Artificial Boundary Conditions and Finite Difference Approximations for a Time-fractional Diffusion-wave Equation on two-dimensional and three-dimensional Unbounded Spatial Domain

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Abstract

We consider the numerical solution of the time-fractional diffusion-wave equation on two-dimensional and three-dimensional unbounded spatial domains. Introduce an artificial boundary and find the exact and approximate artificial boundary conditions for the given problem, which lead to a bounded computational domain. Using the exact or approximating boundary conditions on the artificial boundary, the original problem is reduced to an initial-boundary-value problem on the bounded computational domain which is respectively equivalent to or approximates the original problem. Finite difference methods are used to solve the reduced problems on the bounded computational domain and the stability of these finite difference methods are proved. The numerical results demonstrate that the method given in this paper is effective and feasible.

Keywords: Time-fractional diffusion-wave equation, unbounded spatial domain, artificial boundary conditions, numerical solution
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1. Introduction

The time-fractional diffusion-wave equation is a mathematical model of a wide range of important physical phenomena, including ordinary diffusion, dispersive anomalous diffusion ([16, 50, 62]), Pipkin's viscoelasticity ([35, 52, 53, 45]), colloid, glassy and porous materials, in fractals, percolation clusters ([29, 58]), biological systems ([44]), random and disordered media ([48, 49]), finance ([47]), quantum mechanics ([28]), and constant-Q seismic-wave propagation ([5, 6, 46]).

Many authors have tried to construct analytical solutions to problems of time-fractional differential equations. This has been done for example in [55, 59] for the time-fractional diffusion-wave equations, in which the corresponding Green's functions and their properties are obtained in terms of Fox functions. In [17, 18] the similarity method and Laplace transform techniques are used to obtain the scale-invariant solution of time-fractional diffusion-wave

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equations in terms of the Wright function. The solutions to the time-fractional advection-dispersion equation in the whole space and the half space is given in [30] by resorting to Fourier-Laplace transforms. Non-central-symmetric solutions in a sphere are constructed by using the Laplace transform for time and the Legendre transform, the finite Fourier transform and the finite Hankel transform in space in [54]. Advances in obtaining analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations can be found in [31, 32, 60].

However, it is usually not possible to obtain the analytic solution for the general case. Instead, Liu et al. ([40]) use a first-order finite difference scheme in both time and space direction, and some stability conditions are derived. In [57] a finite difference scheme for the fractional diffusion-wave equation is proposed. A compact difference scheme for solving the fractional diffusion-wave equation is presented in [11], and a numerical method using the decomposition method for this equation is considered in [1], while the implicit numerical scheme for a fractional diffusion equation is considered in [36]. The paper [9] describes and analyzes the time-discretization of time-fractional diffusion-wave equations by convolution quadrature. The backward Euler scheme is used to discretize the first-order time derivative and the L^1 scheme is used to approximate the fractional-order time derivative. The implicit difference approximation for the two-dimensional space-time fractional diffusion equation is considered in [63]. A finite difference scheme in time and the Legendre spectral method in space for the time-fractional diffusion equation are employed in [38] and a convergence rate of $(2-\alpha)$-order in time and spectral accuracy in space of the method was rigorously proved. Then in [37], a space-time spectral method for the time fractional diffusion equation is presented and the well-posedness of this method by introducing a well-suited variational formulation is established. The spectral accuracy of the method is proven by providing a priori error estimate. A conservative difference approximation for the time-fractional diffusion equation is proposed and analyzed in detail in [56]. Other numerical approaches to the time-fractional diffusion equation can be found in [4, 7, 8, 39, 41, 42, 51, 61, 64] and the references therein.

In this paper, we consider the numerical solution of the time-fractional diffusion-wave equation in two-dimensional and three-dimensional unbounded domain. The unboundedness of the domain is one essential difficulty for finding the numerical solution of the given problems. The artificial boundary method [12, 13, 22, 23, 24] is a powerful tool for the numerical solution of initial-boundary-value problems on unbounded domains. By introducing an artificial boundary, the given domain is divided into two parts, a finite computational domain and an infinite domain. A suitable boundary condition is imposed on the artificial boundary, such that the solution of the problem with the suitable boundary condition on the artificial boundary in the finite computational domain is a good approximation of the original problem.

In [15, 14], the exact artificial boundary condition (ABC) for the one-dimensional time-fractional sub-diffusion equation $(0 < \alpha < 1)$ is derived by resorting to Laplace transform techniques. Then the finite difference methods are used to solve the reduced problem on bounded domain. The convergence rate and stability are also established in these two papers. Similar techniques for deriving the absorbing boundary conditions of Engquist and Majda are used in [10] to obtain the absorbing boundary condition for the two-dimensional time-fractional wave equation with $1 < \alpha < 2$. By a joint application of the Laplace transform and the Fourier series expansion, an exact and some approximating artificial boundary conditions for fractional sub-diffusion equation on two-dimensional unbounded domains are given in[20]. In[3], the authors construct an exact ABC and a series approximate ABCs for time-fractional diffusion wave equations ($0 < \alpha < 2$) on an unbounded 2D spatial domain and we use two finite difference methods to solve the reduced problem on bounded computational domain. The stability of the two difference methods are proved. In [65, 66, 67], by using spherical coordinates and the solutions of Bessel equation, the exact boundary conditions for the Schrödinger equation in $\mathbb{R}^2$ and $\mathbb{R}^3$ are constructed on given artificial boundary. In the present paper, we derive the exact and a series of approximate artificial boundary conditions for the time-fractional diffusion-wave equations ($0 < \alpha < 2$) in two-dimensional and three-dimensional unbounded spatial domain by extending the ideas for constructing ABCs for 2D parabolic equation[21], 2D wave equation[27], 3D parabolic equation[25] and 3D wave equation[26] on unbounded spatial domains.

The paper is organized as follows. In Section 2, the exact boundary condition is derived on the given artificial boundary Γ for the time-fractional diffusion-wave equation on an unbounded two-dimensional spatial domain; that is, the relationship between $\frac{\partial u}{\partial n}\big|_\Gamma$ and $\frac{\partial u}{\partial t}\big|_\Gamma$ is established. Moreover, a series of artificial boundary conditions with high accuracy is obtained. By means of the artificial boundary conditions, a family of approximate problems for the original problem on the bounded computational domain is constructed. And the ABCs for the time-fractional diffusion-wave equation on an unbounded three-dimensional spatial domain are give in section 3. Section 4 contains the stability estimates for the two finite difference schemes for 2D case. To demonstrate the accuracy and efficiency of our ABCs, numerical examples are given in Section 5. Finally, we add some concluding remarks in Section 6.

2. The artificial boundary condition

Let $D \subset \mathbb{R}^2$ denote a bounded domain, namely $D \subset B(0,a) = \{x \in \mathbb{R}^2 \mid ||x|| \leq a\}$ with $a > 0$. Suppose

$$D^c = \mathbb{R}^2 \backslash \overline{D}, \quad \Omega_c^T = D^c \times (0,T], \quad \Gamma_0 = \partial D \times (0,T]$$

Consider the following initial-boundary value problem:

$$\begin{cases} {}^cD_t^\alpha u(x,t) = \Delta u(x,t) + f(x,t), & (x,t) \in \Omega_c^T, \quad 0 < \alpha < 1, \\ u(x)\big|_{t=0} = \varphi(x), & x \in D^c, \\ u(x,t)\big|_{\Gamma_0} = g(x,t), & (x,t) \in \Gamma_0, \\ u \to 0, & \|x\| \to +\infty; \end{cases} \tag{2.1a}$$

$$\begin{cases} {}^cD_t^\alpha u(x,t) = \Delta u(x,t) + f(x,t), & (x,t) \in \Omega_c^T, \quad 1 < \alpha < 2, \\ u(x)\big|_{t=0} = \varphi(x), & x \in D^c, \\ u_t(x)\big|_{t=0} = \varrho(x), & x \in D^c, \\ u(x,t)\big|_{\Gamma_0} = g(x,t), & (x,t) \in \Gamma_0, \\ u \to 0, & \|x\| \to +\infty; \end{cases} \tag{2.1b}$$

where ${}^cD_t^\alpha$ denotes the Caputo fractional derivative of order α with respect to t defined by

$$
{}^cD_t^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{1}{(t-\tau)^\alpha} \frac{\partial u(x,\tau)}{\partial \tau} d\tau, & 0 < \alpha < 1, \\ \\ \dfrac{1}{\Gamma(2-\alpha)} \displaystyle\int_0^t \frac{1}{(t-\tau)^{\alpha-1}} \frac{\partial^2 u(x,\tau)}{\partial \tau^2} d\tau, & 1 < \alpha < 2, \end{cases} \tag{2.2}
$$

$f(x,t), g(x,t), \varphi(x)$ and $\rho(x)$ are given smooth functions and $f(x,t)$, $\varphi(x), \rho(x)$ vanish outside the ball $B(0,a)$, namely

$$
f(x,t) = 0, \quad \varphi(x) = 0, \quad \rho(x) = 0, \quad \text{if } ||x|| \geq a
$$

We introduce an artificial boundary $\Gamma = \{(x,t) \mid ||x|| = b, 0 < t \leq T\}$ with $b > a$ to divide the domain Ω_c^T into two parts,

$$
\begin{aligned}
\Omega_i^T &= \{(x,t) \mid x \in D^c \text{ and } ||x|| < b, 0 < t < T\} \\
\Omega_e^T &= \{(x,t) \mid ||x|| \geq b, 0 < t \leq T\}
\end{aligned}
$$

If we can seek a suitable boundary condition on Γ, the problem (2.1a)-(2.1b) can be reduced to the bounded computational domain Ω_i^T. We will consider the case $0 < \alpha < 1$ and $1 < \alpha < 2$ in the following subsection respectively.

2.1. ABCs for the case $0 < \alpha < 1$

In the polar coordinate, the restriction of the solution $u(r,\theta,t)$ of problem (2.1a) on the unbounded domain satisfies

$$
\begin{aligned}
{}^cD_t^\alpha u(x,t) &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad (r,\theta,t) \in \Omega_e^T, && (2.3) \\
u\big|_{r=b} &= u(b,\theta,t), && (2.4) \\
u|_{t=0} &= 0, && (2.5) \\
u &\to 0, \quad \text{when} \quad r \to +\infty. && (2.6)
\end{aligned}
$$

where $\Omega_e^T = \{r > b, \theta \in [0,2\pi], t \in [0,T]\}$.

Since $u(b,\theta,t)$ is unknown, the problem (2.3)-(2.6) is an incompletely posed problem; it can't be solved independently. If $u(b,\theta,\phi,t)$ is given, the problem (2.3)-(2.6) is well posed, so the solution $u(r,\theta,t)$ of (2.3)-(2.6) can be given by $u(b,\theta,t)$.

Let

$$
\begin{aligned}
u(b,\theta,t) &= a_0(t) + \sum_{n=1}^{\infty} (a_n(t)\cos n\theta + b_n(t)\sin n\theta), && (2.7) \\
a_0(t) &= \frac{1}{2\pi}\int_0^{2\pi} u(b,\theta,t)d\theta, && (2.8) \\
a_n(t) &= \frac{1}{\pi}\int_0^{2\pi} u(b,\theta,t)\cos n\theta d\theta, \ n = 1,2,\cdots, && (2.9) \\
b_n(t) &= \frac{1}{\pi}\int_0^{2\pi} u(b,\theta,t)\sin n\theta d\theta, \ n = 1,2,\cdots. && (2.10)
\end{aligned}
$$

Let the solution of problem (2.3)–(2.6), $u(r,\theta,t)$, be

$$u(r,\theta,t) = u_0(r,t) + \sum_{n=1}^{\infty}(u_n(r,t)\cos n\theta + v_n(r,t)\sin n\theta), \tag{2.11}$$

where

$$\begin{aligned} u_n(r,t) &= \frac{1}{\pi}\int_0^{2\pi} u(r,\theta,t)\cos n\theta d\theta, && (2.12)\\ v_n(r,t) &= \frac{1}{\pi}\int_0^{2\pi} u(r,\theta,t)\sin n\theta d\theta. && (2.13) \end{aligned}$$

Substituting (2.11) into (2.3), we obtain:

(i) $u_0(r,t)$ satisfies the following initial-boundary value problem:

$$\begin{aligned} {}^cD_t^\alpha u_0(t) &= \frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r}\frac{\partial u_0}{\partial r}, \quad r > b, 0 < t \le T && (2.14)\\ u_0\big|_{r=b} &= a_0(t), && (2.15)\\ u_0\big|_{t=0} &= 0, && (2.16)\\ u_0 \to 0, &\quad \text{when } \ r \to +\infty. && (2.17) \end{aligned}$$

(ii) $u_n(r,t)$ (or $v_n(r,t)$) satisfies the following initial-boundary value problem:

$$\begin{aligned} {}^cD_t^\alpha \mathcal{Q}_n &= \frac{\partial^2 \mathcal{Q}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{Q}_n}{\partial r} - \frac{n^2}{r^2}\mathcal{Q}_n, \quad r > b, 0 < t \le T, && (2.18)\\ \mathcal{Q}_n|_{r=b} &= a_n(t)(\text{or } b_n(t)), && (2.19)\\ \mathcal{Q}_n|_{t=0} &= 0, && (2.20)\\ \mathcal{Q}_n \to 0, &\quad \text{when } \ r \to +\infty. && (2.21) \end{aligned}$$

Before solving the above equations, we need the following Duhamel's theorem for time-fractional diffusion-wave equation (2.1a)-(2.1b),

Theorem 2.1. *Suppose that $V(x,t,\tau), 0 \le \tau \le t, x \in \mathbb{R}^n$, is a solution of the following initial-boundary value problem,*

$$\begin{cases} {}^cD_t^\alpha V(x,t,\tau) = \Delta V(x,t,\tau) + f(x,t,\tau), & t > \tau, x \in \Omega \subset \mathbb{R}^n, \quad 0 < \alpha < 1,\\ V(x,0,\tau) = h(x,\tau), & x \in \Omega,\\ V(x,t,\tau)\big|_{\partial\Omega} = g(x,\tau). \end{cases} \tag{2.22}$$

Then

$$u(x,t) = \frac{\partial}{\partial t}\int_0^t V(x,t-\tau,\tau)d\tau \tag{2.23}$$

is a solution of the following initial-boundary value problem,

$$\begin{cases} {}^cD_t^\alpha u(x,t) = \Delta u(x,t) + f(x,t), & t>0,\ x\in\Omega\subset\mathbb{R}^n,\ 0<\alpha<1, \\ u(x)\big|_{t=0} = h(x), & x\in\Omega, \\ u(x,t)\big|_{\partial\Omega} = g(x,t), & t>0. \end{cases} \tag{2.24}$$

Proof: Applying the Laplace transformation with respect to t to the first two equations in problem (2.24), in view of the initial condition, and the fact

$$\mathcal{L}\{{}^cD_t^\alpha f\}(s) = s^\alpha \mathcal{L}\{f\}(s) - s^{\alpha-1}f(0), \alpha\in(0,1],$$

with Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt,$$

we arrive at

$$\begin{cases} s^\alpha \tilde{u}(x,s) - s^{\alpha-1}h(x) = \Delta\tilde{u}(x,s) + \tilde{f}(x,s), & x\in\Omega \\ \tilde{u}(x,s)\big|_{\partial\Omega} = \tilde{g}(x,s), \end{cases} \tag{2.25}$$

where $\tilde{u}, \tilde{f}, \tilde{g}$ are the Laplace transforms of u, f, and g, respectively.

Let $v(x,s,\tau)$ represent the Laplace transformation of $V(x,t,\tau)$ with respect to t, and $\tilde{v}(x,s)$ the iterated Laplace transform of $V(x,t,\tau)$ with respect to t and τ,

$$\tilde{v}(x,s) = \int_0^\infty e^{-s\tau}d\tau \int_0^\infty e^{-st}V(x,t,\tau)dt. \tag{2.26}$$

Then applying the Laplace transformation with respect to t, it follows from (2.22) that,

$$\begin{cases} s^\alpha v(x,s,\tau) - s^{\alpha-1}h(x) = \Delta v(x,s,\tau) + \frac{1}{s}f(x,\tau), & x\in\Omega, \\ v(x,s,\tau)\big|_{\partial\Omega} = \frac{1}{s}g(x,\tau), \end{cases} \tag{2.27}$$

and consequently, on applying the Laplace transform with respect to τ to the above equations, that

$$\begin{cases} s^\alpha \tilde{v}(x,s) - s^{\alpha-2}h(x) = \Delta\tilde{v}(x,s) + \frac{1}{s}\tilde{f}(x,s), & x\in\Omega \\ \tilde{v}(x,s)\big|_{\partial\Omega} = \frac{1}{s}g(x,s), \end{cases} \tag{2.28}$$

Comparing (2.28) with (2.22), we find these two problems are equivalent and the function $s\tilde{v}(x,s)$ is a solution of (2.22). Thus, if the solution of (2.22) is unique, it follows that

$$\tilde{u}(x,s) = s\tilde{v}(x,s) \tag{2.29}$$

According to the fact,

$$\mathcal{L}\big\{V^*(x,t)\big\} = \mathcal{L}\Big\{\int_0^t V(x,t-\tau,\tau)d\tau\Big\} = \tilde{v}(x,s)$$

and $V^*(x,t) = \int_0^t V(x, t-\tau, \tau) = 0$ satisfies $V^*(x,0) = 0$, we obtain

$$\mathcal{L}\Big\{\frac{\partial V^*}{\partial t}\Big\} = s\tilde{v}(x,s).$$

Hence, on performing the inverse Laplace transformation, we get

$$u(x,t) = \frac{\partial}{\partial t}\int_0^t V(x, t-\tau, \tau)d\tau. \tag{2.30}$$

This end the proof. The case for the time-fractional wave equation can be proved similarly.

We now consider the initial-boundary value problem (2.18)–(2.21). First consider the following simplified problem:

$$\begin{aligned} {}^cD_t^\alpha G_n &= \frac{\partial^2 G_n}{\partial r^2} + \frac{1}{r}\frac{\partial G_n}{\partial r} - \frac{n^2}{r^2}G_n, \quad r > b, 0 < t \le T & (2.31)\\ G_n\Big|_{r=b} &= 1, & (2.32)\\ G_n\Big|_{t=0} &= 0, & (2.33)\\ G_n \to 0, &\quad \text{when } \ r \to +\infty. & (2.34) \end{aligned}$$

Let

$$G_n(r,t) = E_\alpha(-\mu^2 t^\alpha)w(r). \tag{2.35}$$

Substituting (2.35) into(2.31), we have

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + (\mu^2 - \frac{n^2}{r^2})w = 0. \tag{2.36}$$

this is the Bessel equation of order n, there are two independent solutions $J_n(\mu r)$ and $Y_n(\mu r)$:

Let

$$G_*(r,t) = \frac{2}{\pi}\int_0^\infty E_\alpha(-\mu^2 t^\alpha)\frac{J_n(\mu r)Y_n(\mu b) - Y_n(\mu b)J_n(\mu r)}{J_n^2(\mu b) + Y_n^2(\mu b)}\frac{d\mu}{\mu}. \tag{2.37}$$

where J_n and Y_n is the Bessel functions of the first kind and the Bessel function of the second kind, respectively, and E_α is the Mittag-Leffler functions defined by

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \ \alpha > 0, z \in \mathbb{C}. \tag{2.38}$$

It is straight to check that $G_*(r,t)$ satisfies the equation (2.31) and

$$\begin{aligned} G_*(r,t)|_{r=b} &= 0,\\ G_*(r,t)|_{t=0} &= \lim_{t\to+0} G_*(r,t)\\ &= \frac{2}{\pi}\int_0^\infty \frac{J_n(\mu r)Y_n(\mu b) - Y_n(\mu b)J_n(\mu r)}{J_n^2(\mu b) + Y_n^2(\mu b)}\frac{d\mu}{\mu}\\ &= -\left(\frac{b}{r}\right)^n, \quad r > b. \end{aligned}$$

The last equality is from [[19], formula 6.542]. Let

$$G_n(r,t) = \left(\frac{b}{r}\right)^n + G_*(r,t), \tag{2.39}$$

then $G_n(r,t)$ is the solution of problem(2.31)–(2.34).

Applying the Laplace transformation with respect to t to the equations in (2.31)-(2.34), we arrive at

$$\begin{cases} s^\alpha \tilde{G}_n(r,s) = s^2\tilde{G}_n(r,s)) + \frac{s}{r}\tilde{G}_n(r,s) - \frac{n^2}{r^2}\tilde{G}_n(r,s), \quad r > b, \\ \tilde{G}_n(r,s)\big|_{r=b} = \frac{1}{s}, \end{cases} \tag{2.40}$$

Simliarly, applying the Laplace transformation to (2.18)-(2.21), we get

$$\begin{cases} s^\alpha \tilde{\mathcal{Q}}_n(r,s) = s^2\tilde{\mathcal{Q}}_n(r,s)) + \frac{s}{r}\tilde{\mathcal{Q}}_n(r,s) - \frac{n^2}{r^2}\tilde{\mathcal{Q}}_n(r,s), \quad r > b, \\ \tilde{\mathcal{Q}}_n(r,s)\big|_{r=b} = \tilde{a}_n(s)\big(\text{or } \tilde{b}_n(s)\big), \end{cases} \tag{2.41}$$

It follows the solution of the new equations above is

$$\begin{cases} \tilde{u}_n(r,s) = s\tilde{G}_n(r,s)\tilde{a}_n(s) \\ \tilde{v}_n(r,s) = s\tilde{G}_n(r,s)\tilde{b}_n(s) \end{cases} \tag{2.42}$$

So that, by Duhamel's theorem we obtain the solution $u_n(r,t)$ (or $v_n(r,t)$) of problem (2.18)–(2.21)

$$\begin{aligned} u_n(r,t) &= \int_0^t a_n(\lambda)\frac{\partial}{\partial t}G_n(r,t-\lambda)d\lambda \\ &= -\int_0^t a_n(\lambda)\frac{\partial}{\partial \lambda}G_n(r,t-\lambda)d\lambda \\ &= -a_n(\lambda)G_n(r,t-\lambda)\Big|_{\lambda=0}^{\lambda=t} + \int_0^t \frac{da_n(d\lambda)}{\lambda}G_n(r,t-\lambda)d\lambda \\ &= \int_0^t \frac{da_n(\lambda)}{d\lambda}G_n(r,t-\lambda)d\lambda. \end{aligned} \tag{2.43}$$

Similarly we have

$$v_n(r,t) = \int_0^t \frac{db_n(\lambda)}{d\lambda}G_n(r,t-\lambda)d\lambda. \tag{2.44}$$

Furthermore we get

$$\frac{\partial u_n}{\partial r}\Big|_{r=b} = \int_0^t \frac{da_n(\lambda)}{d\lambda}\frac{\partial G_n(r,t-\lambda)}{\partial r}d\lambda\Big|_{r=b} \tag{2.45}$$

$$= -\frac{n}{b}a_n(t) - \frac{4}{\pi^2 b}\int_0^t \frac{da_n(\lambda)}{d\lambda}\left[\int_0^\infty \frac{E_\alpha(-\mu^2(t-\lambda)^\alpha)d\mu}{\mu[J_n^2(\mu b)+Y_n^2(\mu b)]}\right]d\lambda \tag{2.46}$$

The last equality in the above equations is from the Wronskian relation

$$J_\nu(z)Y_\nu'(z) - J_\nu'(z)Y_\nu(z) = \frac{2}{\pi z}. \tag{2.47}$$

Let

$$H_n(b,t) = \frac{4}{\pi^2 b}\int_0^\infty \frac{E_\alpha(-\mu^2 t^\alpha)}{\mu[J_n^2(\mu b) + Y_n^2(\mu b)]}d\mu. \tag{2.48}$$

Combining (3.35) and (3.36), we have

$$\frac{\partial u_n}{\partial r}\Big|_{r=b} \quad = \quad -\frac{n}{b}a_n(t) - \int_0^t \frac{\partial a_n(\lambda)}{\partial \lambda}H_n(b,t-\lambda)d\lambda,\ n = 0,1,2,\cdots. \tag{2.49}$$

Similarly we obtain

$$\frac{\partial v_n}{\partial r}\Big|_{r=b} = -\frac{n}{b}b_n(t) - \int_0^t \frac{\partial b_n(\lambda)}{\partial \lambda}H_n(b,t-\lambda)d\lambda,\ n = 1,2,\cdots. \tag{2.50}$$

By expansion (2.11) with

$$\begin{aligned}
u_0(r,t) &= \frac{1}{2\pi}\int_0^{2\pi} u(r,\phi,t)d\phi,\\
u_n(r,t) &= \frac{1}{\pi}\int_0^{2\pi} u(r,\phi,t)\cos n\phi d\phi,\\
v_n(r,t) &= \frac{1}{\pi}\int_0^{2\pi} u(r,\phi,t)\sin n\phi d\phi,
\end{aligned} \tag{2.51}$$

$$\begin{aligned}
\frac{\partial u}{\partial r}\Big|_{r=b} =& \frac{\partial u_0}{\partial r}\Big|_{r=b} + \sum_{n=1}^\infty\Big(\frac{\partial u_n}{\partial r}\Big|_{r=b}\cos n\theta + \frac{\partial v_n}{\partial r}\Big|_{r=b}\sin n\theta\Big)\\
=& -\sum_{n=1}^\infty \frac{n}{b}(u_n\big|_{r=b}\cos n\theta + v_n\big|_{r=b}\sin n\theta)\\
&- \sum_{n=0}^\infty \int_0^t\Big[\frac{\partial u_n(b,\lambda)}{\partial \lambda}\cos n\theta + \frac{\partial v_n(b,\lambda)}{\partial\lambda}\sin n\theta\Big]H_n(t-\lambda)d\lambda.
\end{aligned} \tag{2.52}$$

Finally, we obtain the exact boundary condition on the artificial boundary Γ:

$$\begin{aligned}
\frac{\partial u}{\partial r}(r,\theta,t)\Big|_{r=b} =& -\frac{1}{2\pi}\int_0^t\int_0^{2\pi}\frac{\partial u}{\partial\lambda}(b,\phi,\lambda)d\phi H_0(t-\lambda)d\lambda\\
&- \frac{1}{\pi}\int_0^t\sum_{n=1}^\infty\int_0^{2\pi}\frac{\partial u}{\partial \lambda}(b,\phi,\lambda)\cos n(\phi-\theta)d\phi H_n(t-\lambda)d\lambda\\
&- \frac{1}{b\pi}\sum_{n=1}^\infty n\int_0^{2\pi} u(b,\phi,t)\cos n(\phi-\theta)d\phi\\
\equiv& \mathcal{K}_\infty(u,\frac{\partial u}{\partial t})(b,\theta,t).
\end{aligned} \tag{2.53}$$

Remark 2.1. *For $\alpha = 1$, $E_1(-\mu^2 t) = e^{-\mu^2 t}$, then the exact boundary conditions (2.53) gives the exact boundary conditions for parabolic equations on 2D unbounded domain with an circle artificial boundary in [21].*

By the boundary condition (2.53), the original problem (2.1a) is reduced to the following equivalent problem on bounded domain Ω_i^T,

$$\begin{aligned} {}^cD_t^\alpha u(x,t) &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + f(r,\theta,t), \quad (r,\theta,t) \in \Omega_i^T, \\ u\big|_\Gamma &= \mathcal{K}_\infty(u, \frac{\partial u}{\partial t})(b,\theta,t), \\ u|_{t=0} &= \varphi(x), \\ u\Big|_{\Gamma_0} &= g(r,\theta,t). \end{aligned} \tag{2.54}$$

Furthermore, we take the first few terms of the above summation; we obtain a series of approximating artificial boundary conditions on Γ:

$$\begin{aligned} \frac{\partial u^N}{\partial r}(r,\theta,t)\big|_{r=b} =& -\frac{1}{2\pi}\int_0^t \int_0^{2\pi} \frac{\partial u^N}{\partial \lambda}(b,\phi,\lambda)d\phi H_0(t-\lambda)d\lambda \\ & - \frac{1}{\pi}\int_0^t \sum_{n=1}^N \int_0^{2\pi} \frac{\partial u^N}{\partial \lambda}(b,\phi,\lambda)\cos n(\phi-\theta)d\phi H_n(t-\lambda)d\lambda \\ & - \frac{1}{b\pi}\sum_{n=1}^N n\int_0^{2\pi} u^N(b,\phi,t)\cos n(\phi-\theta)d\phi \\ \equiv& \mathcal{K}_N(u^N, \frac{\partial u^N}{\partial t})(b,\theta,t), \quad \text{for } N = 0,1,2,\cdots. \end{aligned} \tag{2.55}$$

By the boundary conditions (2.55), the original problem (2.1a) is reduced to a series of approximation problem on bounded domain Ω_i^T:

$$\begin{aligned} {}^cD_t^\alpha u^N(x,t) &= \frac{\partial^2 u^N}{\partial r^2} + \frac{1}{r}\frac{\partial u^N}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^N}{\partial \theta^2} + f(r,\theta,t), \quad (r,\theta,t) \in \Omega_i^T, \\ u^N\big|_\Gamma &= \mathcal{K}_N(u, \frac{\partial u^N}{\partial t})(b,\theta,t), \\ u^N|_{t=0} &= \varphi(x), \\ u^N\Big|_{\Gamma_0} &= g(r,\theta,t), \end{aligned} \tag{2.56}$$

2.2. ABCs for the case $1 < \alpha < 2$

In the following, we consider the artificial boundary condition for the time-fractional wave equation ($1 < \alpha < 2$). In the polar coordinate, the restriction of the solution $u(r,\theta,t)$

of problem (2.1a) on the unbounded domain satisfies

$$
\begin{aligned}
{}^cD_t^\alpha u(x,t) &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{2}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad (r,\theta,t) \in \Omega_e^T, && (2.57)\\
u\big|_{r=b} &= u(b,\theta,\phi,t), && (2.58)\\
u\big|_{t=0} &= 0, && (2.59)\\
\frac{\partial u}{\partial t}\Big|_{t=0} &= 0, && (2.60)\\
u &\to 0, \quad \text{when} \quad r \to +\infty. && (2.61)
\end{aligned}
$$

where $\Omega_e^T = \{r > b, \theta \in [0,2\pi], t \in [0,T]\}$, $u(b,\theta,t)$ and $u(r,\theta,t)$ is defined by (2.7) and (2.11),respectively.

Substituting (2.11) into (2.57), we obtain:

(i) $u_0(r,t)$ satisfies the following initial-boundary value problem:

$$
\begin{aligned}
{}^cD_t^\alpha u_0(t) &= \frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r}\frac{\partial u_0}{\partial r}, \quad r > b, 0 < t \le T && (2.62)\\
u_0\big|_{r=b} &= a_0(t), && (2.63)\\
u_0\big|_{t=0} &= 0, && (2.64)\\
\frac{\partial u_0}{\partial t}\Big|_{t=0} &= 0, && (2.65)\\
u_0 \to 0, &\quad \text{when} \quad r \to +\infty. && (2.66)
\end{aligned}
$$

(ii) $u_n(r,t)$ (or $v_n(r,t)$) satisfies the following initial-boundary value problem:

$$
\begin{aligned}
{}^cD_t^\alpha \mathcal{Q}_n &= \frac{\partial^2 \mathcal{Q}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{Q}_n}{\partial r} - \frac{n^2}{r^2}\mathcal{Q}_n, \quad r > b, 0 < t \le T, && (2.67)\\
\mathcal{Q}_n\big|_{r=b} &= a_n(t)(\text{or } b_n(t)), && (2.68)\\
\mathcal{Q}_n\big|_{t=0} &= 0, && (2.69)\\
\frac{\partial \mathcal{Q}_n}{\partial t}\Big|_{t=0} &= 0, && (2.70)\\
\mathcal{Q}_n \to 0 &\quad \text{when} \quad r \to +\infty. && (2.71)
\end{aligned}
$$

Consider the following simplified problem:

$$
\begin{aligned}
{}^cD_t^\alpha G_n &= \frac{\partial^2 G_n}{\partial r^2} + \frac{1}{r}\frac{\partial G_n}{\partial r} - \frac{n^2}{r^2}G_n, \quad r > b, 0 < t \le T && (2.72)\\
G_n\big|_{r=b} &= 1, && (2.73)\\
\frac{\partial G_n}{\partial t}\Big|_{t=0} &= 0, && (2.74)\\
G_n\big|_{t=0} &= 0, && (2.75)\\
G_n \to 0, &\quad \text{when} \quad r \to +\infty. && (2.76)
\end{aligned}
$$

As in the case $0 < \alpha < 1$, $G_*(r,t)$ satisfies the equation (2.72), and

$$
\begin{aligned}
G_*(r,t)\Big|_{r=b} &= 0,\\
G_*(r,t)\Big|_{t=0} &= -\Big(\frac{b}{r}\Big)^n,\\
\frac{\partial G_*(r,t)}{\partial t}\Big|_{t=0} &= \lim_{t\to 0+}\frac{\partial G_*(r,t)}{\partial t}\\
&= \lim_{t\to 0+}\frac{2}{\pi}\int_0^\infty \frac{\partial E_\alpha(-\mu^2 t^\alpha)}{\partial t}\frac{J_n(\mu r)Y_n(\mu b)-Y_n(\mu b)J_n(\mu r)}{J_n^2(\mu b)+Y_n^2(\mu b)}\frac{d\mu}{\mu}\\
&= \lim_{t\to 0+}\frac{2}{\pi}\int_0^\infty \Big(-\mu^2 t^{\alpha-1}E_{\alpha,\alpha}(-\mu^2 t^\alpha)\Big)\frac{J_n(\mu r)Y_n(\mu b)-Y_n(\mu b)J_n(\mu r)}{J_n^2(\mu b)+Y_n^2(\mu b)}\frac{d\mu}{\mu}\\
&= 0,
\end{aligned}
$$

where $E_{\alpha,\alpha}$ is the general Mittag-Leffler function defined by

$$
E_{\alpha,\beta}(z) \equiv \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}. \tag{2.77}
$$

Then

$$
G_n(r,t) = \Big(\frac{b}{r}\Big)^n + G_*(r,t),
$$

is the solution of problem(2.72)–(2.76). Similarly, we obtain the exact boundary condition on the artificial boundary Γ for $1 < \alpha < 2$:

$$
\begin{aligned}
\frac{\partial u}{\partial r}(r,\theta,t)\Big|_{r=b} =& -\frac{1}{2\pi}\int_0^t\int_0^{2\pi}\frac{\partial u}{\partial \lambda}(b,\phi,\lambda)d\phi H_0(t-\lambda)d\lambda\\
&-\frac{1}{\pi}\int_0^t\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial u}{\partial\lambda}(b,\phi,\lambda)\cos n(\phi-\theta)d\phi H_n(t-\lambda)d\lambda\\
&-\frac{1}{b\pi}\sum_{n=1}^{\infty}n\int_0^{2\pi}u(b,\phi,t)\cos n(\phi-\theta)d\phi\\
\equiv&\mathcal{K}_\infty(u,\frac{\partial u}{\partial t})(b,\theta,t).
\end{aligned} \tag{2.78}
$$

And the corresponding approximate ABCs for $1 < \alpha < 2$,

$$
\begin{aligned}
\frac{\partial u^N}{\partial r}(r,\theta,t)\Big|_{r=b} =& -\frac{1}{2\pi}\int_0^t\int_0^{2\pi}\frac{\partial u^N}{\partial \lambda}(b,\phi,\lambda)d\phi H_0(t-\lambda)d\lambda\\
&-\frac{1}{\pi}\int_0^t\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial u^N}{\partial\lambda}(b,\phi,\lambda)\cos n(\phi-\theta)d\phi H_n(t-\lambda)d\lambda\\
&-\frac{1}{b\pi}\sum_{n=1}^{N}n\int_0^{2\pi}u^N(b,\phi,t)\cos n(\phi-\theta)d\phi\\
\equiv&\mathcal{K}_N(u^N,\frac{\partial u^N}{\partial t})(b,\theta,t),\quad \text{for } N=0,1,2,\cdots.
\end{aligned} \tag{2.79}
$$

Remark 2.2. *For $\alpha = 2$, $E_2(-\mu^2 t^2) = cos(\mu t)$, then the ABCs (2.78) and (2.79) are the ABCs for 2D wave equation which givens in [27].*

3. The artificial boundary condition for 3D diffusion-wave equation

Let $D \subset \mathbb{R}^3$ denote a bounded domain, namely $D \subset B(0,a) = \{x \in \mathbb{R}^3 \mid ||x|| \le a\}$ with $a > 0$. Suppose

$$D^c = \mathbb{R}^3 \backslash \overline{D}, \quad \Omega_c^T = D^c \times (0,T], \quad \Gamma_0 = \partial D \times (0,T]$$

Consider the following initial-boundary value problem:

$$\begin{cases} {}^cD_t^\alpha u(x,t) = \Delta u(x,t) + f(x,t), & (x,t) \in \Omega_c^T, \quad 0 < \alpha < 1, \\ u(x)\big|_{t=0} = \varphi(x), & x \in D^c, \\ u(x,t)\big|_{\Gamma_0} = g(x,t), & (x,t) \in \Gamma_0, \\ u \to 0, & \|x\| \to +\infty; \end{cases} \tag{3.1a}$$

$$\begin{cases} {}^cD_t^\alpha u(x,t) = \Delta u(x,t) + f(x,t), & (x,t) \in \Omega_c^T, \quad 1 < \alpha < 2, \\ u(x)\big|_{t=0} = \varphi(x), & x \in D^c, \\ u_t(x)\big|_{t=0} = \varrho(x), & x \in D^c, \\ u(x,t)\big|_{\Gamma_0} = g(x,t), & (x,t) \in \Gamma_0, \\ u \to 0, & \|x\| \to +\infty; \end{cases} \tag{3.1b}$$

$f(x,t), g(x,t), \varphi(x)$ and $\rho(x)$ are given smooth functions and $f(x,t)$, $\varphi(x), \rho(x)$ vanish outside the ball $B(0,a)$, namely

$$f(x,t) = 0, \quad \varphi(x) = 0, \quad \rho(x) = 0, \quad \text{if } ||x|| \ge a.$$

We introduce an artificial boundary $\Gamma = \{(x,t) \mid ||x|| = b, 0 < t \le T\}$ with $b > a$ to divide the domain Ω_c^T into two parts,

$$\begin{aligned} \Omega_i^T &= \{(x,t) \mid x \in D^c \text{ and } ||x|| < b, 0 < t < T\} \\ \Omega_e^T &= \{(x,t) \mid ||x|| \ge b, 0 < t \le T\} \end{aligned}$$

If we can seek a suitable boundary condition on Γ, the problem (3.1a) can be reduced to the bounded computational domain Ω_i^T. In the spherical coordinate, the restriction of the solution $u(r,\theta,\phi,t)$ of problem (3.1a) on the unbounded domain satisfies

$${}^cD_t^\alpha u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial u}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}, \quad (r,\theta,\phi,t) \in \Omega_e^T \tag{3.2}$$

$$u|_{r=b} = u(b,\theta,\phi,t), \tag{3.3}$$

$$u|_{t=0} = 0, \tag{3.4}$$

$$u \to 0 \text{ when } r \to +\infty. \tag{3.5}$$

where $\Omega_e^T = \{r > b, \theta \in [0,\pi], \phi \in [0,2\pi], t \in [0,T]\}$.

Since $u(b,\theta,\phi,t)$ is unknown, the problem (3.2)-(3.5) is an uncompleted posed problem; it can't be solved independently. If $u(b,\theta,\phi,t)$ is given, the problem (3.2)-(3.5) is well posed, so the solution $u(r,\theta,\phi,t)$ of (3.2)-(3.5) can be given by $u(b,\theta,\phi,t)$.

Let

$$u(b,\theta,\phi,t) = \frac{a_{00}(t)}{2}+\sum_{n=1}^{\infty}\Bigg\{\frac{a_{n0}(t)}{2}P_n^0(\cos\theta) + \sum_{m=1}^{n}P_n^m(\cos\theta)\Big(a_{nm}(t)\cos m\phi+b_{nm}(t)\sin m\phi\Big)\Bigg\}, \tag{3.6}$$

where $P_n^m(\cos\theta),\ (n=1,2,\cdots,m=1,2,\cdots,n)$ are the Associate *Legendre* functions[2] and

$$a_{nm}(t) = \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\int_0^{\pi}u(b,\xi,\psi,t)P_n^m(\cos\xi)\cos m\psi\sin\xi d\xi d\psi, \tag{3.7}$$

$$b_{nm}(t) = \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\int_0^{\pi}u(b,\xi,\psi,t)P_n^m(\cos\xi)\sin m\psi\sin\xi d\xi d\psi. \tag{3.8}$$

Let the solution of problem (3.2)–(3.5), $u(r,\theta,\phi,t)$, be

$$u(r,\theta,\phi,t) = \frac{u_{00}(r,t)}{2}+\sum_{n=1}^{\infty}\Bigg\{\frac{u_{n0}(r,t)}{2}P_n^0(\cos\theta) + \sum_{m=1}^{n}P_n^m(\cos\theta)\Big(u_{nm}(r,t)\cos m\phi+v_{nm}(r,t)\sin m\phi\Big)\Bigg\}. \tag{3.9}$$

Substituting (3.9) into (3.2), we obtain:
(i) $u_{00}(r,t)$ satisfies the following initial-boundary value problem:

$$\begin{aligned}
{}^cD_t^\alpha u_{00} &= \frac{\partial^2 u_{00}}{\partial r^2}+\frac{2}{r}\frac{\partial u_{00}}{\partial r}, \quad r>b, 0<t\le T,\ 0<\alpha<1, && (3.10)\\
u_{00}|_{r=b} &= a_{00}(t), && (3.11)\\
u_{00}|_{t=0} &= 0, && (3.12)\\
u_{00}\to 0 &\quad \text{when}\ \ r\to+\infty. && (3.13)
\end{aligned}$$

(ii) $u_{nm}(r,t)$ (or $v_{nm}(r,t)$) satisfies the following initial-boundary value problem:

$$\begin{aligned}
{}^cD_t^\alpha \mathcal{Q}_n &= \frac{\partial^2 \mathcal{Q}_n}{\partial r^2}+\frac{2}{r}\frac{\partial \mathcal{Q}_n}{\partial r}-\frac{n(n+1)}{r^2}\mathcal{Q}_n,\ \ r>b, 0<t\le T, 0<\alpha<1, && (3.14)\\
\mathcal{Q}_n|_{r=b} &= a_{nm}(t)(\text{or } b_{nm}(t)), && (3.15)\\
\mathcal{Q}_n|_{t=0} &= 0, && (3.16)\\
\mathcal{Q}_n\to 0 &\quad \text{when}\ \ r\to+\infty. && (3.17)
\end{aligned}$$

Let $v_{00}(r,t)=ru_{00}(r,t)$, then from problem (3.10)–(3.13), we have :

$$\begin{aligned}
{}^cD_t^\alpha v_{00} &= \frac{\partial^2 u_{00}}{\partial r^2}, \quad r>b, 0<t\le T,\ 0<\alpha<1, && (3.18)\\
v_{00}|_{r=b} &= ba_{00}(t), && (3.19)\\
v_{00}|_{t=0} &= 0, && (3.20)\\
v_{00}\to 0 &\quad \text{when}\ \ r\to+\infty. && (3.21)
\end{aligned}$$

Then form we have

$$\frac{\partial u_{00}}{\partial r}(b,t) = -\frac{1}{b}u_{00}\Big|_{r=b} - {}^cD_t^{\alpha/2}u_{00}\Big|_{r=b}. \tag{3.22}$$

We now consider the initial-boundary value problem (3.14)–(3.17). First consider the following simplified problem:

$$\begin{aligned} {}^cD_t^{\alpha}G_n &= \frac{\partial^2 G_n}{\partial r^2} + \frac{2}{r}\frac{\partial G_n}{\partial r} - \frac{n(n+1)}{r^2}G_n, \quad r > b, 0 < t \le T & (3.23)\\ G_n|_{r=b} &= 1, & (3.24)\\ G_n|_{t=0} &= 0, & (3.25)\\ G_n \to 0 \quad & \text{when} \ \ r \to +\infty. & (3.26) \end{aligned}$$

Let

$$G_n(r,t) = E_\alpha(-\mu^2 t^\alpha)w(r). \tag{3.27}$$

Substituting (3.27) into (3.23), we have

$$\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r} + (\mu^2 - \frac{n(n+1)}{r^2})w = 0. \tag{3.28}$$

The equation (3.28) has two independent solutions:

$$\left.\begin{aligned} w_1(\mu r) &= \sqrt{\tfrac{\pi}{2\mu r}}J_{n+1/2}(\mu r),\\ w_2(\mu r) &= \sqrt{\tfrac{\pi}{2\mu r}}Y_{n+1/2}(\mu r). \end{aligned}\right\} \tag{3.29}$$

Let

$$G_*(r,t) = \frac{2}{\pi}\int_0^\infty E_\alpha(-\mu^2 t^\alpha)\frac{w_1(\mu r)w_2(\mu b) - w_1(\mu b)w_2(\mu r)}{w_1^2(\mu b) + w_2^2(\mu b)}\frac{d\mu}{\mu}. \tag{3.30}$$

It is straight to check that $G_*(r,t)$ satisfies the equation (3.23) and

$$\begin{aligned} G_*(r,t)|_{r=b} &= 0,\\ G_*(r,t)|_{t=0} &= \lim_{t\to+0} G_*(r,t)\\ &= \frac{2}{\pi}\int_0^\infty \frac{w_1(\mu r)w_2(\mu b) - w_1(\mu b)w_2(\mu r)}{w_1^2(\mu b) + w_2^2(\mu b)}\frac{d\mu}{\mu}\\ &= \frac{2}{\pi}\left(\frac{b}{r}\right)^{1/2}\int_0^\infty \frac{J_{n+1/2}(\mu r)Y_{n+1/2}(\mu b) - Y_{n+1/2}(\mu r)J_{n+1/2}(\mu b)}{J^2_{n+1/2}(\mu b) + Y^2_{n+1/2}(\mu b)}\frac{d\mu}{\mu}\\ &= -\left(\frac{b}{r}\right)^{n+1}, \quad r > b. \end{aligned}$$

Let

$$G_n(r,t) = \left(\frac{b}{r}\right)^{n+1} + G_*(r,t), \tag{3.31}$$

then $G_n(r,t)$ is the solution of problem (3.23)–(3.26). By Duhamel's theorem we obtain the solution $u_{nm}(r,t)$ (or $v_{nm}(r,t)$) of problem (3.14)–(3.17)

$$\begin{aligned} u_{nm}(r,t) &= \int_0^t a_{nm}(\lambda)\frac{\partial}{\partial t}G_n(r,t-\lambda)d\lambda \\ &= -\int_0^t a_{nm}(\lambda)\frac{\partial}{\partial \lambda}G_n(r,t-\lambda)d\lambda \\ &= -a_{nm}(\lambda)G_n(r,t-\lambda)\Big|_{\lambda=0}^{\lambda=t} + \int_0^t \frac{da_{nm}(d\lambda)}{\lambda}G_n(r,t-\lambda)d\lambda \\ &= \int_0^t \frac{da_{nm}(\lambda)}{d\lambda}G_n(r,t-\lambda)d\lambda. \end{aligned} \tag{3.32}$$

Similarly we have

$$v_{nm}(r,t) = \int_0^t \frac{db_{nm}(\lambda)}{d\lambda}G_n(r,t-\lambda)d\lambda. \tag{3.33}$$

Furthermore we get

$$\frac{\partial u_{nm}}{\partial r}\Big|_{r=b} = \int_0^t \frac{da_{nm}(\lambda)}{d\lambda}\frac{\partial G_n(r,t-\lambda)}{\partial r}d\lambda\Big|_{r=b} \tag{3.34}$$

$$= -\frac{n+1}{b}a_{nm}(t) - \frac{4}{\pi^2 b}\int_0^t \frac{da_{nm}(\lambda)}{d\lambda}\left[\int_0^\infty \frac{E_\alpha(-\mu^2(t-\lambda)^\alpha)\,d\mu}{\mu[J^2_{n+1/2}(\mu b)+Y^2_{n+1/2}(\mu b)]}\right]d\lambda \tag{3.35}$$

On the other hand,

$$\begin{aligned} \frac{4\sqrt{t}}{\pi^2 b}\int_0^\infty \frac{E_\alpha(-\mu^2 t^\alpha)}{\mu[J^2_{n+1/2}(\mu b)+Y^2_{n+1/2}(\mu b)]}d\mu &= \frac{4}{\pi^2}\frac{\sqrt{t}}{b}\int_0^\infty \frac{E_\alpha\big(-(\xi^2/b^2)t^\alpha\big)}{J^2_{n+1/2}(\xi)+Y^2_{n+1/2}(\xi)}\frac{d\xi}{\xi} \\ &\equiv H_{n+1/2}(t/b^2). \end{aligned} \tag{3.36}$$

Combining (3.35) and (3.36), we have

$$\frac{\partial u_{nm}}{\partial r}\Big|_{r=b} = -\frac{n+1}{b}a_{nm}(t) - \int_0^t \frac{\partial a_{nm}(\lambda)}{\partial \lambda}\frac{H_{n+1/2}(\frac{t-\lambda}{b^2})}{\sqrt{t-\lambda}}d\lambda. \tag{3.37}$$

Similarly we obtain

$$\frac{\partial v_{nm}}{\partial r}\Big|_{r=b} = -\frac{n+1}{b}b_{nm}(t) - \int_0^t \frac{\partial b_{nm}(\lambda)}{\partial \lambda}\frac{H_{n+1/2}(\frac{t-\lambda}{b^2})}{\sqrt{t-\lambda}}d\lambda. \tag{3.38}$$

From (3.9), we have the artificial boundary condition on Γ_b:

$$\begin{aligned}
\frac{\partial u}{\partial r}\Big|_{r=b} &= \frac{1}{2}\frac{\partial u_{00}}{\partial r}\Big|_{r=b} \\
&\quad + \sum_{n=1}^{\infty}\left[\frac{1}{2}\frac{\partial u_{n0}}{\partial r}P_n^0(\cos\theta) + \sum_{m=1}^{n}P_n^m(\cos\theta)\left(\frac{\partial u_{nm}}{\partial r}\cos m\phi + \frac{\partial v_{nm}}{\partial r}\sin m\phi\right)\right]_{r=b} \\
&= -\frac{1}{2b}a_{00}(t) - {}^cD_t^{\alpha/2}a_{00}(t) + \sum_{m=1}^{n}\left[-\frac{n+1}{b}\Big(a_{nm}(t)\cos m\phi + b_{nm}(t)\sin m\phi\Big)\right. \\
&\quad + \sum_{n=1}^{\infty}\left\{\frac{1}{2}\left[-\frac{n+1}{b}a_{n0}(t) - \int_0^t \frac{da_{n0}(\lambda)}{d\lambda}\frac{H_{n+1/2}(\frac{t-\lambda}{b^2})}{\sqrt{t-\lambda}}d\lambda\right]P_n^0(\cos\theta)\right. \\
&\quad \left.\left. - \int_0^t \left(\frac{da_{nm}(\lambda)}{d\lambda}\cos m\phi + \frac{db_{nm}(\lambda)}{d\lambda}\sin m\phi\right)\frac{H_{n+1/2}(\frac{t-\lambda}{b^2})}{\sqrt{t-\lambda}}d\lambda\right]P_n^m(\cos\theta)\right\} \\
&\equiv \Phi_\infty\left(u\big|_\Gamma, \frac{\partial u}{\partial t}\big|_\Gamma\right). \qquad (3.39)
\end{aligned}$$

where $\Gamma = \{(x,t)\mid ||x|| = b, 0 < t \le T\}$. By the addition theorem of *Legendre* functions [2]:

$$P_n(\cos\gamma) = P_n^0(\cos\xi)P_n^0(\cos\theta) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\xi)P_n^m(\cos\theta)\cos m(\psi-\phi),$$

where

$$\cos\gamma = \cos\xi\cos\theta + \sin\xi\sin\theta\cos(\psi-\phi),$$

we can simplify the formula in (3.39):

$$\begin{aligned}
\frac{\partial u}{\partial r}\Big|_{r=b} &= -\frac{1}{4\pi b}\int_S u(b,\xi,\psi,t)dS_{\xi,\psi} - \frac{1}{4\pi^{3/2}}\int_S {}^cD_t^{\alpha/2}u(b,\xi,\psi)dS_{\xi,\psi} \\
&\quad -\sum_{n=1}^{\infty}\left\{\frac{(n+1)(2n+1)}{4\pi b}\int_S u(b,\xi,\psi,t)P_n(\cos\gamma)dS_{\xi,\psi}\right. \\
&\quad \left. + \frac{2n+1}{4\pi}\int_0^t\int_S \frac{\partial u(b,\xi,\psi,\lambda)}{\partial\lambda}P_n(\cos\gamma)dS_{\xi,\psi}\frac{H_{n+1/2}(\frac{t-\lambda}{b^2})}{\sqrt{t-\lambda}}d\lambda\right\} \\
&\equiv \Phi_\infty\left(u\big|_\Gamma, \frac{\partial u}{\partial t}\big|_\Gamma\right). \qquad (3.40)
\end{aligned}$$

This is the exact boundary condition satisfied by the solution of problem (3.1a). Therefore, the problem (3.1a) is equivalent to the following initial-boundary value problem on the bounded domain $\Omega_i^T = \{(r,\theta,\phi,t)\in\Omega^T \,|\, r<b\}$ and $\Omega_i = \{(r,\theta,\phi)\in D^c, r<b\}$.

$$\begin{aligned}
{}^cD_t^\alpha u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2},\ (r,\theta,\phi,t)\in\Omega_i^T, \qquad (3.41) \\
u|_{\Gamma_0} &= g(\theta,\phi,t), \qquad (3.42) \\
\frac{\partial u}{\partial r}\Big|_{r=b} &= \Phi_\infty\left(u\big|_\Gamma, \frac{\partial u}{\partial t}\big|_\Gamma\right), \qquad (3.43) \\
u|_{t=0} &= 0, \qquad (3.44)
\end{aligned}$$

If we take the first few terms of the above summation, namely for $N = 0, 1, 2, \cdots$

$$\begin{aligned}
\frac{\partial u^N}{\partial r}\Big|_{r=b} &= -\frac{1}{4\pi b}\int_S u^N(b,\xi,\psi,t)dS_{\xi,\psi} - \frac{1}{4\pi^{3/2}}\int_S {}^cD_t^\alpha u^N(b,\xi,\psi,t)dS_{\xi,\psi} \\
&\quad -\sum_{n=1}^{N}\left\{\frac{(n+1)(2n+1)}{4\pi b}\int_S u^N(b,\xi,\psi,t)P_n(\cos r)dS_{\xi,\psi}\right. \\
&\quad \left. +\frac{2n+1}{4\pi}\int_0^t\int_S \frac{\partial u^N(b,\xi,\psi,\lambda)}{\partial\lambda}P_n(\cos r)dS_{\xi,\psi}\frac{H_{n+1/2}(\frac{t-\lambda}{b^2})}{\sqrt{t-\lambda}}d\lambda\right\} \\
&\equiv \Phi_N\left(u^N\big|_\Gamma, \frac{\partial u^N}{\partial t}\big|_\Gamma\right). \qquad (3.45)
\end{aligned}$$

Using the boundary condition (3.45) instead of (3.43), we obtain a series of approximate problems. And the ABCs for $1 < \alpha < 2$ can be derived similarly.

Remark 3.1. *When $\alpha = 1$, the ABCs (3.44) and (3.45) are the ABCs for parabolic equation on 3D unbounded domain which are given in [25]. If $\alpha = 2$, the ABCs are the ABCs for wave equations on 3D unbounded domains in [27].*

4. Stability analysis of the reduced problems on the bounded computational domain

We firstly concentrate on the approximate problem for $N = 0, 1, 2, \cdots$,

$$\begin{aligned}
{}^cD_t^\alpha u^N &= \triangle u^N + f, \quad (x,t)\in\Omega_i^T,\ 0<\alpha<1, && (4.1)\\
u^N|_{\Gamma_0} &= g(\theta,\phi,t), && (4.2)\\
\frac{\partial u^N}{\partial r}\Big|_{r=b} &= \begin{cases}\mathcal{K}_N\left(u^N\big|_\Gamma, \frac{\partial u}{\partial t}\big|_\Gamma\right), & x\in\mathbb{R}^2,\\ \Phi_N\left(u^N\big|_\Gamma, \frac{\partial u}{\partial t}\big|_\Gamma\right), & x\in\mathbb{R}^3,\end{cases} && (4.3)\\
u^N|_{t=0} &= \varphi(x). && (4.4)
\end{aligned}$$

Suppose that $u^N(r,\theta,\phi,t)$ is a solution of problem (4.1)-(4.4). The stability estimate of the approximate problem (4.1)-(4.4) is based on the following lemmas.

Lemma 4.1. *If $v \in C[0,T]$, ${}^cD_t^\alpha v(t) \in C[0,T]$ $(0<\alpha<1)$ and $v(0)=0$, then*

$$\int_0^t v(s)\,{}^cD_s^\alpha v(s)ds \geq 0,\ 0 < t \leq T. \qquad (4.5)$$

The proof of this lemma is given in [37] and [33], see lemma 3.4 in p.247.

Lemma 4.2. *The following inequality holds:*

$$\int_0^t\int_0^{2\pi}\mathcal{K}_N\left(u^N\big|_\Gamma, \frac{\partial u^N}{\partial s}\big|_\Gamma\right)u^N\big|_\Gamma b\,d\theta ds \leq 0, \qquad (4.6)$$

$$\int_0^t\int_0^{2\pi}\int_0^{\pi}\Phi_N\left(u^N\big|_\Gamma, \frac{\partial u^N}{\partial s}\big|_\Gamma\right)u^N\big|_\Gamma b^2\sin\theta d\theta d\phi ds \leq 0. \qquad (4.7)$$

Proof: We only prove inequality (4.7), (4.6) can be proved similarly. Recall that

$$\begin{aligned} u(b,\theta,\phi,t) &= \frac{a_{00}(t)}{2} + \sum_{n=1}^{\infty}\left\{\frac{a_{n0}(t)}{2}P_n^0(\cos\theta)\right. \\ &\quad \left. + \sum_{m=1}^{n} P_n^m(\cos\theta)\Big(a_{nm}(t)\cos m\phi + b_{nm}(t)\sin m\phi\Big)\right\} \end{aligned} \tag{4.8}$$

Substituting (4.8) into (3.45), we can get

$$\begin{aligned} \Phi_N(u\big|_\Gamma, \frac{\partial u}{\partial t}\big|_\Gamma) &\equiv -\frac{1}{2b}a_{00}(t) - {}^cD_t^{\alpha/2}a_{00}(t) \\ &\quad + \sum_{n=1}^{N}\left\{\frac{1}{2}\left[-\frac{n+1}{b}a_{n0}(t) - \int_0^t \frac{da_{n0}(\lambda)}{d\lambda}\frac{H_{n+1/2}(\frac{t-\lambda}{b^2})}{\sqrt{t-\lambda}}d\lambda\right]P_n^0(\cos\theta)\right. \\ &\quad + \sum_{m=1}^{n}\left[-\frac{n+1}{b}\Big(a_{nm}(t)\cos m\phi + b_{nm}(t)\sin m\phi\Big)\right. \\ &\quad \left.\left. - \int_0^t\left(\frac{da_{nm}(\lambda)}{d\lambda}\cos m\phi + \frac{db_{nm}(\lambda)}{d\lambda}\sin m\phi\right)\frac{H_{n+1/2}(\frac{t-\lambda}{b^2})}{\sqrt{t-\lambda}}d\lambda\right]P_n^m(\cos\theta)\right\} \\ &\equiv W_0(a_{00}) + \sum_{n=1}^{N}\frac{W_n(a_{n0}(t))}{2}P_n^0(\cos\theta) + \\ &\quad \sum_{n=1}^{N}\sum_{m=1}^{n}\Big\{W_n(a_{nm}(t))\cos m\phi + W_n(b_{nm}(t))\sin m\phi\Big\}P_n^m(\cos\theta) \end{aligned} \tag{4.9}$$

with

$$\begin{aligned} W_0(f(t)) &= -\frac{1}{2b}f(t) - {}^cD_t^{\alpha/2}f(t), \\ W_n(f(t)) &= -\frac{n+1}{b}f(t) - \int_0^t \frac{df(\lambda)}{d\lambda}\frac{H_{n+1/2}(\frac{t-\lambda}{b^2})}{\sqrt{t-\lambda}}d\lambda. \end{aligned}$$

Combining (4.8) and (4.9), we obtain

$$\begin{aligned} &\int_0^t\int_0^{2\pi}\int_0^{\pi}\Phi_N\left(u\big|_\Gamma, \frac{\partial u}{\partial s}\big|_\Gamma\right)u\big|_\Gamma b^2\sin\theta d\theta d\phi d\tau = \\ &2\pi b^2\int_0^t\left\{W_0(a_{00}(s))a_{00}(s) + \sum_{n=1}^{N}\frac{1}{2n+1}W_n(a_{n0}(s))a_{n0}(s) + \right. \\ &\left.\sum_{n=1}^{N}\sum_{m=1}^{n}\frac{(n+m)!}{(2n+1)(n-m)!}\Big\{W_n(a_{nm}(s))a_{nm}(s) + W_n(b_{nm}(s))b_{nm}(s)\Big\}\right\}d\tau. \end{aligned} \tag{4.10}$$

On the other hand, we consider the following auxiliary problem on the domain $\{(r,t) \mid b \leq$

$r < +\infty, 0 \le t \le T\}$ for $n = 0, 1, 2, \cdots, N$:

$$
\begin{aligned}
{}^cD_t^\alpha G_n &= \frac{\partial^2 G_n}{\partial r^2} + \frac{2}{r}\frac{\partial G_n}{\partial r} - \frac{n(n+1)}{r^2} G_n, \quad r > b, 0 < t \le T && (4.11)\\
G_n|_{r=b} &= a_{nm}(t), && (4.12)\\
G_n|_{t=0} &= 0, && (4.13)\\
G_n \to 0 & \quad \text{when} \quad r \to +\infty. && (4.14)
\end{aligned}
$$

From (3.37) we have

$$\frac{\partial G_n}{\partial r}\Big|_{r=b} = W_n(a_{nm}(t)) \tag{4.15}$$

Multiplying $r^2 G_n(r,t)$ on the equation (4.10), integrating by parts on $[b, +\infty) \times [0, t]$ and using (4.13)-(4.14), we have

$$
\begin{aligned}
& \int_b^\infty \int_0^t {}^cD_s^\alpha G_n(r,s) G_n(r,s) r^2 dr \\
= \quad & -\int_0^t b^2 \frac{\partial G_n(b,s)}{\partial r} G_n(b,s) ds - \int_0^t \int_b^\infty \Big[(r\frac{\partial G_n(r,s)}{\partial r})^2 + (n^2+n) G_n^2(r,s) \Big] drds.
\end{aligned}
$$

Namely

$$
\begin{aligned}
& \int_b^\infty \int_0^t \Big\{ {}^cD_s^\alpha G_n(r,s) G_n(r,s) r^2 + r^2 (\frac{\partial G_n(r,s)}{\partial r})^2 + n(n+1) G_n^2(r,s) \Big\} drds \\
= \quad & -\int_0^t b^2 \frac{\partial G_n(b,s)}{\partial r} G_n(b,s) ds. && (4.16)
\end{aligned}
$$

Combining Lemma 4.1 with the above equality, we obtain

$$\int_0^t \frac{\partial G_n(b,s)}{\partial r} G_n(b,s) ds = \int_0^t W_n(a_{nm}(s)) a_{nm}(s) ds \le 0. \tag{4.17}$$

Similarly we have

$$\int_0^t W_n(b_{nm}(s)) b_{nm}(s) ds \le 0. \tag{4.18}$$

Combining (4.10), (4.17) and (4.18), we complete the proof of the lemma.

For the approximate problem (4.1)-(4.4), we have the following stability estimate:

Theorem 4.1. *Suppose that $u^N(x,t)$ is a solution of the approximate reduced problem (4.1)-(4.4) and $v^n(x,t) := u^N(x,t) - h(x)$. Then the following stability estimate holds,*

$$
\begin{aligned}
& \int_0^t \int_{\Omega_i} v^N(x,s) {}^cD_s^\alpha v^N(x,s) dxds + \int_0^t \int_{\Omega_i} \big|\nabla v^N(x,s)\big|^2 dxds \\
& \qquad \le C\Big(\int_{\Omega_i} \big|\nabla \varphi(x)\big|^2 dx + \int_0^t \int_{\Omega_i} f^2(x,s) dxds \Big).
\end{aligned} \tag{4.19}
$$

with the constant C depending only on t.

Proof: Since

$$v^N(x,t) := u^N(x,t) - \varphi(x), \tag{4.20}$$

it follows from the initial conditions that

$$v^N(x,0) = 0,\ \forall x \in \Omega_i, \tag{4.21}$$

Multiplying the equation (4.1) by the functions $v^N(x,t)$ and integrating on $\Omega_i \times [0,t]$, we arrive at

$$\begin{aligned}&\int_{\Omega_i}\int_0^t v^N(x,s)^c D_s^\alpha v^N(x,s)dsdx\\ &= \int_0^t\int_{\Omega_i} v^N(x,s)\Delta u^N(x,s)dxds + \int_0^t\int_{\Omega_i} v^N(x,s)f(x,s)dxds.\end{aligned} \tag{4.22}$$

Since

$$\begin{aligned}&\int_0^t\int_{\Omega_i} v^N(x,s)\Delta u^N(x,s)dxds\\ =\ & -\int_0^t\int_{\Omega_i}\nabla v^N(x,s)\cdot\nabla u^N(x,s)dxds + \int_0^t\int_{\partial\Omega_i}\frac{\partial u^N}{\partial n}(x,s)v^N(x,s)dxds\\ =\ & -\int_0^t\int_{\Omega_i}\left|\nabla v^N(x,s)\right|^2dxds - \int_0^t\int_{\Omega_i}\nabla\varphi(x)\cdot\nabla v^N(x,s)dxds\\ & + \int_0^t\int_\Gamma \Phi_N\left(u^N\big|_\Gamma, \frac{\partial u^N}{\partial s}\big|_\Gamma\right)u^N(x,s)dxds.\end{aligned}$$

On the other hand,

$$\left|\int_0^t\int_{\Omega_i}\nabla\varphi(x)\nabla v^N(x,s)dxds\right| \le \frac{1}{2}\int_0^t\int_{\Omega_i}\Big(\left|\nabla\varphi(x)\right|^2 + \left|\nabla v^N(x,s)\right|\Big)dxds, \tag{4.23}$$

and

$$\begin{aligned}&\left|\int_0^t\int_{\Omega_i} v^N(x,s)f(x,s)dxds\right|\\ \le\ & \frac{\delta}{2}\int_0^t\int_{\Omega_i}\left(v^N(x,s)\right)^2dxds + \frac{1}{2\delta}\int_0^t\int_{\Omega_i} f^2(x,s)dxds\\ \le\ & \frac{\delta C}{2}\int_0^t\int_{\Omega_i}\left|\nabla v^N(x,s)\right|^2dxds + \frac{1}{2\delta}\int_0^t\int_{\Omega_i} f^2(x,s)dxds,\ \forall\delta > 0.\end{aligned}$$

This implies that

$$\begin{aligned}&\int_{\Omega_i}\int_0^t v^N(x,s)^c D_s^\alpha v^N(x,s)dsdx + \left(\frac{1}{2} - \frac{\delta C}{2}\right)\int_0^t\int_{\Omega_i}\left|\nabla v^N(x,s)\right|^2dxds\\ &+ \int_0^t\int_\Gamma \Phi_N\left(u^N\big|_\Gamma, \frac{\partial u^N}{\partial s}\big|_\Gamma\right)u^N(x,s)dxds.\\ &\le \frac{1}{2}\int_0^t\int_{\Omega_i}\left|\nabla\varphi(x)\right|^2dxds + \frac{1}{2\delta}\int_0^t\int_{\Omega_i} f^2(x,s)dxds\\ &= \frac{t}{2}\int_{\Omega_i}\left|\nabla\varphi(x)\right|^2dx + \frac{1}{2\delta}\int_0^t\int_\Omega f^2(x,s)dxds.\end{aligned} \tag{4.24}$$

Taking $\delta = \frac{1}{2c}$ in (4.24), we complete the proof.

The stability estimate (4.19) implies the uniqueness of the solution of the problem (4.1)-(4.4) for any positive integer N.

Theorem 4.2. *For any given positive integer N, the problem (4.1)-(4.4) at most has one solution.*

For the case $1 < \alpha < 2$, the reduce problem on bounded computational domain

$$\begin{aligned}
{}^cD_t^\alpha u^N &= \triangle u^N, \quad (x,t) \in \Omega_i^T,\ 1<\alpha<2, &(4.25)\\
u^N|_{\Gamma_0} &= g(\theta,\phi,t), &(4.26)\\
\frac{\partial u^N}{\partial r}\Big|_{r=b} &= \begin{cases} \mathcal{K}_N\left(u^N\big|_\Gamma, \frac{\partial u}{\partial t}\big|_\Gamma\right), & x\in\mathbb{R}^2,\\ \Phi_N\left(u^N\big|_\Gamma, \frac{\partial u}{\partial t}\big|_\Gamma\right), & x\in\mathbb{R}^3,\end{cases} &(4.27)\\
u^N|_{t=0} &= \varphi(x), &(4.28)\\
\frac{\partial u^N}{\partial t}|_{t=0} &= \varrho(x). &(4.29)
\end{aligned}$$

Similarly, for problem (4.25)-(4.29), we have the following estimate

Theorem 4.3. *Suppose that $u^N(x,t)$ is a solution of the approximate reduced problem (4.25)-(3dwave5) and $v^N(x,t) := u^N(x,t) - tk(x)$. Then the following stability estimate holds:*

$$\begin{aligned}
&\int_{\Omega_i}\int_0^t v_s^N(x,s){}^cD_s^{\alpha-1}v_s^N(x,s)dxds + \int_{\Omega_i}\left|\nabla v^N(x,t)\right|^2 dx\\
\le&C\Big(\int_0^t\int_{\Omega_i} f_s^2(x,s)dxds + \int_{\Omega_i}\left[\left|\nabla\varphi(x)\right|^2 + \left|\nabla\rho(x)\right|^2 + f^2(x,t)\right]dx\Big),
\end{aligned} \tag{4.30}$$

with the constant C depending only on t.

From the stability estimate (4.30), immediately, we obtain the uniqueness of the problem (4.25)-(4.29) for any positive integer N.

Theorem 4.4. *For any given positive integer N, the problem (4.25)-(4.29), at most has one solution.*

5. Analysis of the 2D difference scheme

In this section, we shall construct a finite difference scheme to solve the 2D reduced problem on bounded domains. The spatial mesh size in the r and θ direction is $h_1 = b/I$ and $h_2 = 2\pi/J$ respectively, and the temporal step size is $\tau = T/K$. Let $r_i = ih_1(0 \le i \le I)$, $\theta_j = jh_2\,(0 \le j \le J)$, $t_k = k\tau\,(0 \le k \le K)$, $\Omega_h = \{(r_i,\theta_j)|0 \le i \le I, 0 \le j \le J\}$ and $\Omega_t = \{t_k|0 \le k \le K\}$. The finite computational domain $\Omega_i \times [0,T]$ is covered by $\Omega_h \times \Omega_t$. For the grid function $u = \{u_{ij}^k|0 \le i \le I, 0 \le j \le J, 0 \le k \le K\}$,we introduce the following notations.

$$\begin{aligned}
\delta_r u_{i-\frac12,j}^k &= \frac{u_{ij}^k - u_{i-1,j}^k}{h_1}, \quad \delta_t u_{ij}^{k-\frac12} = \frac{u_{ij}^k - u_{ij}^{k-1}}{\tau},\\
u_{ij}^{k-\frac12} &= \frac12(u_{ij}^k + u_{ij}^{k-1}), \delta_r^2 u_{ij}^k = \frac{\delta_r u_{i+\frac12,j}^k - \delta_r u_{i-\frac12,j}^k}{h_1}.
\end{aligned}$$

The finite differences $\delta_\theta u^k_{i,j-\frac{1}{2}}$ and $\delta^2_\theta u^k_{ij}$ are defined in a similar way.

We first consider the stability of difference scheme for time-fractional diffusion equations, corresponding to $0 < \alpha < 1$.

5.1. Stability of the difference scheme for $0 < \alpha < 1$

For any given function $f(t)$, its ${}^cD^\alpha_t f(t)\big|_{t=t_k}$ is approximated by let

$$ {}^cD^\alpha_t f(t_k) \approx D^\alpha_\tau f(t_k) = \frac{(\tau)^{-\alpha}}{\Gamma(2-\alpha)}\Big[a^{(\alpha)}_0 f^k - \sum_{l=1}^{k-1}(a^{(\alpha)}_{k-l-1} - a^{(\alpha)}_{k-l})f^l - a^{(\alpha)}_{k-1}f^0\Big], \tag{5.1}$$

where

$$ a^{(\alpha)}_l = (l+1)^{1-\alpha} - l^{1-\alpha}, \quad l \geq 0,\ 0<\alpha<1. \tag{5.2}$$

For this type of time discretization, the following lemma is true.

Lemma 5.1 ([57]). *Suppose that $f \in C^2[0,t_k]$ and let*

$$\overline{R}(f(t_k)) :=^c D^\alpha_t f(t_k) - D^\alpha_\tau f(t_k).$$

Then

$$ |\overline{R}(f(t_k))| \leq \frac{1}{\Gamma(2-\alpha)}\Big[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})\Big]\|f''\|_\infty \tau^{2-\alpha}, \tag{5.3}$$

This lemma reveals that the accuracy of D^α_τ for ${}^cD^\alpha_t$ is of $O(\tau^{2-\alpha})$ for all $f \in C^2[0,t_k]$.

For the calculation of the ABCs, we need the following lemma.

Lemma 5.2 ([34]). *Let $\rho, \mu, \gamma, \lambda \in \mathbb{C}$, $Re(\rho), Re(\mu) > 0$. Then*

$$ \int_0^z t^{\mu-1}E^\gamma_{\rho,\mu}(\lambda t^\rho)dt = z^\mu E^\gamma_{\rho,\mu+1}(\lambda z^\rho). \tag{5.4}$$

At each time step t_k, we need to compute the following integral in the ABCs (2.55):

$$ \int_0^{t_k}\int_0^{2\pi}\frac{\partial u}{\partial\lambda}(b,\phi,\lambda)\cos n(\phi-\theta)d\phi H_n(t_k-\lambda)d\lambda,\ n=0,1,\cdots,N, \tag{5.5}$$

$$ \int_0^{2\pi} u(b,\phi,t_k)\cos n(\phi-\theta)d\phi,\ n=0,1,\cdots,N. \tag{5.6}$$

The computation for integral (5.6) is relatively simple.

$$\begin{aligned}
&\int_0^{2\pi} u(b,\phi,t_k)\cos n(\phi-\theta_j)d\phi \\
=\ & \sum_{m=1}^{J}\int_{\theta_{m-1}}^{\theta_m}\left[u^{k-\frac{1}{2}}_{I,m}\left(\frac{\phi-\theta_{m-1}}{h_2}\right) + u^{k-\frac{1}{2}}_{I,m-1}\left(\frac{\theta_m-\phi}{h_2}\right)\right]\cos n(\phi-\theta_j)d\phi \\
=\ & \frac{1}{n^2h_2}\sum_{m=1}^{J} u^{k-\frac{1}{2}}_{I,m}\left[2\cos n(\theta_m-\theta_j) - \cos n(\theta_{m+1}-\theta_j) - \cos n(\theta_{m-1}-\theta_j)\right] \\
\equiv\ & \sum_{m=1}^{J}\beta^{(n)}_{mj}u^{k-\frac{1}{2}}_{I,m},
\end{aligned}$$

where

$$\beta_{mj}^{(n)} = \frac{2\cos n(\theta_m - \theta_j) - \cos n(\theta_{m+1} - \theta_j) - \cos n(\theta_{m-1} - \theta_j)}{n^2 h_2}.$$

For integral (5.5), when $n > 0$, we have

$$\begin{aligned}
&\int_0^{t_k}\int_0^{2\pi}\frac{\partial u}{\partial\lambda}(b,\phi,\lambda)\cos n(\phi-\theta)d\phi H_n(t_k-\lambda)d\lambda\\
= \;& \sum_{l=1}^{k}\int_{t_{l-1}}^{t_l}\sum_{m=1}^{J}\int_{\theta_{m-1}}^{\theta_m}\left[\delta_t u_{I,m}^{l-\frac{1}{2}}\left(\frac{\phi-\theta_{m-1}}{h_2}\right)+\delta_t u_{I,m-1}^{l-\frac{1}{2}}\left(\frac{\theta_m-\phi}{h_2}\right)\right]\cos n(\phi-\theta_j)d\phi H_n(t-\lambda)d\lambda\\
= \;& \sum_{l=1}^{k}\sum_{m=1}^{J}\beta_{mj}^{(n)}\delta_t u_{I,m}^{l-\frac{1}{2}}\int_{t_{l-1}}^{t_l}H_n(t_k-\lambda)d\lambda,
\end{aligned}$$

while $n = 0$, we calculate that

$$\begin{aligned}
&\int_0^{t_k}\int_0^{2\pi}\frac{\partial u}{\partial\lambda}(b,\phi,\lambda)d\phi H_0(t_k-\lambda)d\lambda\\
= \;& \sum_{l=1}^{k}\int_{t_{l-1}}^{t_l}\sum_{m=1}^{J}\int_{\theta_{m-1}}^{\theta_m}\left[\delta_t u_{I,m}^{l-\frac{1}{2}}\left(\frac{\phi-\theta_{m-1}}{h_2}\right)+\delta_t u_{I,m-1}^{l-\frac{1}{2}}\left(\frac{\theta_m-\phi}{h_2}\right)\right]d\phi H_0(t_k-\lambda)d\lambda\\
= \;& \frac{h_2}{2}\sum_{l=1}^{k}\sum_{m=1}^{J}\left(\delta_t u_{I,m}^{l-\frac{1}{2}}+u_{I,m-1}^{l-\frac{1}{2}}\right)\int_{t_{l-1}}^{t_l}H_0(t_k-\lambda)d\lambda\\
= \;& h_2\sum_{l=1}^{k}\sum_{m=1}^{J}\delta_t u_{I,m}^{l-\frac{1}{2}}\int_{t_{l-1}}^{t_l}H_0(t_k-\lambda)d\lambda.
\end{aligned}$$

For the integral

$$\begin{aligned}
&\int_{t_{l-1}}^{t_l}H_n(t_k-\lambda)d\lambda = \int_{t_{l-1}}^{t_l}\frac{4}{\pi^2 b}\int_0^\infty\frac{E_\alpha(-\mu^2(t_k-\lambda)^\alpha)}{\mu[J_n^2(\mu b)+Y_n^2(\mu b)]}d\mu\\
= \;& \frac{4}{\pi^2 b}\int_0^\infty\frac{1}{\mu[J_n^2(\mu b)+Y_n^2(\mu b)]}\int_{t_{l-1}}^{t_l}E_\alpha(-\mu^2(t_k-\lambda)^\alpha)d\lambda d\mu\\
\overset{\text{by}(5.4)}{=} \;& \frac{4}{\pi^2 b}\int_0^\infty\left\{\frac{(t_k-t_{l-1})E_\alpha(-\mu^2(t_k-t_{l-1})^\alpha)}{\mu[J_n^2(\mu b)+Y_n^2(\mu b)]}-\frac{(t_k-t_l)E_\alpha(-\mu^2(t_k-t_l)^\alpha)}{\mu[J_n^2(\mu b)+Y_n^2(\mu b)]}\right\}d\mu\\
\equiv \;& c_{k-l}^{(n)}.
\end{aligned}$$

At the boundary point (x_I, θ_j), we have

$$\begin{aligned}
&\mathcal{K}_N\Big(u,\frac{\partial u}{\partial t}\Big)(\theta_j,t_k)\\
\approx \;& -\frac{h_2}{2\pi}\sum_{l=1}^{k}\sum_{m=1}^{J}c_{k-l}^{(0)}\delta_t u_{I,m}^{l-\frac{1}{2}}-\frac{1}{\pi}\sum_{n=1}^{N}\sum_{l=1}^{k}\sum_{m=1}^{J}c_{k-l}^{(n)}\beta_{mj}^{(n)}\delta_t u_{I,m}^{l-\frac{1}{2}}-\frac{1}{b\pi}\sum_{n=1}^{N}n\sum_{m=1}^{J}\beta_{mj}^{(n)}u_{I,m}^{k-\frac{1}{2}}\\
\equiv \;& B_N(U_{I,j}^{k-\frac{1}{2}},\delta_t U_{I,j}^{k-\frac{1}{2}}), \qquad (5.7)
\end{aligned}$$

where grid function $U_{ij}^k = u(r_i, \theta_j, t_k)$.

Applying the discretization formula (5.1) for the time-fractional derivative, the second-order central difference scheme and the backward difference scheme for spatial approximation, we obtain

$$D_\tau^\alpha U_{ij}^{k-\frac{1}{2}} = (\delta_r^2 + \frac{1}{r_i}\delta_r + \frac{1}{r_i^2}\delta_\theta^2)U_{ij}^{k-\frac{1}{2}} + f_{ij}^{k-\frac{1}{2}} + R_{ij}^{k-\frac{1}{2}}, (r_i, \theta_j, t_k) \in \Omega_h \times \Omega_t, \tag{5.8a}$$

$$U_{ij}^0 = \varphi_{ij}, \quad 0 \le i \le I,\ 0 \le j \le J, \tag{5.8b}$$

$$U_{i,J}^{k-\frac{1}{2}} = U_{i,0}^{k-\frac{1}{2}}, \quad 0 \le i \le I,\ 1 \le k \le K, \tag{5.8c}$$

$$\delta_r U_{I-\frac{1}{2},j}^{k-\frac{1}{2}} = B_N(U_{I,j}^{k-\frac{1}{2}}, \delta_t U_{I,j}^{k-\frac{1}{2}}) + Q_{I,j}^{k-\frac{1}{2}}, \quad 1 \le j \le J-1,\ 1 \le k \le K, \tag{5.8d}$$

If the solution $u(x,t)$ is smooth enough, there exists a positive constant C such that

$$|R_{ij}^{k-\frac{1}{2}}| \le C\big(h + \tau^{2-\alpha}\big),\ |Q_{I,j}^{k-\frac{1}{2}}| \le C(h_1 + \tau),$$

where $h = \max\{h_1, h_2\}$.

Omitting the truncation errors in (5.8a)-(5.8d), we construct a finite difference scheme for the reduced problem (2.56):

$$D_\tau^\alpha u_{ij}^{k-\frac{1}{2}} = (\delta_r^2 + \frac{1}{r_i}\delta_r + \frac{1}{r_i^2}\delta_\theta^2)u_{ij}^{k-\frac{1}{2}} + f_{ij}^{k-\frac{1}{2}}, \quad (r_i, \theta_j, t_k) \in \Omega_h \times \Omega_t, \tag{5.9a}$$

$$u_{ij}^0 = \varphi_{ij}, \quad 0 \le i \le I,\ 0 \le j \le J, \tag{5.9b}$$

$$u_{i,J}^{k-\frac{1}{2}} = u_{i,0}^{k-\frac{1}{2}}, \quad 0 \le i \le I,\ 1 \le k \le K, \tag{5.9c}$$

$$\delta_r u_{I-\frac{1}{2},j}^{k-\frac{1}{2}} = B_N(u_{I,j}^{k-\frac{1}{2}}, \delta_t u_{I,j}^{k-\frac{1}{2}}), \quad 1 \le j \le J-1,\ 1 \le k \le K, \tag{5.9d}$$

For any grid function $v = \{v_{ij} | 0 \le i \le I, 0 \le j \le J\}$, we define

$$\|v\| = \Big(h_1 h_2 \sum_{i=1}^{I-1}\sum_{j=1}^{J-1} r_i |v_{ij}|^2\Big)^{1/2}, \tag{5.10}$$

$$\|\delta_r v\| = \Big(h_1 h_2 \sum_{i=2}^{I}\sum_{j=1}^{J-1} r_i |\delta_r v_{i-\frac{1}{2},j}|^2\Big)^{1/2}, \tag{5.11}$$

$$\|\delta_\theta v\| = \Big(h_1 h_2 \sum_{i=1}^{I-1}\sum_{j=1}^{J} \frac{|\delta_\theta v_{i,j-\frac{1}{2}}|^2}{r_i}\Big)^{1/2}, \tag{5.12}$$

$$\|\nabla_h v\| = \Big(\|\delta_r v\|^2 + \|\delta_\theta v\|^2\Big)^{1/2}, \tag{5.13}$$

$$\|\delta_t v| = \Big(h_1 h_2 \sum_{i=1}^{I-1}\sum_{j=1}^{J-1} r_i |\delta_t v_{ij}|^2\Big)^{1/2}. \tag{5.14}$$

We include now some lemmas to establish our results,

Lemma 5.3 ([57])**.** *For any grid function u, the following inequality holds,*

$$\tau\sum_{l=1}^{k}(D_\tau^\alpha u^l,u^l)\geq\frac{t_k^{-\alpha}}{2\Gamma(1-\alpha)}\tau\sum_{l=1}^{k}\|u^l\|^2-\frac{t_k^{1-\alpha}}{2\Gamma(2-\alpha)}\|u^0\|^2. \tag{5.15}$$

Multiplying (5.9a) by $h_1h_2\tau r_iu_{ij}^{k-1/2}$ and summing up for i from 1 to $I-1$, j from 1 to $J-1$, and k from 1 to K, we obtain

$$h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}r_i\Big[(D_\tau^\alpha u_{ij}^{k-\frac{1}{2}})u_{ij}^{k-\frac{1}{2}}-(\delta_r^2u_{ij}^{k-\frac{1}{2}}+\frac{1}{r_i}\delta_ru_{i,j}^{k-\frac{1}{2}}+\frac{1}{r_i^2}\delta_\theta^2u_{ij}^{k-\frac{1}{2}}+f_{ij}^{k-\frac{1}{2}})u_{ij}^{k-\frac{1}{2}}\Big]=0. \tag{5.16}$$

Using Lemma 5.3, we have

$$\begin{aligned}&h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}r_i(D_\tau^\alpha u_{ij}^{k-\frac{1}{2}})u_{ij}^{k-\frac{1}{2}}\\&\geq\frac{t_K^{-\alpha}\tau}{2\Gamma(1-\alpha)}\sum_{k=1}^{K}\|u^{k-\frac{1}{2}}\|^2-\frac{t_K^{1-\alpha}}{2\Gamma(2-\alpha)}\|u^{-\frac{1}{2}}\|^2.\end{aligned} \tag{5.17}$$

where $u_{ij}^{-1/2}=\frac{1}{2}u_{ij}^0$ and we set $u_{ij}^{-1}=0$.

Using the boundary condition (5.9c), $u_{i,J}^{k-\frac{1}{2}}=u_{i,0}^{k-\frac{1}{2}}$, we find

$$\begin{aligned}&h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}r_i(\frac{1}{r_i^2}\delta_\theta^2u_{ij}^{k-\frac{1}{2}})u_{ij}^{k-\frac{1}{2}}\\&=-h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J}\frac{1}{r_i}(\delta_\theta u_{i,j-\frac{1}{2}}^{k-\frac{1}{2}})^2=-\tau\sum_{k=1}^{K}\|\delta_\theta u^{k-\frac{1}{2}}\|^2.\end{aligned} \tag{5.18}$$

On the other hand, we have

$$\begin{aligned}&h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}r_i(\delta_r^2u_{ij}^{k-\frac{1}{2}}+\frac{1}{r_i}\delta_ru_{ij}^{k-\frac{1}{2}})u_{ij}^{k-\frac{1}{2}}\\=\;&h_1h_2\tau\sum_{k=1}^{K}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\frac{1}{h}\Big[iu_{i+1,j}^{k-\frac{1}{2}}u_{ij}^{k-\frac{1}{2}}-(2i-1)(u_{ij}^{k-\frac{1}{2}})^2+(i-1)u_{i-1,j}^{k-\frac{1}{2}}u_{ij}^{k-\frac{1}{2}}\Big]\\=\;&-h_1h_2\tau\sum_{k=1}^{K}\sum_{i=2}^{I}\sum_{j=1}^{J-1}r_i(\delta_ru_{i-\frac{1}{2},j}^{k-\frac{1}{2}})^2+r_{I-1}h_2\tau\sum_{k=1}^{K}\sum_{j=1}^{J-1}\Big((\delta_ru_{I-\frac{1}{2},j}^{k-\frac{1}{2}})u_{I,j}^{k-\frac{1}{2}}\Big).\end{aligned} \tag{5.19}$$

Using the boundary condition (5.9d), we can see that

$$\sum_{k=1}^{K}\sum_{j=1}^{J-1}(\delta_xu_{I-\frac{1}{2},j}^{k-\frac{1}{2}})u_{I,j}^{k-\frac{1}{2}}=\sum_{k=1}^{K}\sum_{j=1}^{J-1}B_N\left(u_{I,j}^{j-\frac{1}{2}},\delta_tu_{I,j}^{k-\frac{1}{2}}\right)\cdot u_{I,j}^{k-\frac{1}{2}}, \tag{5.20}$$

We will prove the following lemma.

Lemma 5.4.

$$\sum_{k=1}^{K}\sum_{j=1}^{J-1} B_N(u_{I,j}^{k-\frac{1}{2}}, \delta_t u_{I,j}^{k-\frac{1}{2}}) \cdot u_{I,j}^{k-\frac{1}{2}} \le 0. \tag{5.21}$$

Proof: The inequality (4.6) is equivalent to

$$\int_0^t \int_0^{2\pi} \mathcal{K}_N \left(u\big|_\Gamma, \frac{\partial u}{\partial s}\big|_\Gamma \right) u\big|_\Gamma d\theta ds \le 0. \tag{5.22}$$

For the left part of above inequality, we have

$$\begin{aligned} & \int_0^t \int_0^{2\pi} \mathcal{K}_N \left(u\big|_\Gamma, \frac{\partial u}{\partial s}\big|_\Gamma \right) u\big|_\Gamma d\theta ds \\ = & \sum_{k=1}^{K}\sum_{j=1}^{J} \int_{t_{k-1}}^{t_k} \int_{\theta_{j-1}}^{\theta_j} u(b,\theta,s) \cdot \Bigg(-\frac{1}{2\pi} \sum_{l=1}^{k}\sum_{m=1}^{J} \int_{t_{l-1}}^{t_l} \int_{\theta_{m-1}}^{\theta_m} \frac{\partial u}{\partial \lambda}(b,\phi,\lambda) d\phi H_0(t_k-\lambda) d\lambda \\ & -\frac{1}{\pi} \sum_{n=1}^{N}\sum_{l=1}^{k}\sum_{m=1}^{J} \int_{t_{l-1}}^{t_l} \int_{\theta_{m-1}}^{\theta_m} \frac{\partial u}{\partial \lambda}(b,\phi,\lambda) \cos n(\theta_j-\phi) d\phi H_n(t_k-\lambda) d\lambda \\ & -\frac{1}{b\pi} \sum_{n=1}^{N} n \sum_{m=1}^{J} \int_{\theta_{m-1}}^{\theta_m} u(b,\phi,t) \cos n(\theta_j-\phi) d\phi \Bigg) d\theta ds \le 0. \end{aligned} \tag{5.23}$$

Using bilinear interpolation to construct a continuous function $u(\theta,t)$ which in $[\theta_{j-1},\theta_j] \times [t_{k-1},t_k]$ has the form

$$\begin{aligned} u(x,t) =& u_{I,j}^k \frac{(t-t_{k-1})(\theta-\theta_{j-1})}{h_2\cdot\tau} - u_{I,j-1}^k \frac{(t-t_{k-1})(\theta-\theta_j)}{h_2\cdot\tau} \\ & - u_{I,j}^{k-1} \frac{(t-t_k)(\theta-\theta_{j-1})}{h_2\cdot\tau} + u_{I,j-1}^{k-1} \frac{(t-t_k)(\theta-\theta_j)}{h_2\cdot\tau}. \end{aligned} \tag{5.24}$$

Thus, in $[\theta_{j-1},\theta_j] \times [t_{k-1},t_k]$, we have

$$\frac{\partial u}{\partial t}(b,\theta,t) = \delta_t u_{I,j}^{k-\frac{1}{2}} \left(\frac{\theta-\theta_{j-1}}{h_2} \right) + \delta_t u_{I,j}^{k-\frac{1}{2}} \left(\frac{\theta_j-\theta}{h_2} \right). \tag{5.25}$$

Substituting (5.24) and (5.25) into (5.23) , we obtain

$$\begin{aligned} & -\frac{1}{2\pi} \sum_{k=1}^{K}\sum_{j=1}^{J} \int_{t_{k-1}}^{t_k} \int_{\theta_{j-1}}^{\theta_j} u(b,\theta,s) \left(\sum_{l=1}^{k}\sum_{m=1}^{J} \int_{t_{l-1}}^{t_l} \int_{\theta_{m-1}}^{\theta_m} \frac{\partial u}{\partial \lambda}(b,\phi,\lambda) d\phi H_0(t_k-\lambda) d\lambda \right) d\theta ds \\ = & -\frac{1}{2\pi} \sum_{k=1}^{K}\sum_{j=1}^{J} \frac{h_2\tau}{4} (u_{Ij}^k + u_{I,j-1}^k + u_{Ij}^{k-1} + u_{I,j-1}^{k-1}) \left(\sum_{l=1}^{K}\sum_{m=1}^{J} \frac{h_2}{2} c_{k-l}^{(0)} (\delta_t u_{I,j}^{k-\frac{1}{2}} + \delta_t u_{I,j}^{k-\frac{1}{2}}) \right) \\ = & h_2\tau \sum_{k=1}^{K}\sum_{j=1}^{J} u_{Ij}^{k-\frac{1}{2}} \left(-\frac{h_2}{2\pi} \sum_{l=1}^{k}\sum_{m=1}^{J} c_{k-l}^{(0)} \delta_t u_{I,j}^{k-\frac{1}{2}} \right). \end{aligned} \tag{5.26}$$

Similarly, we have

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{j=1}^{J}\int_{t_{k-1}}^{t_k}\int_{\theta_{j-1}}^{\theta_j} u(b,\theta,s)\\
&\cdot\left(-\frac{1}{\pi}\sum_{n=1}^{N}\sum_{l=1}^{k}\sum_{m=1}^{J}\int_{t_{l-1}}^{t_l}\int_{\theta_{m-1}}^{\theta_m}\frac{\partial u}{\partial\lambda}(b,\phi,\lambda)\cos n(\theta_j-\phi)d\phi H_n(t_k-\lambda)d\lambda\right)d\theta ds\\
&= h_2\tau\sum_{k=1}^{K}\sum_{j=1}^{J}u_{I,j}^{k-\frac{1}{2}}\left(-\frac{1}{\pi}\sum_{n=1}^{N}\sum_{l=1}^{k}\sum_{m=1}^{J}c_{k-l}^{(n)}\beta_{mj}^{(n)}\delta_t u_{I,m}^{k-\frac{1}{2}}\right), \qquad (5.27)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{j=1}^{J}\int_{t_{k-1}}^{t_k}\int_{\theta_{j-1}}^{\theta_j} u(b,\theta,s)\left(-\frac{1}{b\pi}\sum_{n=1}^{N}n\sum_{m=1}^{J}\int_{\theta_{m-1}}^{\theta_m}u(b,\phi,t)\cos n(\theta_j-\phi)d\phi\right)d\theta ds\\
= \;& h_2\tau\sum_{k=1}^{K}\sum_{j=1}^{J}\left(-\frac{1}{b\pi}\sum_{n=1}^{N}n\sum_{m=1}^{J}\beta_{mj}^{(n)}u_{I,m}^{k-\frac{1}{2}}\right). \qquad (5.28)
\end{aligned}
$$

Combing (5.23) and (5.26)-(5.28), we obtain

$$
\sum_{k=1}^{K}\sum_{j=1}^{J-1}u_{I,j}^{k-\frac{1}{2}}\cdot B_N(\delta_t u_{I,j}^{k-\frac{1}{2}})\leq 0.
$$

Hence the proof is complete. $\square$

In the following theorem we establish the stability of our finite difference scheme.

Theorem 5.1. *Let* $\left\{u_{ij}^{k-\frac{1}{2}}|0\leq i\leq I, 0\leq j\leq J, k\geq 0\right\}$ *be the solution of the finite difference scheme (5.9a)-(5.9d). Then we have*

$$
\tau\sum_{k=1}^{K}\|\nabla_h u^{k-\frac{1}{2}}\|^2+\frac{t_K^{-\alpha}\tau}{4\Gamma(1-\alpha)}\sum_{k=1}^{K}\|u^{k-\frac{1}{2}}\|^2\leq\frac{t_K^{1-\alpha}}{4\Gamma(2-\alpha)}\|\varphi(x)\|^2+\Gamma(1-\alpha)t_K^{\alpha}\tau\sum_{k=1}^{K}\|f^{k-\frac{1}{2}}\|^2. \qquad (5.29)
$$

Proof: Combining (5.44)-(5.21), we get

$$
\begin{aligned}
&h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}r_i(\delta_r^2u_{ij}^{k-\frac{1}{2}}+\frac{1}{r_i}\delta_r u_{ij}^{k-\frac{1}{2}})u_{ij}^{k-\frac{1}{2}}\\
\leq\;& -h_1h_2\tau\sum_{k=1}^{K}\sum_{i=2}^{I}\sum_{j=1}^{J-1}r_i(\delta_r u_{i-\frac{1}{2},j}^{k-\frac{1}{2}})^2=-\tau\sum_{k=1}^{K}\|\delta_r u^{k-\frac{1}{2}}\|^2. \qquad (5.30)
\end{aligned}
$$

For the term

$$\begin{aligned}
& h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1} r_i u_{ij}^{k-\frac{1}{2}} f_{ij}^{k-\frac{1}{2}} \\
\leq\ & h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1} r_i\Big[\frac{t_K^{-\alpha}}{4\Gamma(1-\alpha)}(u_{ij}^{k-\frac{1}{2}})^2+\Gamma(1-\alpha)t_K^{\alpha}(f_{ij}^{k-\frac{1}{2}})^2\Big] \\
=\ & \frac{t_K^{-\alpha}}{4\Gamma(1-\alpha)}\tau\sum_{k=1}^{K}\|u^{k-\frac{1}{2}}\|^2+\Gamma(1-\alpha)t_N^{-\alpha}\tau\sum_{k=1}^{K}\|f^{k-\frac{1}{2}}\|^2. \qquad (5.31)
\end{aligned}$$

Combining (5.16)-(5.43) with (5.30)-(5.31), we find

$$\tau\sum_{k=1}^{K}\left(\|\delta_r u^{k-\frac{1}{2}}\|^2+\|\delta_\theta u^{k-\frac{1}{2}}\|^2\right)+\frac{t_K^{-\alpha}}{4\Gamma(1-\alpha)}\tau\sum_{k=1}^{K}\|u^{k-\frac{1}{2}}\|^2 \qquad (5.32)$$

$$\leq \frac{t_K^{1-\alpha}}{4\Gamma(2-\alpha)}\|u^0\|^2+\Gamma(1-\alpha)t_K^{\alpha}\tau\sum_{k=1}^{K}\|f^{k-\frac{1}{2}}\|^2. \qquad (5.33)$$

The the proof is completed. □

Since the difference scheme (5.9a)-(5.9d) is a system of linear algebraic equation at each time level, it is easy to obtain the following result.

Lemma 5.5. *The difference scheme (5.9a)-(5.9d) has a unique solution.*

5.2. Stability of the difference scheme for $1<\alpha<2$

For the time discretization, we use

$$\begin{aligned}
D_\tau^\alpha f(t_k) &= \frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\Big[b_0\delta_t f^{k-\frac{1}{2}}-\sum_{m=1}^{k-1}(b_{k-m-1}-b_{k-m})\delta_t f^{m-\frac{1}{2}}-b_{k-1}q\Big], \\
&:= \tilde{D}_\tau^\alpha \delta_t f^{k-\frac{1}{2}},
\end{aligned} \qquad (5.34)$$

where

$$b_m=\frac{\tau^{2-\alpha}}{2-\alpha}\Big[(m+1)^{2-\alpha}-m^{2-\alpha}\Big],\ q=\frac{\partial f}{\partial t}\Big|_{t=t_0}. \qquad (5.35)$$

to approximate ${}^cD_t^\alpha f(t_k)$.

In order to derive the accuracy of this time discretization, we recall the following lemma.

Lemma 5.6 ([57])**.** *Suppose that* $f\in C^3[0,t_k]$*. For* $1<\alpha<2$*, let*

$$\overline{R}(f(t_k)) :={}^cD_t^\alpha f(t_k)-D_\tau^\alpha f(t_k),$$

then

$$|\overline{R}(f(t_k))|\leq\frac{1}{\Gamma(3-\alpha)}\Big[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-(1+2^{1-\alpha})\Big]\|f^{(3)}\|_\infty\tau^{3-\alpha}, \qquad (5.36)$$

This lemma shows that the accuracy of $\tilde{D}_\tau^\alpha$ for ${}^cD_t^\alpha$ is of $O(\tau^{3-\alpha})$, provided we have $f \in C^3[0, t_k]$ and $1 < \alpha < 2$.

We use the following difference scheme to approximate (4.25)-(4.29):

$$\tilde{D}_\tau^\alpha \delta_t u_{ij}^{k-\frac{1}{2}} = (\delta_r^2 + \delta_\theta^2) u_{ij}^{k-\frac{1}{2}} + f_{ij}^{k-\frac{1}{2}}, \quad (r_i, \theta_j, t_k) \in \Omega_h \times \Omega_t, \tag{5.37a}$$

$$u_{ij}^0 = \varphi_{ij}, \quad 0 \le i \le I,\ 0 \le j \le J, \tag{5.37b}$$

$$\delta_t u_{ij}^{\frac{1}{2}} = \varrho_{ij}, \quad 0 \le i \le I,\ 0 \le j \le J, \tag{5.37c}$$

$$u_{i,J}^{k-\frac{1}{2}} = u_{i,0}^{k-\frac{1}{2}}, \quad 0 \le i \le I,\ 1 \le k \le K, \tag{5.37d}$$

$$\delta_r u_{I-\frac{1}{2},j}^{k-\frac{1}{2}} = B_N(u_{Ij}^{k-\frac{1}{2}}, \delta_t u_{Ij}^{k-\frac{1}{2}}), \quad 1 \le j \le J-1,\ 1 \le k \le K. \tag{5.37e}$$

In order to prove the stability of this difference scheme, we shall require the following lemma.

Lemma 5.7 ([57]). *For any grid function u the following inequality holds:*

$$\tau \sum_{l=1}^{k} (\tilde{D}_\tau^\alpha u^l, u^l) \ge \frac{t_k^{1-\alpha}}{2\Gamma(2-\alpha)} \tau \sum_{l=1}^{k} \|u^l\|^2 - \frac{t_k^{2-\alpha}}{2\Gamma(3-\alpha)} \|u^0\|^2. \tag{5.38}$$

In analogy to Lemma 5.4 we have the inequalities described in the following lemma.

Lemma 5.8.

$$\sum_{k=1}^{K} \sum_{j=1}^{J-1} B_N(u_{Ij}^{k-\frac{1}{2}}, \delta_t u_{Ij}^{k-\frac{1}{2}}) \big(\delta_t u_{Ij}^{k-\frac{1}{2}}\big) \le 0, \tag{5.39}$$

Using Lemma 5.7, Lemma 5.8, we can prove the following theorem:

Theorem 5.2. *Let $\left\{u_{ij}^k | 0 \le i \le I, 0 \le j \le J, k \ge 0\right\}$ be the solution of the finite difference scheme (5.37a)-(5.37e). Then we have*

$$\|\nabla_h u^K\|^2 + \frac{t_K^{1-\alpha}}{2\Gamma(2-\alpha)} \tau \sum_{k=1}^{K} \|\delta_t u^{k-\frac{1}{2}}\|^2 \le \|\nabla_h \phi\|^2 + \frac{t_K^{2-\alpha}}{\Gamma(3-\alpha)} \|\varphi\|^2 + 2\Gamma(2-\alpha) t_K^{\alpha-1} \tau \sum_{k=1}^{K} \|f^{k-\frac{1}{2}}\|^2. \tag{5.40}$$

Proof: Multiplying (5.37a) by $h_1 h_2 \tau r_i \delta_t u_{ij}^{k-\frac{1}{2}}$ and summing up for i from 1 to $I-1$, for j from 1 to $J-1$, and for k from 1 to K, we obtain

$$h_1 h_2 \tau \sum_{k=1}^{K} \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} r_i \Big[\tilde{D}_\tau^\alpha \delta_t u_{ij}^{k-\frac{1}{2}} - (\delta_r^2 u_{ij}^{k-\frac{1}{2}} + \frac{1}{r_i} \delta_r u_{i,j}^{k-\frac{1}{2}} + \frac{1}{r_i^2} \delta_\theta^2 u_{ij}^{k-\frac{1}{2}} + f_{ij}^{k-\frac{1}{2}})\Big] \delta_t u_{ij}^{k-\frac{1}{2}} = 0. \tag{5.41}$$

By lemma 5.7, we obtain

$$
\begin{aligned}
& h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1} r_i\Big[(\tilde{D}_\tau^\alpha \delta_t u_{ij}^{k-\frac{1}{2}})\delta_t u_{ij}^{k-\frac{1}{2}}]\\
\geq\ & \frac{t_K^{1-\alpha}}{2\Gamma(2-\alpha)}\tau\sum_{k=1}^{K}\|\delta_t u^{k-\frac{1}{2}}\|^2-\frac{t_K^{2-\alpha}}{2\Gamma(3-\alpha)}\|\delta_t u^{\frac{1}{2}}\|^2\\
=\ & \frac{t_K^{1-\alpha}}{2\Gamma(2-\alpha)}\tau\sum_{k=1}^{K}\|\delta_t u^{k-\frac{1}{2}}\|^2-\frac{t_K^{2-\alpha}}{2\Gamma(3-\alpha)}\|\varrho\|^2. \qquad (5.42)
\end{aligned}
$$

Using the boundary condition (5.9c), $u_{i,J}^{k-\frac{1}{2}}=u_{i,0}^{k-\frac{1}{2}}$, we find

$$
\begin{aligned}
& h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1} r_i(\frac{1}{r_i^2}\delta_\theta^2 u_{ij}^{k-\frac{1}{2}})\delta_t u_{ij}^{k-\frac{1}{2}}\\
& = -h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J}\frac{1}{r_i}(\delta_\theta u_{i,j-\frac{1}{2}}^{k-\frac{1}{2}})\delta_t(\delta_\theta u_{i,j-\frac{1}{2}}^{k-\frac{1}{2}}) = -\frac{1}{2}\big(\|\delta_\theta u^K\|^2-\|\delta_\theta\varphi\|^2\big). \qquad (5.43)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
& h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1} r_i(\delta_r^2 u_{ij}^{k-\frac{1}{2}}+\frac{1}{r_i}\delta_r u_{ij}^{k-\frac{1}{2}})\delta_t u_{ij}^{k-\frac{1}{2}}\\
=\ & -\frac{1}{2}\left(\|\delta_r u^K\|^2-\|\delta_r\varphi\|^2\right)+r_{I-1}h_2\tau\sum_{k=1}^{K}\sum_{j=1}^{J-1}\Big((\delta_r u_{I-\frac{1}{2},j}^{k-\frac{1}{2}})\delta_t u_{I,j}^{k-\frac{1}{2}}\Big). \qquad (5.44)
\end{aligned}
$$

Using the boundary condition (5.37e) and Lemma 5.8, we can see that

$$
\sum_{k=1}^{K}\sum_{j=1}^{J-1}(\delta_x u_{I-\frac{1}{2},j}^{k-\frac{1}{2}})\delta_t u_{I,j}^{k-\frac{1}{2}} = \sum_{k=1}^{K}\sum_{j=1}^{J-1} B_N\left(u_{I,j}^{j-\frac{1}{2}},\delta_t u_{I,j}^{k-\frac{1}{2}}\right)\cdot\delta_t u_{I,j}^{k-\frac{1}{2}}\leq 0. \qquad (5.45)
$$

In addition,

$$
\begin{aligned}
& h_1h_2\tau\sum_{k=1}^{K}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left[(\delta_t u_{ij}^{k-\frac{1}{2}})f_{ij}^{k-\frac{1}{2}}\right]\\
& \leq h_1h_2\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\tau\sum_{k=1}^{K}\Big[\frac{1}{\Gamma(2-\alpha)}\frac{1}{4}t_K^{1-\alpha}(\delta_t u_{ij}^{k-\frac{1}{2}})^2+\Gamma(2-\alpha)t_K^{\alpha-1}(f_{ij}^{k-\frac{1}{2}})^2\Big]\\
& = \frac{1}{\Gamma(2-\alpha)}\frac{1}{4}t_K^{1-\alpha}\tau\sum_{k=1}^{K}\|\delta_t u^{k-\frac{1}{2}}\|^2+\Gamma(2-\alpha)t_K^{\alpha-1}\tau\sum_{k=1}^{K}\|f\|^{k-\frac{1}{2}}. \qquad (5.46)
\end{aligned}
$$

Substituting (5.42)-(5.46) into (5.41), we obtain

$$
\begin{aligned}
&\frac{t_K^{1-\alpha}}{2\Gamma(2-\alpha)}\tau\sum_{k=1}^{K}\|\delta_t u^{k-\frac{1}{2}}\|^2-\frac{t_K^{2-\alpha}}{2\Gamma(3-\alpha)}\|\varphi\|^2\\
&\leq -\frac{1}{2}(\|\nabla_h u^K\|^2-\|\nabla_h\phi\|)+\frac{1}{\Gamma(2-\alpha)}\frac{1}{4}t_K^{1-\alpha}\tau\sum_{k=1}^{K}\|\delta_t u^{k-\frac{1}{2}}\|^2\\
&+\Gamma(2-\alpha)t_K^{\alpha-1}\tau\sum_{k=1}^{K}\|f\|^{k-\frac{1}{2}},
\end{aligned}
$$

namely,

$$
\|\nabla_h u^K\|^2+\frac{t_K^{1-\alpha}}{2\Gamma(2-\alpha)}\tau\sum_{k=1}^{K}\|\delta_t u^{k-\frac{1}{2}}\|^2\leq\|\nabla_h\phi\|^2+\frac{t_K^{2-\alpha}}{\Gamma(3-\alpha)}\|\varphi\|^2+2\Gamma(2-\alpha)t_K^{\alpha-1}\tau\sum_{k=1}^{K}\|f^{k-\frac{1}{2}}\|^2.
$$

This completes the proof. □

Theorem 5.3. *The difference scheme (5.37a)-(5.37e) is uniquely solvable.*

6. Numerical Examples

In this section, we we give numerical examples to verify the ABCs and the numerical method. We only give the results in 2D. We use the fast algorithm of Lubich and Schädle [43] to compute our ABCs (2.55) and (2.79). By doing so we can compute a temporal convolution over N_t successive time steps with $O(N_t\log N_t)$ operations and $O(\log N_t)$ memory.

Example 6.1. *We first consider the time-fractional diffusion equation*

$$
\begin{cases}
{}^cD_t^\alpha u(\mathbf{x},t)=\Delta u(\mathbf{x},t)f, & \mathbf{x}\in\mathbb{R}^2, 0<t\leq T,\ 0<\alpha<1,\\
u(\mathbf{x},0)=\phi(\mathbf{x}), & \mathbf{x}\in\mathbb{R}^2,\\
u\to 0, & \|\mathbf{x}\|\to+\infty;
\end{cases}
\tag{6.1}
$$

with $f(\mathbf{x},t)$ and $\phi(\mathbf{x})$ defined by the following cases:

Case 1: $f(\mathbf{x},t)=0$, $\phi(\mathbf{x})=E_\alpha(-|x^2-1.44y^2|^\alpha)$;

Case 2: $f(\mathbf{x},t)=\begin{cases}(x^2+y^2-3)E_\alpha(-t^\alpha), & x^2+y^2\leq 3,\\ 0, & x^2+y^2>3;\end{cases}$

$\phi(\mathbf{x})=E_\alpha(-(x^2+y^2))$.

The computational domain is chosen as $\Omega_i=\{0<r\leq 4, 0<t\leq 4\}$, namely, the artificial boundary is the circle $r=4$. Because the exact solution of this problem is unknown, we compute a *reference solution* "$u^\infty(x,t)$" by solving the problem on the domain $\{0<r\leq 15, 0<t\leq 4\}$, using corresponding artificial boundary conditions with $N=20$ on the boundaries. The spatial mesh size is $h_1=h_2=0.001$ and the time step is $\tau=0.00001$.

In Table 1 we show the relative error $||u^\infty - u_h||_1/||u^\infty||_1$ in the Case 1 for $\alpha = 0.5$, with $h = 1/M$ and time step $\tau = 0.001$, on the computational domain at $t = 4$. If we only use one term ($N = 1$) in (2.55), we cannot obtain a satisfactory approximate solution (cf. the first column in Table 1). However, as N increases the approximate solutions become more and more accurate. In particular, for $N = 6$, the numerical solution approximates the exact solution very well and exhibits the optimal convergence rate.

Table 1: Example 1: The relative error for different approximate ABCs in the Case 1.

M	N=1	N=2	N=3	N=4	N=5	N=6
8	4.3213e-1	2.7412e-1	2.6322e-1	2.5743e-1	2.5032e-1	2.4934e-1
16	2.6501e-1	1.4423e-1	1.1232e-1	8.4574e-2	8.2233e-2	8.1215e-2
32	1.4021e-1	.8.1764e-2	4.1333e-2	2.1394e-2	2.0541e-2	2.0408e-2
64	1.2018e-1	4.1323e-2	1.0342e-2	6.0233e-3	5.0104e-3	5.0143e-3
128	9.7652e-2	2.5204e-2	4.4315e-3	1.6432e-3	1.6092e-3	1.2415e-3
256	8.4532e-2	1.4221e-2	1.8063e-3	5.1039e-4	4.0144e-4	3.0135e-4
512	8.4361e-2	9.6553e-3	1.4337e-3	1.9404e-4	1.2944e-4	7.5044e-5

Figure 1 shows the error decay rate on different N for relatively big M in the Case 1. We can see that the error decay rate is of $O(e^{-\gamma(N+1)})$, where $0 < \gamma < 1$.

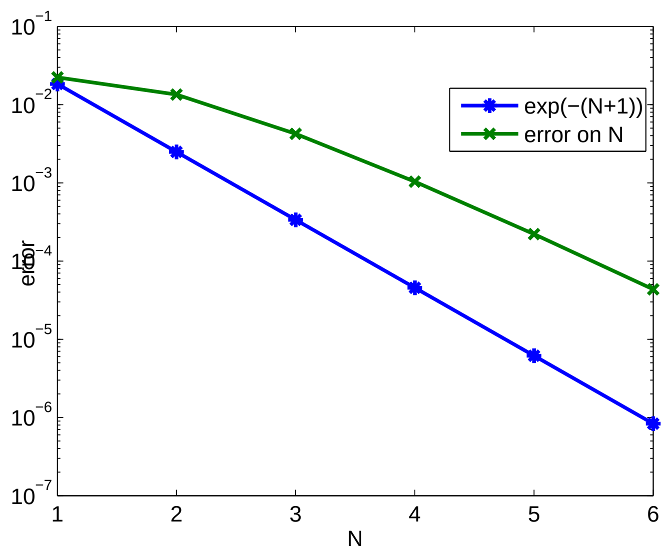

Figure 1: Example 1: error decay rate on different N in the Case 1.

In Figure 2, we show the evolution of the numerical solution in the Case 1 at the x axis, namely $\theta = 0$ in polar coordinates ($0 \le t \le 4$) corresponding to $N = 6$ in (2.55), with spatial mesh size $h_1 = h_2 = 1/512$ and time step $\tau = 0.001$. We observe that there are no reflections at the artificial boundaries compared to the discretizaton errors.

In Figure 3, we give the numerical solution time in the the Case 2 $t = 3$ corresponding to $N = 6$ in (2.55), with spatial mesh size $h_1 = h_2 = 1/512$ and time step $\tau = 0.001$. There are no reflections at the artificial boundaries.

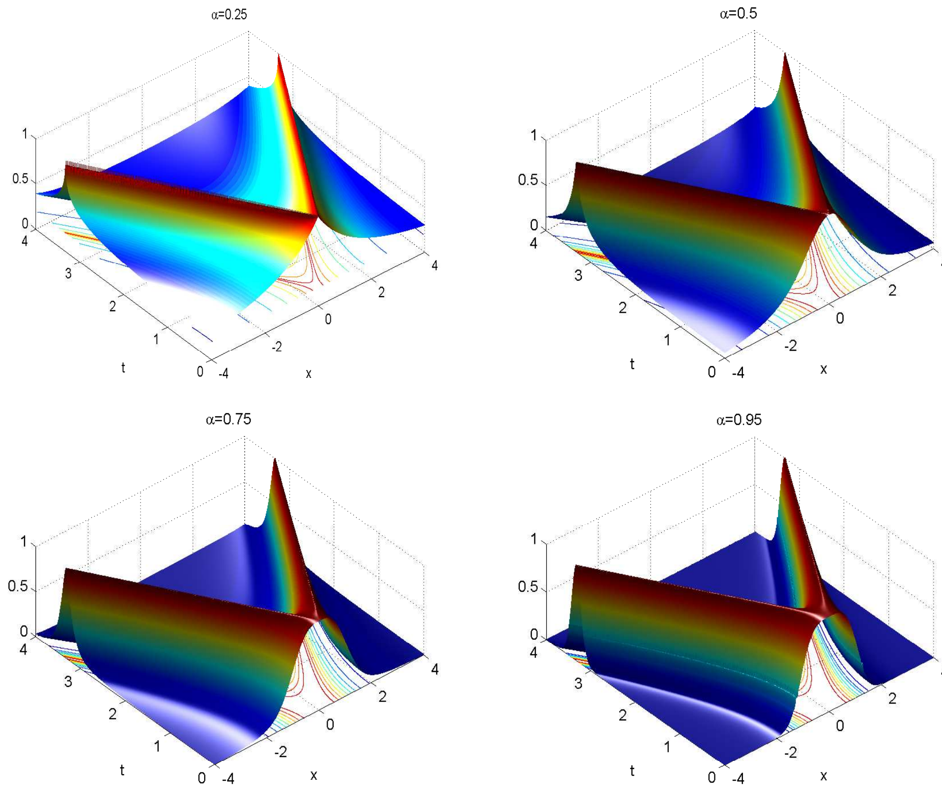

Figure 2: Example 1: The evolution of the solution for different α in the Case 1.

Example 6.2. *Here, we solve the time-fractional wave equation*

$$\begin{cases} {}^cD_t^\alpha u(\mathbf{x},t) = \Delta u(\mathbf{x},t) + f(\mathbf{x},t), & \mathbf{x} \in \mathbb{R}^2, t \in (0,T], \quad 1 < \alpha < 2, \\ u(\mathbf{x},0) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2, \\ u_t(\mathbf{x},0) = \varphi(\mathbf{x}), & x \in \mathbb{R}^{\nvDash} \\ u \to 0, & \|\mathbf{x}\| \to +\infty; \end{cases} \tag{6.2}$$

with the following initial conditions and source terms:

Case 1: $\phi(\mathbf{x}) = E_\alpha(-2|\mathbf{x}|^{2\alpha}), \varphi(\mathbf{x}) = 2|\mathbf{x}|^\alpha E_\alpha(-2|\mathbf{x}|^{2\alpha})$*;* $f(\mathbf{x},t) = 0$*;*

Case 2: $f(\mathbf{x},t) = 0$*,* $\phi(\mathbf{x}) = E_\alpha(-|x^2 - 1.44y^2|^\alpha), \varphi(x) = \alpha E_\alpha(-|x^2 - 1.44y^2|^\alpha)$*;*

Case 3: $f(\mathbf{x},t) = \begin{cases} (x^2+y^2-3)E_\alpha(-t^\alpha), & x^2+y^2 \le 3, \\ 0, & x^2+y^2 > 3; \end{cases},$

$\phi(\mathbf{x}) = E_\alpha(-r^2)$*,* $\varphi(\mathbf{x}) = (x+y)E_\alpha(r^2)$*;*

The computational domain is chosen as $\Omega_i = \{0 < r \le 4, 0 < t \le 4\}$, namely, the artificial boundary is the circle $r = 4$. Because the exact solutions of these problems are unknown, we compute a *reference solution* "$u^\infty(x,t)$" by solving these problems on the

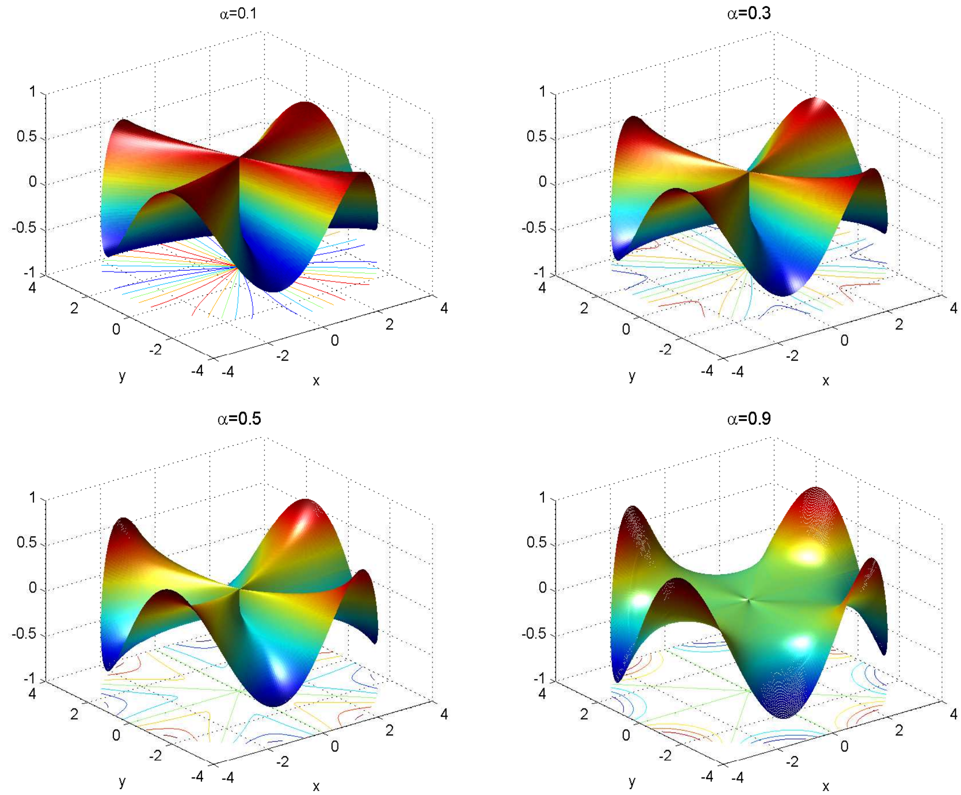

Figure 3: Example 1: The evolution of the solution for different α in the Case 2.

domain $\{0 < r \leq 20\} \times [0, 4]$ with corresponding ABCs with $N = 20$ on the boundaries, and spatial mesh size $h_1 = h_2 = 0.001$ and the time step $\tau = 0.00001$.

In the Table 2 we present the l^1 error of our numerical solution u_h in the Case 1 compared to the reference solution "u^∞" with $N = 6$ for different $1 < \alpha < 2$. In the practical computation, we fix the time step size $\tau = 0.001$ and let $h_1 = h_2 = 1/M$ decay. For $N = 6$, the approximate ABC is accurate enough so that it has no effect on the convergence rate of the numerical discretization in the space variables. Similar numerical results have been found for the Case 2 and Case 3.

Table 2: Example 2: The relative error for different α with $N = 6$in the Case 1.

M	$\alpha = 1.1$	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.9$	$\alpha = 1.99$
8	2.7032e-1	2.7122e-1	2.7132e-1	2.7103e-1	2.7032e-1	2.7048e-1
16	9.8154e-2	9.8024e-2	9.8031e-2	9.8101e-2	9.8003e-2	9.7959e-2
32	2.6311e-2	2.6204e-2	2.6102e-2	2.6093e-2	2.5972e-2	2.5908e-2
64	6.5602e-3	6.5401e-3	6.5200e-3	6.5024e-3	6.4931e-3	6.4904e-3
128	1.6401e-3	1.6350e-3	1.6300e-3	1.6231e-3	1.6230e-3	1.6226e-3
256	4.1001e-4	4.0852e-4	4.0750e-5	4.0575e-4	4.0575e-4	4.0565e-4

In the following figures, the numerical solutions by our method for the different cases is presented. There are no reflections at the artificial boundaries with respect to the discretization errors, which means that our ABCs are very accurate. When α is close to 1, the behavior of the solution is more like diffusion, while for α is close to 2 it is more wave-like.

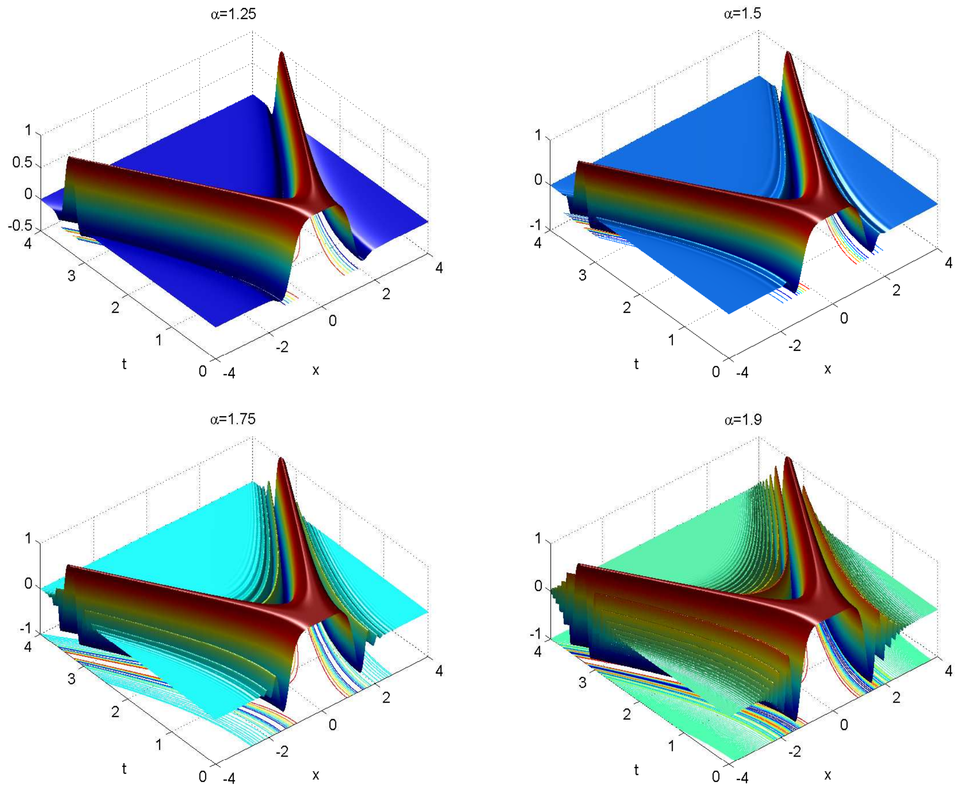

Figure 4: Example 2: The evolution of the solution for different α in the Case 1.

In realistic simulations, the errors of the numerical solutions may depend on the location of the artificial boundaries. To obtain insight into the relation between the errors and the location of the artificial boundaries, we selected different artificial boundaries, namely $r = 3, 4, 5, 6, 7$ with $N = 6$ in the case 1. The l^1-error of the numerical solution at $(x, y) = (2, 2)$ is shown in Table 3. We conclude that the location of the artificial boundaries appears to have little effect on the accuracy of our ABCs.

Table 3: Example 2: The error for different artificial boundaries in the Case 1.

M	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$
8	1.6128e-1	1.4573e-1	1.3416e-1	1.3152e-1	1.3149e-1
16	4.5032e-2	4.2621e-2	4.0831e-2	3.8628e-2	3.8593e-2
32	1.1244e-2	1.0605e-3	1.0206e-3	9.6534e-3	9.6525e-3
64	2.5314e-3	2.5051e-3	2.5052e-3	2.4131e-3	2.4134e-3
128	6.3218e-4	6.2631e-4	6.2513e-4	6.0032e-4	6.0032e-4
256	1.5805e-4	1.5607e-4	1.5628e-5	1.5008e-4	1.5008e-4

There are two ways to improve the accuracy of the ABCs: one is to extend the computational domain, while another is to increase N (the number of terms in the truncated

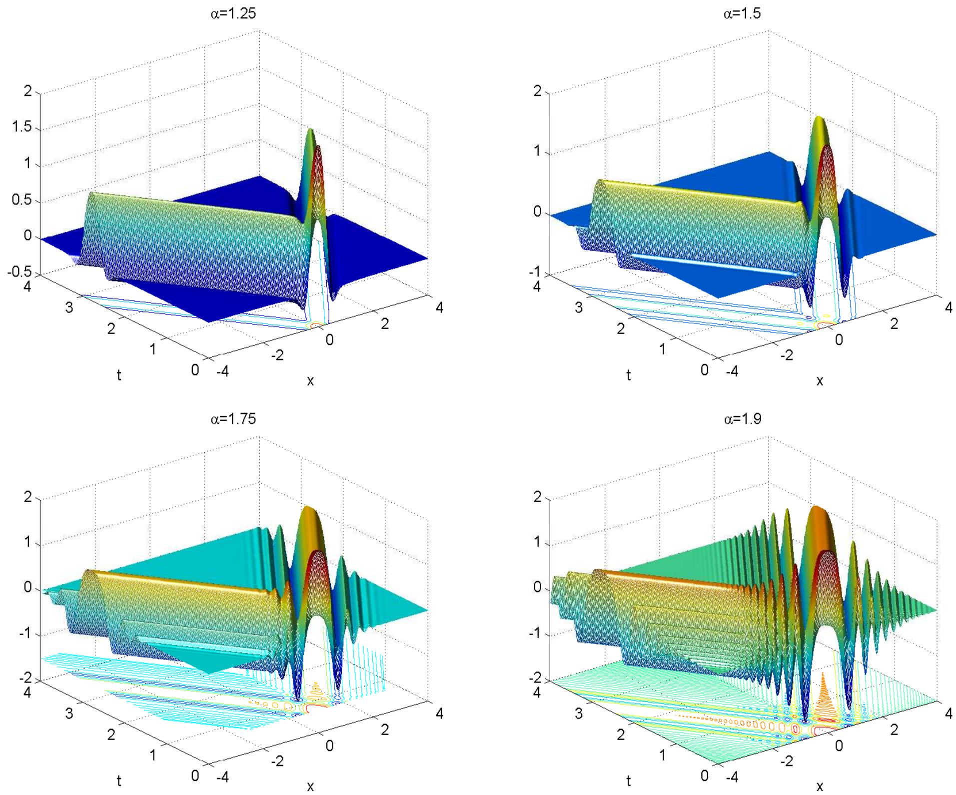

Figure 5: Example 2: The evolution of the solution for different α in the Case 2.

ABCs). For the same numerical tolerance, we compare the computational time of the two approaches. The spatial mesh size is chosen to be $h_1 = h_2 = 1/256$ and $t = 3$ in the Case 1. From the numerical results in Table 4, we conclude that in order to reduce the computational cost one can choose a smaller computational domain and larger N. This is an advantage of our highly accurate ABCs.

Table 4: Computational time for different r and N in the Case 1

tolerance	r	N	CPU time	r	N	CPU time	r	N	CPU time
1.0e-3	3	4	11s	4	3	18s	5	2	45s
1.0e-4	3	6	18s	4	5	36s	5	3	98s
1.0e-5	3	7	39s	4	6	82s	5	4	245s

7. Conclusion

The numerical simulation of the time-fractional diffusion-wave equation on unbounded spatial domains in $\mathbb{R}^2$ has been studied. By introducing an artificial boundary and the deriving the exact and a family of approximate artificial boundary conditions, the original problem is reduced to an equivalent problem or a sequence of approximate problems on the

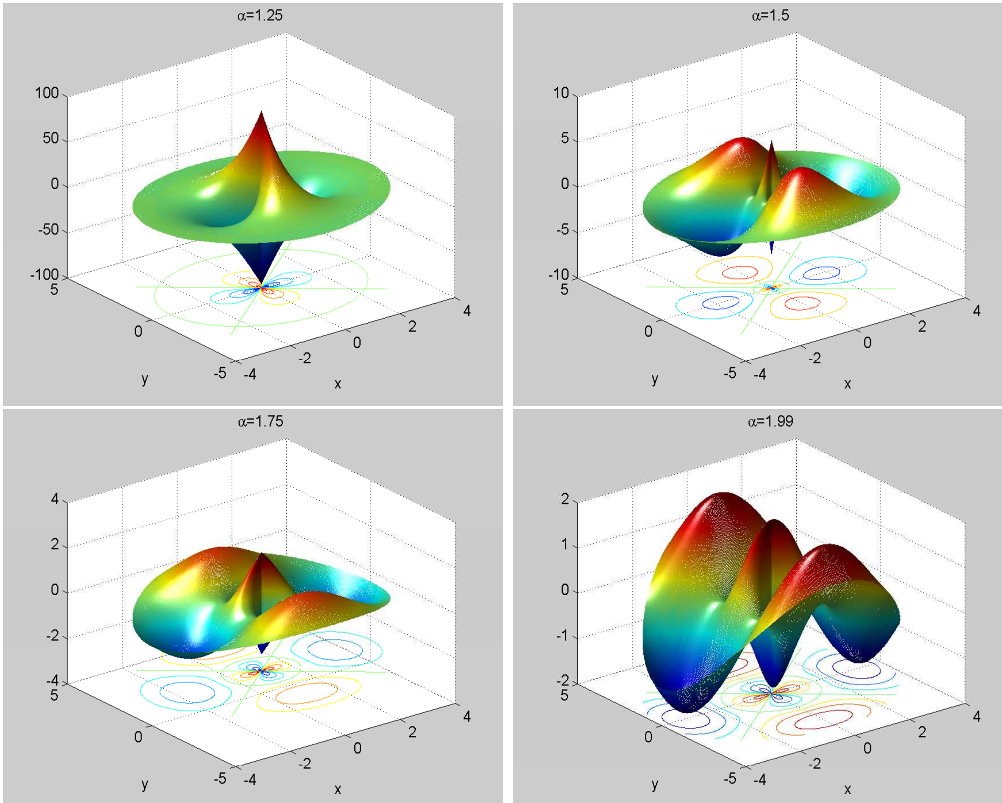

Figure 6: Example 2: The solution for different α in the Case 3 at $t = 3$.

bounded computational domain. The good performance of the numerical examples shows that the given method is feasible and effective.

In future work we will extend our method to time-fractional Schrödinger equations in $\mathbb{R}^n$ ($n = 2, 3$).

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References

[1] K. Al-Khaled and S. Momani, An approximate solution for a fractional diffusion-wave equation using the decomposition method, Appl. Math. Comput. 165 (2005) 473–483.

[2] L. C. Andrews, *Special functions of mathematics for engineers.* second Edition, McGraw-Hill Inc. 1992.

[3] H. Brunner, H. Han and D. Yin, Artiticial boundary conditions and finite difference approximations for a time-fractional diffusion-wave equation on a two-dimensional unbounded spatial domain, J. Comput. Phys. 276 (2014) 541–562.

[4] H. Brunner, L. Ling and M. Yamamoto, Numerical simulations of 2D fractional subdiffusion problems, J. Comp. Phys. 229 (2010) 6613–6622.

[5] M. Caputo, Linear models of dissipation whose Q is almost frequency independent. I, Geophys. J. R. Astr. Soc. 13 (1967) 529–539.

[6] M. Caputo, A model for the fatigue in elastic materials with frequency independent Q, J. Acoust. Soc. Am. 66 (1979) 176–179.

[7] C. Chen, F. Liu, I. Turner and V. Anh, Numerical schemes and multivariate extrapolation of a two-dimensional anomalous sub-diffusion equation, Numer. Algorithms. 54 (2010) 1–12.

[8] J. Chen, F. Liu, V. Anh, S. Shen, Q. Liu and C. Liao, The analytical solution and numerical solution of the fractional diffusion-wave equation with damping, Appl. Math. Comp. 219, (2012), 1737–1748.

[9] E. Cuesta, C. Lubich and C. Palencia, Convolution quadrature time discretization of fractional diffusion-wave equations, Math. Comp. 75 (2006) 673–696.

[10] J. R. Dea, Absorbing boundary conditions for the fractional wave equation, Appl. Math. Comput. 219 (2013) 9810–9820.

[11] R. Du, W.R. Cao and Z.Z. Sun, A compact difference scheme for the fractional diffusion-wave equation, Appl. Math. Modelling 34 (2010) 2998–3007.

[12] B. Engquist and A. Majda, Absorbing boundary conditions for the numerical simulation of waves, Math. Comp. 31 (629–651) 1977.

[13] K. Feng, Asymptotic radiation conditions for reduced wave equations, J. Comp. Math. 2 (1984) 130–138.

[14] G. Gao and Z. Sun, The finite difference approximation for a class of fractional subdiffusion equations on a space unbounded domain, J. Comput. Phys. 236 (2013) 443–460.

[15] G. Gao, Z. Sun and Y. Zhang, A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions, J. Comput. Phys. 231 (2012) 2865–2879.

[16] M. Giona and H. E. Roman, Fractional diffusion equation for transport equation in random media, Physica A 211 (1992) 13–24.

[17] R. Gorenflo, Y. Luchko and F. Mainardi, Wright function as scale-invariant solutions of the diffusion-wave equation, J. Comput. Appl. Math. 118 (2000) 175–191.

[18] R. Gorenflo, F. Mainardi, D. Moretti and P. Paradisi, Time fractional diffusion: a discrete random walk approach, Nonlinear Dynam. 29 (2002) 129–143.

[19] I. S. Graashteyn and I. M. Ryzhik, *Table of integrals, series and products,* sixth Edition. Academic Press.

[20] R. Ghaffari and S. Hosseini, Obtaining artificial boundary conditions for fractional sub-diffusion equation on space two-dimensional unbounded domains, Comput. & Math. Appl. 68 (2014) 13–26.

[21] H. Han and Z. Huang, Exact and approximating boundary conditions for the parabolic problems on unbounded domains. Comput. Math. Appl. 44(2002) 655–666.

[22] H. Han and X. Wu, Approximation of infinite boundary conditions and its applications to finite element methods, J. Comp. Math. 3 (1985) 179–192.

[23] H. Han and X. Wu, The mixed finite element method for Stokes equations on unbounded domains, J. Systems Sci. Math. Sci. 5 (1985) 121–132.

[24] H. Han and X. Wu, Artificial Boundary Method, Tsinghua University Press, Beijing and Springer, Heidelberg-New York-Dordrecht-London, 2013.

[25] H. Han and D. Yin, Numerical solutions of parabolic problems on unbounded 3-D spatial domain, J. Comp. Math. 23 (2005) 449–462.

[26] H. Han and C. Zheng, Exact nonreflection boundary conditions for acoustic problems in three dimensions, J. Comp. Math. 21 (2003) 15–24.

[27] H. Han and C. Zheng, Exact nonreflection boundary conditions for exterior problems of the hyperbolic equation, Chinese J. Comput. Phys. 22(2005) 95–107.

[28] R. Herrmann, Fractional Calculus: An Introduction for Physicists, World Scientific, Hackensack, NJ, 2011.

[29] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[30] F. Huang and F. Liu, The time fractional diffusion and advection-dispersion equation, ANZIAM J. 46 (2005) 1–14.

[31] H. Jiang, F. Liu, I. Turner and K. Burrage, Analytical solutions for the generalized multi-term time-fractional diffusion-wave/diffusion equation in a finite domain, Comput. Math. Appl. 64 (2012), 3377–3388.

[32] H. Jiang, F. Liu, M.M. Meerschaert and R. McGough, Fundamental solutions for the multi-term modified power law wave equations in a finite domain, Electronic Journal of Mathematical Analysis and Applications, 1(1) (2013), 55–66.

[33] J. Kemppainen and K. Ruotsalainen, Boundary integral solution of the time-fractional diffusion equation, Integr. Equ. Oper. Theory. 64 (2009) 239–249.

[34] A. Kilbas, M. Saigo and K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators, Integral Transforms Spec. Funct., 15(1) (2004), 31–49.

[35] A. Kreis and A. Pipkin, Viscoelastic pulse propagation and stable probability distributions, Quart. Appl. Math. 44 (1986) 353–360

[36] T.A.M. Langlands and B.I. Henry, The accuracy and stability of an implicit solution method for the fractional diffusion equation, J. Comput. Phys. 205 (2005) 719–736.

[37] X. Li and C. Xu, A space-time spectral method for the time fractional diffusion equation, SIAM J. Numer. Anal. 47 (2009) 2108–2131.

[38] Y. Lin and C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, J. Comput. Phys. 225 (2007) 1533–1552.

[39] F. Liu, M.M. Meerschaert, R. McGough, P. Zhuang and Q. Liu, Numerical methods for solving the multi-term time fractional wave equations, Fract. Calc. Appl. Anal. 16(1) (2013), 9–25.

[40] F. Liu, S. Shen, V. Anh and I. Turner, Analysis of a discrete non-Markovian random walk approximation for the time fractional diffusion equation, ANZIAM J. 46 (E) (2005) 488–504.

[41] Q. Liu, F. Liu, I. Turner and V. Anh, Finite element approximation for the modified anomalous subdiffusion process, Appl. Math. Modelling, 35(8), (2011), 4103–4116.

[42] Q. Liu, Y. Gu, P. Zhuang, F. Liu and Y. Nie, An implicit RBF meshless approach for time fractional diffusion equations, Comput. Mech., 48, (2011), 1–12.

[43] C. Lubich and A. Schädle, Fast convolution for nonreflecting boundary conditions, SIAM J. Sci. Comput. 24 (2002) 161–182.

[44] R. L. Magin, Fractional Calculus in Bioengineering. Begell House Publishers Inc., Connecticut, 2006.

[45] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, Imperial College Press, London, 2010.

[46] F. Mainardi and M. Tomirotti, Seismic pulse propagation with constant Q and stable probability distributions, Annali Geofis. 40 (1997) 1311–1328.

[47] B. B. Mandelbrot, The Fractal Geometry of Nature, Freeman, San Fransicon, 1983.

[48] R. Metzler and J. Klafter, The random walks guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.

[49] R. Metzler and J. Klafter, The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A: Math. Gen. 37 (2004) R161–R208.

[50] R. Nigmatullin, The realization of the generalized transfer in a medium with fractal geometry, Physica Status Solidi B, 133 (1986) 425–430.

[51] Z. Odibat and S. Momani, Approximate solutions for boundary-value problems of time-fractional wave equation, Appl. Math. Comput. 181 (2006) 767–774.

[52] A. Pipkin, Lectures on Viscoelasticity Theory, 2nd ed., Springer-Verlag, New York, 1986.

[53] A. Pipkin, Asymptotic behaviour of viscoelastic waves, Quart. J. Mech. Appl. Math. 41 (1988) 51–64.

[54] Y. Povstenko, Non-central-symmetric solution to time-fractional diffusion-wave equation in a sphere under Dirichlet boundary condition, Fract. Calc. Appl. Anal., 15 (2012) 253–265.

[55] W. R. Schneider and W. Wyss, Fractional diffusion and wave equations, J. Math. Phys. 30 (1989) 34–144.

[56] S. Shen, F. Liu, V. Anh and I. Turner, Detailed analysis of a conservative difference approximation for the time-fractional diffusion equation, J. Appl. Math. Comput. 22 (2006) 1–19.

[57] Z. Z. Sun X. Wu, A fully discrete difference scheme for a diffusion-wave system, Appl. Numer. Math. 56 (2006) 193–209.

[58] B. J. West, M. Bologna and P. Grigolini, Physics of Fractal Operators, Springer, New York, 2003.

[59] W. Wyss, The fractional diffusion equation, J. Math. Phys. 27 (1986) 2782–2785.

[60] H. Ye, F. Liu, I. Turner, V. Anh, and K. Burrage, Series expansion solutions for the multi-term time and space fractional partial differential equations in two and three dimensions, Eur. Phys. J., Special Topics 222 (2013) 1901–1914.

[61] S. B. Yuste, Weighted average finite difference methods for fractional diffusion equations, J. Comput. Phys. 216 (2006) 264–274.

[62] G.M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, Phys. Rep. 371 (2002) 461–580.

[63] P. Zhuang and F. Liu, Implicit difference approximation for the two-dimensional space-time fractional diffusion equation, J. Appl. Math. Comput. 25 (2007) 269–282.

[64] P. Zhuang, F. Liu, V. Anh and I. Turner, New solution and analytical techniques of the implicit numerical methods for the anomalous sub-diffusion equation, SIAM J. Numer. Anal. 46(2) (2008) 1079–1095.

[65] M. Heinen and H.-J. Kull, Numerical calculation of strong-field laser-atom interaction: An approach with perfect reflection-free radiation boundary conditions Laser Physics 20, 581-590 (2010).

[66] H. Han and Z. Huang, Exact artificial boundary conditions for Schrodinger equation in R2 , Comm. Math. Sci. 2 (2004), 79-94.

[67] H. Han, D. Yin and Z. Huang Numerical solutions of Schrodinger equations in R3 , Numer. Meth. Partial Diff. Eqs. 23 (2006), 511-533.