Distinguishing maximally entangled states by one-way local operations and classical communication

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In this paper, we mainly study the local indistinguishability of mutually orthogonal bipartite maximally entangled states. We construct sets of fewer than d orthogonal maximally entangled states which are not distinguished by one-way local operations and classical communication (LOCC) in the Hilbert space of $d \otimes d$. The proof, based on the Fourier transform of an additive group, is very simple but quite effective. Simultaneously, our results give a general unified upper bound for the minimum number of one-way LOCC indistinguishable maximally entangled states. This improves previous results which only showed sets of $N \geqslant d-2$ such states. Finally, our results also show that previous conjectures in Zhang *et al.* [Z.-C. Zhang, Q.-Y. Wen, F. Gao, G.-J. Tian, and T.-Q. Cao, Quant. Info. Proc. **13**, 795 (2014)] are indeed correct.

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I. INTRODUCTION

In quantum information theory, the question of local distinguishability of orthogonal quantum states has received wide attention in recent years [1–12]. Maximally entangled states are of considerable importance in quantum information theory and foundations of quantum mechanics because of their role in quantum communication primitives such as quantum teleportation and superdense coding as well as demonstrating maximal violations of Bell inequalities. Thus, studying the local distinguishability of maximally entangled states is very meaningful and has also attracted much attention [13–26]. As is known, $d+1$ or more maximally entangled states in $d \otimes d$ are not perfectly locally distinguishable [13–15,27]. Therefore, one of the main interesting questions is whether a set of $N \leqslant d$ orthogonal maximally entangled states in $d \otimes d$ can be perfectly distinguished by local operations and classical communication (LOCC) for all $d \geqslant 4$.

To help answer this question, some results have been presented in [22–25]. First, Bandyopadhyay *et al.* [22] gave examples of sets of d or $d-1$ maximally entangled states in $d \otimes d$ for $d = 4,5,6$ that cannot be perfectly distinguished by one-way LOCC. Then, Yu *et al.* [23] constructed four locally indistinguishable maximally entangled states by positive partial transpose (PPT) operations in $4 \otimes 4$. Furthermore, Cosentino [24] used semidefinite programming to prove there is a set of d maximally entangled states that is not perfectly distinguishable by PPT operations in $d \otimes d, d = 2^n$. Recently, we presented the argument that there exist examples of sets of $d-1$ or $d-2$ maximally entangled states in $d \otimes d$ for $d = 7,8,9,10$ that are not perfectly distinguishable by one-way LOCC [25].

All the above results have only given some locally indistinguishable maximally entangled states in special quantum systems. They do not discuss the local indistinguishability in general quantum systems. Naturally, finding the general locally indistinguishable maximally entangled states in $d \otimes d$ is still meaningful and interesting.

In this paper, we focus on constructing the general one-way LOCC indistinguishable orthogonal maximally entangled states in $d \otimes d$. Fortunately, when d is odd, we find there are $\frac{d+5}{2}$ orthogonal maximally entangled states in $d \otimes d$ that cannot be distinguished by one-way LOCC. And when d is even, we construct $\frac{d+4}{2}$ orthogonal maximally entangled states in $d \otimes d$ that are not distinguishable by one-way LOCC. Furthermore, for one-way LOCC indistinguishability of the states, we present a very simple effective proof method which is based on the Fourier transform of an additive group. Last, we present some examples for a smaller number of maximally entangled states that cannot be perfectly distinguished by one-way LOCC. Our results show that the conjectures in [25] are right.

The rest of this paper is organized as follows. In Sec. II, we review some previous results. In Sec. III, the local indistinguishability of maximally entangled states is discussed. And we prove that there are $\frac{d+5}{2}$ or $\frac{d+4}{2}$ one-way LOCC indistinguishable orthogonal maximally entangled states in $d \otimes d$, where d is odd or even, respectively. Finally, in Sec. IV, we draw the conclusion.

II. PRELIMINARIES

In this section, we introduce some definitions and results which are used in the following parts.

In $d \otimes d, d^2$ generalized Bell states can be expressed as

$$\left|\Psi_{nm}^{(d)}\right\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{\frac{2\pi i j n}{d}} |j\rangle \otimes |j \oplus_d m\rangle \tag{1}$$

for $n,m = 0,1,\ldots,d-1$, where $j \oplus_d m \equiv (j+m) \mathrm{mod} d$. The standard maximally entangled state $|\Phi^+\rangle$ in $d \otimes d$ is $|\Psi_{00}^{(d)}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \otimes |j\rangle$. It is easy to verify that

$$\left(I \otimes U_{nm}^{(d)}\right)\left|\Psi_{00}^{(d)}\right\rangle = \left|\Psi_{nm}^{(d)}\right\rangle \tag{2}$$

where

$$U_{nm}^{(d)} = \sum_{j=0}^{d-1} e^{\frac{2\pi i j n}{d}} |j \oplus_d m\rangle\langle j|. \tag{3}$$

$U_{nm}^{(d)}$ is a $d \times d$ unitary matrix for $n,m = 0,1,\ldots,d-1$.

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Definition 1. In $d\otimes d$, there are a total of n sets of one-way LOCC indistinguishable maximally entangled states $S_k^{(d)}=\{|\Psi_i^{(d)}\rangle=(I\otimes U_i)|\Psi_{00}^{(d)}\rangle, i=0,\ldots,N-1\}$, $k=1,2,\ldots n$, where U_i is a general $d\times d$ unitary matrix. We define $f(d)=\min\{|S_k^{(d)}|,k=1,2,\ldots n\}$, which is the minimum number of one-way LOCC indistinguishable maximally entangled states in $d\otimes d$.

Lemma 1 [25]. In $d\otimes d, N\leqslant d$ number of pairwise orthogonal maximally entangled states $|\Psi_{n_im_i}^{(d)}\rangle$ (for $i=1,2,\ldots,N$), taken from the set given in Eq. (1), can be perfectly distinguished by one-way LOCC, if and only if there exists at least one state $|\alpha^{(d)}\rangle$ for which the states $U_{n_1m_1}^{(d)}|\alpha^{(d)}\rangle, U_{n_2m_2}^{(d)}|\alpha^{(d)}\rangle,\ldots, U_{n_Nm_N}^{(d)}|\alpha^{(d)}\rangle$ are pairwise orthogonal, where $U_{n_im_i}^{(d)}$'s are given by Eq. (3).

Lemma 2. Letting $F:\mathcal{Z}_d\rightarrow\mathbb{C}$ be a complex valued function of additive group $\mathcal{Z}_d=\{0,1,\ldots,d-1\}$, the Fourier transform of F is $M:\mathcal{Z}_d\rightarrow\mathbb{C}$, where $M(n)=\sum_{j=0}^{d-1}\omega_d^{jn}F(j), 0\leqslant n\leqslant d-1, \omega_d=e^{\frac{2\pi i}{d}}$. Defining the subset of $\mathcal{Z}_d$ is $A=\{n\in\mathcal{Z}_d: M(n)=0\}$, so then we know $F(j)=\frac{1}{d}\sum_{n\in\mathcal{Z}_d\setminus A}\omega_d^{-jn}M(n), 0\leqslant j\leqslant d-1$.

Proof. Using the Fourier inverse transform, we get $F(j)=\frac{1}{d}\sum_{n\in\mathcal{Z}_d}\omega_d^{-jn}M(n)=\frac{1}{d}\sum_{n\in\mathcal{Z}_d\setminus A}\omega_d^{-jn}M(n), \omega_d=e^{\frac{2\pi i}{d}}$, $0\leqslant j\leqslant d-1$. ■

Now we are ready to present our main results as follows.

III. ONE-WAY LOCC INDISTINGUISHABLE ORTHOGONAL MAXIMALLY ENTANGLED STATES

In this section, we construct the orthogonal maximally entangled states in the quantum systems of $d\otimes d$ and prove these states are indistinguishable by one-way LOCC. Then, we present an upper bound for the minimum number of one-way LOCC indistinguishable orthogonal maximally entangled states as follows.

Theorem 1. In $d\otimes d, d\geqslant 2$, (1) when d is odd, $f(d)\leqslant\frac{d+5}{2}$; (2) when d is even, $f(d)\leqslant\frac{d+4}{2}$, where $f(d)$ is the minimum number of one-way LOCC indistinguishable maximally entangled states.

Proof. (1) When d is odd, letting $N=\frac{d+5}{2}$, we consider the set $S^{(d)}=\{|\Psi_{n_im_i}^{(d)}\rangle=(I\otimes U_{n_im_i})|\Psi_{00}^{(d)}\rangle, i=0,\ldots,N-1\}$, where $(n_i,m_i)=(i,0)$ for $i=0,\ldots,N-3$, $(n_{N-2},m_{N-2})=(0,1),(n_{N-1},m_{N-1})=(0,d-1)$. We will prove that $S^{(d)}$ is one-way LOCC indistinguishable. According to Lemma 1, we only need to prove there is not any state $|\alpha^{(d)}\rangle$ for which the states $U_{n_1m_1}|\alpha^{(d)}\rangle, U_{n_2m_2}|\alpha^{(d)}\rangle,\ldots, U_{n_Nm_N}|\alpha^{(d)}\rangle$ are pairwise orthogonal.

Suppose there exists a normalized vector $|\alpha\rangle=\sum_{j=0}^{d-1}\alpha_j|j\rangle$ satisfying the normalization condition $\sum_{j=0}^{d-1}|\alpha_j|^2=1$ such that the states $U_{n_1m_1}|\alpha^{(d)}\rangle, U_{n_2m_2}|\alpha^{(d)}\rangle,\ldots,U_{n_Nm_N}|\alpha^{(d)}\rangle$ are pairwise orthogonal. Therefore, for $i=1,\ldots,\frac{d-1}{2}$, we get some equations as follows:

$$\langle\alpha^{(d)}|U_{n_0m_0}^{\dagger}U_{n_im_i}|\alpha^{(d)}\rangle=\sum_{j=0}^{d-1}\omega_d^{ij}\alpha_j\alpha_j^*=0, \tag{4}$$

$$\langle\alpha^{(d)}|U_{n_im_i}^{\dagger}U_{n_0m_0}|\alpha^{(d)}\rangle=\sum_{j=0}^{d-1}\omega_d^{-ij}\alpha_j\alpha_j^*=0. \tag{5}$$

From Eqs. (4) and (5), we know $\sum_{j=0}^{d-1}\omega_d^{ij}\alpha_j\alpha_j^*=0, i=1,\ldots,d-1$. According to Lemma 2, we have that $\alpha_j\alpha_j^*=a$ is a constant, $j=0,\ldots,d-1$. Then, $\sum_{j=0}^{d-1}\alpha_j\alpha_j^*=da=1$. Thus, $\alpha_j\alpha_j^*=\frac{1}{d}\neq 0, \alpha_j\neq 0, j=0,\ldots,d-1$.

For $i=0,\ldots,\frac{d-1}{2}$, because the states $U_{n_1m_1}|\alpha^{(d)}\rangle$, $U_{n_2m_2}|\alpha^{(d)}\rangle,\ldots,U_{n_Nm_N}|\alpha^{(d)}\rangle$ are pairwise orthogonal, we also get

$$\langle\alpha^{(d)}|U_{n_im_i}^{\dagger}U_{n_{N-2}m_{N-2}}|\alpha^{(d)}\rangle=\sum_{j=0}^{d-1}\omega_d^{-ij}\alpha_j\alpha_{j+1}^*=0, \tag{6}$$

$$\langle\alpha^{(d)}|U_{n_{N-1}m_{N-1}}^{\dagger}U_{n_im_i}|\alpha^{(d)}\rangle=\sum_{j=0}^{d-1}\omega_d^{ij}\alpha_j\alpha_{j+1}^*=0. \tag{7}$$

Thus, for $i=0,\ldots,d-1$, $\sum_{j=0}^{d-1}\omega_d^{ij}\alpha_j\alpha_{j+1}^*=0$. According to Lemma 2, we get $\alpha_j\alpha_{j+1}^*=0, j=0,\ldots,d-1$. This is a contradiction with $\alpha_j\neq 0, j=0,\ldots,d-1$. Therefore, $S^{(d)}$ is not perfectly distinguished by one-way LOCC. Then, we know $f(d)\leqslant|S^{(d)}|=\frac{d+5}{2}$, where d is odd.

(2) When d is even, letting $N=\frac{d+4}{2}$, we construct the set $S'^{(d)}=\{|\Psi_{n_im_i}^{(d)}\rangle=(I\otimes U_{n_im_i})|\Psi_{00}^{(d)}\rangle, i=0,\ldots,N-1\}$, where $(n_i,m_i)=(i,0)$ for $i=0,\ldots,N-2, (n_{N-1},m_{N-1})=(0,\frac{d}{2})$. In the same way, we can prove $S'^{(d)}$ cannot be distinguished by one-way LOCC.

Suppose there exists a normalized vector $|\alpha\rangle=\sum_{j=0}^{d-1}\alpha_j|j\rangle$ satisfying the normalization condition $\sum_{j=0}^{d-1}|\alpha_j|^2=1$ such that the states $U_{n_1m_1}|\alpha^{(d)}\rangle, U_{n_2m_2}|\alpha^{(d)}\rangle,\ldots,U_{n_Nm_N}|\alpha^{(d)}\rangle$ are pairwise orthogonal. Then, for $i=1,\ldots,\frac{d}{2}$, we know

$$\langle\alpha^{(d)}|U_{n_0m_0}^{\dagger}U_{n_im_i}|\alpha^{(d)}\rangle=\sum_{j=0}^{d-1}\omega_d^{ij}\alpha_j\alpha_j^*=0, \tag{8}$$

$$\langle\alpha^{(d)}|U_{n_im_i}^{\dagger}U_{n_0m_0}|\alpha^{(d)}\rangle=\sum_{j=0}^{d-1}\omega_d^{-ij}\alpha_j\alpha_j^*=0. \tag{9}$$

From Eqs. (8) and (9), we know $\sum_{j=0}^{d-1}\omega_d^{ij}\alpha_j\alpha_j^*=0$, $i=1,\ldots,d-1$. According to Lemma 2, we can know $\alpha_j\alpha_j^*=a$ is a constant, $j=0,\ldots,d-1$. Then, $\sum_{j=0}^{d-1}\alpha_j\alpha_j^*=da=1$. Thus, $\alpha_j\alpha_j^*=\frac{1}{d}\neq 0, \alpha_j\neq 0, j=0,\ldots,d-1$.

For $i=0,\ldots,\frac{d}{2}$, we also get

$$\langle\alpha^{(d)}|U_{n_im_i}^{\dagger}U_{n_{N-1}m_{N-1}}|\alpha^{(d)}\rangle=\sum_{j=0}^{d-1}\omega_d^{-ij}\alpha_j\alpha_{j+\frac{d}{2}}^*=0, \tag{10}$$

$$\begin{aligned}\langle\alpha^{(d)}|U_{n_{N-1}m_{N-1}}^{\dagger}U_{n_im_i}|\alpha^{(d)}\rangle&=\sum_{j=0}^{d-1}\omega_d^{ij}\alpha_j\alpha_{j-\frac{d}{2}}^*\\&=\sum_{j=0}^{d-1}\omega_d^{ij}\alpha_j\alpha_{j+\frac{d}{2}}^*=0.\end{aligned} \tag{11}$$

Then, we know $\sum_{j=0}^{d-1}\omega_d^{ij}\alpha_j\alpha_{j+\frac{d}{2}}^*=0$ for $i=0,\ldots,d-1$. According to Lemma 2, we get $\alpha_j\alpha_{j+\frac{d}{2}}^*=0, j=0,\ldots,d-1$. This is a contradiction with $\alpha_j\neq 0, j=0,\ldots,d-1$. Thus, $S'^{(d)}$ is not perfectly distinguished by one-way LOCC. That is to say, $f(d) \leqslant |S'^{(d)}| = \frac{d+4}{2}$, where d is even. This completes the proof. ■

From Theorem 1, we know $f(2) \leqslant \frac{2+4}{2} = 3, f(3) \leqslant \frac{3+5}{2} = 4$. It is known that any two orthogonal maximally entangled states may be perfectly distinguished with one-way LOCC [2], so $f(2) = 3$. From [5], we know that any three mutually orthogonal maximally entangled states can be distinguished with one-way LOCC in $3 \otimes 3$, so $f(3) = 4$. Our results give a general unified upper bound for the minimum number of maximally entangled states that are not perfectly distinguishable by one-way LOCC. However, the upper bound may be not a supremum for some high dimensional systems.

According to Theorem 1, we get $f(5) \leqslant \frac{5+5}{2} = 5$. In fact, there are four one-way LOCC indistinguishable maximally entangled states in $5 \otimes 5$ [22]. Thus, $f(5) \leqslant 4$. In addition, for some higher dimensional systems, we find the upper bound in Theorem 1 is loose and not a supremum. For example, $S^{(10)} = \{|\Psi_{00}^{(10)}\rangle, |\Psi_{10}^{(10)}\rangle, |\Psi_{30}^{(10)}\rangle, |\Psi_{60}^{(10)}\rangle, |\Psi_{05}^{(10)}\rangle, |\Psi_{55}^{(10)}\rangle\}$ is a set of one-way LOCC indistinguishable maximally entangled states in $10 \otimes 10$. For the proof, we give a simple explication as follows.

Applying the same method with Theorem 1, we suppose there exists a normalized vector $|\alpha\rangle = \sum_{j=0}^{9} \alpha_j |j\rangle$ satisfying the normalization condition $\sum_{j=0}^{9} |\alpha_j|^2 = 1$ such that the states $U_{00}|\alpha^{(10)}\rangle, U_{01}|\alpha^{(10)}\rangle, \ldots, U_{55}|\alpha^{(10)}\rangle$ are pairwise orthogonal. Then, we get $\sum_{j=0}^{9} \omega_{10}^{ij} \alpha_j \alpha_j^* = 0, i = 1, \ldots, 9$ and $\sum_{j=0}^{9} \omega_{10}^{ij} \alpha_j \alpha_{j+5}^* = 0, i = 0, \ldots, 9$. According to Lemma 2, we know $\alpha_j \alpha_j^* = a$ is a constant and $\alpha_j \alpha_{j+5}^* = 0, j = 0, \ldots, 9$. Because of $\sum_{j=0}^{9} |\alpha_j|^2 = 1$, we know $\alpha_j \alpha_j^* = a \neq 0, j = 0, \ldots, 9$. This means that there is a contradiction. Therefore, $S^{(10)}$ is not perfectly distinguished by one-way LOCC. Then, we know $f(10) \leqslant |S^{(10)}| = 6 < \frac{10+4}{2} = 7$.

In the same way, we have that $S^{(12)} = \{|\Psi_{00}^{(12)}\rangle, |\Psi_{10}^{(12)}\rangle, |\Psi_{30}^{(12)}\rangle, |\Psi_{40}^{(12)}\rangle, |\Psi_{80}^{(12)}\rangle, |\Psi_{06}^{(12)}\rangle, |\Psi_{66}^{(12)}\rangle\}$ and $S^{(14)} = \{|\Psi_{00}^{(14)}\rangle, |\Psi_{10}^{(14)}\rangle, |\Psi_{30}^{(14)}\rangle, |\Psi_{50}^{(14)}\rangle, |\Psi_{60}^{(14)}\rangle, |\Psi_{07}^{(14)}\rangle, |\Psi_{77}^{(14)}\rangle\}$ are also two sets of one-way LOCC indistinguishable maximally entangled states in $12 \otimes 12$ and $14 \otimes 14$, respectively. Therefore, $f(12) \leqslant |S^{(12)}| = 7 < \frac{12+4}{2} = 8, f(14) \leqslant |S^{(14)}| = 7 < \frac{14+4}{2} = 9$. Therefore, we understand there may be a smaller upper bound for the number of one-way LOCC indistinguishable orthogonal maximally entangled states in some high dimensional systems.

IV. CONCLUSION

In summary, we have extended previous known results about the indistinguishability of orthogonal maximally entangled states by one-way LOCC in $d \otimes d$. These results provide insight into the indistinguishability of maximally entangled states by one-way LOCC. We hope that our results can lead to a better understanding of the nonlocality of maximally entangled states. It is left as an interesting open question whether there is a supremum for the number of indistinguishable maximally entangled states by one-way LOCC.

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