreducible singular M-matrix, there is a vector $\mathbf{w} \in \mathbb{R}^n_{>0}$ such that $\mathbf{w}^\top A = \mathbf{0}^\top$. Then $z_2(\mathbf{w}^\top\mathbf{x}) = -\mathbf{w}^\top A\mathbf{z}_1 = 0$. Since $\mathbf{w}, \mathbf{x} \in \mathbb{R}^n_{>0}$, $\mathbf{w}^\top\mathbf{x} > 0$ and hence $z_2 = 0$. From (3.1), we have $A\mathbf{z}_1 = \mathbf{0}$ and $\mathbf{y}^\top\mathbf{z}_1 = 0$. Since A is an irreducible singular M-matrix, $\mathbf{z}_1 > 0$, and hence $\mathbf{y}^\top\mathbf{z}_1 > 0$, a contradiction. This completes the proof. ☐

3.1. Computing the Z-eigenpair of nonnegative tensors. Given a tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$, let $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1 \in \mathbb{R}^{[m,n]}_{>0}$ be a symmetric rank-1 tensor, where $\mathbf{x}_1 \in \mathbb{R}^n_{>0}$ is generic. Let

$$\mathbf{x}_0 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} > 0, \qquad \lambda_0 = \|\mathbf{x}_1\| \left(\mathbf{x}_1^\top \mathbf{x}_0\right)^{m-1} = \|\mathbf{x}_1\|^m > 0. \tag{3.2}$$

It follows from Lemma 2.2 (i) that $(\lambda_0, \mathbf{x}_0) \in \mathbb{R}^{n+1}_{>0}$ is a Z-eigenpair of $\mathcal{A}_0$ and $\mathbf{x}_0$ is the unique Z-eigenvector of $\mathcal{A}_0$ in $\mathbb{R}^n_{\geqslant 0}$. Then $F^0_{\mathrm{Z}}(\mathbf{x}_0, \lambda_0) = H_{\mathrm{Z}}(\mathbf{x}_0, \lambda_0, 0) = \mathbf{0}$, where F^0_{Z} is defined in (2.5) and H_{Z} is defined in (2.8) with the symmetric rank-1 tensor $\mathcal{A}_0$.

Suppose that $(\mathbf{x}_*, \lambda_*, t_*) \in \mathbb{R}^n_{>0} \times \mathbb{R}_{>0} \times [0, 1)$ is a solution of $H_{\mathrm{Z}}(\mathbf{x}, \lambda, t) = \mathbf{0}$. The Jacobian matrix of H_{Z} at $(\mathbf{x}_*, \lambda_*, t_*)$ has the form

$$\mathscr{D}_{\mathbf{x},\lambda,t} H_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*, t_*) = [\mathscr{D}_{\mathbf{x},\lambda} H_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*, t_*) \mid \mathscr{D}_t H_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*, t_*)], \tag{3.3a}$$

where

$$\mathscr{D}_{\mathbf{x},\lambda} H_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*, t_*) = \left[\begin{array}{c|c} A_{t_*} - \lambda_* I_n & -\mathbf{x}_* \\ \hline 2\mathbf{x}_*^\top & 0 \end{array}\right] \in \mathbb{R}^{(n+1)\times(n+1)}, \tag{3.3b}$$

$$\mathscr{D}_t H_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*, t_*) = \left[\begin{array}{c} (\mathcal{A} - \mathcal{A}_0)\mathbf{x}_*^{m-1} \\ \hline 0 \end{array}\right] \in \mathbb{R}^{n+1} \tag{3.3c}$$

with

$$\begin{aligned} A_{t_*} &\equiv \mathscr{D}_{\mathbf{x}}(\mathcal{A}(t_*)\mathbf{x}^{m-1})|_{\mathbf{x}=\mathbf{x}_*} \\ &= \sum_{k=2}^{m} \mathcal{A}(t_*) \times_2 \mathbf{x}_* \cdots \times_{k-1} \mathbf{x}_* \times_{k+1} \mathbf{x}_* \cdots \times_m \mathbf{x}_* \in \mathbb{R}^{n\times n}. \end{aligned} \tag{3.4}$$

Since $\mathbf{x}_* > 0$, it follows from (3.4) that $A_{t_*} > 0$ and $A_{t_*}\mathbf{x}_* = (m-1)\mathcal{A}(t_*)\mathbf{x}_*^{m-1} = (m-1)\lambda_*\mathbf{x}_*$. Hence, $-(A_{t_*} - (m-1)\lambda_* I_n)$ is a singular M-matrix. The leading submatrix of (3.3b), $A_{t_*} - \lambda_* I_n$, has at least one positive real eigenvalue when $m > 2$.

Next, we show that $\mathbf{0} \in \mathbb{R}^{n+1}$ is a regular value of $H_{\mathrm{Z}} : \mathbb{R}^{n+1}_{>0} \times [0, 1) \to \mathbb{R}^{n+1}$.

THEOREM 3.2. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ and $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1$, where $\mathbf{x}_1 \in \mathbb{R}^n_{>0}$ is generic. Then $\mathbf{0} \in \mathbb{R}^{n+1}$ is a regular value of the homotopy function $H_{\mathrm{Z}} : \mathbb{R}^{n+1}_{>0} \times [0, 1) \to \mathbb{R}^{n+1}$ in* (2.8).

Proof. Let $P : \mathbb{R}^{n+1}_{>0} \times (0,1) \times \mathbb{R}^n_{>0} \to \mathbb{R}^{n+1}$ be defined by $P(\mathbf{u}, t, \mathbf{c}) = H_{\mathrm{Z}}(\mathbf{u}, t)$, where $\mathbf{u} = (\mathbf{x}, \lambda)$ and $H_{\mathrm{Z}}(\mathbf{u}, t)$ is given in (2.8) with $\mathcal{A}_0 = \mathbf{c} \circ \cdots \circ \mathbf{c} \in \mathbb{R}^{[m,n]}_{>0}$. Now we show that $\mathbf{0} \in \mathbb{R}^{n+1}$ is a regular value of P. Let $(\mathbf{u}_*, t_*, \mathbf{c}_*) \in \mathbb{R}^{n+1}_{>0} \times (0,1) \times \mathbb{R}^n_{>0}$ be a solution of $P(\mathbf{u}, t, \mathbf{c}) = \mathbf{0}$, then $\mathscr{D}_{\mathbf{u},t,\mathbf{c}} P(\mathbf{u}_*, t_*, \mathbf{c}_*) = [\mathscr{D}_{\mathbf{u},t} P(\mathbf{u}_*, t_*, \mathbf{c}_*) | \mathscr{D}_{\mathbf{c}} P(\mathbf{u}_*, t_*, \mathbf{c}_*)]$, where $\mathscr{D}_{\mathbf{u},t} P(\mathbf{u}_*, t_*, \mathbf{c}_*) = \mathscr{D}_{\mathbf{x},\lambda,t} H_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*, t_*)$ is given in (3.3) with $\mathbf{u}_* = (\mathbf{x}_*, \lambda_*)$ and $\mathcal{A}_0 = \mathbf{c}_* \circ \cdots \circ \mathbf{c}_*$, and

$$\mathscr{D}_{\mathbf{c}} P(\mathbf{u}_*, t_*, \mathbf{c}_*) = (1 - t_*) \left[\begin{array}{c} (\mathbf{c}_*^\top \mathbf{x}_*)^{m-1} I_n + (m-1)(\mathbf{c}_*^\top \mathbf{x}_*)^{m-2} \mathbf{c}_* \mathbf{x}_*^\top \\ \hline \mathbf{0}^\top \end{array} \right] \in \mathbb{R}^{(n+1)\times n}.$$

Since $\mathbf{x}_*, \mathbf{c}_* \in \mathbb{R}^n_{>0}$, the matrix $(\mathbf{c}_*^\top \mathbf{x}_*)^{m-1} I_n + (m-1)(\mathbf{c}_*^\top \mathbf{x}_*)^{m-2} \mathbf{c}_* \mathbf{x}_*^\top$ is invertible. From (3.3), the last row of the matrix $\mathscr{D}_{\mathbf{u},t} P(\mathbf{u}_*, t_*, \mathbf{c}_*)$ is nonzero, and hence $\operatorname{rank}(\mathscr{D}_{\mathbf{u},t,\mathbf{c}} P(\mathbf{u}_*, t_*, \mathbf{c}_*)) = n+1$. That is, $\mathbf{0}$ is a regular value of P. It follows from the Parameterized Sard's Theorem (Theorem 2.3) that for almost all $\mathbf{x}_1 \in \mathbb{R}^n_{>0}$, $\mathbf{0} \in \mathbb{R}^{n+1}$ is a regular value of $H_{\mathrm{Z}} : \mathbb{R}^{n+1}_{>0} \times (0,1) \to \mathbb{R}^{n+1}$ with $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1$.

It remains to show that $\mathbf{0}$ is a regular value of $F^0_{\mathrm{Z}}(\mathbf{x}, \lambda) \equiv H_{\mathrm{Z}}(\mathbf{x}, \lambda, 0)$ on $\mathbb{R}^{n+1}_{\geqslant 0}$. For almost all $\mathbf{x}_1 \in \mathbb{R}^n_{>0}$, the system of polynomial equations $H_{\mathrm{Z}}(\mathbf{x}, \lambda, 0) = \mathbf{0}$ has only one solution $(\mathbf{x}_0, \lambda_0)$ in $\mathbb{R}^{n+1}_{>0}$, where $\mathbf{x}_0$ and $\lambda_0 = \|\mathbf{x}_1\|^m$ are given in (3.2). Now, we show that the Jacobian matrix

$$\mathscr{D}_{\mathbf{x},\lambda} H_{\mathrm{Z}}(\mathbf{x}_0, \lambda_0, 0) = \left[\begin{array}{c|c} A_0 - \lambda_0 I_n & -\mathbf{x}_0 \\ \hline 2\mathbf{x}_0^\top & 0 \end{array} \right]$$

is invertible, where $A_0 \equiv \mathscr{D}_{\mathbf{x}}(\mathcal{A}_0 \mathbf{x}^{m-1})|_{\mathbf{x}=\mathbf{x}_0} = (m-1)\|\mathbf{x}_1\|^{m-2} \mathbf{x}_1 \mathbf{x}_1^\top$ is given in (3.4). It is easily seen that $A_0 > 0$ has only one nonzero eigenvalue $(m-1)\|\mathbf{x}_1\|^m$ corresponding eigenvector $\mathbf{x}_1$. Then $A_0 - \lambda_0 I_n$ is nonsingular matrix when $m > 2$.

- If $m > 2$, then the value $\mathbf{x}_0^\top (A_0 - \lambda_0 I_n)^{-1} \mathbf{x}_0 = \frac{1}{(m-2)\|\mathbf{x}_0\|^m} \neq 0$, hence, $\mathscr{D}_{\mathbf{x},\lambda} H_{\mathrm{Z}}(\mathbf{x}_0, \lambda_0, 0)$ is invertible.
- If $m = 2$, then from Lemma 3.1, we obtain that $\mathscr{D}_{\mathbf{x},\lambda} H_{\mathrm{Z}}(\mathbf{x}_0, \lambda_0, 0)$ is invertible.

Since $\mathbf{0}$ is also a regular value of $H_{\mathrm{Z}}(\cdot, 0)$ on $\mathbb{R}^{n+1}_{\geqslant 0}$, $\mathbf{0}$ is a regular value of $H_{\mathrm{Z}} : \mathbb{R}^{n+1}_{>0} \times [0,1) \to \mathbb{R}^{n+1}$. ◻

From Theorem 3.2 and the implicit function theorem, we know that the equation $H_{\mathrm{Z}}(\mathbf{x}, \lambda, t) = \mathbf{0}$ has a solution curve $\mathbf{w}(s)$ with initial $\mathbf{w}(0) = (\mathbf{x}_0, \lambda_0, 0) \equiv (\mathbf{x}_1/\|\mathbf{x}_1\|, \|\mathbf{x}_1\|^m, 0)$,

$$\mathbf{w}(s) \equiv (\mathbf{x}(s), \lambda(s), t(s)) \in \mathbb{R}^n_{>0} \times \mathbb{R}_{>0} \times [0,1) \text{ for } s \in [0, s_{max}), \tag{3.5}$$

which can be parameterized by arc-length s, where s_{max} is the largest arc-length such that $\mathbf{w}(s) \in \mathbb{R}^n_{>0} \times \mathbb{R}_{>0} \times [0,1)$. Note that this curve, $\mathbf{w}(s)$ for $s \in [0, s_{max})$, has no bifurcation and is bounded (by (2.3)). This curve, $t(s)$ for $s \in [0, s_{max})$, may have turning points at some parameters s. The following proposition shows that the turning point will happen when $A_{t(s)} - \lambda(s) I$ is singular.

Proposition 3.3. *Let $m > 2$, $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ and $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1 \in \mathbb{R}^{[m,n]}_{>0}$, where $\mathbf{x}_1 \in \mathbb{R}^n_{>0}$ is generic. Suppose that $\mathbf{w}(s) = (\mathbf{x}(s), \lambda(s), t(s))$ defined in (3.5) is the solution curve of (2.8). If $t(s)$ has turning point at $s_* \in [0, s_{max})$ then $A_{t(s_*)} - \lambda(s_*) I$ is singular, where $A_{t(s_*)} \equiv \mathscr{D}_{\mathbf{x}}(\mathcal{A}(t(s_*))\mathbf{x}^{m-1})|_{\mathbf{x}=\mathbf{x}(s_*)}$.*

Proof. Suppose $A_{t(s_*)} - \lambda(s_*) I$ is invertible. Denote $(\mathbf{x}_*, \lambda_*, t_*) = (\mathbf{x}(s_*), \lambda(s_*), t(s_*))$. Since $(A_{t_*} - \lambda_* I)\mathbf{x}_* = (m-1)\mathcal{A}(t_*)\mathbf{x}_*^{m-1} - \lambda_* \mathbf{x}_* = (m-2)\lambda_* \mathbf{x}_*$, $m > 2$ and $\lambda_* \neq 0$, we have $(A_{t_*} - \lambda_* I)^{-1} \mathbf{x}_* = \frac{1}{(m-2)\lambda_*} \mathbf{x}_*$. Hence, $\alpha \equiv \mathbf{x}_*^\top (A_{t_*} - \lambda_* I)^{-1} \mathbf{x}_* = \frac{1}{(m-2)\lambda_*} > 0$. Then the Jacobian matrix $\mathscr{D}_{\mathbf{x},\lambda} H_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*, t_*)$ in (3.3b) is invertible. By the implicit function theorem, the solution curve $\mathbf{w}(s)$ can be parametrized by t when t approximates $t_* = t(s_*)$, a contradiction. □

THEOREM 3.4. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ and $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1 \in \mathbb{R}^{[m,n]}_{>0}$, where $\mathbf{x}_1 \in \mathbb{R}^n_{>0}$ is generic. Suppose that $\mathbf{w}(s) = (\mathbf{x}(s), \lambda(s), t(s))$ in* (3.5) *is the solution curve of* (2.8). *Then $\lim_{s \to s_{max}^-} t(s) = 1$.*

Proof. Let $\mathbf{w}(s) \in \mathbb{R}^{n+1}_{>0} \times [0, 1)$ for $s \in [0, s_{max})$ be the solution curve of $H_{\mathrm{Z}}(\mathbf{x}, \lambda, t) = \mathbf{0}$. Suppose that the sequence $\{s_k\}_{k=1}^\infty \subset [0, s_{max})$ is increasing and $\lim_{k\to\infty} s_k = s_{max}$. Now, we show that $\lim_{k\to\infty} t(s_k) = 1$.

Suppose that $\lim_{k\to\infty} t(s_k) \neq 1$, then there exists a subsequence $\{s_{k_\ell}\}_{\ell=1}^\infty$ of $\{s_k\}_{k=1}^\infty$ such that $\lim_{\ell\to\infty} t(s_{k_\ell}) = t_* \neq 1$. It follows from (2.3) that the set $(\mathbf{x}(s_{k_\ell}), \lambda(s_{k_\ell})) \in \mathbb{R}^n_{>0} \times \mathbb{R}_{>0}$ is bounded, then there is a subsequence $\{s_{k_{\hat{\ell}}}\}_{\hat{\ell}=1}^\infty$ of $\{s_{k_\ell}\}_{\ell=1}^\infty$ such that $\lim_{\hat{\ell}\to\infty}(\mathbf{x}(s_{k_{\hat{\ell}}}), \lambda(s_{k_{\hat{\ell}}})) = (\mathbf{x}_*, \lambda_*) \in \mathbb{R}^n_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$. It is easily seen that $H_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*, t_*) = \mathbf{0}$, where $t_* \in [0, 1)$. From Theorem 2.7, we have $\mathbf{x}_* > 0$ and $\lambda_* > 0$.

Case1: If $t_* \in (0, 1)$, then the solution $(\mathbf{x}_*, \lambda_*, t_*)$ is in the set $\mathbb{R}^n_{>0} \times \mathbb{R}_{>0} \times [0, 1)$. It follows from Theorem 3.2 that the equation $H_{\mathrm{Z}}(\mathbf{w}) = \mathbf{0}$ has a solution curve in a certain neighborhood of $\mathbf{w}(s_{max}) \equiv (\mathbf{x}_*, \lambda_*, t_*)$. This is a contradiction because s_{max} is the largest arc-length such that $\mathbf{w}(s) \in \mathbb{R}^n_{>0} \times \mathbb{R}_{>0} \times [0, 1)$.

Case2: If $t_* = 0$, from Lemma 2.2, we obtain that $(\mathbf{x}_*, \lambda_*) = (\mathbf{x}_0, \lambda_0)$, where $(\mathbf{x}_0, \lambda_0)$ is defined in (3.2). It has been shown in Theorem 3.2 that the Jacobian matrix $\mathscr{D}_{\mathbf{x},\lambda} H_{\mathrm{Z}}(\mathbf{x}_0, \lambda_0, 0)$ defined in (3.3b) is invertible. By implicit function theorem, the solution curve $\mathbf{w}(s)$ in (3.5) can be parameterized by t when t approximates 0 and there is no solution of $H_{\mathrm{Z}}(\mathbf{x}, \lambda, t) = \mathbf{0}$ in $B_\rho((\mathbf{x}_0, \lambda_0, 0))$ other than $\mathbf{w}(s)$. This is contradiction.

Hence, $\lim_{s\to s_{max}^-} t(s) = 1$. □

Theorem 3.4 shows that $t(s) \to 1$ as $s \to s_{max}^-$. Next, we will investigate the limit point of the curve, $(\mathbf{x}(s), \lambda(s))$, as $s \to s_{max}^-$, where $(\mathbf{x}(s), \lambda(s))$ is defined in (3.5).

THEOREM 3.5. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ and $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1 \in \mathbb{R}^{[m,n]}_{>0}$, where $\mathbf{x}_1 \in \mathbb{R}^n_{>0}$ is generic. Then the solution curve $\mathbf{w}(s) = (\mathbf{x}(s), \lambda(s), t(s))$ for $s \in [0, s_{max})$ defined in* (3.5) *satisfies the following properties.*

(i) There exist a sequence $\{s_k\}_{k=1}^\infty \subset [0, s_{max})$ and an accumulation point $\lambda_* \geqslant 0$ such that $\lim_{k\to\infty} s_k = s_{max}$ and $\lim_{k\to\infty} \lambda(s_k) = \lambda_*$;

(ii) For every such accumulation point λ_*, there exists a vector $\mathbf{x}_* \geqslant 0$ such that the pair $(\lambda_*, \mathbf{x}_*)$ is a Z-eigenpair of $\mathcal{A}$, i.e., $F_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*) = \mathbf{0}$;

(iii) If $\mathcal{A}$ is weakly symmetric, then the eigenvalue curve $\lambda(s)$ converges to $\lambda_* \geqslant 0$ as $s \to s_{max}^-$;

(iv) Let $(\mathbf{x}_*, \lambda_*)$ be such accumulation point of the curve $(\mathbf{x}(s), \lambda(s))$ for $s \in [0, s_{\max})$. If $(\mathbf{x}_*, \lambda_*)$ is an isolated solution of $F_{\mathrm{Z}}(\mathbf{x}, \lambda) = \mathbf{0}$, then

$$\lim_{s\to s_{max}^-} (\mathbf{x}(s), \lambda(s)) = (\mathbf{x}_*, \lambda_*).$$

Proof. (i) Using the fact that the set $\{\lambda(s) \mid s \in [0, s_{max})\} \subset \mathbb{R}_{>0}$ is bounded, the assertion (i) can be obtained.

(ii) Suppose that $\lambda(s_k) \to \lambda_*$. Since $\mathbf{x}(s_k) \in \{\mathbf{x} \in \mathbb{R}^n_{>0} \mid \|\mathbf{x}\| = 1\}$ for each k, there is a subsequence $\{s_{k_\ell}\}_{\ell=1}^\infty$ of $\{s_k\}_{k=1}^\infty$ such that $\mathbf{x}(s_{k_\ell}) \to \mathbf{x}_* \geqslant 0$ with $\|\mathbf{x}_*\| = 1$ as $\ell \to \infty$. Using the fact that $H_{\mathrm{Z}}(\mathbf{x}(s_{k_\ell}), \lambda(s_{k_\ell}), t(s_{k_\ell})) = \mathbf{0}$ and $\lim_{\ell\to\infty} t(s_{k_\ell}) = 1$ (see Theorem 3.4), we have $H_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*, 1) = F_{\mathrm{Z}}(\mathbf{x}_*, \lambda_*) = \mathbf{0}$. Hence, the pair $(\lambda_*, \mathbf{x}_*)$ is a Z-eigenpair of $\mathcal{A}$.

(iii) Suppose that $\lambda(s)$ for $s \in [0, s_{max})$ does not converge as $s \to s_{max}^-$. Then $\lambda(s)$ has two different accumulation points, λ_*^1 and λ_*^2 (say $\lambda_*^1 < \lambda_*^2$), as $s \to s_{max}^-$. Since the eigenvalue curve $\lambda(s) \in \mathbb{R}_{>0}$ is continuous for $s \in [0, s_{max})$, we obtain that each point $\lambda_* \in [\lambda_*^1, \lambda_*^2]$, λ_* is an accumulation point. From (i) and (ii), we obtain that the tensor $\mathcal{A}$ has infinitely many Z-eigenvalues. This is a contradiction because $\mathcal{A}$ is weakly symmetric, $\mathcal{A}$ has only finitely many Z-eigenvalues (see Proposition 3.10 in [4]). Hence, $\lim_{s\to s_{max}^-} \lambda(s) = \lambda_* \geqslant 0$.

(iv) Suppose $(\mathbf{x}(s), \lambda(s))$ for $s \in [0, s_{max})$ has another accumulation point $(\hat{\mathbf{x}}_*, \hat{\lambda}_*)$ such that $\delta_* = \|(\mathbf{x}_* - \hat{\mathbf{x}}_*, \lambda_* - \hat{\lambda}_*)\| > 0$. Then for each $\epsilon > 0$, by continuity of $(\mathbf{x}(s), \lambda(s))$, there exists an increasing sequence $\{s_k\} \subset [0, s_{max})$ such that $\lim_{k\to\infty} s_k = s_{max}$ and

$$0 < \min\{\delta_*/2, \epsilon/3\} \leqslant \|(\mathbf{x}(s_k) - \mathbf{x}_*, \lambda(s_k) - \lambda_*)\| \leqslant \epsilon/2 \text{ for each } k = 1, 2, \ldots.$$

Since the sequence $(\mathbf{x}(s_k), \lambda(s_k))$ is bounded, there exists an accumulation point $(\tilde{\mathbf{x}}_*, \tilde{\lambda}_*)$ of the sequence $\{(\mathbf{x}(s_k), \lambda(s_k))\}_{k=1}^\infty$ such that $0 < \|(\tilde{\mathbf{x}}_* - \mathbf{x}_*, \tilde{\lambda}_* - \lambda_*)\| < \epsilon$. From (i) and (ii), $(\tilde{\mathbf{x}}_*, \tilde{\lambda}_*)$ is a solution of $F_{\mathrm{Z}}(\mathbf{x}, \lambda) = \mathbf{0}$. This is a contradiction because $(\mathbf{x}_*, \lambda_*)$ is an isolated solution of $F_{\mathrm{Z}}(\mathbf{x}, \lambda) = \mathbf{0}$. ☐

Note that the condition of Theorem 3.5 (iv) holds generically. We conjecture that the convergence of solution curve, $(\mathbf{x}(s), \lambda(s))$ as $s \to s_{max}^-$, is guaranteed even without this condition.

In the following, we show the degree of F_{Z}^0 on $\mathbb{R}_{>0}^{n+1}$ for $\mathbf{p} = \mathbf{0}$ is only dependent on the dimension n, where F_{Z}^0 is defined in (2.5).

Lemma 3.6. *Let* $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1 \in \mathbb{R}_{>0}^{[m,n]}$, *where* $\mathbf{x}_1 \in \mathbb{R}_{>0}^n$ *is generic. Then* $\deg(F_{\mathrm{Z}}^0, \mathbb{R}_{>0}^{n+1}, \mathbf{0}) = (-1)^{n-1}$, *where* F_{Z}^0 *is defined in* (2.5).

Proof. From Lemma 2.2 (i), we know that $(\mathbf{x}_0, \lambda_0) = (\mathbf{x}_1/\|\mathbf{x}_1\|, \|\mathbf{x}_1\|^m)$ is the unique solution of $F_{\mathrm{Z}}^0(\mathbf{x}, \lambda) = \mathbf{0}$ on $\mathbb{R}_{\geqslant 0}^{n+1}$. Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix such that $Q\mathbf{x}_0 = \mathbf{e}_n \equiv [0, \cdots, 0, 1]^\top$ and $\widehat{Q} = \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 1 \end{array}\right]$. Then the Jacobian matrix

$$\begin{aligned}
\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}^0(\mathbf{x}_0, \lambda_0) &= \left[\begin{array}{c|c} (m-1)\|\mathbf{x}_1\|^{m-2}\mathbf{x}_1\mathbf{x}_1^\top - \lambda_0 I_n & -\mathbf{x}_0 \\ \hline 2\mathbf{x}_0^\top & 0 \end{array}\right] \\
&= \widehat{Q}^\top \left[\begin{array}{c|c} (m-1)\|\mathbf{x}_1\|^m \mathbf{e}_n\mathbf{e}_n^\top - \|\mathbf{x}_1\|^m I_n & -\mathbf{e}_n \\ \hline 2\mathbf{e}_n^\top & 0 \end{array}\right] \widehat{Q} \\
&= \widehat{Q}^\top \left[(-\|\mathbf{x}_1\|^m I_{n-1}) \oplus \left[\begin{array}{c|c} (m-2)\|\mathbf{x}_1\|^m & -1 \\ \hline 2 & 0 \end{array}\right]\right] \widehat{Q}.
\end{aligned}$$

So, $\deg(F_{\mathrm{Z}}^0, \mathbb{R}_{>0}^{n+1}, \mathbf{0}) = \mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}^0(\mathbf{x}_0, \lambda_0))) = \mathrm{Sgn}\left((-1)^{n-1} 2\|\mathbf{x}_1\|^{m(n-1)}\right) = (-1)^{n-1}$. ☐

Theorem 3.7. *Suppose that* $\mathcal{A} \in \mathbb{R}_{\geqslant 0}^{[m,n]}$ *is irreducible. Then* $\deg(F_{\mathrm{Z}}, \mathbb{R}_{>0}^{n+1}, \mathbf{0}) = (-1)^{n-1}$, *where* F_{Z} *is defined in* (2.5).

Proof. Let $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1 \in \mathbb{R}_{>0}^{[m,n]}$ with $\mathbf{x}_1 \in \mathbb{R}_{>0}^n$ being generic. Let $\mathbf{u} = (\mathbf{x}^\top, \lambda)^\top \in \mathbb{R}^{n+1}$ and $H_{\mathrm{Z}}(\mathbf{u}, t)$ be defined in (2.8). Then $F_{\mathrm{Z}}^0(\mathbf{u}) = H_{\mathrm{Z}}(\mathbf{u}, 0)$ and $F_{\mathrm{Z}}(\mathbf{u}) = H_{\mathrm{Z}}(\mathbf{u}, 1)$. From (2.3), there exists continuous function, $\rho(t)$ for $t \in [0, 1]$, such that $(1-t)\mathcal{A}_0 + t\mathcal{A}$ has no Z-eigenpair on the set $\mathbb{R}_{\geqslant 0}^{n+1} \backslash \Omega(t)$, where $\Omega(t) \equiv \{\mathbf{u} \in \mathbb{R}_{>0}^{n+1} \mid \|\mathbf{u}\| < \rho(t)\}$. Let $\rho = \max_{t\in[0,1]} \rho(t)$ and $\Omega \equiv \{\mathbf{u} \in \mathbb{R}_{>0}^{n+1} \mid \|\mathbf{u}\| < \rho\}$ be a bounded open set. Then for all $t \in [0, 1]$, $H_{\mathrm{Z}}(\mathbf{x}, \lambda, t) \neq \mathbf{0}$ on the set $\mathbb{R}_{\geqslant 0}^{n+1} \backslash \Omega$. Since $\mathcal{A}$ is irreducible, $\mathcal{A}$ has no Z-eigenpair on $\partial(\mathbb{R}_{>0}^{n+1})$. From Theorem 2.7, we obtain that $H_{\mathrm{Z}}(\mathbf{u},t)\neq\mathbf{0}$ on $\partial\Omega\times[0,1]$. It follows from the Homotopy Invariance of Degree theorem (Theorem 2.5) and Lemma 3.6 that $\deg(F_{\mathrm{Z}},\mathbb{R}^{n+1}_{>0},\mathbf{0})=\deg(F_{\mathrm{Z}},\Omega,\mathbf{0})=\deg(F^0_{\mathrm{Z}},\Omega,\mathbf{0})=\deg(F^0_{\mathrm{Z}},\mathbb{R}^{n+1}_{>0},\mathbf{0})=(-1)^{n-1}$. □

The following result can be obtained from Theorem 3.7 directly.

COROLLARY 3.8. *Let $\mathcal{A}\in\mathbb{R}^{[m,n]}_{\geqslant 0}$ be irreducible. Suppose that all solutions of $F_{\mathrm{Z}}(\mathbf{x},\lambda)=\mathbf{0}$ in $\mathbb{R}^{n+1}_{\geqslant 0}$ are isolated, where F_{Z} is defined in (2.5). Then the number of positive Z-eigenpairs of $\mathcal{A}$, counting multiplicities, is $2k+1$ for some integer $k\geqslant 0$.*

3.2. Computing the H-eigenpair of nonnegative tensors. Given a tensor $\mathcal{A}\in\mathbb{R}^{[m,n]}_{\geqslant 0}$, let $\mathcal{A}_0\in\mathbb{R}^{[m,n]}_{>0}$ in (2.4) be a rank-1 positive tensor. Let

$$\lambda_0=\prod_{k=2}^{m}\left(\mathbf{x}_k^\top\mathbf{x}_1^{[1/(m-1)]}\right)>0,\quad \mathbf{x}_0=\frac{\mathbf{x}_1^{[1/(m-1)]}}{\|\mathbf{x}_1^{[1/(m-1)]}\|}>0. \tag{3.6}$$

It follows from Lemma 2.2 (*ii*) that $(\lambda_0,\mathbf{x}_0)\in\mathbb{R}^{n+1}_{>0}$ is a H-eigenpair of $\mathcal{A}_0$ and $\mathbf{x}_0$ is the unique H-eigenvector of $\mathcal{A}_0$ in $\mathbb{R}^n_{\geqslant 0}$. Then $F^0_{\mathrm{H}}(\mathbf{x}_0,\lambda_0)=H_{\mathrm{H}}(\mathbf{x}_0,\lambda_0,0)=\mathbf{0}$, where F^0_{H} and H_{H} are defined in (2.6) and (2.9), respectively.

LEMMA 3.9. *Suppose that $(\mathbf{x}_*,\lambda_*,t_*)\in\mathbb{R}^{n+1}_{>0}\times[0,1)$ is a solution of $H_{\mathrm{H}}(\mathbf{x},\lambda,t)=\mathbf{0}$. Then the Jacobian matrix $\mathscr{D}_{\mathbf{x},\lambda}H_{\mathrm{H}}(\mathbf{x}_*,\lambda_*,t_*)\in\mathbb{R}^{(n+1)\times(n+1)}$ is invertible.*

Proof. Suppose that $(\mathbf{x}_*,\lambda_*,t_*)\in\mathbb{R}^{n+1}_{>0}\times[0,1)$ is a solution of $H_{\mathrm{H}}(\mathbf{x},\lambda,t)=\mathbf{0}$, we have $\mathcal{A}(t_*)\mathbf{x}_*^{m-1}=\lambda_*\mathbf{x}_*^{[m-1]}$, where $\mathcal{A}(t_*)\in\mathbb{R}^{[m,n]}_{>0}$ is defined in (2.7). Then

$$\mathscr{D}_{\mathbf{x},\lambda}H_{\mathrm{H}}(\mathbf{x}_*,\lambda_*,t_*)=\left[\begin{array}{c|c} A_{t_*}-(m-1)\lambda_*[\![\mathbf{x}_*^{[m-2]}]\!] & -\mathbf{x}_*^{[m-1]}\\ \hline 2\mathbf{x}_*^\top & 0\end{array}\right]\in\mathbb{R}^{(n+1)\times(n+1)},$$

where $[\![\mathbf{x}]\!]$ denotes a squared diagonal matrix with the elements of vector $\mathbf{x}$ on the main diagonal, and $A_{t_*}=\mathscr{D}_{\mathbf{x}}(\mathcal{A}(t_*)\mathbf{x}^{m-1})|_{\mathbf{x}=\mathbf{x}_*}$ is given in (3.4). Using the fact that $\mathcal{A}(t_*)>0$ and $\mathbf{x}_*>0$, we obtain $A_{t_*}>0$. Since $A_{t_*}\mathbf{x}_*=(m-1)\mathcal{A}(t_*)\mathbf{x}_*^{m-1}=(m-1)\lambda_*\mathbf{x}_*^{[m-1]}$, the matrix $-(A_{t_*}-(m-1)\lambda_*[\![\mathbf{x}_*^{[m-2]}]\!])$ is singular M-matrix with singular vector $\mathbf{x}_*$. It follows from Lemma 3.1 that $\mathscr{D}_{\mathbf{x},\lambda}H_{\mathrm{H}}(\mathbf{x}_*,\lambda_*,t_*)$ is invertible. □

Since $(\mathbf{x}_0,\lambda_0,0)$ is a solution of $H_{\mathrm{H}}(\mathbf{x},\lambda,t)=\mathbf{0}$, where λ_0 and $\mathbf{x}_0$ are given in (3.6). By Lemma 3.9 and the implicit function theorem, we know that the equation $H_{\mathrm{H}}(\mathbf{x},\lambda,t)=\mathbf{0}$ has a solution curve with initial $(\mathbf{x}_0,\lambda_0,0)$, $\mathbf{w}(t)=(\mathbf{x}(t),\lambda(t),t)$ for $t\in[0,1)$ and $\mathbf{w}(0)=(\mathbf{x}_0,\lambda_0,0)$, which can be parameterized by t. Let

$$C_{(\mathbf{x}_0,\lambda_0)}=\{(\mathbf{x}(t),\lambda(t),t)|t\in[0,1)\},\ \text{with}\ (\mathbf{x}(0),\lambda(0),0)=(\mathbf{x}_0,\lambda_0,0) \tag{3.7}$$

be the set of solution curve $\mathbf{w}(t)$ for $t\in[0,1)$. The following theorem shows that $C_{(\mathbf{x}_0,\lambda_0)}$ is the solution set of $H_{\mathrm{H}}(\mathbf{x},\lambda,t)=\mathbf{0}$ in $\mathbb{R}^{n+1}_{\geqslant 0}\times[0,1)$.

THEOREM 3.10. *$C_{(\mathbf{x}_0,\lambda_0)}$ is the solution set of $H_{\mathrm{H}}(\mathbf{x},\lambda,t)=\mathbf{0}$ in $\mathbb{R}^{n+1}_{\geqslant 0}\times[0,1)$.*

Proof. Suppose that $(\mathbf{x}_*,\lambda_*,t_*)\in\mathbb{R}^{n+1}_{\geqslant 0}\times[0,1)$ is a solution of $H_{\mathrm{H}}(\mathbf{x},\lambda,t)=\mathbf{0}$ and $(\mathbf{x}_*,\lambda_*,t_*)\neq(\mathbf{x}(t_*),\lambda(t_*),t_*)\in C_{(\mathbf{x}_0,\lambda_0)}$. By Lemma 3.9 and the implicit function theorem, there is a solution curve, $\mathbf{w}_*(t)=(\mathbf{x}_*(t),\lambda_*(t),t)$ for $t\in[0,t_*]$ with $\mathbf{w}_*(t_*)=(\mathbf{x}_*,\lambda_*,t_*)$, which is parameterized by t. Let $C_{(\mathbf{x}_*,\lambda_*)}=\{(\mathbf{x}_*(t),\lambda_*(t),t)|t\in[0,t_*]\}$ be the set of the solution curve. Then from Lemma 3.9, we have $C_{(\mathbf{x}_*,\lambda_*)}\bigcap C_{(\mathbf{x}_0,\lambda_0)}=\emptyset$. From Theorem 2.7, we obtain that $C_{(\mathbf{x}_*,\lambda_*)}\subseteq\mathbb{R}^{n+1}_{\geqslant 0}\times[0,1)$. Since $(\mathbf{x}_*(0),\lambda_*(0))$ satisfies $F^0_{\mathrm{H}}(\mathbf{x},\lambda)=H_{\mathrm{H}}(\mathbf{x},\lambda,0)=\mathbf{0}$ and hence, $(\lambda_*(0),\mathbf{x}_*(0))\in\mathbb{R}^{n+1}_{\geqslant 0}$ is a H-eigenpair of the rank-1 tensor $\mathcal{A}_0$. It follows from Lemma 2.2 (*ii*) that $(\lambda_*(0),\mathbf{x}_*(0))=$

$(\lambda_0, \mathbf{x}_0)$, where λ_0 and $\mathbf{x}_0$ are defined in (3.6). This is a contradiction because $C_{(\mathbf{x}_*,\lambda_*)} \bigcap C_{(\mathbf{x}_0,\lambda_0)} = \emptyset$. $\square$

Suppose that $(\mathbf{x}(t), \lambda(t), t)$ for $t \in [0, 1)$ is the solution curve of $H_{\mathrm{H}}(\mathbf{x}, \lambda, t) = \mathbf{0}$. Since $\|\mathbf{x}(t)\| = 1$ and $\mathcal{A}(t)$ in (2.7) is bounded for $t \in [0, 1)$, then $\|(\lambda(t), \mathbf{x}(t))\|$ is bounded for $t \in [0, 1)$. Suppose that $(\mathbf{x}_*, \lambda_*, 1)$ is an accumulation point of $C_{(\mathbf{x}_0,\lambda_0)}$. Then $\mathbf{x}_* \in \mathbb{R}^n_{\geqslant 0}$ with $\|\mathbf{x}_*\| = 1$, $\lambda_* \geqslant 0$ and $F_{\mathrm{H}}(\mathbf{x}_*, \lambda_*) = H_{\mathrm{H}}(\mathbf{x}_*, \lambda_*, 1) = \mathbf{0}$. Hence, $(\lambda_*, \mathbf{x}_*)$ is a H-eigenpair of $\mathcal{A}$. In [15], the authors shown that the eigenvalues of tensor $\mathcal{A}$ are the roots of a nonzero polynomial, hence, the eigenvalues of $\mathcal{A}$ are isolated and we have $\lim_{t\to 1^-} \lambda(t) = \lambda_*$. Let

$$\Gamma_{\lambda_*} = \{\mathbf{x}_* \in \mathbb{R}^n_{\geqslant 0} | (\mathbf{x}_*, \lambda_*, 1) \text{ is an accumulation point of } C_{(\mathbf{x}_0,\lambda_0)}\}.$$

Then $\Gamma_{\lambda_*} \neq \emptyset$ is connected. It is easily seen that for each $\mathbf{x}_* \in \Gamma_{\lambda_*}$, $(\lambda_*, \mathbf{x}_*)$ is a H-eigenpair of $\mathcal{A}$. The following theorem can be obtained directly.

Theorem 3.11. *Let $\mathbf{x}_* \in \Gamma_{\lambda_*}$. If $(\mathbf{x}_*, \lambda_*)$ is an isolated solution of $F_{\mathrm{H}}(\mathbf{x}, \lambda) = \mathbf{0}$ then $\Gamma_{\lambda_*} = \{\mathbf{x}_*\}$ and $\lim_{t\to 1^-}(\mathbf{x}(t), \lambda(t)) = (\mathbf{x}_*, \lambda_*)$, where $(\mathbf{x}(t), \lambda(t), t)$ for $t \in [0, 1)$ is the solution curve of $H_{\mathrm{H}}(\mathbf{x}, \lambda, t) = \mathbf{0}$.*

Theorem 3.12. *Let $\mathcal{A} \geqslant 0$ be weakly irreducible, then the nonnegative solution of $F_{\mathrm{H}}(\mathbf{x}, \lambda) = \mathbf{0}$ is isolated and hence $\lim_{t\to 1^-}(\mathbf{x}(t), \lambda(t)) = (\mathbf{x}_*, \lambda_*)$.*

Proof. Let $\mathbf{x}_* \in \Gamma_{\lambda_*}$. From Theorem 3.11, it suffices to show that $(\mathbf{x}_*, \lambda_*)$ is an isolated solution of $F_{\mathrm{H}}(\mathbf{x}, \lambda) = \mathbf{0}$. The Jacobian matrix of F_{H} at $(\mathbf{x}_*, \lambda_*)$ is

$$\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{H}}(\mathbf{x}_*, \lambda_*) = \left[\begin{array}{c|c} A_1 - (m-1)\lambda_* [\![\mathbf{x}_*^{[m-2]}]\!] & -\mathbf{x}_*^{[m-1]} \\ \hline 2\mathbf{x}_*^\top & 0 \end{array}\right] \in \mathbb{R}^{(n+1)\times(n+1)},$$

where $A_1 = \mathscr{D}_{\mathbf{x}}(\mathcal{A}\mathbf{x}^{m-1})|_{\mathbf{x}=\mathbf{x}_*} \in \mathbb{R}^{n\times n}_{\geqslant 0}$ is given in (3.4). Since $\mathcal{A}$ is weakly irreducible, $\lambda_* > 0$ and $\mathbf{x}_* > 0$. We next show that A_1 is invertible before proving that $\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{H}}(\mathbf{x}_*, \lambda_*)$ is invertible.

Suppose that $A_1 = [A^1_{i,j}]$ is reducible, then there exists a nonempty proper subset $S \subset \{1, 2, \cdots, n\}$ such that $A^1_{i,j} = 0$, $\forall\, i \in S$, $\forall\, j \notin S$, which implies

$$\begin{aligned} 0 = A^1_{i,j} &= \left(\sum_{k=2}^{m} \mathcal{A} \times_2 \mathbf{x}_* \cdots \times_{k-1} \mathbf{x}_* \times_{k+1} \mathbf{x}_* \cdots \times_m \mathbf{x}_*\right)_{i,j} \\ &\geq (\mathcal{A} \times_2 \mathbf{x}_* \cdots \times_{j-1} \mathbf{x}_* \times_{j+1} \mathbf{x}_* \cdots \times_m \mathbf{x}_*)_{i,j}\,. \end{aligned} \tag{3.8}$$

Since $\mathcal{A}$ is weakly irreducible, there exist $i \in S$ and $i_2, \ldots, i_m$ with at least one $j = i_q \notin S$ such that $\mathcal{A}_{i,i_2,\ldots,i_{q-1},j,i_{q+1}\ldots,i_m} \neq 0$. It follows from $\mathcal{A} \geqslant 0$, $\mathbf{x}_* > 0$ and (3.8) that $(\mathcal{A} \times_2 \mathbf{x}_* \cdots \times_{i_q-1} \mathbf{x}_* \times_{i_q+1} \mathbf{x}_* \cdots \times_m \mathbf{x}_*)_{i,j} > 0$ and hence $A^1_{i,j} > 0$, where $i \in S$ and $j \notin S$. This is a contradiction. So, $A_1 \geqslant 0$ is irreducible.

Using the fact that $(\lambda_*, \mathbf{x}_*) \in \mathbb{R}^{n+1}_{>0}$ is H-eigenpair of $\mathcal{A}$, we obtain that $(A_1 - (m-1)\lambda_* [\![\mathbf{x}_*^{[m-2]}]\!])\mathbf{x}_* = \mathbf{0}$, i.e., $-(A_1 - (m-1)\lambda_* [\![\mathbf{x}_*^{[m-2]}]\!])$ is irreducible singular M-matrix. It follows from Lemma 3.1 that the Jacobian matrix $\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{H}}(\mathbf{x}_*, \lambda_*)$ is invertible. $\square$

Theorem 3.12 shows that for each weakly irreducible nonnegative tensor $\mathcal{A}$, we can compute the unique positive H-eigenpair $(\lambda_*, \mathbf{x}_*)$ of $\mathcal{A}$ by tracing the solution curve of $H_{\mathrm{H}}(\mathbf{x}, \lambda, t) = \mathbf{0}$ in (2.9) with initial $(\mathbf{x}_0, \lambda_0, 0)$, where λ_0 and $\mathbf{x}_0$ are defined in (3.6).

4. Algorithms. A continuation method usually follows the solution curves of $H(\mathbf{u},t)=\mathbf{0}$ with prediction and correction steps, where $H:\mathbb{R}^{n+1}\to\mathbb{R}^n$ is a continuously differentiable function. In this section, we propose homotopy continuation methods to compute the nonnegative Z-eigenpair and H-eigenpair of a tensor $\mathcal{A}\in\mathbb{R}^{[m,n]}_{\geqslant 0}$. In addition, if $\mathcal{A}$ is irreducible, a novel continuation method is proposed to compute an odd number of positive Z-eigenpairs of $\mathcal{A}$.

4.1. Pseudo-arclength continuation method for computing an odd number of Z-eigenpairs of irreducible nonnegative tensors. Given a nonnegative tensor $\mathcal{A}\in\mathbb{R}^{[m,n]}_{\geqslant 0}$. Let $\mathcal{A}_0=\mathbf{x}_1\circ\cdots\circ\mathbf{x}_1\in\mathbb{R}^{[m,n]}_{>0}$, where $\mathbf{x}_1\in\mathbb{R}^n_{>0}$ is generic. Then $(\lambda_0,\mathbf{x}_0)=(\|\mathbf{x}_1\|^m,\frac{\mathbf{x}_1}{\|\mathbf{x}_1\|})$ is the positive Z-eigenpair of $\mathcal{A}_0$. Theorems 3.2 and 3.4 show that the solution curve of $H_{\mathrm{Z}}(\mathbf{x},\lambda,t)=\mathbf{0}$, $\mathbf{w}(s)=(\mathbf{x}(s),\lambda(s),t(s))$ for $s\in[0,s_{max})$, has no bifurcation and $t(s)\to 1^-$ as $s\to s_{max}^-$, where the function H_{Z} is defined in (2.8). This solution curve may have turning points at some parameters $s\in[0,s_{max})$, it is natural to employ pseudo-arclength continuation method (see [19]) for tracking the solution curve of $H_{\mathrm{Z}}(\mathbf{x},\lambda,t)=\mathbf{0}$ with initial $(\mathbf{x}_0,\lambda_0,0)$. Denote $\mathbf{w}=(\mathbf{x}^\top,\lambda,t)^\top\in\mathbb{R}^{n+2}$. Then $H_{\mathrm{Z}}(\mathbf{w})=H_{\mathrm{Z}}(\mathbf{x},\lambda,t)=\mathbf{0}$. Theorem 3.5 (iv) shows that if the roots of $F_{\mathrm{Z}}(\mathbf{x},\lambda)=H_{\mathrm{Z}}(\mathbf{x},\lambda,1)=\mathbf{0}$ in $\mathbb{R}^{n+1}_{\geqslant 0}$ are isolated, then we can compute a nonnegative Z-eigenpair of $\mathcal{A}$ by tracking the solution curve with initial $\mathbf{w}_0=(\mathbf{x}_0,\lambda_0,0)$. To follow the solution curve, we use the prediction-correction process. The prediction and correction steps are described as follows.

- Prediction step: Suppose that $\mathbf{w}_i\in\mathbb{R}^{n+1}_{\geqslant 0}$ is a point lying (approximately) on a solution curve of $H_{\mathrm{Z}}(\mathbf{w})=\mathbf{0}$. The Euler predictor
$$\mathbf{w}_{i+1,1}=\mathbf{w}_i+\Delta s_i\dot{\mathbf{w}}_i$$
is used to predict a new point. Here, $\dot{\mathbf{w}}_i\in\mathbb{R}^{n+2}$ is the unit tangent vector of the solution curve of $H_{\mathrm{Z}}(\mathbf{w})=\mathbf{0}$ at $\mathbf{w}_i$ and $\Delta s_i>0$ is a suitable step length. Let $\mathscr{D}_{\mathbf{w}}H_{\mathrm{Z}}(\mathbf{w}_i)\in\mathbb{R}^{(n+1)\times(n+2)}$ in (3.3a) be the Jacobian matrix of H_{Z} at $\mathbf{w}=\mathbf{w}_i$. The unit tangent vector $\dot{\mathbf{w}}_i$ should satisfy the linear system $\mathscr{D}_{\mathbf{w}}H_{\mathrm{Z}}(\mathbf{w}_i)\dot{\mathbf{w}}_i=\mathbf{0}$ and $\dot{\mathbf{w}}_i^\top\dot{\mathbf{w}}_{i-1}>0$ if $i\geqslant 1$. If $i=0$, we choose the unit tangent vector $\dot{\mathbf{w}}_0$ such that the last component of $\dot{\mathbf{w}}_0$ is positive.
- Correction step: Let $c_i=\dot{\mathbf{w}}_i^\top\mathbf{w}_{i+1,1}$ be a constant. We use Newton's method to compute the approximate solution of system
$$\begin{cases} H_{\mathrm{Z}}(\mathbf{w})=\mathbf{0},\\ \dot{\mathbf{w}}_i^\top\mathbf{w}-c_i=0,\end{cases}$$
with initial value $\mathbf{w}_{i+1,1}$. The iteration $\mathbf{w}_{i+1,\ell+1}=\mathbf{w}_{i+1,\ell}+\delta_\ell$ is computed for $\ell=1,2,\ldots$, where δ_ℓ satisfies the linear system
$$\begin{bmatrix}\mathscr{D}_{\mathbf{w}}H_{\mathrm{Z}}(\mathbf{w}_{i+1,\ell})\\ \dot{\mathbf{w}}_i^\top\end{bmatrix}\delta_\ell=-\begin{bmatrix}H_{\mathrm{Z}}(\mathbf{w}_{i+1,\ell})\\ \dot{\mathbf{w}}_i^\top\mathbf{w}_{i+1,\ell}-c_i\end{bmatrix}.$$
If $\{\mathbf{w}_{i+1,\ell}\}$ converges until $\ell=\ell_\infty$, then we accept $\mathbf{w}_{i+1}=\mathbf{w}_{i+1,\ell_\infty}$ as a new approximation to the solution curve of $H_{\mathrm{Z}}(\mathbf{w})=\mathbf{0}$.

Suppose that $\mathcal{A}\in\mathbb{R}^{[m,n]}_{\geqslant 0}$ is irreducible and all solutions of $F_{\mathrm{Z}}(\mathbf{x},\lambda)=\mathbf{0}$ in $\mathbb{R}^{n+1}_{\geqslant 0}$ are isolated. Then Corollary 3.8 shows that the number of positive Z-eigenpairs of $\mathcal{A}$, counting multiplicities, is odd. In the following, we propose a novel algorithm for computing an odd number of positive Z-eigenpairs. The following theorem is useful to construct the algorithm.

Theorem 4.1. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ be irreducible. Suppose that $\mathbf{0}$ is a regular value of $F_{\mathrm{Z}} : \mathbb{R}^{n+1}_{\geqslant 0} \to \mathbb{R}^{n+1}$ which is defined in (2.5). Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n_{>0}$ be generic and $H_{\mathrm{Z},1}(\mathbf{x}, \lambda, t)$ and $H_{\mathrm{Z},2}(\mathbf{x}, \lambda, t)$ be the homotopy functions constructed in (2.8) with $\mathcal{A}_{0,1} = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1$ and $\mathcal{A}_{0,2} = \mathbf{x}_2 \circ \cdots \circ \mathbf{x}_2$, respectively. Assume that $(\mathbf{x}_{*,1}, \lambda_{*,1})$ and $(\mathbf{x}_{*,2}, \lambda_{*,2})$ are accumulation points of solution curves of $H_{\mathrm{Z},1}(\mathbf{x}, \lambda, t) = \mathbf{0}$ and $H_{\mathrm{Z},2}(\mathbf{x}, \lambda, t) = \mathbf{0}$, respectively. If $(\mathbf{x}_{*,1}, \lambda_{*,1}) \neq (\mathbf{x}_{*,2}, \lambda_{*,2})$, then*

(i) $\mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\mathbf{x}_{*,1}, \lambda_{*,1}))) = \mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\mathbf{x}_{*,2}, \lambda_{*,2}))) = (-1)^{n-1}$*;*

(ii) there exists a smooth solution curve of $H_{\mathrm{Z},1}(\mathbf{x}, \lambda, t) = \mathbf{0}$,

$$\hat{\mathbf{w}}(s) \equiv (\hat{\mathbf{x}}(s), \hat{\lambda}(s), \hat{t}(s)) \in \mathbb{R}^{n+1}_{>0} \times [0,1] \quad \textit{for} \quad s \in [0, \hat{s}_{max})$$

with initial $\hat{\mathbf{w}}(0) = (\mathbf{x}_{*,2}, \lambda_{*,2}, 1)$, where $\hat{s}_{max}$ is the largest arc-length such that $\hat{\mathbf{w}}(s) \in \mathbb{R}^{n+1}_{>0} \times [0,1)$;

(iii) $\lim_{s \to \hat{s}^-_{max}} \hat{t}(s) = 1$*;*

(iv) $\lim_{s \to \hat{s}^-_{max}} (\hat{\mathbf{x}}(s), \hat{\lambda}(s)) = (\hat{\mathbf{x}}_*, \hat{\lambda}_*) \in \mathbb{R}^{n+1}_{>0}$*. Then $(\hat{\lambda}_*, \hat{\mathbf{x}}_*)$ is a Z-eigenpair of $\mathcal{A}$ and $(\hat{\mathbf{x}}_*, \hat{\lambda}_*) \neq (\mathbf{x}^i_*, \lambda^i_*)$, for $i = 1, 2$. In fact,* $\mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\hat{\mathbf{x}}_*, \hat{\lambda}_*))) = (-1)^n$.

Proof. (i) From the definitions of $H_{\mathrm{Z},1}$ and $H_{\mathrm{Z},2}$, we have $H_{\mathrm{Z},1}(\mathbf{x}, \lambda, 0) = F^0_{\mathrm{Z},1}(\mathbf{x}, \lambda)$, $H_{\mathrm{Z},2}(\mathbf{x}, \lambda, 0) = F^0_{\mathrm{Z},2}(\mathbf{x}, \lambda)$ and $H_{\mathrm{Z},1}(\mathbf{x}, \lambda, 1) = H_{\mathrm{Z},2}(\mathbf{x}, \lambda, 1) = F_{\mathrm{Z}}(\mathbf{x}, \lambda)$, where $F^0_{\mathrm{Z},1}$ and $F^0_{\mathrm{Z},2}$ are of the form in (2.5) with $\mathcal{A}_0 = \mathcal{A}_{0,1}$ and $\mathcal{A}_0 = \mathcal{A}_{0,2}$, respectively. For $i = 1, 2$, let $\mathbf{w}_i(s) \equiv (\mathbf{x}_i(s), \lambda_i(s), t_i(s)) \in \mathbb{R}^{n+1}_{>0} \times [0, 1)$ for $s \in [0, s_{i,max})$ be the solution curve of $H_{\mathrm{Z},i}(\mathbf{x}, \lambda, t) = \mathbf{0}$ with initial $\mathbf{w}_i(0) = (\mathbf{x}_i/\|\mathbf{x}_i\|, \|\mathbf{x}_i\|^m, 0)$. Here $s_{i,max}$ is the largest arc-length such that $\mathbf{w}_i(s) \in \mathbb{R}^{n+1}_{>0} \times [0, 1)$. Since $\mathbf{0}$ is a regular value of F_{Z}, from Theorem 3.5 (iv), we have $\lim_{s \to s^-_{1,max}} \mathbf{w}_1(s) = (\mathbf{x}_{*,1}, \lambda_{*,1}, 1)$ and $\lim_{s \to s^-_{2,max}} \mathbf{w}_2(s) = (\mathbf{x}_{*,2}, \lambda_{*,2}, 1)$. By the Homotopy Invariance of Degree theorem (Theorem 2.5) and Lemma 3.6, we obtain that

$$\mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\mathbf{x}_{*,1}, \lambda_{*,1}))) = \mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F^0_{\mathrm{Z},1}(\mathbf{x}_1/\|\mathbf{x}_1\|, \|\mathbf{x}_1\|^m))) = (-1)^{n-1}, \text{ and}$$
$$\mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\mathbf{x}_{*,2}, \lambda_{*,2}))) = \mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F^0_{\mathrm{Z},2}(\mathbf{x}_2/\|\mathbf{x}_2\|, \|\mathbf{x}_2\|^m))) = (-1)^{n-1}.$$

(ii) Since $\mathbf{0}$ is a regular value of F_{Z} and $(\mathbf{x}_{*,2}, \lambda_{*,2})$ is a solution of $F_{\mathrm{Z}}(\mathbf{x}, \lambda) = \mathbf{0}$, $\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\mathbf{x}_{*,2}, \lambda_{*,2})$ is invertible. From Theorem 3.2, this assertion can be obtained.

(iii) Since the equation $H_{\mathrm{Z},1}(\mathbf{x}, \lambda, 0) = F^0_{\mathrm{Z},1}(\mathbf{x}, \lambda) = \mathbf{0}$ has only one solution $(\mathbf{x}_0, \lambda_0) = (\mathbf{x}_1/\|\mathbf{x}_1\|, \|\mathbf{x}_1\|^m)$ in $\mathbb{R}^{n+1}_{>0}$ and $(\mathbf{x}_{*,1}, \lambda_{*,1}, 1)$ is the accumulation point of the set $\{\mathbf{w}_1(s) \mid s \in [0, s_{1,max})\}$, we obtain that $\hat{t}(s)$ does not converge to 0 as $s \to \hat{s}^-_{max}$. The proof of $\lim_{s \to \hat{s}^-_{max}} \hat{t}(s) = 1$ is similar to the proof of Theorem 3.4.

(iv) Since $\mathbf{0}$ is a regular value of F_{Z}, all solutions of $F_{\mathrm{Z}}(\mathbf{x}, \lambda) = \mathbf{0}$ in $\mathbb{R}^{n+1}_{\geqslant 0}$ are isolated. It follows from Theorem 3.5 (iv) that $\lim_{s \to \hat{s}^-_{max}} (\hat{\mathbf{x}}(s), \hat{\lambda}(s)) = (\hat{\mathbf{x}}_*, \hat{\lambda}_*)$. It is easily seen that $(\hat{\mathbf{x}}_*, \hat{\lambda}_*)$ is a Z-eigenpair of $\mathcal{A}$ and $(\hat{\mathbf{x}}_*, \hat{\lambda}_*) \in \mathbb{R}^{n+1}_{>0}$ because $\mathcal{A}$ be irreducible. Since $\mathbf{0}$ is a regular value of F_{Z}, we have $(\hat{\mathbf{x}}_*, \hat{\lambda}_*) \neq (\mathbf{x}_{*,1}, \lambda_{*,1})$ and $(\hat{\mathbf{x}}_*, \hat{\lambda}_*) \neq (\mathbf{x}_{*,2}, \lambda_{*,2})$. Using the Homotopy Invariance of Degree theorem (Theorem 2.5), we have $\mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\hat{\mathbf{x}}_*, \hat{\lambda}_*))) = -\mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\mathbf{x}_{*,2}, \lambda_{*,2}))) = (-1)^n$. ◻

Now, we can develop an algorithm for computing an odd number of positive Z-eigenpairs of an irreducible nonnegative tensor $\mathcal{A}$. The flowchart of this algorithm is shown in Figure 4.1.

Remark 4.2. *When $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ is irreducible, if all solutions in $\mathbb{R}^{n+1}_{\geqslant 0}$ of $F_{\mathrm{Z}}(\mathbf{x}, \lambda) = \mathbf{0}$ are isolated, then the algorithm shown in Figure 4.1 is guaranteed to*

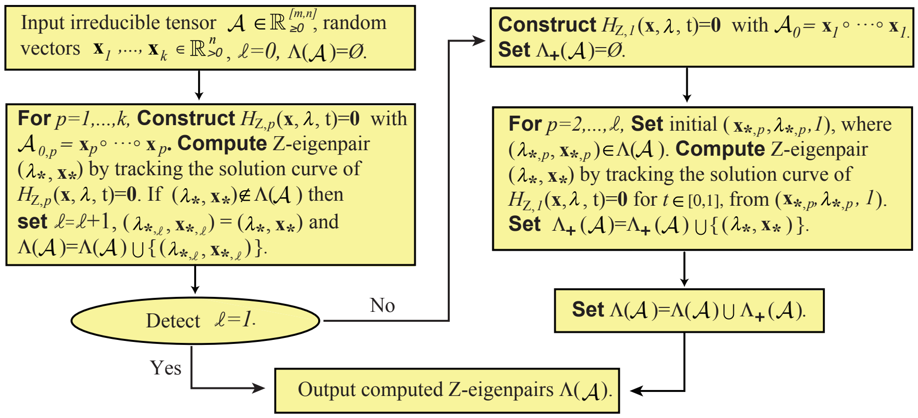

FIG. 4.1. *The flowchart of the pseudo-arclength continuation method for computing an odd number of positive Z-eigenpairs of an irreducible nonnegative tensor $\mathcal{A}$.*

compute an odd number of positive Z-eigenpairs, counting multiplicities. In addition, if $\mathbf{0} \in \mathbb{R}^{n+1}$ is a regular value of F_{Z}, then those positive Z-eigenpairs are distinct.

4.2. Parameter continuation method for computing H-eigenpair of nonnegative tensors. Given a nonnegative tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$. Let $\mathcal{A}_0 \in \mathbb{R}^{[m,n]}_{>0}$ in (2.4) be a rank-1 tensor and let $(\lambda_0, \mathbf{x}_0)$ in (3.6) be the positive H-eigenpair of $\mathcal{A}_0$. Theorem 3.10 shows that $C_{(\mathbf{x}_0,\lambda_0)}$ defined in (3.7) is the solution set of the homotopy $H_{\mathrm{H}}(\mathbf{x}, \lambda, t) = \mathbf{0}$ in $\mathbb{R}^{n+1}_{\geqslant 0} \times [0, 1)$, where the function H_{H} is defined in (2.9). Here, the solution set $C_{(\mathbf{x}_0,\lambda_0)}$ can be parameterized by $t \in [0, 1)$. In addition, Theorem 3.11 shows that if $F_{\mathrm{H}}(\mathbf{x}, \lambda) = \mathbf{0}$ has only isolated solution in $\mathbb{R}^{n+1}_{\geqslant 0}$, then a nonnegative H-eigenpair of $\mathcal{A}$ can be computed by tracking the curve $C_{(\mathbf{x}_0,\lambda_0)}$. It is natural to employ parameter continuation method for tracking the solution curve of $H_{\mathrm{H}}(\mathbf{x}, \lambda, t) = \mathbf{0}$ with initial $(\mathbf{x}_0, \lambda_0, 0)$. Denote $\mathbf{u} = (\mathbf{x}^\top, \lambda)^\top \in \mathbb{R}^{n+1}$. Then $H_{\mathrm{H}}(\mathbf{u}, t) \equiv H_{\mathrm{H}}(\mathbf{x}, \lambda, t) = \mathbf{0}$. Parameter continuation method (see [19]) takes a prediction-correction approach. The prediction and correction steps are described as follows.

- Prediction step: Suppose that $(\mathbf{u}_i, t_i)$ is a point lying (approximately) on a solution curve of $H_{\mathrm{H}}(\mathbf{u}, t) = \mathbf{0}$. The Euler predictor $\mathbf{u}_{i+1,1} = \mathbf{u}_i + \Delta t_i \dot{\mathbf{u}}_i$ is used to predict a new point. Here, $\Delta t_i > 0$ is a suitable step length satisfying $t_{i+1} = t_i + \Delta t_i \leqslant 1$ and $\dot{\mathbf{u}}_i$ satisfies the linear system $\mathscr{D}_{\mathbf{u}} H_{\mathrm{H}}(\mathbf{u}_i, t_i)\dot{\mathbf{u}}_i = -\mathscr{D}_t H_{\mathrm{H}}(\mathbf{u}_i, t_i)$.
- Correction step: Let $t = t_{i+1}$ be fixed. We use Newton's method to compute the approximate solution of $H_{\mathrm{H}}(\mathbf{u}, t_{i+1}) = \mathbf{0}$ with initial value $\mathbf{u}_{i+1,1}$. The iteration $\mathbf{u}_{i+1,\ell+1} = \mathbf{u}_{i+1,\ell} + \delta_\ell$ is computed for $\ell = 1, 2, \dots$, where δ_ℓ satisfies the linear system $\mathscr{D}_{\mathbf{u}} H_{\mathrm{H}}(\mathbf{u}_{i+1,\ell}, t_{i+1})\delta_\ell = -H_{\mathrm{H}}(\mathbf{u}_{i+1,\ell}, t_{i+1})$. If $\{\mathbf{u}_{i+1,\ell}\}$ converges until $\ell = \ell_\infty$, then we set $\mathbf{u}_{i+1} = \mathbf{u}_{i+1,\ell_\infty}$ and accept $(\mathbf{u}_{i+1}, t_{i+1})$ as a new approximation to the solution curve of $H_{\mathrm{H}}(\mathbf{u}, t) = \mathbf{0}$.

REMARK 4.3. *If $\mathcal{A} \geqslant 0$ is weakly irreducible, then Theorem 3.12 shows that we can compute the unique positive H-eigenpair, $(\lambda_*, \mathbf{x}_*)$, of $\mathcal{A}$ by tracking the solution curve. Note that the positive H-eigenvalue λ_* is the largest H-eigenvalue of $\mathcal{A}$.*

4.3. Comparison to other methods. In this section, we compare the numerical schemes, SS-HOPM (for Z-eigenpair) and NQZ, NNI (for H-eigenpair), with continuation method. The main difference is that those three schemes are iteration methods. For the computational complexity, SS-HOPM, NQZ and NNI require one evaluation of a vector in the form $\mathcal{A}\mathbf{x}^{m-1}$ for each iteration. Thus, the computational complexity for each iteration is $O(n^m)$.

Continuation methods are guaranteed to compute nonnegative Z-eigenpair and H-eigenpairs of a tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$. In prediction and correction steps of continuation method, we need evaluate the Jacobian matrix and residual of homotopy function, which is the dominant computational complexity in continuation method. With the help of rank-1 tensor $\mathcal{A}_0$ in (2.4), we can compute $\mathcal{A}(t_*)\mathbf{x}_*^{m-1}$ as follows:

$$\mathcal{A}(t_*)\mathbf{x}_*^{m-1} = (1-t_*)\left(\prod_{k=2}^{m}(\mathbf{x}_k^\top \mathbf{x}_*)\right)\mathbf{x}_1 + t_*\mathcal{A}\mathbf{x}_*^{m-1}. \tag{4.1}$$

The computational complexities of $\mathcal{A}(t_*)\mathbf{x}_*^{m-1}$ and of $\mathcal{A}\mathbf{x}_*^{m-1}$ are almost the same, which are $O(n^m)$. The Jacobian matrix $A_{t_*} \equiv \mathscr{D}_{\mathbf{x}}(\mathcal{A}(t_*)\mathbf{x}^{m-1})|_{\mathbf{x}=\mathbf{x}_*}$ requires to compute matrices $\mathcal{A} \times_2 \mathbf{x}_* \cdots \times_{k-1} \mathbf{x}_* \times_{k+1} \mathbf{x}_* \cdots \times_m \mathbf{x}_*$ for $k = 2, \ldots, m$. If the tensor $\mathcal{A}$ is semi-symmetric,[2] then we have $\mathscr{D}_{\mathbf{x}}(\mathcal{A}\mathbf{x}^{m-1})|_{\mathbf{x}=\mathbf{x}_*} = (m-1)\mathcal{A}\mathbf{x}_*^{m-2}$, which is a precursor of $\mathcal{A}\mathbf{x}_*^{m-1}$. Note that a symmetric tensor[3] is semi-symmetric. When $\mathcal{A}$ is semi-symmetric, we may choose the rank-1 tensor $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1 \in \mathbb{R}^{[m,n]}_{>0}$ as a symmetric tensor. Then the computational complexity of each prediction step or each iteration of Newton's method in correction step of continuation method is $O(n^m)$ by using the formula (4.1). When $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ is not a semi-symmetric, Ni and Qi [26] shown that there exists a semi-symmetric $\mathcal{A}_s \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ such that $\mathcal{A}\mathbf{x}^{m-1} = \mathcal{A}_s\mathbf{x}^{m-1}$ for each $\mathbf{x} \in \mathbb{R}^n$. The computational complexity of constructing the semi-symmetric $\mathcal{A}_s \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ is $O(n^m)$. Hence, it is more efficient if we replace the tensor $\mathcal{A}$ to a semi-symmetric $\mathcal{A}_s \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ before employing continuation method.

In the following, we itemized the sufficient conditions for the convergence of numerical schemes, SS-HOPM, NQZ, NNI and continuation method.

- For computing Z-eigenpairs of a tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$:
  - SS-HOPM [18] is guaranteed to compute the Z-eigenpairs of a *real symmetric tensor* $\mathcal{A}$, which is closely related to optimal rank-1 approximation of $\mathcal{A}$. In addition, if $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ is nonnegative symmetric, then SS-HOPM is guaranteed to find a nonnegative Z-eigenpair of $\mathcal{A}$. The convergence of SS-HOPM appears to be linear.
  - Continuation method is guaranteed to find a nonnegative Z-eigenpair of $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$ if $F_{\mathrm{Z}}(\mathbf{x}, \lambda) = \mathbf{0}$ has only isolated solution in $\mathbb{R}^{n+1}_{\geqslant 0}$ (see Theorem 3.5).
- For computing H-eigenpair of a tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$:
  - NQZ [25, 33] is a power method for computing the largest H-eigenvalue of $\mathcal{A}$. The convergence of NQZ appears to be linear for weakly primitive tensors.
  - NNI [22, 23] is guaranteed to compute the largest H-eigenvalue of a weakly irreducible nonnegative tensor. The convergence rate is quadratic when it is near convergence. However, the initial monotone convergence of NNI may be quite slow.
  - Continuation method is guaranteed to compute the largest H-eigenvalue of $\mathcal{A}$ if all solutions of $F_{\mathrm{H}}(\mathbf{x}, \lambda) = \mathbf{0}$ in $\mathbb{R}^{n+1}_{\geqslant 0}$ are isolated.

[2] $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called semi-symmetric if $\mathcal{A}_{i,j_2,\cdots,j_m} = \mathcal{A}_{i,i_2,\cdots,i_m}$, where $1 \leqslant i_1 \leqslant n$, $j_2, \cdots, j_m$ is any permutation of $i_2, \cdots, i_m$, $1 \leqslant i_2, \cdots, i_m \leqslant n$.

[3] $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called symmetric if $\mathcal{A}_{j_1,j_2,\cdots,j_m} = \mathcal{A}_{i_1,i_2,\cdots,i_m}$, where $j_1, j_2, \cdots, j_m$ is any permutation of $i_1, i_2, \cdots, i_m$, for $1 \leqslant i_1, i_2, \cdots, i_m \leqslant n$.

Note that if $\mathcal{A}$ is weakly primitive then $\mathcal{A}$ is weakly irreducible and if $\mathcal{A}$ is weakly irreducible then the solution of $F_{\mathrm{H}}(\mathbf{x},\lambda)=\mathbf{0}$ in $\mathbb{R}^{n+1}_{\geqslant 0}$ is unique and isolated.

5. Numerical experiments. In this section, we present some numerical results to support our theory. All numerical tests were performed using MATLAB 2014a on a Mac Pro with 3.7 GHz Quad-Core Intel Xeon E5 and 32 GB memory. In the following numerical results, "Steps" denotes the number of steps (a step = a prediction step + a correction step) of continuation method to achieve the solution, "#(Eval)" denotes the number of evaluations of $\mathcal{A}\mathbf{x}^{m-1}$, "Res" denotes the residual, $\|F_{\mathrm{Z}}(\mathbf{x}_*,\lambda_*)\|$ (or $\|F_{\mathrm{H}}(\mathbf{x}_*,\lambda_*)\|$), when the Z-eigenpair (or H-eigenpair), $(\lambda_*,\mathbf{x}_*)$, is computed and #(TP) denotes the number of turning points of the solution curve. The maximum number of evaluations allowed is 2000 for NQZ, SS-HOPM and NNI.

5.1. Numerical results for computing Z-eigenpairs. We first apply continuation method to compute Z-eigenpairs of the mth-order n-dimensional signless Laplacian tensor [12, 13].

EXAMPLE 5.1. *Consider the signless Laplacian tensor* $\mathcal{A}=\mathcal{D}+\mathcal{C}\in\mathbb{R}^{[m,n]}_{\geqslant 0}$ *of an* m*-uniform connected hypergraph [12, 13], where* $\mathcal{D}$ *is the diagonal tensor with diagonal element* $\mathcal{D}_{i,\cdots,i}$ *equal to the degree of vertex* i *for each* i*, and* $\mathcal{C}$ *is the adjacency tensor defined in [12, 13, 14] which is symmetric. Consider the edge set* $E=\{\{i-m+2,i-m+3,\ldots,i,i+1\},i=m-1,\ldots,n\}$ *in [23], where* $n+1$ *is identified with* 1*. The corresponding tensor* $\mathcal{A}$ *is weakly primitive (and thus weakly irreducible).*

Given a signless Laplacian tensor $\mathcal{A}\in\mathbb{R}^{[m,n]}_{\geqslant 0}$*, let* $\mathcal{A}_0=\mathbf{x}_1\circ\cdots\circ\mathbf{x}_1$*, where* $\mathbf{x}_1\in\mathbb{R}^n_{>0}$ *is generic with* $\|\mathbf{x}_1\|\in[0.9,1.1]$*. It follows from* (3.2) *that* $(\lambda_0,\mathbf{x}_0)=(\|\mathbf{x}_1\|^m,\frac{\mathbf{x}_1}{\|\mathbf{x}_1\|})$ *is the unique positive Z-eigenpair of* $\mathcal{A}_0$*. Table 5.1 reports the results obtained by tracking the solution curve of* $H_{\mathrm{Z}}(\mathbf{x},\lambda,t)=\mathbf{0}$ *by pseudo-arclength continuation method for various of* m *and* n*. From Table 5.1, we can see that the solution curve of* $H_{\mathrm{Z}}(\mathbf{x},\lambda,t)=\mathbf{0}$ *with initial* $(\mathbf{x}_0,\lambda_0,0)$ *has two turning points for each test case. The numbers of Steps and #(Eval) increase when the distance between those two turning points increases. In this example, the number of evaluations, #(Eval), is at most 176. Figure 5.1 shows the bifurcation diagram of the solution curve of* $H_{\mathrm{Z}}(\mathbf{x},\lambda,t)=\mathbf{0}$ *for the case* $m=5$ *and* $n=20$*. The corresponding eigenvectors,* $\mathbf{x}(s)$*, are attached near to the solution curve.*

TABLE 5.1
Numerical results for Example 5.1.

Tensor $\mathcal{A}$		Continuation method					
m	n	Steps	#(Eval)	Res	#(TP)	turning points (t)	
3	20	12	67	7.70e-11	2	0.405	0.385
3	50	18	94	2.68e-12	2	0.391	0.309
3	100	20	106	2.26e-20	2	0.444	0.210
4	20	18	91	1.64e-16	2	0.276	0.177
4	50	28	128	3.42e-11	2	0.372	0.0577
4	100	34	160	4.52e-11	2	0.642	0.0666
5	20	22	103	4.90e-18	2	0.244	0.0781
5	50	32	148	1.48e-12	2	0.662	0.0276
5	100	40	176	1.47e-13	2	0.846	0.0105

Corollary 3.8 shows that the number of positive Z-eigenpairs of an irreducible tensor $\widehat{\mathcal{A}}$*, counting multiplicities, is odd. Since the tensor* $\mathcal{A}$ *constructed in this example is weakly irreducible, we set* $\widehat{\mathcal{A}}=\mathcal{A}+10^{-5}\mathcal{E}$*, where* $\mathcal{E}$ *is the tensor with all entries equal to 1. Employ the algorithm shown in Figure 4.1 to the irreducible tensor* $\widehat{\mathcal{A}}$*. In*

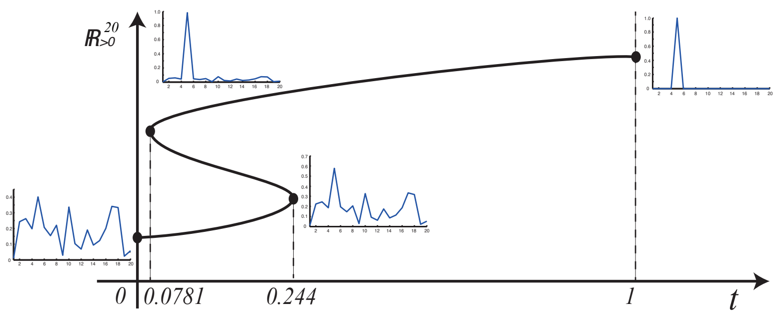

FIG. 5.1. *The bifurcation diagram of the solution curve of* $H_{\mathrm{Z}}(\mathbf{x},\lambda,t)=\mathbf{0}$ *with* $\mathcal{A}\in\mathbb{R}_{\geqslant 0}^{[5,20]}$. *The corresponding eigenvectors are attached near to the solution curve (Example 5.1).*

the following numerical tests, we consider the case $m=4$ *and* $n=20$*. For a fixed tensor* $\widehat{\mathcal{A}}\in\mathbb{R}_{>0}^{[4,20]}$*, we run 100 trials of the algorithm using* k *random initial vectors* $\mathbf{x}_1,\ldots,\mathbf{x}_k\in\mathbb{R}_{>0}^{20}$*. Figure 5.2 reports the number of occurrences (over 100 trials) for the numbers of computed positive Z-eigenpairs of* $\widehat{\mathcal{A}}$ *in terms of* $k=50$ *and* 70.

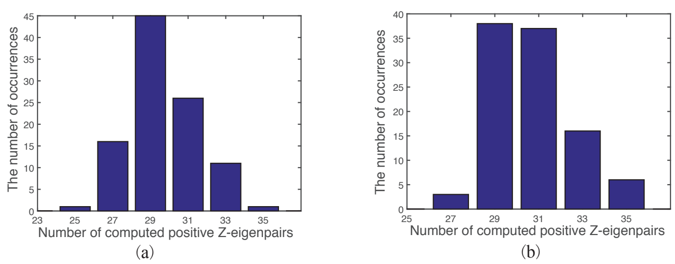

FIG. 5.2. *The number of occurrences (over 100 trials) for the numbers of computed positive Z-eigenpairs of* $\widehat{\mathcal{A}}\in\mathbb{R}_{>0}^{[4,20]}$ *by using* $k=50$ *(a) and* $k=70$ *(b) random vectors (Example 5.1).*

EXAMPLE 5.2. *Consider the symmetric tensor* $\mathcal{A}(w)=\mathcal{D}+w\mathcal{C}\in\mathbb{R}_{\geqslant 0}^{[4,20]}$*, where* $w\in\mathbb{R}_{>0}$*,* $\mathcal{D}$ *and* $\mathcal{C}$ *are defined in Example 5.1. We employ continuation method and SS-HOPM with shift parameter* $\alpha\in\mathbb{R}$ *to compute positive Z-eigenpair of* $\mathcal{A}$*. Suppose that* $(\lambda_*,\mathbf{x}_*)\in\mathbb{R}_{\geqslant 0}^{21}$ *is a Z-eigenpair of* $\mathcal{A}(w)$*, then* $\lambda_*=\lambda_*\mathbf{x}_*^\top\mathbf{x}_*=\mathbf{x}_*^\top\mathcal{A}(w)\mathbf{x}_*^3=:\mathcal{A}(w)\mathbf{x}_*^4$*. The algorithm SS-HOPM [18] is guaranteed to converge to a local maximum of the optimization problem:*

$$\max_{\mathbf{x}\in\mathbb{R}^{20},\ \|\mathbf{x}\|=1}\mathcal{A}(w)\mathbf{x}^4 \tag{5.1}$$

if the shift $\alpha>\beta(\mathcal{A}(w))$*, where the constant* $\beta(\mathcal{A}(w))$ *is dependent on tensor* $\mathcal{A}(w)$*. Lemma 4.1 in [18] shows that* $\gamma(w)=72(1+w)$ *is an upper bound of* $\beta(\mathcal{A}(w))$*. Choosing* $\alpha>\gamma(w)$ *is guaranteed to work but may slow down convergence. In our numerical experiments, we choose* $w=1,3$ *and* 5*. Note that when* $w=1,3$ *and* 5 *then* $\gamma(w)=144,288$ *and* 432*, respectively. Table 5.2 reports the results obtained by continuation method and SS-HOPM with* $\alpha=\gamma(w)+1$ *and* $\alpha=1$ *in terms of* $w=1,3$ *and* 5*, where we terminate the iteration of SS-HOPM when* $\mathrm{Res}<10^{-10}$*. In this table, we can see that a local maximum value,* $\lambda_*=\mathcal{A}(w)\mathbf{x}_*^4$*, of the optimization problem (5.1) can also be computed by continuation method. The number of evaluations, #(Eval), of continuation method is at most 114 that is much less than the* *number of evaluations of SS-HOPM with $\alpha = \gamma(w) + 1$. SS-HOPM with $\alpha = 1$ works for this example, but there is no theory to guarantee the convergence.*

TABLE 5.2
Numerical results for Example 5.2.

	Continuation method			SS-HOPM($\alpha = \gamma(w) + 1$)			SS-HOPM($\alpha = 1$)		
w	#(Eval)	λ_*	Res	#(Eval)	λ_*	Res	#(Eval)	λ_*	Res
1	91	4	1.64e-16	1211	4	9.78e-11	24	4	8.03e-11
3	114	4	5.56e-19	2000	4	6.61e-07	33	4	9.36e-11
5	112	2.95	5.73e-17	2000	2.95	2.98e-04	2000	2.95	4.27e-05

The next example, we consider a *multilinear PageRank* problem provided in [10]. In multilinear PageRank problem, it needs to compute the positive Z-eigenpair of a stochastic transition tensor,

$$\mathcal{A}(\alpha) = \alpha\mathcal{P} + (1-\alpha)\mathbf{v} \circ \mathbf{e} \circ \cdots \circ \mathbf{e} \in \mathbb{R}_{\geqslant 0}^{[m,n]}, \tag{5.2}$$

where $\mathcal{P}$ is the transition tensor of the higher-order Markov chain, $\mathbf{v} \in \mathbb{R}^n_{\geqslant 0}$ is a stochastic vector, $\mathbf{e} = [1, 1, \cdots, 1]^\top \in \mathbb{R}^n$ and $\alpha \in (0, 1)$.

EXAMPLE 5.3. *We consider stochastic transition $\mathcal{A}(\alpha) \in \mathbb{R}_{\geqslant 0}^{[3,6]}$ has the form in* (5.2)*, where the unfolding of tensor $\mathcal{P}$ is*

$$\left[\ \mathcal{P}(:,:,1) \mid \mathcal{P}(:,:,2) \mid \mathcal{P}(:,:,3) \mid \mathcal{P}(:,:,4) \mid \mathcal{P}(:,:,5) \mid \mathcal{P}(:,:,6)\ \right]$$
$$= \left[\begin{array}{c|c|c|c|c|c}
0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,1 & 1\,0\,0\,0\,0\,0 & 0\,0\,1\,0\,1\,0 & 0\,1\,1\,0\,0\,0 & 0\,0\,0\,0\,0\,1 \\
0\,0\,0\,0\,0\,0 & 0\,0\,1\,0\,0\,0 & 0\,1\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0 & 0\,1\,0\,0\,1\,0 & 0\,0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,1\,0 & 1\,1\,0\,0\,1\,0 \\
0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,0\,0 & 0\,0\,0\,1\,0\,1 & 0\,0\,0\,0\,0\,0 & 0\,1\,0\,1\,0\,0 \\
0\,0\,0\,0\,0\,0 & 0\,1\,0\,0\,0\,1 & 0\,0\,0\,0\,0\,0 & 0\,1\,0\,0\,0\,0 & 0\,0\,0\,1\,0\,0 & 0\,0\,1\,0\,1\,0 \\
1\,1\,1\,1\,1\,1 & 1\,0\,0\,1\,1\,0 & 1\,0\,1\,1\,1\,1 & 1\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,1 & 0\,0\,0\,0\,0\,0
\end{array}\right],$$

and the stochastic vector $\mathbf{v} = \mathbf{e}/6$. SS-HOPM and Newton method fail to converge the nonnegative Z-eigenpair when $\alpha = 0.99$ (see [10]). We employ continuation method to compute the positive Z-eigenpair of $\mathcal{A}(\alpha)$ in terms of $\alpha = 0.9$, 0.99 and 0.999. Table 5.3 reports the numerical results. This table shows that when $\alpha = 0.99$ and 0.999, the solution curves have two turning points, but no turning point occur when $\alpha = 0.9$.

TABLE 5.3
Continuation method for positive Z-eigenpair of $\mathcal{A}(\alpha)$ (Example 5.3).

α	Steps	#(Eval)	Res	#(TP)	turning points (t)	
0.9	4	29	3.55e-16	0	-	-
0.99	13	74	1.13e-16	2	0.999	0.952
0.999	16	90	1.11e-16	2	0.984	0.849

In the following example, we consider a small size irreducible tensor $\mathcal{A} \in \mathbb{R}_{\geqslant 0}^{[4,2]}$, which has three positive Z-eigenpairs. This tensor is provided in [4].

EXAMPLE 5.4. *Let $\mathcal{A} \in \mathbb{R}_{\geqslant 0}^{[4,2]}$ be defined by*

$$\mathcal{A}_{1111} = \mathcal{A}_{2222} = \frac{4}{\sqrt{3}}, \quad \mathcal{A}_{1112} = \mathcal{A}_{1211} = \mathcal{A}_{2111} = 1,$$
$$\mathcal{A}_{1222} = \mathcal{A}_{2122} = \mathcal{A}_{2212} = \mathcal{A}_{2221} = 1, \ \textit{and } \mathcal{A}_{ijkl} = 0 \ \textit{elsewhere}.$$

Obviously, $\mathcal{A}$ is irreducible. The system of polynomials F_Z in (2.5) *has the form*

$$F_{\mathrm{Z}}(\mathbf{x}, \lambda) = \begin{pmatrix} \frac{4}{\sqrt{3}}x_1^3 + 3x_1^2x_2 + x_2^3 - \lambda x_1 \\ \frac{4}{\sqrt{3}}x_2^3 + 3x_1x_2^2 + x_1^3 - \lambda x_2 \\ x_1^2 + x_2^2 - 1 \end{pmatrix} = \mathbf{0},$$

where $\mathbf{x} = (x_1, x_2)^\top$. *[4] shown that* $\mathcal{A}$ *has there positive Z-eigenpairs:*

- $\hat{\lambda}_0 = 2 + \frac{2}{\sqrt{3}} \approx 3.1547$ *with corresponding positive Z-eigenvector* $\hat{\mathbf{x}}_0 = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^\top$;
- $\hat{\lambda}_1 = \hat{\lambda}_2 = \frac{11}{2\sqrt{3}} \approx 3.1754$ *with corresponding positive Z-eigenvectors* $\hat{\mathbf{x}}_1 = \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\right]^\top$ *and* $\hat{\mathbf{x}}_2 = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]^\top$.

That is, $F_Z(\hat{\mathbf{x}}_0, \hat{\lambda}_0) = F_Z(\hat{\mathbf{x}}_1, \hat{\lambda}_1) = F_Z(\hat{\mathbf{x}}_2, \hat{\lambda}_2) = \mathbf{0}$. *The Jacobian matrix of* F_{Z} *is*

$$\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\mathbf{x}, \lambda) = \begin{bmatrix} 4\sqrt{3}x_1^2 + 6x_1x_2 - \lambda & 3x_1^2 + 3x_2^2 & -x_1 \\ 3x_1^2 + 3x_2^2 & 4\sqrt{3}x_2^2 + 6x_1x_2 - \lambda & -x_2 \\ 2x_1 & 2x_2 & 0 \end{bmatrix}.$$

Then we have

$$\begin{aligned} &\mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\hat{\mathbf{x}}_0, \hat{\lambda}_0))) = 1, \ \text{and} \\ &\mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\hat{\mathbf{x}}_1, \hat{\lambda}_1))) = \mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\hat{\mathbf{x}}_2, \hat{\lambda}_2))) = -1, \end{aligned}$$

and hence, $\deg(F_{\mathrm{Z}}, \mathbb{R}^3_{>0}, \mathbf{0}) \equiv \sum_{k=0}^{2} \mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_{\mathrm{Z}}(\hat{\mathbf{x}}_k, \hat{\lambda}_k))) = -1$. *This result has been shown in Theorem 3.7 with* $n = 2$. *For any rank-1 symmetric tensor* $\mathcal{A}_0 \in \mathbb{R}^{[4,2]}_{>0}$, $\deg(F^0_{\mathrm{Z}}, \mathbb{R}^3_{>0}, \mathbf{0}) = -1$ *(see Lemma 3.6), where* F^0_{Z} *is defined in* (2.5). *From Theorem 4.1 (i), we can only compute Z-eigenpairs,* $(\hat{\lambda}_1, \hat{\mathbf{x}}_1)$ *or* $(\hat{\lambda}_2, \hat{\mathbf{x}}_2)$, *by tracking the solution curve of* $H_{\mathrm{Z}}(\mathbf{x}, \lambda, t) = \mathbf{0}$. *Let* $\mathcal{A}_{0,1}$, $\mathcal{A}_{0,2} \in \mathbb{R}^{[4,2]}_{>0}$ *be rank-1 symmetric tensors and two homotopy equations* $H_{\mathrm{Z},1}(\mathbf{x}, \lambda, t) = \mathbf{0}$ *and* $H_{\mathrm{Z},2}(\mathbf{x}, \lambda, t) = \mathbf{0}$ *be constructed in* (2.8). *Suppose that the Z-eigenpairs,* $(\hat{\lambda}_1, \hat{\mathbf{x}}_1)$ *and* $(\hat{\lambda}_2, \hat{\mathbf{x}}_2)$, *can be computed by tracking the solution curves of* $H_{\mathrm{Z},1}(\mathbf{x}, \lambda, t) = \mathbf{0}$ *and* $H_{\mathrm{Z},2}(\mathbf{x}, \lambda, t) = \mathbf{0}$, *respectively. Theorem 4.1 (iv) shows that a new positive Z-eigenpair* $(\lambda_*, \mathbf{x}_*) \in \mathbb{R}^3_{>0}$ *can be computed by tracking the solution curve of* $H_{\mathrm{Z},1}(\mathbf{x}, \lambda, t) = \mathbf{0}$ *with initial* $(\hat{\mathbf{x}}_2, \hat{\lambda}_2, 1)$ *and* $\mathrm{Sgn}(\det(\mathscr{D}_{\mathbf{x},\lambda} F_Z(\mathbf{x}_*, \lambda_*))) = 1$. *Hence,* $(\lambda_*, \mathbf{x}_*) = (\hat{\lambda}_0, \hat{\mathbf{x}}_0)$. *We run 100 trials of the algorithm using* k *random initial vectors* $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^2_{>0}$. *Table 5.4 reports the number of occurrences (over 100 trials) for the numbers of computed Z-eigenpairs of* $\mathcal{A}$ *in terms of* $k = 2$, 5 *and* 8.

Table 5.4

The number of occurrences (over 100 trials) for the numbers of computed positive Z-eigenpairs of $\mathcal{A} \in \mathbb{R}^{[4,2]}_{>0}$ *by using* $k = 2$, 5 *and* 8 *random vectors (Example 5.4).*

	No. of computed Z-eigenpairs	
k	1	3
2	49	51
5	6	94
8	0	100

5.2. Numerical results for computing H-eigenpair. In this section, we then apply continuation method, NQZ and NNI to compute the positive H-eigenpair of the mth-order n-dimensional signless Laplacian tensor [12, 13].

Example 5.5. *Consider a tensor* $\mathcal{A} = \mathcal{D} + \mathcal{C} \in \mathbb{R}^{[m,n]}_{\geqslant 0}$, *where* $\mathcal{D}$ *and* $\mathcal{C}$ *are defined in Example 5.1. Let* $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1$, *where* $\mathbf{x}_1 = \frac{1}{n^{(m-1)/m}}[1, \ldots, 1]^\top \in \mathbb{R}^n_{>0}$. *From* (3.6), *we obtain the unique nonzero H-eigenvalue of* $\mathcal{A}_0$ *is* $\lambda_0 = \prod_{k=2}^{m}(\mathbf{x}_1^\top \mathbf{x}_1^{[1/(m-1)]}) = \prod_{k=2}^{m}(\frac{1}{n^{(m-1)/m}} \cdot \frac{1}{n^{1/m}} \cdot n) = 1$ *and the associated unit positive H-eigenvector is* $\mathbf{x}_0 =$

$\frac{1}{\sqrt{n}}[1,\ldots,1]^\top$. *Table 5.5 reports the results obtained by continuation method, NQZ and NNI for various of* m *and* n*, where we terminate the iteration of NQZ and NNI when* $\mathrm{Res} < 10^{-10}$.

TABLE 5.5
Numerical results for Example 5.5.

Tensor $\mathcal{A}$		Continuation method			NQZ		NNI	
m	n	Steps	#(Eval)	Res	#(Eval)	Res	#(Eval)	Res
3	20	4	18	1.90e-11	240	9.33e-11	7	4.50e-16
3	50	7	36	2.88e-12	1285	9.88e-11	11	7.43e-16
3	100	11	56	3.42e-12	2000	3.03e-06	126	1.24e-12
4	20	4	18	2.94e-14	138	9.03e-11	7	7.28e-11
4	50	6	30	5.25e-11	767	9.80e-11	13	2.61e-14
4	100	8	42	6.91e-11	2000	2.57e-08	92	1.33e-15
5	20	4	19	5.99e-12	91	8.31e-11	8	3.18e-15
5	50	5	25	1.49e-17	531	9.64e-11	9	1.04e-13
5	100	9	45	6.66e-16	1918	9.89e-11	80	3.41e-13

From Table 5.5, we see that the numbers of evaluations, #(Eval), for continuation method are between 18 *to* 56. *The convergence of NQZ [25, 33] is linear and the numbers of evaluations for NQZ are between 91 to 2000. The convergence rate of NNI [22, 23] is quadratic when it is near convergence. However, the initial monotone convergence of NNI with positive parameters* $\{\theta_k\}$ *may be quite slow. In this example, the numbers of evaluations for NNI are between* 7 *and* 126.

REMARK 5.1. *There is no theory to guarantee the convergence of NNI with* $\theta_k = 1$. *In this example, if we employ NNI with* $\theta_k = 1$ *to compute the positive H-eigenpair, it has very nice performance. The number of evaluations for NNI with* $\theta_k = 1$ *is at most* 11.

6. Conclusions. We have presented homotopy continuation method for computing nonnegative Z-/H-eigenpairs of a nonnegative tensor $\mathcal{A}$. A linear homotopy $H(\mathbf{x},\lambda,t) = \mathbf{0}$ is constructed by a target nonnegative tensor $\mathcal{A}$ and a rank-1 initial tensor $\mathcal{A}_0 = \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_1$, where $\mathbf{x}_1 \in \mathbb{R}^n_{>0}$ is generic. It is shown that $H(\mathbf{x},\lambda,t) = \mathbf{0}$ has only one positive solution at $t = 0$ and the solution curve of the linear homotopy starting from the positive solution, $(\mathbf{x}(s),\lambda(s),t(s)) \in \mathbb{R}^{n+1}_{>0} \times [0,1)$ for $s \in [0, s_{max})$, is smooth and $t(s) \to 1^-$ as $s \to s^-_{max}$. Hence, the nonnegative eigenpair can be computed by tracking the solution curve if the nonnegative solutions of $H(\mathbf{x},\lambda,1) = \mathbf{0}$ are isolated. Furthermore, we have shown that the number of positive Z-eigenpairs of an irreducible nonnegative tenor is odd and proposed an algorithm to compute odd number of positive Z-eigenpairs. For computing nonnegative eigenpairs, the norm of the generic positive vector $\mathbf{x}_1$ will affect the distance of two turning points and then, affect the time of computing. How to choose a suitable norm of the generic positive vector $\mathbf{x}_1$ remains an open problem.

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