ON THE CONJECTURE $\mathcal{O}$ OF GGI FOR G/P

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Abstract. In this paper, we show that general homogeneous manifolds G/P satisfy Conjecture $\mathcal{O}$ of Galkin, Golyshev and Iritani which 'underlies' Gamma conjectures I and II of them. Our main tools are the quantum Chevalley formula for G/P and a theory on nonnegative matrices including Perron-Frobenius theorem.

1. INTRODUCTION

Let X be a Fano manifold, i.e., a smooth projective variety whose anti-canonical line bundle is ample. The quantum cohomlogy ring $H^\star(X,\mathbb{C})$[1] of X is a certain deformation of the classical cohomology ring $H^*(X,\mathbb{C})$ (§2.4 below). For $\sigma \in H^\star(X,\mathbb{C})$, define the quantum multiplication operator $[\sigma]$ on $H^\star(X,\mathbb{C})$ by $[\sigma](\tau) = \sigma \star \tau$ for $\tau \in H^\star(X,\mathbb{C})$, where $\star$ denotes the quantum product in $H^\star(X,\mathbb{C})$. Let δ_0 be the absolute value of a maximal modulus eigenvalue of the operator $[c_1(X)]$, where $c_1(X)$ denotes the first Chern class of the tangent bundle of X. In [8], Galkin, Golyshev and Iritani say that X satisfies Conjecture $\mathcal{O}$ if

(1) δ_0 is an eigenvalue of $[c_1(X)]$.
(2) The multiplicity of the eigenvalue δ_0 is one.
(3) If δ is an eigenvalue of $[c_1(TX)]$ such that $|\delta| = \delta_0$, then $\delta = \delta_0\xi$ for some r-th root of unity, where r is the Fano index of X.

In fact, in addition to Conjecture $\mathcal{O}$, Galkin, Golyshev and Iritani proposed two more conjectures called Gamma conjectures I, II, which can be stated under the Conjecture $\mathcal{O}$. Let us briefly introduce Gamma conjectures I, II in order to explain how it underlies them. Consider the quantum connection of Dubrovin

$$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}(c_1(X)\star) + \mu,$$

acting on $H^*(X,\mathbb{C}) \otimes \mathbb{C}[z,z^{-1}]$, where μ is the grading operator on $H^*(X)$ defined by $\mu(\tau) = (k - \frac{\dim X}{2})\tau$ for $\tau \in H^{2k}(X,\mathbb{C})$. This has a regular singularity at $z=\infty$ and an irregular singularity at $z=0$. Flat sections near $z=\infty$ can be constructed through flat sections near $z=0$ classified by their exponential growth order, and they

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[1]We use this notation for the quantum cohomology ring with the multiplication $\star$, and without quantum variables.

are put into correspondence with cohomology classes. To be precise, if X satisfies Conjecture $\mathcal{O}$, we can take a flat section $s_0(z)$ with the smallest asymptotics $\sim e^{-\delta_0/z}$ as $z \to +0$ along $\mathbb{R}_{>0}$. We transport $s_0(z)$ to $z = \infty$ and identify the corresponding class A_X called the principal asymptotic class of X. Then Gamma conjecture I states that the cohomolgy class A_X is equal to the Gamma class $\hat{\Gamma}_X$. Here $\hat{\Gamma}_X := \prod_{i=1}^n \Gamma(1+\vartheta_i) \in H^*(X)$, where ϑ_i are the Chern roots of the tangent bundle TX for $i = 1, ..., n$. Under further assumption of semisimplicity of the ring $H^\star(X)$, we can identify cohomology classes A_δ corresponding to each eigenvalue δ in similar way. The classes A_δ form a basis of $H^*(X, \mathbb{C})$. Then Gamma conjecture II, a refinement of a part of Dubrovin's conjecture ([3]), states that there is an exceptional collection $\{E_\delta \mid \delta \text{ eignevalues of } [c_1(X)]\}$ of the derived category $D^b_{\mathrm{coh}}(X)$ such that for each δ,

$$A_\delta = \hat{\Gamma}_X \mathrm{Ch}(E_\delta),$$

where $\mathrm{Ch}(E_\delta) := \sum_{k=0}^{\dim X} (2\pi \mathbf{i})^k \mathrm{ch}_k(E_\delta)$ is the modified Chern character. In this situation, Gamma conjecture I says that the exceptional object $\mathcal{O}_X$ corresponds to δ_0. See [8] and [3] for details on these materials.

As far as we know, the Conjecture $\mathcal{O}$ has thus far been proved for the ordinary, Lagrangian and orthogonal Grassmannians. For the ordinary Grassmannian, Galkin, Golyshev and Iritani ([8]) proved Conjecture $\mathcal{O}$ together with Gamma conjectures I, II by using the quantum Satake of Golyshev and Manivel ([9]). In fact we notice that there were already two earlier papers proving Conjecture $\mathcal{O}$ for the ordinary Grassmannian. In 2006, Galkin and Golyshev ([7]) gave a very short proof of Conjecture $\mathcal{O}$ using a theorem of Seibert and Tian ([23]) and some elementary considerations. In 2003, Rietsch([21]) gave a full description of eigenvalues and corresponding (simultaneous) eigenvectors of quantum multiplication operators for the Grassmannian, which actually proves Conjecture $\mathcal{O}$, by using a result of Peterson and some combinatorics. Very recently, the first author proved the Conjecture $\mathcal{O}$ for Lagrangian and orthogonal Grassmannian ([2]), following Rietsch ([21]).

As for toric Fano manifolds, Galkin, Golyshev and Iritani([8]) proved Gamma conjectures I, II modulo Conjecture $\mathcal{O}$, and then Galkin ([6]) has made some progress on Conjecture $\mathcal{O}$ by showing that the quantum cohomology ring of a toric Fano manifold contains a field as a direct summand.

It is natural to consider general homogeneous spaces $X = G/P$ as next targets for Gamma conjectures. Indeed, here we prove Conjecture $\mathcal{O}$ for homogeneous spaces as a first step into this project. A scheme of proof of Conjecture $\mathcal{O}$ is to use the so-called quantum Chevalley formula which computes the multiplication $\sigma_1 \star \sigma_2$ of two basis elements with σ_1 or σ_2 in $H^2(X, \mathbb{Z})$, and a theory on nonnegative matrices including Perron-Frobenius theorem.

To be precise, first note that the structure constants of the quantum product in $H^\star(X, \mathbb{C})$ in the basis of Schubert classes are three-point genus zero Gromov-Witten invariants. They are actual counts of holomorphic spheres satisfying appropriate conditions, and are therefore nonnegative. Hence the matrix $M(X)$ of $[c_1(X)]$ with respect to this basis is a nonnegative matrix. Therefore once we prove that $M(X)$ is irreducible, then by the celebrated Perron-Frobenius theorem (§3.1 below), the conditions (1) and (2) are automatically satisfied. We remark that the use of Perron-Frobenius theorem in the proof of (1) and (2) is due to Kaoru Ono ([8, Remark 3.17]). However, the Perron-Frobenius theorem does not assert that the Fano index r of X is equal to the number h of eigenvalues of maximal modulus, which is to be shown for the condition (3). Towards this equality, it is already known that r divides h even for general Fano manifolds by [8, Remark 3.1.3]. Then to show that conversely h divides r, we bring a theory on directed graphs, a disguise of nonnegative matrices, into our situation and construct a certain number of cycles at a fixed vertex in the directed graph in question. The lengths of these cycles are used to show that h, in turn, divides r, and hence $r = h$. This fact together with Proposition 3.3 proves the condition (3). Lastly, we point out that one of the advantages of our approach is that if one of the eigenvalues (of unnecessarily maximal modulus) of $[c_1(X)]$ is obtained, then one can recover other eigenvalues of the same modulus from the known eigenvalue by rotating it by a fixed angle depending on the Fano index of X with the aid of Proposition 3.3.

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2. Quantum cohomology of G/P

We review some basic facts here. Our readers can refer to [10, 11] for the details in the first subsection, and can refer to [5] and references therein for the rest.

2.1. **Notations.** Throughout this paper, G denotes a complex, connected, semisimple, algebraic group, B a fixed Borel subgroup, and T a maximal torus in B. As usual, $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{t}$ denote the Lie algebras of G, B and T, respectively. Denote the set of all roots by R. Then we have a decomposition of root spaces $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. Let $\triangle$ be a set of simple roots and I be an indexing set for $\triangle$. The parabolic subgroups P of G containing B correspond to subsets $\triangle_P$ of $\triangle$. Let $I_P \subset I$ be the indexing subset for $\triangle_P$, and $I^P = I \setminus I_P$. Denote the set of positive respectively negative roots relative to B by R^+ respectively R^-. Let R_P^+ be the subset of positive roots which can be written as sums of roots in $\triangle_P$. Then the Lie algebra $\mathfrak{p}$ has a decomposition of root spaces $\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+ \sqcup (-R_P^+)} \mathfrak{g}_\alpha$.

Let W be the Wely group of G, i.e., $W = N_G(T)/T$. Then W is generated by simple reflections s_i, $i \in I$, where s_i is a simple refection corresponding to α_i. For $u \in W$, the length of u, denoted by $l(u)$, is defined to be the mimimum number of simple reflections whose product is u. The Weyl group W acts on R. Furthermore for any

$\gamma \in R^+$, there exist $w \in W$ and $i \in I$ such that $\gamma = w(\alpha_i)$. We then have a reflection $s_\gamma := ws_iw^{-1}$ and the coroot $\gamma^\vee := w(\alpha_i^\vee)$, which are independent of the choices the choices of w, i. There is a unique element of maximal length in W, denoted by w_0. Then the opposite Borel subgroup B^- is written as $B^- = w_0Bw_0$.

Let W_P be the subgroup of W generated by the generators s_i with $i \in I_P$. Note that the generators s_i with $i \in I_P$ are precisely ones such that $s_i \subset P$. Denote by w_P the longest element of W_P. We will write $[u]$ for cosets uW_P in W/W_P. We will write $[u]$ for cosets uW_P in W/W_P. It is well-known that each coset $[u]$ has a unique representative of minimal length. Let W^P be the subset of W consisting of such representatives in the cosets. Let w_0^P be the minimal length representative in $[w_0]$, and hence w_0^P is the longest element in W^P. The length of $[u]$, denoted as $l([u])$, is defined to be the length of the minimal length representative in the coset $[u]$. The dual of $u \in W^P$, denoted $u^\vee$, is defined to be the minimal length representative of the coset $[w_0u]$. Note that $l(w_0^P) = \dim G/P$ and $l(u^\vee) = l(w_0^P) - l(u)$ for $u \in W^P$. Let 0^P be the element of W^P of minimum length. In fact, $0^P = \mathrm{id}$ is the identity of W, and so $l(0^P) = 0$.

2.2. **Cohomology.** For convenience, throughout we will identify the element $u \in W^P$ with the element $[v] \in W/W_P$ if u is the minimal length representative in $[v]$. For $u \in W^P$, let $X(u) = \overline{BuP/P}$ be the Schubert variety corresponding to u and $Y(u) = \overline{B^-uP/P}$ the opposite Schubert variety corresponding to u. Then $X(u)$ is a subvariety of G/P of dimension $l(u)$ and $Y(u)$ is a subvariety of G/P of codimension $l(u)$. Let $\sigma(u)$ respectively σ_u be the cohomology class $[X(u)]$ respectively $[Y(u)]$. Then we have the following classical results on $H^*(X) = H^*(X, \mathbb{Z})$, where $n = \dim_{\mathbb{C}} G/P$.

(1) For $u \in W^P$, $\sigma_u \in H^{2l(u)}(X)$, $\sigma(u) \in H^{2n-2l(u)}(X)$, and $\sigma_u = \sigma(u^\vee)$.
(2) $H^*(X) = \bigoplus_{u\in W^P} \mathbb{Z}\sigma_u = \bigoplus_{u\in W^P} \mathbb{Z}\sigma(u)$. In particular, $H^2(X) = \bigoplus_{i\in I^P} \mathbb{Z}\sigma_{s_i}$, and $H^{2n-2}(X) = \bigoplus_{i\in I^P} \mathbb{Z}\sigma(s_i)$.
(3) $\int_X \sigma_u \cup \sigma_v = 1$ if $v = u^\vee$, and 0 otherwise.

2.3. **Degrees.** Since the cohomology group $H^{2n-2}(X)$ can be canonically identified with the homology group $H_2(X)$ by Poincaré duality, elements of $H^{2n-2}(X)$ may be referred to as curve classes. By a *degree* d, we mean an effective class in $H^{2n-2}(X)$, i.e., a nonnegative integral linear combination of the Schubert generators $\sigma(s_i)$ with $i \in I^P$. A degree $d = \sum_{i\in I^P} d_i\sigma(s_i)$ can be identified with $(d_i)_{i\in I^P}$.

For $\alpha \in R^+$, write $\alpha = \sum_{i\in I} m_{\alpha,\alpha_i}\alpha_i$ for some $m_{\alpha,\alpha_i} \in \mathbb{Z}_{\geq 0}$. Then we define the *degree* of α as

$$d(\alpha) = \sum_{i\in I^P} m_{\alpha,\alpha_i} \frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} \sigma(s_i).$$

Note that $d(\alpha_i) = \sigma(s_i)$ if $i \in I^P$, and $d(\alpha_i) = 0$ otherwise, since $m_{\alpha_i,\alpha_j} = \delta_{i,j}$ for $i, j \in I$. Let $h_\alpha = \frac{2\alpha}{(\alpha,\alpha)}$ and let ω_{α_i} be the fundamental weight corresponding to α_i, so that h_{α_i} and ω_{α_i} are dual bases for $i \in I$. Then $h_\alpha(\omega_{\alpha_i}) = m_{\alpha,\alpha_i}(\alpha_i,\alpha_i)/(\alpha,\alpha)$, and hence we have

$$d(\alpha) = \sum_{i \in I^P} h_\alpha(\omega_{\alpha_i})\sigma(s_i).$$

We notice that if α is a root, then the aforementioned h_α is identified with the coroot $\alpha^\vee$ via the Killing form $(\cdot,\cdot)$.

Set

$$\rho_P = \frac{1}{2}\sum_{\alpha \in R^+ \setminus R_P^+} \alpha, \quad n_\alpha = 4\frac{(\rho_P,\alpha)}{(\alpha,\alpha)}, \quad \text{and} \quad n_i := n_{\alpha_i} \text{ for } i \in I^P.$$

Lemma 2.1 (Lemma 3.5 of [5])**.** *The first Chern class of $X = G/P$ is given by*

$$c_1(X) = \sum_{i \in I^P} n_i \sigma_{s_i} = 2\sum_{i \in I^P} h_{\alpha_i}(\rho_P)\sigma_{s_i}.$$

Remark 2.2. The coefficient n_i is a positive integer for all $i \in I^P$. The positivity of n_i plays an important role in our proof together with the nonnegativity of the structure constants below (§2.4). In the case $P = B$, we have $R_B^+ = \emptyset$ and $\rho_B = \sum_{i \in I} \omega_{\alpha_i}$.

The Chern number $\int_X c_1(X) \cup d(\alpha)$ of the degree $d(\alpha)$ is equal to n_α. In particular, we have $n_i = \int_{[X(s_i)]} c_1(X) = d(\alpha_i)$ for any $i \in I^P$. We notice that the Fano index of $X = G/P$ is given by

$$r := \text{g.c.d}\,\{n_i \mid i \in I^P\}.$$

2.4. **Quantum cohomology of G/P.** To define the quantum cohomology ring of X, we begin with Gromov-Witten invariants. Given $u, v, w \in W^P$ and $d = \sum d_i \sigma(s_i)$ with $l(u) + l(v) + l(w) = \dim X + \int_X c_1(X) \cdot d$, the *three-pointed, genus zero Gromov-Witten invariant* associated with u, v, w and d, denoted $c_{u,v}^{w,d}$, can be defined as the number of morphisms $f : \mathbb{P}^1 \to X$ of degree d such that (fixed) general translates of $Y(u), Y(v)$ and $Y(w)$ pass through the three points $f(0), f(1)$ and $f(\infty)$, respectively.

For each $i \in I^P$, take a variable q_i, and let $\mathbb{Z}[q]$ be the polynomial ring with indeterminates $q_i, i \in I^P$. We will regard $\mathbb{Z}[q]$ as a graded $\mathbb{Z}$-algebra by assigning to q_i the (complex) degree n_i. For a degree $d = \sum d_i \sigma(s_i)$, let q^d stand for $\prod q_i^{d_i}$. The quantum cohomology ring $qH^\star(X)$ of X, as a $\mathbb{Z}[q]$-module, is defined to be

$$qH^\star(X) = H^*(X) \otimes \mathbb{Z}[q].$$

The Schubert classes σ_u with $u \in W^P$ form a $\mathbb{Z}[q]$-basis for $qH^*(X)$. The multiplication is defined as

$$\sigma_u \star \sigma_v = \sum_d \sum_w c_{u,v}^{w^\vee,d} \sigma_w, \tag{2.1}$$

where the sums are taken over all $w \in W^P$ and degrees d such that $l(u) + l(v) = l(w) + \int_X c_1(X) \cdot d$.

The quantum product of two general Schubert classes σ_u and σ_v are far from completely understood. When either of them is in $H^2(X)$, then the so-called quantum Chevalley formula, due to Peterson [20] and proved by Fulton and Woodward [5], gives an explicit description of the coefficients in (2.1).

Proposition 2.3 (Quantum Chevalley formula)**.** *For any $i \in I^P$ and $u \in W^P$, the quantum product of σ_{s_i} and σ_u is given by*

$$\sigma_{s_i} \star \sigma_u = \sum_\alpha h_\alpha(\omega_{\alpha_i})\sigma_v + \sum_\alpha q^{d(\alpha)} h_\alpha(\omega_{\alpha_i})\sigma_w,$$

where the first sum is over roots $\alpha \in R^+ \setminus R_P^+$ for which $v = us_\alpha \in W^P$ satisfies $l(v) = l(u) + 1$, and the second sum is over roots $\alpha \in R^+ \setminus R_P^+$ for which w is the minimal length representative in $[us_\alpha]$ satisfying $l(w) = l(u) + 1 - n_\alpha$.

Remark 2.4. Since h_{α_i} and ω_{α_i} are dual bases for $i \in I$, by the very definition of R_P^+, if $\alpha \in R_P^+$, then $h_\alpha(\omega_{\alpha_i}) = 0$ for all $i \in I^P$, and if $\alpha \in R^+ \setminus R_P^+$, then there is an $i \in I^P$ such that $h_\alpha(\omega_{\alpha_i}) \neq 0$.

Remark 2.5. We can also replace this parameterization set of the first sum by an apparently larger one: *over roots $\alpha \in R^+$ for which the minimal length representative v of $[us_\alpha]$ satisfies $\ell(v) = \ell(u) + 1$.* Indeed, the natural projection $G/B \to G/P$ induces an injective morphism $H^*(G/P) \hookrightarrow H^*(G/B)$ of algebras, sending a Schubert class σ_u^P ($u \in W^P$) in $H^*(G/P)$ to the Schubert class σ_u^B in $H^*(G/B)$ labeled by the same u. The new parameterization set of the first sum gives the Chevalley formula for $H^*(G/B)$. Due to the injective morphism, the coefficient $h_\alpha(\omega_{\alpha_i})$ is nonzero only if $v = us_\alpha$ itself belongs to W^P and $\alpha \notin R_P^+$.

Corollary 2.6. *For any $i \in I^P$, we have $2 \leq n_i \leq \dim X + 1$.*

Proof. It follows from the definition of n_i that $n_i \geq 2$. Since $\ell(w_0^P) = \dim G/P$, the cup product $\sigma_{s_i} \cup \sigma_{w_0^P} = 0$ vanishes. On the other hand, the quantum product of Schubert classes never vanishes (following from [5, Theorem 9.1]). Therefore by the quantum Chevalley formula, the expansion of $\sigma_{s_i} \star \sigma_{w_0^P}$ contains a class $q^{d(\alpha)}\sigma_w$ for some $\alpha \in R^+ \setminus R_P^+$ which has a positive coefficient at the simple root α_i. It follows that $n_i = \deg(q_i) \leq \deg(q^{d(\alpha)}) + \ell(w) = \ell(s_i) + \ell(w_0^P) = 1 + \dim X$. □

The quantum cohomology ring of X can also be defined without using the quantum variables q_i ($i \in I^P$), which is denoted as $H^\star(X, \mathbb{C})$. In our language, the ring $H^\star(X, \mathbb{C})$ is identified with the specialization of $qH^\star(X, \mathbb{C})$ at $q_i = 1$ for all i, i.e.,

$$H^\star(X, \mathbb{C}) = qH^\star(X, \mathbb{C}) / < q_i - 1 \mid i \in I^P > .$$

Note that $H^\star(X, \mathbb{C})$ is a finite dimensional vector space over $\mathbb{C}$, while $qH^\star(X, \mathbb{C})$ is not over $\mathbb{C}$, but over $\mathbb{C}[q]$.

3. Nonnegative matrices

In this section, we review Perron-Frobenius theory on nonnegative matrices and some related results which will be used later. Details on these materials can be found in [18] and [1].

3.1. **Irreducible matrices.**

Definition. A nonnegative matrix M is said to be *cogredient* to a matrix M' if there is a permutation matrix P such that $M = P^T M' P$.

A nonnegative matrix M is called *reducible* if it is cogredient to a matrix in the form

$$M' = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}, \tag{3.2}$$

where A, D are square submatrices. If it is not reducible, then M is called *irreducible.*

Remark 3.1. (1) Note that if V is a vector space with an ordered basis $\mathcal{B} = \{v_1, ..., v_m\}$ and T is an operator on V, then the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$ is reducible if and only if there is a nontrivial proper coordinate subspace invariant under T, equivalently there is an ordered basis $\mathcal{B}'$ with respect to which $[T]_{\mathcal{B}'}$ is in the form (3.2), where $\mathcal{B}'$ is obtained from $\mathcal{B}$ by reordering elements of $\mathcal{B}$.

(2) Suppose $[T]_{\mathcal{B}}$ is reducible. Let V_0 denote a nontrivial proper coordinate subspace of V invariant under T. We point out that if V_0 contains a basis element $v_i \in \mathcal{B}$, then V_0 contains all basis elements $v_j \in \mathcal{B}$ such that the coefficient b_{ji} of v_j is nonzero in $T(v_i) = \sum_{k=1}^m b_{ki} v_k$. More generally, suppose $T = \sum_{i=1}^l c_i T_i$ for some positive numbers c_i and operators T_i with $[T_i]_{\mathcal{B}}$ nonnegative. Then the coefficient b_{ji} is nonzero if and only if there exists a $1 \le p \le l$ such that the coefficient b_{ji}^p of v_j is nonzero in $T_p(v_i) = \sum_{k=1}^m b_{ki}^p v_k$.

The next two propositions are due to Perron [19] and Frobenius [4].

Proposition 3.2 (See e.g. Theorem 1.4 of Chapter 2 of [1])**.** *Let M be an irreduclble matrix. Then M has a real positive eigenvalue δ_0 of multiplicity one such that*

$$\delta_0 \ge |\delta|$$

for any eigenvalue δ of M. Furthermore, M has a positive eigenvector corresponding to δ_0.

Definition. For an irreducible matrix M, we define the *index of imprimitivity* of M, denoted as $h(M)$, to be the number of eigenvalues of maximal modulus. If $h(M) = 1$, then M is said to be *primitive*; otherwise, it is *imprimitive.*

If M is an irreducible matrix, then eigenvalues of the same modulus are completely determined by one of them.

Proposition 3.3 (See e.g. Theorem 2.20 of Chapter 2 of [1])**.** *Let M be an irreducible matrix with $h(M) = h$. Then the eigenvalues of M of modulus δ_0 are all of multiplicity one, given by the distinct roots of $\lambda^h - \delta_0^h = 0$. Moreover, the set of eigenvalues of M is invariant under rotation by $\frac{2\pi}{h}$.*

Definition. A matrix in the form

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{2,3} & \cdots & 0 & 0 \\ \vdots & & & \ddots & 0 & \vdots \\ 0 & 0 & & \cdots & 0 & A_{k-1,k} \\ A_{k1} & 0 & & \cdots & & 0 \end{bmatrix} \tag{3.3}$$

is said to be *in the superdiagonal* $(m_1, m_2, ..., m_k)$-*block form* if the block $A_{i,i+1}$ is a $(m_i \times m_{i+1})$ matrix for $i = 1, ..., k-1$, and $A_{k,1}$ is a $(m_k \times m_1)$ matrix.

If an irreducible matrix M is cogredient to a matrix M' in the form (3.3), much spectral information of M can be read off from M'. Among them, first comes the index of imprimitivity.

Proposition 3.4 ([17]; see e.g. Theorem 4.1 of Chapter 3 of [18])**.** *Let M be an irreducible matrix with $h(M) = h$. Then M is cogredient to a matrix in the form (3.3) such that all the k blocks $A_{1,2}, \cdots, A_{k-1,k}, A_{k,1}$ are nonzero if and only if k divides h.*

3.2. **Directed graphs.** When we deal with spectral properties of nonnegative matrices, mostly we are only interested in the zero pattern of their entries. One of ways of encoding this pattern is through the so-called directed graph. We list a multiple of definitions related with directed graphs.

Definition. (1) A *directed graph D* consists of data (Ver, Arc), where Ver is a set and Arc is a binary relation on Ver, i.e., a subset of Ver $\times$ Ver. Elements of Ver are called *vertices* and elements of Arc are called *arcs*. For convenience, we assume that Ver $= \{v_1, ..., v_m\}$.

(2) A sequence of arcs $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), (v_{i_3}, v_{i_4}), ..., (v_{i_{k-1}}, v_{i_k})$ in D is called a *path from v_{i_1} to v_{i_k}* which we will denote by PATH$(v_{i_1} : v_{i_k})$, or simply PATH(i_1, i_l). The *length* of a path is the number of arcs in the sequence. A path of length k from a vertex to itself is called a *cycle* of length k.

(3) The *adjacency matrix* [2] of D, denoted $A = A(D) = (a_{i,j})$, is an $m \times m$ square matrix with entries 0 or 1 defined by

$$a_{i,j} = \begin{cases} 1 & \text{if } (v_j, v_i) \in \text{Arc}, \\ 0, & \text{otherwise.} \end{cases}$$

[2]Our definition of the adjacency matrix may be slightly different from ones in some literature. The one in [18] (p.77) is the transpose of ours. Our definition is a bit more intuitive in our situation.

(4) A directed graph D is said to be *associated* with a nonnegative matrix M if the adjacency matrix $A(D)$ has the same zero pattern as M.
(5) A directed graph is *strongly connected* if for any ordered pair (v_i, v_j) with $i \neq j$, there is a path from v_i to v_j
(6) Let D be a strongly connected directed graph. The *index of imprimitivity* of D, denoted as $h(D)$, is defined to be the g.c.d of lengths of all cycles in D

Remark 3.5. A directed graph D can be visualized by a diagram in which an arc (v_i, v_j) is represented by a directed line going from v_i to v_j. The diagram associated to D is also referred to as a directed graph.

Fact 3.6. Let V be a vector space with an ordered basis $\mathcal{B} = \{v_1, ..., v_m\}$, and, for $i = 1, ..., l$, let T_i be an operator on V with $[T_i]_{\mathcal{B}}$ nonnegative. Given positive real numbers $c_1, ..., c_l$, we let $T := \sum_{i=1}^{l} c_i T_i$, and can associate to T a directed graph

$$D(T : \mathcal{B}) = (\mathrm{Ver}(T : \mathcal{B}), \mathrm{Arc}(T : \mathcal{B})).$$

Here $\mathrm{Ver}(T : \mathcal{B}) = \{v_1, ..., v_m\}$, and we define a relation $\mathrm{Arc}(T : \mathcal{B})$ on $\mathrm{Ver}(T : \mathcal{B})$ by

$$(v_i, v_j) \in \mathrm{Arc}(T : \mathcal{B}) \Leftrightarrow b_{ji} \neq 0 \text{ in } T(v_i) = \sum_{k=1}^{m} b_{ki} v_k,$$

$$\Leftrightarrow \exists\ p, 1 \leq p \leq l, \text{ such that } b_{ji}^{p} \neq 0 \text{ in } T_p(v_i) = \sum_{k=1}^{m} b_{ki}^{p} v_k.$$

The matrix $[T]_{\mathcal{B}}$ has the same zero pattern as the adjacency matrix $A(D(T : \mathcal{B}))$, and hence the directed graph $D(T : \mathcal{B})$ is associated with the matrix $[T]_{\mathcal{B}}$. In particular, to any nonnegative matrix M, we can associate the directed graph $D(M) := D(M, \mathcal{B})$ by taking $\mathcal{B}$ to be the standard basis.

The next proposition compares the properties of nonnegative matrices and their assosiated directed graphs; see e.g. Theorems 3.2 and 3.3 of Chapter 4 of [18].

Proposition 3.7. *Let M be a nonnegative matrix.*
(1) *M is irreducible if and only if the associated directed graph $D(M)$ is strongly connected.*
(2) *If M is irreducible, then the index $h(M)$ of imprimitivity of M is equal to the index $h(D(M))$ of imprimitivity of the associated directed graph $D(M)$.*

4. Main result

The quantum cohomology ring $H^\star(X, \mathbb{C})$ is a finite dimensional complex vector space with the Schubert basis $\mathcal{S}$ consisting of Schubert classes σ_u for $u \in W^P$. Arrange elements of $\mathcal{S}$ linearly once and for all to make $\mathcal{S}$ into an ordered basis. We will denote this ordered basis by $\mathcal{S}$, too.

The next lemma is a known fact, and can be found for instance in [13, Lemma 2.7].

Lemma 4.1. *Let $w \in W^P$, and take any reduced decomposition $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ where $\ell = \ell(w)$. Then we have $v := s_{i_2}\cdots s_{i_l} \in W^P$ and $v^{-1}(\alpha_{i_1}) \in R^+ \setminus R_P^+$.*

Proposition 4.2. *Let $M(X)$ be a matrix of the operator $[c_1(X)]$ on $H^\star(X,\mathbb{C})$ with respect to $\mathcal{S}$. Then $M(X)$ is an irreducible, nonnegative, integral matrix.*

Proof. As we can see in Lemma 2.1, the first Chern class $c_1(X)$ of the homogeneous variety $X = G/P$ is a nonnegative integral combination of Schubert divisor classes. By the quantum Chevalley formula, each coefficient in the quantum multiplication $\sigma_{s_i}\star\sigma_u$, being of the form $h_\alpha(\omega_{\alpha_i})$, is again a nonnegative integer. Moreover, $H^\star(X,\mathbb{C})$ is obtained from $qH^*(X,\mathbb{C})$ by taking specializations at $q_i = 1$ for all i. It follows that $M(X)$ is a nonnegative integral matrix.

Suppose $M = M(X)$ is reducible. Then, by definition, there exists a permutation matrix P such that $M = P^TM'P$ for a block-upper triangular matrix M'. It follows that $M^m = P^T(M')^mP$ is reducible for any $m \in \mathbb{Z}_{\geq 0}$, and so is $\sum_{m=0}^n M^m$ (recall $n = \dim_{\mathbb{C}} G/P$). Let V be the nontrivial proper coordinate subspace of $H^\star(X)$ which is invariant under the operator $T := \sum_{m=0}^n([c_1(X)])^m$. By the quantum Chevalley formula, for any $u \in W^P$, the coefficient $b_u(q)$ of σ_u in the quantum product $c_1(X)\star\cdots\star c_1(X)$ belongs to $\mathbb{Z}_{\geq 0}[q]$. Recall $q = (q_i)_{i\in I^P}$, and denote $\vec{1} := (1)_{i\in I^P}$, $\vec{0} := (0)_{i\in I^P}$. First we claim that

1. for any $w \in W^P$, the coefficient $b_w(q) \in \mathbb{Z}_{\geq 0}[q]$ of σ_w in the quantum multiplication $c_1^{\ell(w)}$ of $c_1 := c_1(X)$ has a positive constant term $b_w(\vec{0}) > 0$; therefore the operator T must be of the form $T = \sum_{w\in W^P} a_w(\vec{1})[\sigma_w]$, where $a_w(q) \in \mathbb{Z}_{\geq 0}[q]$ with $a_w(\vec{0}) > 0$, and $[\sigma_w]$ denotes the operator on $H^\star(X,\mathbb{C})$ defined by $[\sigma_w](\beta) = \sigma_w \star \beta|_{q=\vec{1}}$;
2. $\sigma_{w_0^P} \in V$.

To prove (1), we proceed by induction. If $l(w) = 0$, then this is trivial since $w = w_{0^P}$. If $\ell(w) = l$, then we take a reduced decomposition $w = s_{i_1}\cdots s_{i_l}$. By Lemma 4.1, we have $v := s_{i_2}\cdots s_{i_l} \in W^P$ and $\gamma := v^{-1}(\alpha_{i_1}) \in R^+ \setminus R_P^+$, which implies that $h_\gamma(\omega_j) \neq 0$ for some $j \in I^P$. Thus σ_w occurs in the classical part of $\sigma_{s_j}\star\sigma_v$ with a positive coefficient by the Chevalley formula, and so does in $c_1\star\sigma_v$. Clearly, $\ell(v) = \ell(w) - 1$. Therefore the coefficient $b_v(q) \in \mathbb{Z}_{\geq 0}[q]$ of σ_v in $c_1^{\ell(v)}$ satisfies $b_v(\vec{0}) > 0$ by the induction hypothesis, and consequently σ_w occurs in $c_1^{\ell(w)}$ with the same property due to the positivity of quantum multiplication of Schubert classes. To prove (2), we take $w \in W^P$ such that $\sigma_w \in V$. By item (3) in section 2.2, $\sigma_{w_0^P}$ occurs in the classical part of $[\sigma_w](\sigma_{w^\vee}) = \sigma_w\star\sigma_{w^\vee}$, and hence it occurs in $T(\sigma_{w^\vee})$.

Now choose $u \in W^P$ and $d \in H^{2n-2}(X,\mathbb{Z})$ such that $c_{w_0^P,w_0^P}^{u^\vee,d} > 0$. Such elements always exist since the quantum product of two Schubert classes never vanishes [5, Theorem 9.1]. But since we have $c_{u^\vee,w_0^P}^{(0^P)^\vee,d} = c_{w_0^P,w_0^P}^{u^\vee,d} > 0$ by the symmetry of Gromov-Witten invariants, the basis element σ_{0^P} occurs in the quantum product $\sigma_{u^\vee}\star\sigma_{w_0^P}$

and hence in $T(\sigma_{w_0^P})$. Thus $\sigma_{0^P} \in V$, and hence for any $w \in W^P$, σ_w occurs in $T(\sigma_{0^P})$ by noting $[\sigma_w](\sigma_{0^P}) = \sigma_w \star \sigma_{0^P} = \sigma_w$. This implies $V = H^\star(X)$, contradicting the hypothesis $V \subsetneqq H^\star(X)$. Therefore, the matrix $M(X)$ is irreducible. $\square$

Remark 4.3.
1. We note that the proof of Proposition 4.2 works for any quantum multiplication operators of the form $[\sigma] = \sum_{i \in I^P} a_i[\sigma_{s_i}]$ for positive real numbers a_i.
2. We mention that a general idea of proving the irreducibility of $M(X)$ was taken more or less from Lemma 9.3 (p. 384) in [22], where Rietsch used some Peterson's results to show the irreducibility of the matrix $[\sigma]_{\mathcal{S}}$ for a flag manifold X of type A, where $\sigma = \sum_{w \in W^P} \sigma_w \in H^\star(X, \mathbb{C})$.

Recall that $r = \text{g.c.d}\,\{\, n_i \mid i \in I^P \}$ is the Fano index of X, and $\mathcal{S}$ is an ordered basis of $H^\star(X)$. The order on $\mathcal{S}$ induces a linear order on the index set W^P, denoted as $\prec$. Let us make a partition on the set W^P into r subsets. For each $0 \le a \le r-1$, let $W^P(a)$ be the subset of W^P consisting of elements u with $l(u) \equiv a \mod r$, and we assign to each $W^P(a)$ the weight a. Now we define a new linear order $\prec_q$ on W^P as

$$u \prec_q v \Leftrightarrow \begin{cases} u \in W^P(a),\ \ v \in W^P(b),\ \text{ and } \ a > b, \\ \qquad u, v \in W^P(a),\ \text{ and } \ u \prec v. \end{cases}$$

The order $\prec_q$ on W^P naturally makes the Schubert basis for $H^\star(X)$ into an ordered basis, denoted as $\mathcal{S}_q$, in the way that the basis elements σ_u for $u \in W^P(r-1)$ come first, σ_u for $u \in W^P(r-2)$ second, and so on.

Lemma 4.4. *The matrix $M(X)_q$ of the operator $[c_1(X)]$ on $H^\star(X)$ with respect to the ordered basis $\mathcal{S}_q$ is in the superdiagonal $(m_1, m_2, ..., m_r)$-block form (3.3) with $k = r$, where $m_i := |W^P(r-i-1)|$ for $i = 1, ..., r$, and $W^P(-1) := W^P(r-1)$.*

Proof. It is obvious that the blocks $A_{1,2}, \cdots, A_{r-1,r},\ A_{r,1}$ are nonzero. Clearly, if a basis element σ_v appears with a nonzero coefficient in the expansion of $[c_1(X)](\sigma_u)$ in the basis $\mathcal{S}_q$, then by the degree condition of quantum multiplication we have

$$l(v) \equiv l(u) + 1 \mod r.$$

This proves the lemma. $\square$

The next lemma holds for general Fano manifolds ([8, Remark 3.1.3]). In the case of homogeneous spaces $X = G/P$, it also follows immediately from Lemma 4.4 and Proposition 3.4, by noting that $M(X)$ is cogredient to $M(X)_q$.

Lemma 4.5. *The Fano index r of X divides the index of imprimitivity $h(M(X))$ of $M(X)$.*

Definition. For a homogeneous space $X = G/P$, define the directed graph $D(X)$ as

$$D(X) = D([c_1(X)], \mathcal{S}),$$

where $D([c_1(X)], \mathcal{S})$ was defined in Fact 3.6.

Since $M(X)$ is irreducible, then its associated directed graph $D(X)$ is strongly connected by Proposition 3.7, and hence it makes sense to consider the index $h(D(X))$ of imprimitivity of $D(X)$. We will show that $h(D(X))$ divides r.

Let $Q^\vee$ (resp. $Q^\vee_P$) denote the coroot (sub)lattice $\oplus_{\alpha\in\Delta}\mathbb{Z}\alpha^\vee$ (resp. $\oplus_{\alpha\in\Delta_P}\mathbb{Z}\alpha^\vee$). Then we have the natural identifications

$$Q^\vee/Q^\vee_P = H_2(G/P,\mathbb{Z}) = H^{2n-2}(G/P,\mathbb{Z}).$$

Lemma 4.6. *For any $\lambda_P \in Q^\vee/Q^\vee_P$, there exists a unique $\lambda_B \in Q^\vee$ such that $\lambda_P = \lambda_B + Q^\vee_P$ and $\langle\alpha,\lambda_B\rangle \in \{0,-1\}$ for all $\alpha\in R^+_P$ with respect to the natural pairing $\langle\cdot,\cdot\rangle : \mathfrak{h}^*\times\mathfrak{h}\to\mathbb{C}$.*

Given λ_B in the above lemma, we let P' be the parabolic subgroup such that $\Delta_{P'} := \{\alpha\in\Delta_P \mid \langle\alpha,\lambda_B\rangle = 0\}$. Recall that for a parabolic subgroup P, w_P denotes the longest element of W_P. Let us record the following comparison formula which will be used to prove our key lemma. Both Lemma 4.6 and the next proposition are due to Peterson [20], proved by Woodward [25]. Here we are following the equivalent formulation given in [12, Theorem 10.13].

Proposition 4.7. *For any $u,v,w\in W^P$, we have*

$$c_{u,v}^{w^\vee,\lambda_P} = c_{u,v}^{(ww_Pw_{P'})^\vee,\lambda_B},$$

among the structure constants for $qH^\star(G/P)$ and $qH^\star(G/B)$ respectively.

We will need the following lemma. Here we provide an involved combinatorial proof, while a geometric proof by carefully studying the space of lines in G/P (cf. [24] and [15]) is quite desirable and will probably be much simpler.

Lemma 4.8. *For any $i\in I^P$, there exists $u\in W^P$ of length $\ell(u) = n_i - 1$ such that $\sigma_{s_i}\star\sigma_u$ contains the $q_i\sigma_{0^P}$-term with coefficient* 1.

Proof. Let $\lambda_P = \alpha_i^\vee + Q_P^\vee$. The unique lifting $\lambda_B\in Q^\vee$ is a priori, not necessarily a coroot. Nevertheless, let us first assume the next claims hold.

a) There exists $\gamma\in R^+$ of length $\ell(s_\gamma) = 2\langle\rho,\gamma^\vee\rangle - 1$ such that $\lambda_B = \gamma^\vee$ is the coroot of γ, where $\rho := \rho_B$ equals the sum of fundamental weights of (G,Δ).
b) $\ell(w_Pw_{P'}s_\gamma) = \ell(w_Pw_{P'}) + \ell(s_\gamma)$.

We then set $u := w_Pw_{P'}s_\gamma$. Take any $\alpha\in\Delta_P$. If $\langle\alpha,\gamma^\vee\rangle = 0$, then $s_\gamma(\alpha) = \alpha\in\Delta_{P'}$, and hence $u(\alpha) = w_Pw_{P'}(\alpha)\in R^+$, by noting $w_Pw_{P'}\in W^{P'}$. If $\langle\alpha,\gamma^\vee\rangle\neq 0$, then it equals -1 due to the property of $\lambda_B = \gamma^\vee$. On one hand, $u(\alpha)\in R$ is a root and hence must be either purely nonpositive or purely nonnegative combinations of the simple roots; on the other hand, $u(\alpha) = w_Pw_{P'}(\alpha+\gamma) = w_Pw_{P'}(\alpha_i) + w_Pw_{P'}(\alpha+\gamma-\alpha_i) \in \alpha_i + \sum_{\beta\in\Delta_P}\mathbb{Z}\beta$ (since $\alpha+\gamma-\alpha_i\in\sum_{\beta\in\Delta_P}\mathbb{Z}\beta$, $w_Pw_{P'}\in W_P$ and $s_j(\alpha_i)\in\alpha_i+\sum_{\beta\in\Delta_P}\mathbb{Z}\beta$ for all $\alpha_j\in\Delta_P$). Therefore $u(\alpha)\in R^+$. It follows that $u\in W^P$ (by a characterization of minimal length representatives e.g. in [10, section 5.4]). Consequently, we have

$c_{s_i,u}^{(0^P)^\vee,\lambda_P} = c_{s_i,u}^{(w_Pw_{P'})^\vee,\lambda_B} = c_{s_i,u}^{(us_\gamma)^\vee,\gamma^\vee} = 1$ by Proposition 4.7 and the quantum Chevalley formula of $qH^\star(G/B)$.

It remains to prove the claims. Denote by $\Delta(\alpha_i)$ the connected component of α_i in the Dynkin sub-diagram $\{\alpha_i\} \cup \Delta_P$ of Δ, and by $\{\Pi_1, \cdots, \Pi_k\}$ the connected components of Dynkin diagram of $\Delta(\alpha_i) \setminus \{\alpha_i\}$. Let $\omega_{(j)}$ (resp. $\omega'_{(j)}$) denote the longest element in the Weyl subgroup of Π_j (resp. $\Pi_j \cap \Delta_{P'}$), and set $w_j := \omega_{(j)}\omega'_{(j)}$. It follows that $w_iw_j = w_jw_i$ for any i, j, and that $w_Pw_{P'} = w_1 \cdots w_k$. We notice that all the possible (non-empty) Π_j have been listed in [13, Table 3] and [14, Table 2.1]. By the classification, one can see $\lambda_B = \alpha_i^\vee$ in most of the cases, and will be able to precisely describe γ and $w_Pw_{P'}$ in the all the remaining cases. Therefore the claims can be shown by direct calculations. To be precise, we give the details as follows.

For every j, we denote by $Q_{(j)}^\vee := \sum_{\alpha\in\Pi_j} \mathbb{Z}\alpha^\vee$ the coroot sublattice of Π_j, by $R_{(j)}^+ := R^+ \cap (\oplus_{\alpha\in\Pi_j}\mathbb{Z}\alpha)$ the set of positive roots of the root system of Π_j, by $\theta_{(j)}$ the highest root in $R_{(j)}^+$, and by $\alpha_{(j)}^{\rm ad}$ the (unique) simple root in Π_j adjacent to α_i. Given $\beta \in R_{(j)}^+$, we write $\beta = b_\beta\alpha_{(j)}^{\rm ad} + \sum_{\alpha\in\Pi_j\setminus\{\alpha_{(j)}^{\rm ad}\}} a_\alpha\alpha$, and denote by $(\diamond)$ the property:

$$(\diamond): \qquad \forall 1 \le j \le k,\ \langle\alpha_{(j)}^{\rm ad}, \alpha_i^\vee\rangle = -1 \quad \text{and} \quad b_{\theta_{(j)}} = 1.$$

Observe that any positive root β in R_P^+ with $\langle\beta, \alpha_i^\vee\rangle \ne 0$ only if β is in the root subsystem spanned by $\Delta(\alpha_i)\setminus\alpha_i$. It suffices to look into such positive roots. Moreover, any such root β must be in $R_{(j)}^+$ for some unique $1 \le j \le k$, and hence $\beta \le \theta_{(j)}$ with respect to the partial order $\le$ given by $(c_1, \cdots, c_m) \le (d_1, \cdots, d_m)$ iff $c_r \le d_r$ for all $1 \le r \le m$. In particular, we have $0 \le b_\beta \le b_{\theta_{(j)}}$. Thus if property $(\diamond)$ holds, then

$$\langle\beta, \alpha_i^\vee\rangle = \langle b_\beta\alpha_{(j)}^{\rm ad}, \alpha_i^\vee\rangle = -b_\beta \in \{0, -1\}.$$

It follows from the uniqueness in Lemma 4.6 that $\lambda_B = \gamma^\vee$ for $\gamma = \alpha_i$. That is, claim a) holds. Since $w_Pw_{P'}(\alpha_i) = \alpha_i + \sum_{\beta\in\Delta_P}\mathbb{Z}\beta$ and it is a root in R, it follows that $w_Pw_{P'}(\alpha_i) \in R^+$. Hence, $\ell(w_Pw_{P'}s_i) = \ell(w_Pw_{P'}) + 1$. That is, claim b) holds as well. We now check the property $(\diamond)$ across the Lie types.

If G is of Lie type ADE, then the property $(\diamond)$ obviously holds, and hence we are done. (Here we note that Π_j can be at most of type E_6, E_7 but not of type E_8 since $\alpha_i \notin \Pi_j$.)

For G of type $BCFG$, we just need to check the cases when the property $(\diamond)$ does not hold. We notice that $\{\alpha_i\} \cup \Pi_j$ is not of type A for at most one j. If such j exists, we say $j = k$ without loss of generality. Write $\lambda_B = \alpha_i^\vee + \sum_{j=1}^k \lambda_j$ where $\lambda_j \in Q_{(j)}^\vee$. Then $\gamma_j^\vee := \alpha_i^\vee + \lambda_j$ satisfies $\gamma_j^\vee \in \alpha_i^\vee + Q_{(j)}^\vee$ and $\langle\beta, \gamma_j^\vee\rangle = \langle\beta, \lambda_B\rangle \in \{0, -1\}$ for any $\beta \in R_{(j)}^+$. It follows from the uniqueness in Lemma 4.6 that $\lambda_j = 0$ for $j < k$, in which case $\{\alpha_i\} \cup \Pi_j$ is of type A. Thus $\lambda_B = \gamma_k^\vee$.

Suppose that G is of type B and the property $(\diamond)$ does not hold. Then $\Pi_k \cup \{\alpha_i\}$ must also be of type B with α_i being the (unique) short simple root at an end of the associated Dynkin diagram. Therefore we can rename $\Pi_k \cup \{\alpha_i\} = \{\beta_1, \cdots, \beta_{m+1}\}$

such that it is of type B_{m+1} in the standard way and $\alpha_i = \beta_{m+1}$ is the short simple root. For any $\beta \in R^+_{(k)}$, $\langle \beta, \beta_m^\vee + \beta_{m+1}^\vee\rangle$ is equal to 0 if β has no nonzero coefficient at β_{m-1}, or equal to $\langle \beta_{m-1}, \beta_m^\vee + \beta_{m+1}^\vee\rangle = -1$ otherwise (where we use the property of Π_k being of type A_m). By the uniqueness in Lemma 4.6, we have $\lambda_B = \beta_m^\vee + \beta_{m+1}^\vee = s_{\beta_{m+1}}(\beta_m^\vee)$. Hence, $\lambda_B = \gamma^\vee$ for $\gamma = s_{\beta_{m+1}}(\beta_m) \in R^+$. Consequently, $s_\gamma = s_{\beta m+1} s_{\beta_m} s_{\beta_{m+1}}$ and $w_k = s_{\beta_2} \cdots s_{\beta_m} s_{\beta_1} \cdots s_{\beta_{m-1}}$ (where we mean $w_k = \mathrm{id}$ if $m = 1$). Clearly, $\ell(w_k s_\gamma) = \ell(w_k) + \ell(s_\gamma) = \ell(w_k) + 2\langle \rho, \gamma^\vee\rangle - 1$ by a direct calculation. Therefore the claims hold by noting $w_j = \mathrm{id}$ for $j = 1, \cdots, k-1$.

Suppose that G is of type G_2 and the property $(\diamond)$ does not hold. Then $\Pi_k \cup \{\alpha_i\} = \{\beta_1, \beta_2\}$, and $\alpha_i = \beta_2$ is the short root. Everything is the same as the type B case with $m = 1$ therein, except that $w_k = s_{\beta_1}$ for G_2. Clearly, the claims hold.

Suppose that G is of type C and the property $(\diamond)$ does not hold. Then $\Pi_k \cup \{\alpha_i\}$ must be of type C with α_i being a short simple root at an end of the associated Dynkin diagram. Thus we can rename $\Pi_k \cup \{\alpha_i\} = \{\beta_1, \cdots, \beta_{m+1}\}$ such that it is of type C_{m+1} in the standard way, $\alpha_i = \beta_{m+1}$ is a short simple root and β_1 is the long simple root. Note $\langle \alpha, \sum_{j=1}^{m+1} \beta_j^\vee\rangle = 0$ for any $\alpha \in \Pi_k$. By the uniqueness, we have $\lambda_B = \sum_{j=1}^{m+1} \beta_j^\vee = (s_{\beta_{m+1}} \cdots s_{\beta_2}(\beta_1))^\vee$, and hence $w_k = \mathrm{id}$. The claims follow obviously.

Suppose that G is of type F_4. We may assume that $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is already in the standard way such that α_1, α_2 are the long simple roots, and α_1, α_4 are the two ends of the Dynkin diagram. There are 16 possibilities of Δ_P in total, which can be easily discussed as follows.

(1) Case $\alpha_1 \notin \Delta_P$. If $\alpha_i \in \{\alpha_1, \alpha_2\}$, then the property $(\diamond)$ holds, and hence we are done. Otherwise, $\alpha_i \in \{\alpha_3, \alpha_4\}$, and this is reduced to type C case, so that we are done again.
(2) Case $\alpha_1 \in \Delta_P$ and $\alpha_2 \notin \Delta_P$. Then either the property $(\diamond)$ holds or it is reduced to type C case.
(3) Case $\alpha_1 \in \Delta_P$ and $\alpha_2 \in \Delta_P$. If $\alpha_3 \notin \Delta_P$, then either the property $(\diamond)$ holds or it is reduced to type B case. If $\alpha_3 \in \Delta_P$, then $\alpha_i = \alpha_4$. By computing $\langle \beta, \sum_{j=2}^4 \alpha_j^\vee\rangle$ for all $\beta \in R_P^+$, we conclude $\lambda_B = \sum_{j=2}^4 \alpha_j^\vee = s_4 s_3(\alpha_2^\vee) = \gamma^\vee$ for $\gamma = s_4 s_3(\alpha_2) \in R^+$, by the uniqueness of λ_B. Consequently, $w_k = s_1 s_2 s_3 s_2 s_1$, and hence the claims follow by direct calculations.

The nonvanishing of $c_{s_i,u}^{(0^P)^\vee, \lambda_P}$ implies that $\ell(u) = \deg(q_i) + \ell(0^P) - \ell(s_i) = n_i - 1$. □

Remark 4.9. Given $\gamma \in R^+$, the property $\ell(\gamma) = 2\langle \rho, \gamma^\vee\rangle - 1$ always holds whenever the Dynkin diagram of G is simply laced (i.e. of Lie type ADE) [16, Lemma 3.2]. In general, the property requires a nontrivial condition [13, Lemma 3.8].

Lemma 4.10. *For each $i \in I^P$, there is a cycle Ξ_i of length n_i in $D(X)$ through the fixed vertex σ_{0^P}.*

Proof. By Lemma 4.8, there exists $u \in W^P$ of length $n_i - 1$ such that σ_{0^P} occurs in $\sigma_{s_i} \star \sigma_u$, and hence occurs in the expansion $[c_1(X)](\sigma_u)$. Thus there is a path

PATH$(u : 0^P)$ of length 1 from the vertex σ_u to the vertex σ_{0^P}. Now take a reduced decomposition $u = s_{j_1} s_{j_2} \cdots s_{j_{n_i-1}}$, set $v_m := s_{j_m} s_{j_{m+1}} \cdots s_{j_{n_i-1}}$ for $1 \leq m \leq n_i - 1$, and denote $v_0 = v_{n_i} := 0^P$. Then we have $v_m \in W^P$ and $v_{m+1}^{-1}(\alpha_{j_m}) \in R^+ \setminus R_P^+$ for all $1 \leq m \leq n_i - 1$ by Lemma 4.1. Consequently, σ_{v_m} occurs in $[c_1(X)](\sigma_{v_{m+1}})$ by the (quantum) Chevalley formula, resulting in a path PATH$(v_{m+1} : v_m)$ of length 1 for all $1 \leq m \leq n_i - 1$. Clearly, the join of paths

$$\mathrm{PATH}(v_{n_i} : v_{n_i-1}) \sqcup_{v_{n_i-1}} \cdots \sqcup_{v_2} \mathrm{PATH}(v_2 : v_1) \sqcup_{v_1} \mathrm{PATH}(u : 0^P)$$

is a cycle of length n_i through σ_{0^P}, by noting that $v_1 = u$ and $v_{n_i} = 0^P$. $\square$

Recall that $h(D(X))$ is the greatest common divisor (g.c.d) of the lengths of all cycles of $D(X)$. Therefore it divides the g.c.d of the lengths of the cycles $\{\Xi_i \mid i \in I^P\}$ through σ_{0^P}. Namely, $h(D(X))$ divides g.c.d.$\{n_i \mid i \in I^P\} = r$ by Lemma 4.10 and the definition of the Fano index r. Since $h(D(X)) = h(M(X)_q) = h(M(X))$, $h(M(X))$ divides r. This fact together with Lemma 4.5 implies

Corollary 4.11. *The imprimitivity index $h(M(X))$ of the irreducible matrix $M(X)$ is equal to the Fano index r.*

Theorem 4.12. *Homogeneous spaces $X = G/P$ satisfy Conjecture $\mathcal{O}$.*

Proof. Since $M(X)$ is irreducible, Condition (1) and (2) are satisfied by Proposition 3.2. Condition (3) follows from Corollary 4.11 and Proposition 3.3. $\square$

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