

EQUIVARIANT PIERI RULES FOR ISOTROPIC GRASSMANNIANS

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ABSTRACT. We give a Pieri rule for the torus-equivariant cohomology of (sub-maximal) Grassmannians of Lie types B , C , and D . To the authors' best knowledge, our rule is the first manifestly positive formula, beyond the equivariant Chevalley formula. We also give a simple proof of the equivariant Pieri rule for the ordinary (type A) Grassmannian.

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1. INTRODUCTION

Let V be an N -dimensional complex vector space equipped with a symmetric or skew-symmetric bilinear form ω , which can be either trivial or non-degenerate. The Grassmannians $IG_\omega(m, N)$ of classical Lie type parameterize m -dimensional isotropic vector subspaces of V . The cohomology ring of an isotropic Grassmannian $X = IG_\omega(m, N)$, or more generally of a homogeneous variety, has an additive basis of Schubert classes represented by Schubert subvarieties X_λ . One of the central problems of Schubert calculus is to find a manifestly positive formula for the structure constants of the cup product of two Schubert cohomology classes, or equivalently, for the triple intersection numbers of three Schubert subvarieties in general position. Such a positive formula, called a Littlewood-Richardson rule, has deep connections to various subjects, including geometry, combinatorics and representation theory.

An isotropic Grassmannian X can be written as a quotient of a classical complex simple Lie group G by a maximal parabolic subgroup P (with two notable

exceptions of Lie type D_n , described in Section 7). Fix a choice of maximal complex torus T and a Borel subgroup B with $T \subset B \subset P$. The Schubert varieties X_λ (relative to B) are closures of B -orbits, and hence are T -stable. They give a basis $[X_\lambda]^T$ for the T -equivariant cohomology $H_T^*(X)$ as a $H_T^*(\text{pt})$ -module. The structure coefficients $N_{\lambda,\mu}^\nu$ in the equivariant product,

$$[X_\lambda]^T \cdot [X_\mu]^T = \sum_{\nu} N_{\lambda,\mu}^\nu [X_\nu]^T,$$

are homogeneous polynomials which satisfy a positivity condition conjectured by Peterson [40] and proved by Graham [20]. In particular, they are *Graham-positive*, meaning they are polynomials in the negative simple roots, with nonnegative integer coefficients. These equivariant structure coefficients carry much more information than the triple intersection numbers of Schubert varieties, and are more challenging to study. When the bilinear form ω is trivial, i.e., when $X = Gr(m, N)$ is a type A Grassmannian, there has been extensive work on equivariant Littlewood-Richardson rules [26, 37, 28, 46]. However, for Grassmannians of Lie type other than A , there have been no manifestly Graham-positive formulas, to the authors' best knowledge, except for the equivariant Chevalley formula which concerns multiplication by Schubert divisors ([27, 8], see also e.g. [29, Theorem 11.1.7 (i)]). We remark that an effective (but non-positive) algorithm for computing the structure coefficients for general G/P is given in [35] (see also [39, 26] for the type A case). Remarkably, a manifestly positive equivariant Littlewood-Richardson rule has recently been given by Buch for two-step partial flag varieties of type A [10].

In the present paper, we give for the first time an *equivariant Pieri rule* for Grassmannians of Lie types B , C , and D , as well as a new proof of the Pieri rule in type A . Such a rule concerns products with the special Schubert classes $[X_p]^T$, which are related to the equivariant Chern classes of the tautological quotient bundle, and generate the T -equivariant cohomology ring. Using geometric methods, we give a manifestly positive formula for the structure coefficients $N_{\lambda,p}^\mu$ of the equivariant multiplication $[X_\lambda]^T \cdot [X_p]^T$. For type A Grassmannians $X = Gr(m, N)$, the equivariant Pieri rule has been even more extensively studied than the more general equivariant Littlewood-Richardson rule (see e.g. [38, 30, 31, 19, 18]). Nevertheless, we give a new proof that reveals an interesting reduction of arbitrary Pieri coefficients to much simpler ones. Namely, we prove that any Pieri coefficient $N_{\lambda,p}^\mu(Gr(m, N))$ is equal to a Pieri coefficient of the form $N_{\nu,p'}^\nu(Gr(m', N))$ in a possibly different Grassmannian $Gr(m', N)$. Such a reduction has also been made in [21] and [42] in a combinatorial way, but our argument is much simpler and can explain its geometric origin. Each coefficient $N_{\nu,p'}^\nu(Gr(m', N))$ is the restriction of a special equivariant Schubert class $[X_{p'}]^T$ to a T -fixed point of $Gr(m', N)$. There have been several manifestly positive formulas for these *restriction coefficients* (see [27, 2, 5] for general G/P ; [22, 25] for Lagrangian and maximal orthogonal Grassmannians; and for example [14] for type A Grassmannians). For completeness, we include one more restriction formula in the appendix. For isotropic Grassmannians $X = IG_\omega(m, N)$ of Lie types B and C , we use geometric arguments to reduce the Pieri coefficients $N_{\lambda,p}^\mu(X)$ to sums of specializations of restriction coefficients $N_{\nu,p'}^\nu(Gr(m', N))$. In the type D case, we succeed in a similar way for most of the Pieri coefficients, and can reduce the rest to appropriate restriction coefficients $N_{\nu,p'}^\nu(X')$ with respect to an isotropic Grassmannian X' of type D . We remark

that in the case of the complete flag variety of Lie type A , an equivariant Pieri rule with respect to a distinct set of special Schubert classes is contained in [32].

To state our formula precisely, we will parametrize Schubert varieties by *Schubert symbols* (also called *index sets* in [12], or *jump sequences*), which gives a uniform description for all classical Lie types. Schubert symbols for a Grassmannian $IG_\omega(m, N)$ are subsets $\lambda = \{\lambda_1 < \lambda_2 < \dots < \lambda_m\}$ of the integer interval $[1, N]$ which in addition satisfy $\lambda_i + \lambda_j \neq N + 1$ for all i, j if ω is non-degenerate. We denote by $|\lambda|$ the codimension of the Schubert variety X_λ in X . We also need the combinatorial relation $\lambda \rightarrow \mu$ between Schubert symbols λ and μ , which says roughly that the cohomology class $[X_\mu]$ occurs in some cohomological Pieri product involving $[X_\lambda]$. Let us now restrict our attention to a Grassmannian of type C_n ; i.e. we let $N = 2n$ and ω be non-degenerate and skew-symmetric. We adopt the notation $SG(m, 2n)$ to refer to this symplectic Grassmannian. Given $\lambda \rightarrow \mu$, the pair (λ, μ) defines two combinatorial sets $\mathcal{L}_{\lambda, \mu}$ and $\mathcal{Q}_{\lambda, \mu}$ (see Sections 4 - 7 for precise descriptions), which index certain hyperplanes and quadratic hypersurfaces in \mathbb{P}^{2n-1} , respectively. We adopt the Bourbaki [7] expression for simple roots α_i (resp. $\hat{\alpha}_j$) of type C_n (resp. A_{2n-1}) in terms of weights: $\alpha_n = 2t_n$ and $\alpha_i = t_i - t_{i+1}$ for $1 \leq i \leq n - 1$ (resp. $\hat{\alpha}_j = \hat{t}_j - \hat{t}_{j+1}$ for $1 \leq j \leq 2n - 1$). The inclusion of T into a maximal complex torus of $GL(N, \mathbb{C}) \supset G$ induces a ring homomorphism $F : \mathbb{Z}[\hat{t}_1, \dots, \hat{t}_{2n}] \rightarrow \mathbb{Z}[t_1, \dots, t_n]$, defined by $F(\hat{t}_j) = t_j$ if $j \leq n$ and $F(\hat{t}_j) = -t_{2n+1-j}$ otherwise. Conveniently, the homomorphism F sends simple roots of type A_{2n-1} to simple roots of type C_n . It follows that F sends Graham-positive polynomials of type A_{2n-1} to Graham-positive polynomials of type C_n . Using the specialization F , we therefore have

Theorem 1.1 (Equivariant Pieri rule for $SG(m, 2n)$). *Let λ be a Schubert symbol for $SG(m, 2n)$ and $1 \leq p \leq 2n - m$. In $H_T^*(SG(m, 2n))$, we have*

$$[X_\lambda]^T \cdot [X_p]^T = \sum N_{\lambda, p}^\mu [X_\mu]^T,$$

where the sum is over Schubert symbols μ satisfying $\lambda \rightarrow \mu$ and $|\mu| \leq p + |\lambda|$. Furthermore, each coefficient $N_{\lambda, p}^\mu$ is a sum of $2^{\#\mathcal{Q}_{\lambda, \mu}}$ specializations of restriction coefficients for the type A Grassmannian $Gr(m', 2n)$:

$$N_{\lambda, p}^\mu(SG(m, 2n)) = \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}} F\left(N_{\nu_{\mathcal{I}}, p'}^{\nu_{\mathcal{I}}}(Gr(m', 2n))\right),$$

where $m' = m + |\mu| - |\lambda|$, $p' = p + |\lambda| - |\mu|$, and each $\nu_{\mathcal{I}}$ is an associated Schubert symbol for $Gr(m', 2n)$ (defined explicitly in Section 5).

In particular if $|\mu| = p + |\lambda|$, then $p' = 0$. As a consequence, the coefficient $N_{\lambda, p}^\mu$ is a summation of $2^{\#\mathcal{Q}_{\lambda, \mu}}$ copies of the constant polynomial 1. This reproduces the ordinary Pieri rule of Buch, Kresch and Tamvakis [12]. For the odd orthogonal Grassmannian $OG(m, 2n+1)$ (i.e. the type B case), an equivariant Pieri coefficient $N_{\lambda, p}^\mu(OG(m, 2n+1))$ is generally not a multiple of another type C equivariant Pieri coefficient, in contrast to the case of ordinary cohomology [6] (see also [4, section 3.1]). Nevertheless, we give a manifestly positive Pieri formula for it as well as for the type D Grassmannian $OG(m, 2n)$. We refer our readers to **Theorems 6.2** and **7.4** for the precise statements.

Our equivariant Pieri rules are obtained by geometric arguments, which include two major steps. Let us consider the natural projections

$$IG_\omega(m, N) \xleftarrow{\pi} IF_\omega(1, m; N) \xrightarrow{\psi} IG_\omega(1, N),$$

where $IF_\omega(1, m; N)$ denotes the corresponding two-step isotropic flag variety. Let $Y_{\lambda, \mu}$ denote the Richardson variety given by the intersection of the Schubert variety X_λ with the opposite Schubert variety labeled by μ , let $Z_{\lambda, \mu}$ be the projected Richardson variety $\psi(\pi^{-1}(Y_{\lambda, \mu}))$, and let L_p be a subvariety of $IG_\omega(1, N) \subset \mathbb{P}^{N-1}$ with the property that $X_p = \pi(\psi^{-1}(L_p))$. Finally, for any variety Y , let \int_Y^T denote the equivariant pushforward along the morphism $Y \rightarrow \text{pt}$. When X is of type A or C , the natural injection $\iota : IG_\omega(1, N) \rightarrow \mathbb{P}^{N-1}$ is the identity isomorphism, and our first step is to write $N_{\lambda, p}^\mu$ as the integral of an equivariant cohomology class in \mathbb{P}^{N-1} via the projection formula:

$$N_{\lambda, p}^\mu = \int_X^T [Y_{\lambda, \mu}]^T \cdot [X_p]^T = \int_{\mathbb{P}^{N-1}}^T [Z_{\lambda, \mu}]^T \cdot [L_p]^T.$$

When X is of type B or D , the injection ι is no longer surjective, but a more involved analysis still works. Such a technique has led to a Pieri rule for the ordinary cohomology of isotropic Grassmannians [43, 12] as well as triple intersection formulas in K-theoretic Schubert calculus [13, 15, 41]. In equivariant cohomology, it reduces $N_{\lambda, p}^\mu$ to an easier calculation in $H_T^*(\mathbb{P}^{N-1})$. However, any direct computation of $\int_{\mathbb{P}^{N-1}}^T [Z_{\lambda, \mu}]^T \cdot [L_p]^T$ involves sign cancelations, and fails to be manifestly positive.

Our key observation is that the projected Richardson variety $Z_{\lambda, \mu} \subset \mathbb{P}^{N-1}$ can be degenerated into $2^{\#\mathcal{Q}_{\lambda, \mu}}$ linear subvarieties $Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}'}}$, indexed by subsets $\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}$. These in turn can be interpreted as projections to \mathbb{P}^{N-1} of Richardson varieties $Y_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}'}}$ in a type A Grassmannian $X' := Gr(m', N)$. Applying the projection formula in the reverse direction, we reduce $\int_{\mathbb{P}^{N-1}}^T [Z_{\lambda, \mu}]^T \cdot [L_p]^T$ to a sum of quantities that are easy to compute, with positivity apparent:

$$\int_{\mathbb{P}^{N-1}}^T [Z_{\lambda, \mu}]^T \cdot [L_p]^T = \sum_{\nu_{\mathcal{I}}} \int_{\mathbb{P}^{N-1}}^T [Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}'}}]^T \cdot [L_{p'}]^T = \sum_{\nu_{\mathcal{I}}} \int_{X'}^T [Y_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}'}}]^T \cdot [X_{p'}]^T.$$

In particular, each subvariety $Y_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}'}}$ is simply the T -fixed point in X' corresponding to the Schubert symbol $\nu_{\mathcal{I}}$, and $\int_{X'}^T [Y_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}'}}]^T \cdot [X_{p'}]^T$ is the (specialized) restriction of $[X_{p'}]^T$ to that T -fixed point.

In addition to the equivariant Pieri rules, there is another important component to the equivariant Schubert calculus for isotropic Grassmannians, namely the equivariant Giambelli formulas which express an arbitrary Schubert class as a polynomial in special equivariant Schubert classes or Chern classes (see [30, 36, 25, 47, 24, 23, 44, 45] and references therein). The torus-equivariant cohomology ring of an isotropic Grassmannian (or more generally of a homogeneous variety) behaves more simply than its ordinary cohomology ring, in the sense that it is essentially determined by the equivariant Chevalley formula of multiplication by divisor classes due to Mihalcea's criterion [35]. This has led to nice applications on the Giambelli-type formula for type A flag varieties by Lam and Shimozono [33]. However, it is far from obvious how one can obtain applications to full Pieri-type formulas from this criterion or the Giambelli-type formulas.

The paper is organized as follows. In section 2, we introduce some basic notions for Grassmannians across classical Lie types. In section 3, we review basic properties of the torus-equivariant cohomology. In sections 4-7, we give the equivariant Pieri rules for Grassmannians of type A , C , B and D respectively. Finally in the appendix, we include a manifestly positive formula for the restriction coefficients for type A Grassmannians.

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2. GRASSMANNIANS OF CLASSICAL LIE TYPES

Let V be an N -dimensional complex vector space equipped with a bilinear form ω , and denote

$$IG_\omega(m, N) := \{\Sigma \leq V : \dim_{\mathbb{C}} \Sigma = m, \omega(\mathbf{v}, \mathbf{w}) = 0 \forall \mathbf{v}, \mathbf{w} \in \Sigma\}.$$

Throughout this paper, we will consider a Grassmannian variety $X = IG_\omega(m, N)$ of Lie type A_{n-1} , B_n , C_n or D_n , characterized by the following properties and notations respectively:

- A_{n-1} : $\omega(\cdot, \cdot) \equiv 0$. Namely $X = Gr(m, n)$ is an ordinary Grassmannian, where $N = n$.
- B_n : ω is non-degenerate and symmetric, and N is odd. Then $X = OG(m, 2n+1)$ is called an odd orthogonal Grassmannian, where $N = 2n + 1$.
- C_n : ω is non-degenerate and skew-symmetric, and N is even. Then $X = SG(m, 2n)$ is called a symplectic Grassmannian, where $N = 2n$.
- D_n : ω is non-degenerate and symmetric, and N is even. Then $X = OG(m, 2n)$ is called an even orthogonal Grassmannian, where $N = 2n$.

In all of these cases, we will assume $m \leq n$. When $m = n$ we refer to X as a Lagrangian Grassmannian in type C_n and a maximal orthogonal Grassmannian in types B_n and D_n . The Grassmannian X is a smooth projective variety of complex dimension $m(n - m)$ in the type A_{n-1} case, $2m(n - m) + \frac{m(m+1)}{2}$ in types B_n and C_n , and $2m(n - m) + \frac{m(m-1)}{2}$ in type D_n .

Take an isomorphism $V \cong \mathbb{C}^N$ by specifying a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ of V which in addition satisfies $\omega(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i+j, N+1}$ for all $i \leq j$ if ω is non-degenerate. Define a complete (isotropic) flag E_\bullet by $E_j := \langle \mathbf{e}_1, \dots, \mathbf{e}_j \rangle$, the span of the first j basis vectors. Let $[1, N]$ denote the set of integers $\{1, 2, \dots, N\}$. A Schubert symbol for X is a subset $\lambda = \{\lambda_1 < \lambda_2 < \dots < \lambda_m\}$ of $[1, N]$ which in addition satisfies $\lambda_i + \lambda_j \neq N + 1$ for all $i \leq j$ if ω is non-degenerate. The set of Schubert symbols for X is denoted by $\mathfrak{S}(X)$.

The Schubert subvarieties of X (relative to E_\bullet) are parameterized by Schubert symbols as follows (see [12, 34] for identifications with alternate parametrizations). The Grassmannian $X = IG_\omega(m, V)$ admits a transitive action of an appropriate reductive complex Lie group G of rank n (with the exception of $X = OG(n, 2n)$, where the action is transitive only when restricted to one of the two connected components). Precisely, $G = GL(n, \mathbb{C})$, $SO(2n + 1, \mathbb{C})$, $Sp(2n, \mathbb{C})$ or $SO(2n, \mathbb{C})$, according to whether X is of type A_{n-1} , B_n , C_n or D_n . The stabilizer of E_\bullet

in the G -action is a Borel subgroup B of G , which contains a maximal complex torus $T \cong (\mathbb{C}^*)^n$ with eigenvectors $\mathbf{e}_1, \dots, \mathbf{e}_N$. Each Schubert symbol λ indexes a T -fixed point $\Sigma_\lambda := \langle \mathbf{e}_{\lambda_1}, \dots, \mathbf{e}_{\lambda_m} \rangle$, whose B -orbit closure is the Schubert variety $X_\lambda := \overline{B \cdot \Sigma_\lambda}$. Now we let B^- denote the opposite Borel subgroup intersecting B at T , and define the opposite Schubert variety $X^\lambda := \overline{B^- \cdot \Sigma_\lambda}$. We denote the codimension of X_λ in X (equivalently the dimension of X^λ) by $|\lambda|$, which is given by the formula:

$$|\lambda| = \begin{cases} \dim X - \sum_{j=1}^m (\lambda_j - j) & \text{for type } A_{n-1}, \\ \dim X - \sum_{j=1}^m (\lambda_j - j - \#\{i < j : \lambda_i + \lambda_j > N + 1\}) & \text{for type } C_n, \\ \dim X - \sum_{j=1}^m (\lambda_j - j - \#\{i \leq j : \lambda_i + \lambda_j > N + 1\}) & \text{for types } B_n, D_n. \end{cases}$$

We note that in types A_{n-1} , B_n and C_n the Schubert varieties X_λ and X^λ have alternative characterizations by the *Schubert conditions*:

$$\begin{aligned} X_\lambda &= \{\Sigma \in X : \dim(\Sigma \cap E_{\lambda_j}) \geq j \text{ for } 1 \leq j \leq m\}, \text{ and} \\ X^\lambda &= \{\Sigma \in X : \dim(\Sigma \cap \langle \mathbf{e}_{\lambda_j}, \mathbf{e}_{\lambda_{j+1}}, \dots, \mathbf{e}_N \rangle) \geq m + 1 - j \text{ for } 1 \leq j \leq m\}. \end{aligned}$$

For an analagous characterization in type D_n , see [11, Proposition A.2].

Among the Schubert varieties X_λ , there are special Schubert varieties X_{s_p} , also denoted simply as X_p , which have codimension p and are determined by the single condition $\{\Sigma \in X : \dim(\Sigma \cap E_{n_p}) \geq 1\}$. Here p is a positive integer with $p \leq N - m$ if X is of type A or C , and $p \leq N - m - 1$ otherwise. The special Schubert symbol s_p , together with the integer n_p will be specified when we discuss the equivariant Pieri rules individually later.

Given Schubert symbols $\lambda = \{\lambda_1 < \dots < \lambda_m\}$ and $\mu = \{\mu_1 < \dots < \mu_m\}$, we write $\lambda \leq \mu$ if $\lambda_j \leq \mu_j$ for $1 \leq j \leq m$. In types B and C , we have $\lambda \leq \mu$ if and only if $X_\lambda \subset X_\mu$. In Section 7, we define a stronger relation \preceq on Schubert symbols which coincides with the relation $X_\lambda \subset X_\mu$ in type D . The *Richardson variety* $Y_{\lambda, \mu} := X_\lambda \cap X^\mu$ has dimension $|\mu| - |\lambda|$, and is nonempty if and only if $\mu \leq \lambda$ (resp. $\mu \preceq \lambda$ in type D). In particular, $Y_{\lambda, \lambda}$ consists of a single T -fixed point Σ_λ .

Given $\mu \leq \lambda$, we define the associated *Richardson diagram* by

$$D_X(\lambda, \mu) := \{(j, c) : \mu_j \leq c \leq \lambda_j\} \subset [1, m] \times [1, N].$$

We will simply denote $D_X(\lambda, \mu)$ as $D(\lambda, \mu)$ whenever there is no confusion, and will represent this set visually as an $m \times N$ matrix with stars for every entry in $D(\lambda, \mu)$ and zeros elsewhere. We say $c \in [1, N]$ is a *zero column* of $D(\lambda, \mu)$ if $\lambda_j < c < \mu_{j+1}$ for some $j \in [0, m]$, where we set $\lambda_0 = 0$ and $\mu_{m+1} = N + 1$ for convenience. In types B , C , and D , we will also define the notion of a *cut* in $D(\lambda, \mu)$. As we will see, zero columns and cuts will be further used to define combinatorial sets $\mathcal{L}_{\lambda, \mu}$ and $\mathcal{Q}_{\lambda, \mu}$, which index certain hyperplanes and quadratic hypersurfaces in \mathbb{P}^{N-1} .

3. TORUS EQUIVARIANT COHOMOLOGY

The aim of this paper is to give an equivariant Pieri rule for $X = IG_\omega(m, N)$. In this section we review some basic properties of the torus-equivariant cohomology, and do the first step of our reductions.

3.1. Basic properties of $H_T^*(X)$. We refer the readers to [16, 29] and references therein for the facts mentioned here. Recall that X has a transitive G -action, and $T \subset G$ is a fixed maximal complex torus. Let pt denote a point equipped with a trivial T -action. The T -equivariant cohomology $H_T^*(\text{pt})$ is given by

$$H_T^*(\text{pt}) = \Lambda := \mathbb{Z}[t_1, \dots, t_n],$$

where each t_i is defined as follows. Let χ_i denote the character that sends $(z_1, \dots, z_n) \in T = (\mathbb{C}^*)^n$ to z_i . Note that χ_i induces a one-dimensional representation \mathbb{C}_{χ_i} of T , defined by $(z_1, \dots, z_n) \cdot v \mapsto z_i v$ for any $v \in \mathbb{C}$. We then define $t_i := c_1^T(\mathbb{C}_{\chi_i})$, for $1 \leq i \leq n$, where \mathbb{C}_{χ_i} is treated as a T -equivariant line bundle over the point pt .¹

The T -equivariant cohomology $H_T^*(\cdot)$ is a contravariant functor from complex T -spaces to graded Λ -algebras. For a T -invariant subvariety Y of X , we denote the natural inclusion and projection, respectively, by

$$\iota_Y : Y \hookrightarrow X; \quad \rho_Y : Y \longrightarrow \{\text{pt}\}.$$

Both ι_Y and ρ_Y are proper maps, and hence induce equivariant pushforwards $\iota_{Y,*} : H_T^*(Y) \rightarrow H_T^*(X)$ and $\rho_{Y,*} : H_T^*(Y) \rightarrow H_T^*(\text{pt}) = \Lambda$ respectively. We will henceforth denote the map $\rho_{Y,*}$ by \int_Y^T , as in the introduction. The subvariety Y determines an equivariant cohomology class $[Y]^T \in H_T^{2\text{codim}Y}(X)$ under $\iota_{Y,*}$.

We notice that the Schubert subvarieties X_λ and X^λ are all T -invariant. In fact, $H_T^*(X)$ is a free Λ -module and the sets $\{[X_\lambda]^T\}$ and $\{[X^\lambda]^T\}$ form Λ -bases for $H_T^*(X)$. These bases are dual with respect to the equivariant pushforward to the point:

$$\int_X^T [X_\lambda]^T \cdot [X^\mu]^T = \delta_{\lambda,\mu}.$$

The structure coefficients in the equivariant product,

$$[X_\lambda]^T \cdot [X_\mu]^T = \sum_{\nu} N_{\lambda,\mu}^{\nu}(X) [X_\nu]^T,$$

are simply denoted $N_{\lambda,\mu}^{\nu} = N_{\lambda,\mu}^{\nu}(X)$ when there is no confusion, and are given by

$$N_{\lambda,\mu}^{\nu} = \int_X^T [X_\lambda]^T \cdot [X_\mu]^T \cdot [X^\nu]^T.$$

They are homogeneous polynomials of degree $(|\lambda| + |\mu| - |\nu|)$ in the negative simple roots with non-negative integer coefficients [20] (we will specify the simple roots in terms of the weights t_i later).

Let X^T denote the set of T -fixed points Σ_λ in X . The T -equivariant inclusion $\iota_T := \iota_{X^T}$ induces an injective ring morphism

$$\iota_T^* : H_T^*(X) \hookrightarrow H_T^*(X^T) = \bigoplus_{\lambda} \Lambda,$$

which extends to an isomorphism over the fraction field of Λ . We call $N_{\nu,\mu}^{\nu}$ a *restriction coefficient*, due to the following well-known fact [3]:

$$N_{\nu,\mu}^{\nu} = \iota_{\Sigma_\nu}^* [X_\mu]^T.$$

We have $N_{\nu,\mu}^{\nu} = 0$ unless $\Sigma_\nu \in X_\mu$, or equivalently $\nu \leq \mu$. This vanishing is a special case of the fact that $N_{\lambda,\mu}^{\nu} = 0$ unless $\nu \leq \lambda$ and $\nu \leq \mu$ [27].

¹It is also common to use the basis of opposite Schubert varieties together with the identification $t_i := -c_1^T(\mathbb{C}_{\chi_i})$. The resulting structure coefficients will still be Graham-positive (in the sense of being positive polynomials in the negative simple roots).

3.2. First step for the equivariant Pieri rule. Our goal is to give an equivariant Pieri rule for X ; that is, a manifestly positive formula for equivariant multiplication by a special Schubert class,

$$[X_\lambda]^T \cdot [X_p]^T = \sum_{\mu} N_{\lambda,p}^{\mu} [X_\mu]^T.$$

We remark that the Grassmannian $IG_\omega(m, N)$ carries an exact sequence of tautological bundles: $0 \rightarrow \mathcal{S} \rightarrow \mathbb{C}^N \rightarrow \mathcal{Q} \rightarrow 0$. The special equivariant Schubert class $[X_p]^T$ coincides with the relative equivariant Chern class $c_p^T(\mathcal{Q} - E_{n_p})$ (possibly up to a factor of 2 in types B and D), where E_{n_p} is regarded as a trivial vector subbundle of \mathbb{C}^N which is stable under the action of T .

We let $IF_\omega(1, m; N)$ denote the two-step isotropic flag variety, and consider the natural projections π and ψ as below.

$$\begin{array}{ccc} IF_\omega(1, m; N) & \xrightarrow{\psi} & IG_\omega(1, N) \subset \mathbb{P}^{N-1} \\ \pi \downarrow & & \\ X = IG_\omega(m, N) & & \end{array}$$

Let $Z := IG_\omega(1, N) \subset \mathbb{P}^{N-1}$. Recall that $X_p = \{\Sigma \in X : \dim(\Sigma \cap E_{n_p}) \geq 1\}$. It follows that $X_p = \pi(\psi^{-1}(Z \cap \mathbb{P}(E_{n_p})))$. Note that the subvariety $L_p \subset Z$ mentioned in the introduction is precisely $Z \cap \mathbb{P}(E_{n_p})$. Let $Z_{\lambda,\mu}$ be the projected Richardson variety $\psi(\pi^{-1}(Y_{\lambda,\mu}))$. Following [12, §5], we write $\lambda \rightarrow \mu$ if appropriate combinatorial relations are satisfied by the Schubert symbols λ and μ . Precise descriptions of the relation $\lambda \rightarrow \mu$ and the varieties $\mathbb{P}(E_{n_p})$ and $Z_{\lambda,\mu}$ will be postponed to the type-dependent discussions in the next four sections. As mentioned in the introduction, we will achieve our equivariant Pieri rule in two main steps. Let us end this section by the first step:

Proposition 3.1. *Given $\mu \leq \lambda \in \mathfrak{S}(X)$ and $p \in [1, N - m]$ (resp. $[1, N - m - 1]$ in type B or D), we have $N_{\lambda,p}^{\mu}(X) = 0$ unless $\lambda \rightarrow \mu$, $|\mu| \leq |\lambda| + p$, and $\mu \leq S_p$. When $\lambda \rightarrow \mu$, we have*

$$N_{\lambda,p}^{\mu} = \int_Z [Z_{\lambda,\mu}]^T \cdot [Z \cap \mathbb{P}(E_{n_p})]^T.$$

Proof. It is well known that $N_{\lambda,p}^{\mu}$ vanishes unless $|\mu| \leq |\lambda| + p$ and $\mu \leq S_p$ [27]. Given Schubert symbols $\mu \leq \lambda$, we can apply the projection formula to get

$$N_{\lambda,p}^{\mu} = \int_X [Y_{\lambda,\mu}]^T \cdot [X_p]^T = \int_Z \psi_*[\pi^{-1}(Y_{\lambda,\mu})]^T \cdot [Z \cap \mathbb{P}(E_{n_p})]^T.$$

In [12, §5] it is shown that the projection $\psi : \pi^{-1}(Y_{\lambda,\mu}) \rightarrow Z_{\lambda,\mu}$ has positive dimensional fibers when $\lambda \not\rightarrow \mu$, and is a birational isomorphism when $\lambda \rightarrow \mu$. The Gysin pushforward $\psi_*[\pi^{-1}(Y_{\lambda,\mu})]^T$ therefore vanishes unless $\lambda \rightarrow \mu$, in which case it equals $[Z_{\lambda,\mu}]^T$. \square

4. TYPE A PIERI REDUCTION

In this section, we consider the Grassmannian $X = Gr(m, n)$. We will show that each equivariant Pieri coefficient $N_{\lambda,p}^{\mu}(X)$ is equal to a restriction coefficient

$N_{\nu, p'}^\nu(X')$ for a possibly different type A Grassmannian X' . By a result of Graham [20], each coefficient is a homogeneous polynomial in $\mathbb{Z}_{\geq 0}[-\hat{\alpha}_1, \dots, -\hat{\alpha}_{n-1}]$. Here $\hat{\alpha}_i := t_i - t_{i+1} \in H_T^2(\text{pt})$ can be naturally identified with the simple roots of $SL(n, \mathbb{C}) \subset GL(n, \mathbb{C}) = G$. More precisely, each character $\chi_i : T \rightarrow \mathbb{C}^*$ induces a weight $\epsilon_i : \text{Lie}(T) \rightarrow \mathbb{C}$, and $\hat{\alpha}_i$ is identified with the simple root $\epsilon_i - \epsilon_{i+1}$. We will use an analogous identification in types B , C , and D without further comment.

The special Schubert varieties $X_p = X_{S_p}$ are indexed by integers $1 \leq p \leq n - m$, and satisfy $X_p = \pi(\psi^{-1}(\mathbb{P}(E_{n+1-p-m})))$. The special Schubert symbol S_p is given by

$$S_p = \{n + 1 - m - p, n + 2 - m, \dots, n\},$$

which corresponds to the special partition $(p, 0, \dots, 0)$ in the traditional parameterization of Schubert varieties by partitions.

Given Schubert symbols $\mu \leq \lambda$, we let x_1, \dots, x_n denote the basis of V^* dual to $\mathbf{e}_1, \dots, \mathbf{e}_n$. The projected Richardson variety $Z_{\lambda, \mu} = \psi(\pi^{-1}(Y_{\lambda, \mu})) \subset \mathbb{P}^{n-1}$ is defined by the linear equations $x_c = 0$ with c varying over the set

$$\mathcal{L}_{\lambda, \mu} := \{c \in [1, n] : c \text{ is a zero column of } D(\lambda, \mu)\}$$

(see e.g. [17, §9.4] or [15, Lemma 3.3]). We define an associated set

$$\nu := \nu(\lambda, \mu) = [1, n] \setminus \mathcal{L}_{\lambda, \mu}$$

which is simply the complement of $\mathcal{L}_{\lambda, \mu}$. The set ν can naturally be thought of as a Schubert symbol for the Grassmannian $X' := Gr(m', n)$ with $m' := \#\nu$.

We consider the natural projections

$$(4.1) \quad Gr(m', n) \xleftarrow{\pi'} Fl(1, m'; n) \xrightarrow{\psi'} Gr(1, n) = \mathbb{P}^{n-1},$$

and study the projected Richardson variety $Z_{\nu, \nu} := \psi' \circ (\pi')^{-1}(X'_\nu \cap X'^{\nu}) \subset \mathbb{P}^{n-1}$. We have the following trivial but important observation.

Lemma 4.1. *The projected Richardson varieties $Z_{\lambda, \mu}$ and $Z_{\nu, \nu}$ in \mathbb{P}^{n-1} are equal.*

Proof. The diagrams $D_X(\lambda, \mu)$ and $D_{X'}(\nu, \nu)$ have the same zero columns, and hence the same equations define $Z_{\lambda, \mu}$ and $Z_{\nu, \nu}$. \square

Given Schubert symbols $\lambda = \{\lambda_1 < \dots < \lambda_m\}$ and $\mu = \{\mu_1 < \dots < \mu_m\}$ in $\mathfrak{S}(X)$, we write $\lambda \rightarrow \mu$, if both of the following hold: (1) $\mu \leq \lambda$; (2) $\lambda_i < \mu_{i+1}$ for $1 \leq i \leq m - 1$. In other words the Richardson diagram $D(\lambda, \mu)$ is well defined, and none of its columns have more than one star².

Proposition 4.2 (Equivariant Pieri rule for $Gr(m, n)$). *Given Schubert symbols $\lambda \rightarrow \mu$ and a positive integer $p \leq n - m$ such that $|\mu| \leq |\lambda| + p$, we define $p' := |\lambda| + p - |\mu| \geq 0$. We then have*

$$N_{\lambda, p}^\mu(X) = N_{\nu, p'}^\nu(X').$$

Furthermore if $\mu \leq S_p$, then $N_{\lambda, p}^\mu(X) \neq 0$.

Proof. Since $\lambda \rightarrow \mu$, we have $\dim(Y_{\lambda, \mu}) = \#\nu - m = m' - m$. On the other hand, $\dim(Y_{\lambda, \mu}) = |\mu| - |\lambda|$. It follows that $m' - m = p - p'$. Therefore we have

$$\int_X^T [X_\lambda]^T \cdot [X^\mu]^T \cdot [X_p]^T = \int_{\mathbb{P}^{n-1}}^T [Z_{\lambda, \mu}]^T \cdot [\mathbb{P}(E_{n+1-p-m})]^T$$

²Equivalently, letting $P(\lambda)$ and $P(\mu)$ be the corresponding *partitions*, the skew-diagram $P(\mu)/P(\lambda)$ is a horizontal strip.

$$\begin{aligned}
&= \int_{\mathbb{P}^{n-1}}^T [Z_{\nu, \nu}]^T \cdot [\mathbb{P}(E_{n+1-p'-m'})]^T \\
&= \int_{X'}^T [X'_{\nu}]^T \cdot [(X')^{\nu}]^T \cdot [X'_{p'}]^T.
\end{aligned}$$

The first and third equalities follow from Proposition 3.1, and the second equality follows from Lemma 4.1 and the equality $m' + p' = m + p$.

The statement of nonvanishing follows from the fact that $\mu \leq S_p$ if and only if $\mu_1 \leq n + 1 - m - p$, and $\nu \leq S_{p'}$ if and only if $\nu_1 \leq n + 1 - m' - p'$. Since $\nu_1 = \mu_1$ and $-m - p = -m' - p'$, these statements are equivalent. It is well known that $N_{\nu, p'}^{\nu} \neq 0$ exactly when $\nu \leq S_{p'}$ (see e.g. [26]). \square

There have been several manifestly positive formulas for the restriction coefficients $N_{\nu, p'}^{\nu}(X')$ [27, 2, 5, 14], and we include one in Appendix A that uses Schubert symbols. Combining Lemma A.2 with Proposition 4.2 yields an equivariant Pieri rule for the ordinary Grassmannian.

Example 4.3. Let $X = Gr(3, 8)$ and let $\lambda = \{1, 4, 8\}$ and $\mu = \{1, 3, 6\}$. The Richardson diagram $D(\lambda, \mu)$ is then

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & * \end{pmatrix}.$$

The associated set $\nu := \nu(\lambda, \mu) = \{1, 3, 4, 6, 7, 8\}$ is a Schubert symbol for $X' = Gr(6, 8)$. We have $N_{\lambda, 5}^{\mu}(X) = N_{\nu, 2}^{\nu}(X') = (t_2 - t_1)(t_5 - t_1)$, where the final equality follows from Lemma A.2.

5. TYPE C PIERI REDUCTION

In this section, we consider a symplectic Grassmannian $X = SG(m, 2n)$. The special Schubert varieties $X_p = X_{S_p}$ are indexed by integers $1 \leq p \leq 2n - m$. Since $IG_{\omega}(1, 2n) = \mathbb{P}^{2n-1}$, we have $X_p = \pi(\psi^{-1}(\mathbb{P}(E_{n_p})))$, where $n_p := 2n + 1 - m - p$. Furthermore, we have

$$S_p = \begin{cases} \{n_p\} \cup [2n + 2 - m, 2n] & \text{if } n_p > m - 1, \\ (\{n_p\} \cup [2n + 1 - m, 2n]) \setminus \{2n + 1 - n_p\} & \text{if } n_p \leq m - 1. \end{cases}$$

Given Schubert symbols $\mu \leq \lambda \in \mathfrak{S}(X)$, we call $c \in [0, 2n]$ a *cut*³ in the Richardson diagram $D(\lambda, \mu)$ if either $\lambda_j \leq c < \mu_{j+1}$ or $\lambda_j \leq 2n - c < \mu_{j+1}$ for some $j \in [0, m]$. We notice that 0 and $2n$ are always cuts, and set

$$\begin{aligned}
\mathcal{L}_{\lambda, \mu} &:= \{c \in [1, 2n] : \lambda_j < c < \mu_{j+1} \text{ for some } j \in [0, m]\} \\
&\quad \bigcup \{c \in [1, 2n] : \mu_j = 2n + 1 - c = \lambda_j \text{ for some } j \in [1, m]\}, \\
\mathcal{Q}_{\lambda, \mu} &:= \{c \in [2, n] : c \text{ is a cut in } D(\lambda, \mu) \text{ and } c - 1 \text{ is not}\}.
\end{aligned}$$

Let $\{x_j\}$ denote the basis of V^* dual to $\{\mathbf{e}_j\}$. It is shown in [12, §5] that the projected Richardson variety $Z_{\lambda, \mu}$ is the complete intersection in \mathbb{P}^{2n-1} cut out by the polynomials

$$(1) \quad \{x_c : c \in \mathcal{L}_{\lambda, \mu}\}, \text{ and}$$

³Our definition of *cut* differs from [12], wherein c must satisfy $\lambda_j \leq c < \mu_{j+1}$ for some j . However subsequent notions are equivalent.

- (2) $\{x_{d+1}x_{2n-d} + \dots + x_c x_{2n+1-c} : c \in \mathcal{Q}_{\lambda,\mu}\}$, where d is the largest cut less than c .

We therefore let

$$m' := 2n - \#\mathcal{L}_{\lambda,\mu} - \#\mathcal{Q}_{\lambda,\mu} = \dim(Z_{\lambda,\mu}) + 1 \geq 1$$

be the dimension of the affine cone over $Z_{\lambda,\mu}$. Following [12, §5], we write $\lambda \rightarrow \mu$ if the Richardson diagram $D(\lambda, \mu)$ is defined, contains no 2×2 blocks of stars, and whenever it contains two stars in column c , it contains one star in column $2n+1-c$. That is,

Definition 5.1. *Given Schubert symbols λ and μ in $\mathfrak{S}(X)$, we write $\lambda \rightarrow \mu$ when*

- (1) $\mu \leq \lambda$,
- (2) $\lambda_i \leq \mu_{i+1}$ for $1 \leq i \leq m-1$, and
- (3) if $\lambda_i = \mu_{i+1}$ for some i , then $\mu_j < 2n+1-\lambda_i < \lambda_j$ for some j .

Given $\lambda \rightarrow \mu$ in $\mathfrak{S}(X)$, let $\nu(\lambda, \mu) := [1, 2n] \setminus \mathcal{L}_{\lambda,\mu}$. For any subset $\mathcal{I} \subset \mathcal{Q}_{\lambda,\mu}$, we define an associated set

$$(5.1) \quad \nu_{\mathcal{I}}(\lambda, \mu) := \nu(\lambda, \mu) \setminus (\mathcal{I} \cup \{2n+1-c : c \in \mathcal{Q}_{\lambda,\mu} \setminus \mathcal{I}\}),$$

which we simply write as $\nu_{\mathcal{I}}$ whenever there is no confusion with λ, μ . Moreover, we shall naturally think of $\nu_{\mathcal{I}}$ as a Schubert symbol for the type A Grassmannian $X' := Gr(m', 2n)$, by noting that the set $\nu_{\mathcal{I}}$ always has cardinality m' for any \mathcal{I} . We therefore obtain projected Richardson varieties $Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}} \subset \mathbb{P}^{2n-1}$ as defined in the case of type A Grassmannians using (4.1). We note that $Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}$ is the complete intersection in \mathbb{P}^{2n-1} cut out by the linear equations $x_c = 0$ for $c \in \mathcal{L}_{\lambda,\mu}$, $x_c = 0$ for $c \in \mathcal{I}$, and $x_{2n+1-c} = 0$ for $c \in (\mathcal{Q} \setminus \mathcal{I})$.

We have the following important lemma.

Lemma 5.2. *As classes in $H_T^*(\mathbb{P}^{2n-1})$, we have*

$$[Z_{\lambda,\mu}]^T = \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda,\mu}} [Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}]^T.$$

Proof. Let $\zeta := c_1^T(\mathcal{O}_{\mathbb{P}^{2n-1}}(1)) \in H_T^*(\mathbb{P}^{2n-1})$ be the first equivariant Chern class of the dual to the tautological subbundle on \mathbb{P}^{2n-1} . For any $j \in [1, 2n]$, there is a T -invariant hyperplane $Z(x_j)$ defined by the single equation $x_j = 0$. We notice that the natural T -equivariant morphism $\mathcal{O}_{\mathbb{P}^{2n-1}}(-1) \rightarrow \mathbb{C}^{2n} \rightarrow \mathbb{C}\mathbf{e}_j$ defines a T -equivariant section of $\mathcal{O}_{\mathbb{P}^{2n-1}}(1) \otimes \mathbb{C}\mathbf{e}_j$, whose zero set is just $Z(x_j)$. Since

$$\mathbb{C}\mathbf{e}_j = \begin{cases} \rho_X^* \mathbb{C}\chi_j & \text{if } 1 \leq j \leq n, \text{ and} \\ \rho_X^* \mathbb{C}_{-\chi_{2n+1-j}} & \text{if } n+1 \leq j \leq 2n, \end{cases}$$

the equivariant hyperplane class $[Z(x_j)]^T$ is given by

$$[Z(x_j)]^T = c_1^T(\mathcal{O}_{\mathbb{P}^{2n-1}}(1) \otimes \mathbb{C}\mathbf{e}_j) = \begin{cases} \zeta + t_j & \text{for } 1 \leq j \leq n \\ \zeta - t_{2n+1-j} & \text{for } n+1 \leq j \leq 2n. \end{cases}$$

Further details can be found in (for example) [16, Lecture 4]. For any integers $0 < d < c \leq n$, the quadratic polynomial $f_{d,c} := x_{d+1}x_{2n-d} + \dots + x_c x_{2n+1-c}$ defines a nonzero T -equivariant section of the line bundle $\mathcal{O}_{\mathbb{P}^{2n-1}}(1) \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}(1)$ over \mathbb{P}^{2n-1} . It follows that the equivariant quadric class $[Z(f_{d,c})]^T$ satisfies

$$[Z(f_{d,c})]^T = c_1^T(\mathcal{O}_{\mathbb{P}^{2n-1}}(1) \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}(1)) = 2\zeta.$$

Note that $2\zeta = (\zeta + t_c) + (\zeta - t_c)$ for any $c \in [1, n]$. In particular, we have

$$\prod_{c \in \mathcal{Q}_{\lambda, \mu}} (2\zeta) = \prod_{c \in \mathcal{Q}_{\lambda, \mu}} ((\zeta + t_c) + (\zeta - t_c)) = \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}} \left(\prod_{c \in \mathcal{I}} (\zeta + t_c) \prod_{c \in \mathcal{Q}_{\lambda, \mu} \setminus \mathcal{I}} (\zeta - t_c) \right).$$

It follows that

$$\begin{aligned} [Z_{\lambda, \mu}]^T &= \prod_{c \in \mathcal{Q}_{\lambda, \mu}} (2\zeta) \prod_{c \in \mathcal{L}_{\lambda, \mu}} (\zeta + F(\hat{t}_c)) \\ &= \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}} \prod_{c \in \mathcal{I}} (\zeta + t_c) \prod_{c \in \mathcal{Q}_{\lambda, \mu} \setminus \mathcal{I}} (\zeta - t_c) \prod_{c \in \mathcal{L}_{\lambda, \mu}} (\zeta + F(\hat{t}_c)) \\ &= \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}} [Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}]^T, \end{aligned}$$

where the last equality is due to the fact that $Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}$ is defined by the linear equations $x_c = 0$ for $c \in [1, 2n] \setminus \nu_{\mathcal{I}} = \mathcal{I} \cup \{2n+1-c : c \in \mathcal{Q}_{\lambda, \mu} \setminus \mathcal{I}\} \cup \mathcal{L}_{\lambda, \mu}$. \square

The maximal torus $T \subset G = Sp(2n, \mathbb{C}) \subset GL(2n, \mathbb{C})$ acts on $V = \mathbb{C}^{2n}$ by diagonal matrices $\text{diag}\{z_1, z_2, \dots, z_n, z_n^{-1}, \dots, z_2^{-1}, z_1^{-1}\}$. It is embedded into the maximal torus $\hat{T} \subset GL(2n, \mathbb{C})$ of diagonal matrices $\text{diag}\{\hat{z}_1, \dots, \hat{z}_{2n}\}$.

This embedding induces natural morphisms $F : H_{\hat{T}}^*(\text{pt}) = \mathbb{Z}[\hat{t}_1, \dots, \hat{t}_{2n}] \rightarrow \mathbb{Z}[t_1, \dots, t_n] = H_T^*(\text{pt})$ and $\bar{F} : H_{\hat{T}}^*(\mathbb{P}^{2n-1}) \rightarrow H_T^*(\mathbb{P}^{2n-1})$, where F is given by⁴

$$\hat{t}_i \mapsto \begin{cases} t_i & \text{if } i \leq n, \\ -t_{2n+1-i} & \text{if } i \geq n+1. \end{cases}$$

Furthermore, the embedding $T \hookrightarrow \hat{T}$ induces the following commutative diagram of morphisms:

$$\begin{array}{ccc} H_{\hat{T}}^*(\mathbb{P}^{2n-1}) & \xrightarrow{\bar{F}} & H_T^*(\mathbb{P}^{2n-1}) \\ \downarrow f_{\mathbb{P}^{2n-1}}^{\hat{T}} & & \downarrow f_{\mathbb{P}^{2n-1}}^T \\ H_{\hat{T}}^*(\text{pt}) & \xrightarrow{F} & H_T^*(\text{pt}) \end{array} \quad (\star)$$

The simple roots of $GL(2n, \mathbb{C})$ are given by $\hat{\alpha}_i = \hat{t}_i - \hat{t}_{i+1}$ for $i = 1, \dots, 2n-1$, and the simple roots of G are given by $\alpha_n = 2t_n$ and $\alpha_i = t_i - t_{i+1}$ for $i = 1, \dots, n-1$. Clearly, F sends simple roots of $GL(2n, \mathbb{C})$ to simple roots of G . The Pieri coefficients $N_{\lambda, p}^{\mu}$ are elements of $\mathbb{Z}_{\geq 0}[-\alpha_1, \dots, -\alpha_n]$, as proven more generally by Graham [20]. We will express arbitrary Pieri coefficients of type C_n in terms of specializations $F \left(N_{\nu_{\mathcal{I}}, m+p-m'}^{\nu_{\mathcal{I}}} (Gr(m', 2n)) \right)$, resulting in a manifestly positive Pieri formula.

Note that any T -invariant linear subvariety L of \mathbb{P}^{2n-1} is also invariant under the action of \hat{T} . By the Borel construction of equivariant cohomology, it follows immediately that $\bar{F}([L]^{\hat{T}}) = [L]^T$, where $[L]^{\hat{T}}$ denotes the class of L in $H_{\hat{T}}^*(\mathbb{P}^{2n-1})$.

⁴In particular, the map F is determined by $c_1^{\hat{T}}(\hat{M}) \mapsto c_1^T(M)$, where \hat{M} is any one-dimensional representation of \hat{T} and M is the restriction of this representation to T via the embedding $T \hookrightarrow \hat{T}$.

In particular, since \bar{F} is a ring homomorphism, we have

$$(5.2) \quad \bar{F} \left([Z_{\nu_{\mathcal{I}, \mathcal{I}}}]^{\hat{T}} \cdot [\mathbb{P}(E_{n_p})]^{\hat{T}} \right) = [Z_{\nu_{\mathcal{I}, \mathcal{I}}}]^T \cdot [\mathbb{P}(E_{n_p})]^T.$$

We now restate and prove Theorem 1.1, which reduces arbitrary Pieri coefficients for $SG(m, 2n)$ to the specializations of restriction coefficients for $Gr(m', 2n)$ via F , resulting in a manifestly positive Pieri formula.

Theorem 5.3 (Equivariant Pieri rule for $SG(m, 2n)$). *Given $\lambda, \mu \in \mathfrak{S}(X)$ and an integer $1 \leq p \leq 2n - m$, we have $N_{\lambda, p}^\mu = 0$ unless $\lambda \rightarrow \mu$ and $|\mu| \leq |\lambda| + p$. When both hypotheses hold, we define $p' := |\lambda| + p - |\mu|$. We then have*

$$(5.3) \quad N_{\lambda, p}^\mu(X) = \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}} F \left(N_{\nu_{\mathcal{I}, p'}}^{\nu_{\mathcal{I}}} (X') \right).$$

Furthermore if $\mu \leq s_p$, then $N_{\lambda, p}^\mu(X) \neq 0$.

Proof. Due to Proposition 3.1, we can assume $\lambda \rightarrow \mu$ and $|\mu| \leq |\lambda| + p$. By [12, Propostion 5.1], we have $\dim(Y_{\lambda, \mu}) + m = \dim(Z_{\lambda, \mu}) + 1$. Since $\dim(Y_{\lambda, \mu}) = |\mu| - |\lambda|$ and $\dim(Z_{\lambda, \mu}) + 1 = m'$, we have $p' = m + p - m'$. Applying Proposition 3.1 with $n_p := 2n + 1 - p - m$, we have

$$\begin{aligned} N_{\lambda, p}^\mu(X) &= \int_{\mathbb{P}^{2n-1}}^T [Z_{\lambda, \mu}]^T \cdot [\mathbb{P}(E_{2n+1-p-m})]^T \\ &= \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}} \int_{\mathbb{P}^{2n-1}}^T [Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}]^T \cdot [\mathbb{P}(E_{2n+1-p'-m'})]^T && \text{by Lemma 5.2} \\ &= \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}} \int_{\mathbb{P}^{2n-1}}^T \bar{F}([Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}]^{\hat{T}} \cdot [\mathbb{P}(E_{2n+1-p'-m'})]^{\hat{T}}) && \text{by Equation 5.2} \\ &= \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}} F \left(\int_{\mathbb{P}^{2n-1}}^{\hat{T}} [Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}]^{\hat{T}} \cdot [\mathbb{P}(E_{2n+1-p'-m'})]^{\hat{T}} \right) && \text{by diagram } (\star) \\ &= \sum_{\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}} F(N_{\nu_{\mathcal{I}, p'}}^{\nu_{\mathcal{I}}} (X')). \end{aligned}$$

For the nonvanishing statement, note that $\mu_1 = \min(\nu_{\mathcal{I}})$ for any $\mathcal{I} \subset \mathcal{Q}_{\lambda, \mu}$. As in the type A case, it follows that if $\mu \leq s_p$, then $\nu_{\mathcal{I}} \leq s_{p'}$, and hence $F(N_{\nu_{\mathcal{I}, p'}}^{\nu_{\mathcal{I}}} (X')) \neq 0$. $N_{\lambda, p}^\mu(X)$ is therefore a sum of $2^{\#\mathcal{Q}_{\lambda, \mu}}$ nonzero polynomials in $\mathbb{Z}_{>0}[t_2 - t_1, \dots, t_n - t_{n-1}, -2t_n]$, and is itself nonzero. \square

Example 5.4. *Consider $X := SG(3, 8)$ with Schubert symbols $\lambda = \{2, 4, 8\}$ and $\mu = \{1, 3, 5\}$, and let $p = 5$. The Richardson diagram $D(\lambda, \mu)$ is as follows, with the cuts marked by solid lines:*

$$\left(\begin{array}{cc|cc|cc|cc} * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & * \end{array} \right).$$

There are no linear equations defining $Z_{\lambda, \mu}$, and hence the set $\nu(\lambda, \mu) = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Since $\mathcal{Q}_{\lambda, \mu} = \{2, 4\}$, we have the following Schubert symbols for $X' := Gr(6, 8)$:

$$\begin{aligned} \nu_{\{2, 4\}} &= \{1, 3, 5, 6, 7, 8\}, \\ \nu_{\{2\}} &= \{1, 3, 4, 6, 7, 8\}, \end{aligned}$$

$$\begin{aligned}\nu_{\{4\}} &= \{1, 2, 3, 5, 6, 8\}, \text{ and} \\ \nu_{\emptyset} &= \{1, 2, 3, 4, 6, 8\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}N_{\lambda,5}^{\mu}(X) &= F\left(N_{\nu_{\{2,4\},2}^{\nu_{\{2,4\}}}}(X') + N_{\nu_{\{2\},2}^{\nu_{\{2\}}}}(X') + N_{\nu_{\{4\},2}^{\nu_{\{4\}}}}(X') + N_{\nu_{\emptyset},2}^{\nu_{\emptyset}}(X')\right) \\ &= F((\hat{t}_2 - \hat{t}_1)(\hat{t}_4 - \hat{t}_1)) + F((\hat{t}_2 - \hat{t}_1)(\hat{t}_5 - \hat{t}_1)) \\ &\quad + F((\hat{t}_4 - \hat{t}_1)(\hat{t}_7 - \hat{t}_1)) + F((\hat{t}_5 - \hat{t}_1)(\hat{t}_7 - \hat{t}_1)) \\ &= (t_2 - t_1)(t_4 - t_1) + (t_2 - t_1)(-t_4 - t_1) \\ &\quad + (t_4 - t_1)(-t_2 - t_1) + (-t_4 - t_1)(-t_2 - t_1) \\ &= 4t_1^2.\end{aligned}$$

We note that in this example the solution is the square of the highest root. It will be interesting to find appropriate conditions that simplify the intermediate calculation.

Remark 5.5. We mention here that the Pieri formula presented in Theorem 5.3 is just one of several equivalent formulas that can be derived by our methods. In particular, let

$$\mathcal{P} \subset \{c \in [1, n] : \{c, 2n + 1 - c\} \subset \nu(\lambda, \mu)\}$$

be any subset of cardinality $\#\mathcal{Q}_{\lambda,\mu}$. Note that $\mathcal{Q}_{\lambda,\mu}$ is itself such a subset. However, substituting any such \mathcal{P} for $\mathcal{Q}_{\lambda,\mu}$ in (5.3) and (5.1) yields an equivalent formula for $N_{\lambda,p}^{\mu}(X)$. It is likely that the calculation of certain Pieri coefficients could be made simpler by using a formula derived from a subset $\mathcal{P} \neq \mathcal{Q}_{\lambda,\mu}$. We are investigating whether the equivalence of these alternate formulations reveals additional structure of the equivariant cohomology of X .

6. TYPE B PIERI REDUCTION

Throughout this section, we consider an odd orthogonal Grassmannian $X = OG(m, 2n + 1)$. The special Schubert varieties X_p are indexed by integers $1 \leq p \leq 2n - m$, and satisfy $X_p = \{\Sigma \in X : \dim(\Sigma \cap E_{n_p}) \geq 1\}$, where

$$n_p := \begin{cases} 2n + 2 - m - p & \text{if } p \leq n - m, \\ 2n + 1 - m - p & \text{if } p > n - m. \end{cases}$$

Equivalently, we have $X_p = X_{s_p}$, where

$$s_p := \begin{cases} \{n_p\} \cup [2n + 3 - m, 2n + 1] & \text{if } n_p > m - 1, \text{ and} \\ (\{n_p\} \cup [2n + 2 - m, 2n + 1]) \setminus \{2n + 2 - n_p\} & \text{if } n_p \leq m - 1. \end{cases}$$

Definition 6.1. Given Schubert symbols λ and μ in $\mathfrak{S}(X)$, we write $\lambda \rightarrow \mu$ when

- (1) $\mu \leq \lambda$,
- (2) $\lambda_i \leq \mu_{i+1}$ for $1 \leq i \leq m - 1$, and
- (3) if $\lambda_i = \mu_{i+1}$ for some i , then $\mu_j < 2n + 2 - \lambda_i < \lambda_j$ for some j .

Given Schubert symbols $\mu \leq \lambda$ in $\mathfrak{S}(X)$, we call $c \in [0, 2n + 1]$ a *cut* in $D(\lambda, \mu)$ if either $\lambda_j \leq c < \mu_{j+1}$ or $\lambda_j \leq 2n + 1 - c < \mu_{j+1}$ for some $j \in [0, m]$. We notice that 0 and $2n + 1$ are always cuts. Recall that $c \in [1, N]$ is a zero column of $D(\lambda, \mu)$ if $\lambda_j < c < \mu_{j+1}$ for some j . We set

$$\begin{aligned}\mathcal{L}_{\lambda,\mu} &:= \{c \in [1, 2n+1] : c \text{ is a zero column in } D(\lambda, \mu)\} \\ &\quad \cup \{c \in [1, 2n+1] : \mu_j = 2n+2-c = \lambda_j \text{ for some } j \in [1, m]\}, \\ \mathcal{Q}_{\lambda,\mu} &:= \{c \in [2, n+1] : c-1 \text{ is not a cut, and either } c \text{ is a cut or } c = n+1\}.\end{aligned}$$

Let x_1, \dots, x_{2n+1} denote the basis of V^* dual to $\mathbf{e}_1, \dots, \mathbf{e}_{2n+1}$. The projected Richardson variety $Z_{\lambda,\mu}$ is a complete intersection in \mathbb{P}^{2n-1} cut out by the polynomials

- (1) $\{x_c : c \in \mathcal{L}_{\lambda,\mu}\}$, and
- (2) $\{x_{\hat{d}+1}x_{2n+1-\hat{d}} + \dots + x_c x_{2n+2-c} : c \in \mathcal{Q}_{\lambda,\mu}\}$, where \hat{d} is the largest cut less than c .

If $\mathcal{Q}_{\lambda,\mu} \neq \emptyset$, let $\hat{c} \in \mathcal{Q}_{\lambda,\mu}$ be an arbitrary element. Let $Z'_{\lambda,\mu} \subset \mathbb{P}^{2n}$ denote the subvariety cut out by the same polynomials defining $Z_{\lambda,\mu}$ except the quadratic polynomial corresponding to \hat{c} (namely $x_{\hat{d}+1}x_{2n+1-\hat{d}} + \dots + x_{\hat{c}}x_{2n+2-\hat{c}}$, where \hat{d} is the largest element of $\mathcal{Q}_{\lambda,\mu}$ that is less than \hat{c}). Fixing an integer $1 \leq p \leq 2n-m$, we set the following definitions:

$$\begin{aligned}\mathcal{Q}'_{\lambda,\mu} &:= \begin{cases} \mathcal{Q}_{\lambda,\mu} \setminus \{\hat{c}\} & \text{if } p > n-m \text{ and } \mathcal{Q}_{\lambda,\mu} \neq \emptyset, \\ \mathcal{Q}_{\lambda,\mu} & \text{otherwise.} \end{cases} \\ m' &:= 2n+1 - \#\mathcal{L}_{\lambda,\mu} - \#\mathcal{Q}'_{\lambda,\mu}, \\ X' &:= Gr(m', 2n+1).\end{aligned}$$

Let $\nu(\lambda, \mu) := [1, 2n+1] \setminus \mathcal{L}_{\lambda,\mu}$. Let $\nu^+(\lambda, \mu) := \nu(\lambda, \mu) \cup \{n+1\}$. Finally, for each subset $\mathcal{I} \subset \mathcal{Q}'_{\lambda,\mu}$, we let

$$\nu_{\mathcal{I}}(\lambda, \mu) := \nu(\lambda, \mu) \setminus (\mathcal{I} \cup \{2n+2-c : c \in \mathcal{Q}'_{\lambda,\mu} \setminus \mathcal{I}\}).$$

We will simply denote these sets by ν , ν^+ , and $\nu_{\mathcal{I}}$ whenever there is no confusion about λ, μ . Moreover, we will naturally think of $\nu_{\mathcal{I}}$ (resp. ν^+ when $p > n-m$ and $\mathcal{Q}_{\lambda,\mu} = \emptyset$) as a Schubert symbol for the type A Grassmannian X' (resp. $Gr(m'+1, 2n+1)$). We therefore obtain projected Richardson varieties $Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}$ (resp. Z_{ν^+, ν^+}) in \mathbb{P}^{2n} , as defined in the case of type A Grassmannians using (4.1).

The maximal torus $T \subset G = SO(2n+1, \mathbb{C})$ acts on $V = \mathbb{C}^{2n+1}$ by diagonal matrices $\text{diag}\{z_1, z_2, \dots, z_n, 1, z_n^{-1}, \dots, z_2^{-1}, z_1^{-1}\}$. It is embedded into a larger torus $(\mathbb{C}^*)^{2n+1} \subset GL(2n+1, \mathbb{C})$ of diagonal matrices $\text{diag}\{\hat{z}_1, \dots, \hat{z}_{2n+1}\}$. This induces a ring homomorphism $F_B : \mathbb{Z}[\hat{t}_1, \dots, \hat{t}_{2n+1}] \rightarrow \mathbb{Z}[t_1, \dots, t_n]$ defined by

$$\hat{t}_i \mapsto \begin{cases} t_i & \text{if } i \leq n \\ 0 & \text{if } i = n+1 \\ -t_{2n+2-i} & \text{if } i \geq n+2. \end{cases}$$

The simple roots of G are given by $\alpha_n = t_n$ and $\alpha_i = t_i - t_{i+1}$ for $i = 1, \dots, n-1$. It is easy to check that the specialization map F_B sends simple roots of $GL(2n+1, \mathbb{C})$ to simple roots of G . The Pieri coefficients $N_{\lambda,p}^t(X)$ are elements of $\mathbb{Z}_{\geq 0}[-\alpha_1, \dots, -\alpha_n]$ by [20]. Using F_B , we will express arbitrary Pieri coefficients for type B_n in terms of specializations of type A coefficients, resulting in a manifestly positive Pieri formula.

We can now state the main result of this section.

Theorem 6.2 (Equivariant Pieri rule for $OG(m, 2n+1)$). *Given $\lambda, \mu \in \mathfrak{S}(X)$ and an integer $1 \leq p \leq 2n - m$, we have $N_{\lambda,p}^\mu = 0$ unless $\lambda \rightarrow \mu$ and $|\mu| \leq |\lambda| + p$. When both hypotheses hold, we define $p' := |\lambda| + p - |\mu|$. We then have*

$$N_{\lambda,p}^\mu(X) = \begin{cases} \frac{1}{2} F_B \left(N_{\nu^+, p'}^{\nu^+}(\text{Gr}(m'+1, 2n+1)) \right) & \text{if } p > n - m \text{ and } \mathcal{Q}_{\lambda,\mu} = \emptyset, \text{ and} \\ \sum_{\mathcal{I} \subset \mathcal{Q}'} F_B \left(N_{\nu^{\mathcal{I}}, p'}^{\nu^{\mathcal{I}}}(\mathcal{X}') \right) & \text{otherwise.} \end{cases}$$

Furthermore if $\mu \leq s_p$, then $N_{\lambda,p}^\mu(X) \neq 0$.

Consider the $(2n-1)$ dimensional quadric $Q := OG(1, 2n+1)$, with inclusion map $\iota : Q \hookrightarrow \mathbb{P}^{2n}$. By Proposition 3.1, we have $N_{\lambda,p}^\mu = \int_Q^T [Z_{\lambda,\mu}]^T \cdot [Q \cap \mathbb{P}(E_{n_p})]^T$, which we shall reduce further to a calculation on $H_T^*(\mathbb{P}^{2n})$. Note that $\mathbb{P}(E_{n_p})$ is not contained in Q if and only if $p \leq n-m$, and in this case $[Q \cap \mathbb{P}(E_{n_p})]^T = \iota^* [\mathbb{P}(E_{n_p})]^T$. By the following lemma, we can also express $[Q \cap \mathbb{P}(E_{n_p})]^T = [\mathbb{P}(E_{n_p})]^T$ as the pullback of a class in $H_T^*(\mathbb{P}^{2n})$ when $p > n-m$.

Lemma 6.3. *Given $n-m < p \leq 2n-m$, we have*

$$[Q \cap \mathbb{P}(E_{n_p})]^T = \frac{1}{2} \iota^* [\mathbb{P}(E_{n_p} \oplus \langle \mathbf{e}_{n+1} \rangle)]^T.$$

Proof. We thank the referee for the following simplified argument. As in type C , let $\zeta := c_1^T(\mathcal{O}_{\mathbb{P}^{2n}}(1)) \in H_T^*(\mathbb{P}^{2n})$ be the first equivariant Chern class of the dual to the tautological subbundle on \mathbb{P}^{2n} . For any $j \in [1, 2n+1]$, the corresponding equivariant hyperplane class is defined by

$$[Z(x_j)]^T = \zeta + F_B(\hat{t}_j) = \begin{cases} \zeta + t_j & \text{for } 1 \leq j \leq n \\ \zeta & \text{for } j = n+1 \\ \zeta - t_{2n+2-j} & \text{for } n+2 \leq j \leq 2n+1. \end{cases}$$

For any integers $1 \leq d < c \leq n+1$, the corresponding equivariant quadric class is defined by

$$[Z(x_d x_{2n+2-d} + \dots x_c x_{2n+2-c})]^T = 2\zeta.$$

We have

$$\begin{aligned} & \iota_* \iota^* [\mathbb{P}(E_{n_p} \oplus \langle \mathbf{e}_{n+1} \rangle)]^T \\ &= [\mathbb{P}(E_{n_p} \oplus \langle \mathbf{e}_{n+1} \rangle)]^T \cdot \iota_* [Q]^T \\ &= [\mathbb{P}(E_{n_p} \oplus \langle \mathbf{e}_{n+1} \rangle)]^T \cdot 2\zeta \\ &= 2[\mathbb{P}(E_{n_p})]^T \\ &= 2\iota_* [Q \cap \mathbb{P}(E_{n_p})]^T, \end{aligned}$$

where the third equality holds because $\zeta = [Z(x_{n+1})]^T$. Since the pushforward ι_* is injective, the result follows. \square

We can now prove the main result of this section.

Proof of Theorem 6.2. Due to Proposition 3.1, we can assume $\lambda \rightarrow \mu$ and $|\mu| \leq |\lambda| + p$. By [12, Propostion 5.1], we have $\dim(Y_{\lambda,\mu}) + m = \dim(Z_{\lambda,\mu}) + 1$. Since

$\dim(Y_{\lambda,\mu}) = |\mu| - |\lambda|$ and $m' = \dim(Z_{\lambda,\mu}) + 1$ (resp. $\dim(Z_{\lambda,\mu}) + 2$ if $\mathcal{Q}_{\lambda,\mu} \neq \emptyset$ and $p > n - m$), it follows that

$$p' = \begin{cases} m + p - m' + 1 & \text{if } \mathcal{Q}_{\lambda,\mu} \neq \emptyset \text{ and } p > n - m, \\ m + p - m' & \text{otherwise.} \end{cases}$$

We consider the following two cases.

Case 1: Suppose $\mathcal{Q}_{\lambda,\mu} = \emptyset$. In this case $Z_{\lambda,\mu}$ is a linear subvariety of $Q \subset \mathbb{P}^{2n}$ and is equal to $Z_{\nu,\nu}$. Therefore, by Proposition 3.1, we have $N_{\lambda,p}^\mu = \int_Q^T [Z_{\nu,\nu}]^T \cdot [Q \cap \mathbb{P}(E_{n_p})]^T$.

If $p \leq n - m$, then

$$[Q \cap \mathbb{P}(E_{n_p})]^T = \iota^*[\mathbb{P}(E_{2n+2-m-p})]^T = \iota^*[\mathbb{P}(E_{2n+2-m'-p'})]^T.$$

By the projection formula we conclude $N_{\lambda,p}^\mu = F_B(N_{\nu,p'}^\nu(X'))$.

Now, observe that we must have $n+1 \in \mathcal{L}_{\lambda,\mu}$ (since n and $n+1$ are both cuts and $n+1 \notin \lambda \cup \mu$). Recall that $Z(x_{n+1}) \subset \mathbb{P}^{2n}$ is the hyperplane cut out by $x_{n+1} = 0$. If $p > n - m$, then by Lemma 6.3 we have

$$\begin{aligned} N_{\lambda,p}^\mu &= \frac{1}{2} \int_{\mathbb{P}^{2n}}^T [Z_{\nu,\nu}]^T \cdot [\mathbb{P}(E_{n_p} \oplus \langle \mathbf{e}_{n+1} \rangle)]^T \\ &= \frac{1}{2} \int_{\mathbb{P}^{2n}}^T [Z_{\nu^+,\nu^+}]^T \cdot [Z(x_{n+1})]^T \cdot [\mathbb{P}(E_{n_p} \oplus \langle \mathbf{e}_{n+1} \rangle)]^T \\ &= \frac{1}{2} \int_{\mathbb{P}^{2n}}^T [Z_{\nu^+,\nu^+}]^T \cdot [\mathbb{P}(E_{n_p})]^T \\ &= \frac{1}{2} \int_{\mathbb{P}^{2n}}^T [Z_{\nu^+,\nu^+}]^T \cdot [\mathbb{P}(E_{2n+2-(m'+1)-p'})]^T \\ &= \frac{1}{2} F_B(N_{\nu^+,p'}^{\nu^+}(X')). \end{aligned}$$

Case 2: Now suppose $\mathcal{Q}_{\lambda,\mu} \neq \emptyset$. Note that $Z'_{\lambda,\mu} \cap Q = Z_{\lambda,\mu}$ and $\iota^*[Z'_{\lambda,\mu}]^T = [Z_{\lambda,\mu}]^T \in H_T^*(Q)$. As in Case 1, we could use Lemma 6.3 along with the projection formula to move our calculation to \mathbb{P}^{2n} and prove the desired result. However, we will instead use the fact that $[Z_{\lambda,\mu}]^T = \iota^*[Z'_{\lambda,\mu}]^T$ along with the projection formula to move to \mathbb{P}^{2n} . The advantage of this alternate proof is that it generalizes immediately to the type D case, where there is no analogue of Lemma 6.3.

Since $\mathbb{P}(E_{n_p}) \subset Q$ for $p > n - m$, we have

$$\iota_*[\mathbb{P}(E_{n_p}) \cap Q]^T = \begin{cases} [\mathbb{P}(E_{n_p})]^T \cdot [Q]^T & \text{if } p \leq n - m \\ [\mathbb{P}(E_{n_p})]^T & \text{if } p > n - m \end{cases}$$

We therefore have

$$\begin{aligned} N_{\lambda,p}^\mu(X) &= \int_Q^T [Z_{\lambda,\mu}]^T \cdot [\mathbb{P}(E_{n_p}) \cap Q]^T \\ &= \begin{cases} \int_{\mathbb{P}^{2n}}^T [Z'_{\lambda,\mu}]^T \cdot [\mathbb{P}(E_{n_p})]^T \cdot [Q]^T & \text{if } p \leq n - m \\ \int_{\mathbb{P}^{2n}}^T [Z'_{\lambda,\mu}]^T \cdot [\mathbb{P}(E_{n_p})]^T & \text{if } p > n - m \end{cases} \\ &= \begin{cases} \int_{\mathbb{P}^{2n}}^T [Z_{\lambda,\mu}]^T \cdot [\mathbb{P}(E_{n_p})]^T & \text{if } p \leq n - m \\ \int_{\mathbb{P}^{2n}}^T [Z'_{\lambda,\mu}]^T \cdot [\mathbb{P}(E_{n_p})]^T & \text{if } p > n - m \end{cases} \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{P}^{2n}} \sum_{\mathcal{I} \subset \mathcal{Q}'_{\lambda, \mu}} [Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}]^T \cdot [\mathbb{P}(E_{2n+2-m'-p'})]^T \\
&= \sum_{\mathcal{I} \subset \mathcal{Q}'_{\lambda, \mu}} F_B \left(N_{\nu_{\mathcal{I}}, p'}^{\nu_{\mathcal{I}}}(X') \right),
\end{aligned}$$

where the subvarieties $Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}$ are defined as in (4.1). The fact that $[Z_{\lambda, \mu}]^T$ (resp. $[Z'_{\lambda, \mu}]^T$ for $p > n - m$) is equal to $\sum_{\mathcal{I} \subset \mathcal{Q}'_{\lambda, \mu}} [Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}}]^T$ follows from Lemma 5.2. For the nonvanishing statement, let s be any of the Schubert symbols $\nu_{\mathcal{I}}$ in Cases 1 and 3 (resp. ν^+ in Case 2), and note that $\mu_1 = \min(s)$. As in the type A case, it follows that if $\mu \leq s_p$, then $s \leq s_{p'}$, and hence $F_B(N_{s, p'}^s(X')) \neq 0$. In Cases 1 and 3, it follows that $N_{\lambda, p}^{\mu}(X)$ is a sum of $2^{\#\mathcal{Q}'_{\lambda, \mu}}$ polynomials in $\mathbb{Z}_{>0}[t_2 - t_1, \dots, t_n - t_{n-1}, -t_n]$, and in particular is nonzero. \square

Remark 6.4. *When $p > n - m$ and $\mathcal{Q}_{\lambda, \mu} = \emptyset$, the Pieri coefficient $N_{\lambda, p}^{\mu}(X) = \frac{1}{2} F_B(N_{\nu^+, p'}^{\nu^+}(X'))$ is indeed an element of $\mathbb{Z}[t_1, \dots, t_n]$, since 2 divides $\iota^*[\mathbb{P}(E_{n_p} \oplus \langle \mathbf{e}_{n+1} \rangle)]^T$ by Lemma 6.3. By Lemma 7.5, we will see that $N_{\lambda, p}^{\mu}(X)$ is equal to the type B restriction coefficient $N_{\nu, p'}^{\nu}(OG(m', 2n + 1))$. In particular, any restriction coefficient of Lie type B is the specialization of a restriction coefficient of Lie type A, up to a factor of 2.*

Example 6.5. *Let $X := OG(2, 7)$, let $p = 3$, and consider Schubert symbols $\lambda = \{3, 6\}$ and $\mu = \{1, 6\}$. Note that $p > n - m = 1$. We have $\mathcal{L}_{\lambda, \mu} = \{2, 4, 5, 7\}$, $\mathcal{Q}_{\lambda, \mu} = \emptyset$, $m' = 3$, $p' = 2$, $X' = Gr(4, 7)$, and $\nu^+ = \{1, 3, 4, 6\}$. Using Lemma A.2, we calculate $N_{\nu^+, 2}^{\nu^+}(X') = (\hat{t}_5 - \hat{t}_1)(\hat{t}_7 - \hat{t}_1)$. Hence $N_{\lambda, 3}^{\mu} = \frac{1}{2}(-t_3 - t_1)(-2t_1) = (-t_3 - t_1)(-t_1)$.*

7. TYPE D PIERI REDUCTION

Throughout this section, we consider an even orthogonal Grassmannian $X = OG(m, 2n)$, where $m \leq n$. We note that our results apply equally well to $OG(n, 2n)$, which has two connected components, and to $OG(n - 1, 2n)$, which is realized as a homogeneous space by quotienting $SO(2n, \mathbb{C})$ by a *submaximal* parabolic subgroup. For each $1 \leq p \leq 2n - m - 1$, the special Schubert variety X_p has codimension p and satisfies $X_p = \{\Sigma \in X : \dim(\Sigma \cap E_{n_p}) \geq 1\}$, where

$$n_p := \begin{cases} 2n + 1 - m - p & \text{if } p < n - m \\ 2n - m - p & \text{if } p \geq n - m. \end{cases}$$

Equivalently, we have $X_p = X_{s_p}$, where

$$s_p := \begin{cases} \{n_p\} \cup [2n + 2 - m, 2n] & \text{if } n_p > m - 1, \\ (\{n_p\} \cup [2n + 1 - m, 2n]) \setminus \{2n + 2 - n_p\} & \text{if } n_p \leq m - 1, \end{cases}$$

We mention here that there is an additional special Schubert variety \tilde{X}_{n-m} of codimension $n - m$ in X . We will discuss multiplication by the Schubert class $[\tilde{X}_{n-m}]^T$ in Remark 7.7.

Following [11, §A] we make the following definitions. For any Schubert symbol $\lambda \in \mathfrak{S}(X)$, let $[\lambda] = \lambda \cup \{c \in [1, 2n] : 2n + 1 - c \in \lambda\}$. We define $\text{type}(\lambda) \in \{0, 1, 2\}$ as follows. If $n \in [\lambda]$, then we let $\text{type}(\lambda)$ be congruent mod 2 to 1 plus the number

of elements in $[1, n] \setminus \lambda$. In other words, if $\#[1, n] \setminus \lambda$ is even then $\text{type}(\lambda) = 1$, and if $\#[1, n] \setminus \lambda$ is odd then $\text{type}(\lambda) = 2$. Finally, if $n \notin [\lambda]$, we set $\text{type}(\lambda) = 0$.

In type D , the Bruhat order is characterized by a stricter relation on the Schubert symbols.

Definition 7.1. *Given Schubert symbols λ and μ in $\mathfrak{S}(OG(m, 2n))$, we write $\mu \preceq \lambda$ if the following conditions hold:*

- (1) $\mu \leq \lambda$, and
- (2) if there exists $c \in [1, n-1]$ such that $[c+1, n] \subset [\lambda] \cap [\mu]$ and $\#\lambda \cap [1, c] = \#\mu \cap [1, c]$, then we have $\text{type}(\lambda) = \text{type}(\mu)$.

By [11, Proposition A.2], we have $\mu \preceq \lambda$ if and only if $X_\mu \subset X_\lambda$.

Definition 7.2. *Given Schubert symbols λ and μ in $\mathfrak{S}(X)$, we write $\lambda \rightarrow \mu$ when*

- (1) $\mu \preceq \lambda$;
- (2) $\lambda_i \leq \mu_{i+1}$ for $1 \leq i \leq m-1$, unless $\lambda_i = n+1$ and $\mu_{i+1} = n$;
- (3) if $\lambda_i = \mu_{i+1}$ for some i , then $\mu_j < 2n+2-\lambda_i < \lambda_j$ for some j ; and
- (4) it is not the case that $\lambda_i = \mu_{i+1} \in \{n, n+1\}$ for any $1 \leq i \leq m-1$.

Remark 7.3. *Given arbitrary Schubert symbols $\lambda, \mu \in \mathfrak{S}(X)$, the definition of a cut in $D(\lambda, \mu)$ is significantly more complicated in type D (see [41] for details). However, assuming $\lambda \rightarrow \mu$, the situation is simpler.*

Given Schubert symbols $\lambda \rightarrow \mu \in \mathfrak{S}(X)$, we call $c \in [0, 2n]$ a *cut* in $D(\lambda, \mu)$ if either $\lambda_j \leq c < \mu_{j+1}$ or $\lambda_j \leq 2n-c < \mu_{j+1}$ for some $j \in [0, m]$. We notice that 0 and $2n$ are always cuts. Recall that $c \in [1, 2n]$ is a zero column of $D(\lambda, \mu)$ if $\lambda_j < c < \mu_{j+1}$ for some j . The following definitions also hold only for Schubert symbols $\lambda \rightarrow \mu$:

$$\begin{aligned} \mathcal{L}_{\lambda, \mu} &:= \{c \in [1, 2n] : c \text{ is a zero column in } D(\lambda, \mu)\} \\ &\cup \{c \in [1, 2n] : \exists j \in [1, m] \text{ such that } 2n+1-c = \mu_j = \lambda_j \\ &\quad \text{or } 2n+1-c = n+1 = \lambda_j < \mu_{j+1} \\ &\quad \text{or } 2n+1-c = n = \mu_j > \lambda_{j-1}\}, \\ \mathcal{Q}_{\lambda, \mu} &:= \{c \in [2, n] : c-1 \text{ is not a cut, and either } c \text{ is a cut or } c = n\}. \end{aligned}$$

Let x_1, \dots, x_{2n} denote the basis of V^* dual to $\mathbf{e}_1, \dots, \mathbf{e}_{2n}$. The projected Richardson variety $Z_{\lambda, \mu}$ is a complete intersection in \mathbb{P}^{2n-1} cut out by the polynomials

- (1) $\{x_c : c \in \mathcal{L}_{\lambda, \mu}\}$, and
- (2) $\{x_{d+1}x_{2n-d} + \dots + x_c x_{2n+1-c} : c \in \mathcal{Q}_{\lambda, \mu}\}$, where d is the largest cut less than c .

If $\mathcal{Q}_{\lambda, \mu} \neq \emptyset$, let $\hat{c} \in \mathcal{Q}_{\lambda, \mu}$ be an arbitrary element. Let $Z'_{\lambda, \mu} \subset \mathbb{P}^{2n-1}$ denote the subvariety cut out by the same polynomials defining $Z_{\lambda, \mu}$ except the quadratic polynomial corresponding to \hat{c} . Fixing an integer $1 \leq p \leq 2n-m-1$, we set the following definitions:

$$\begin{aligned} \mathcal{Q}'_{\lambda, \mu} &:= \begin{cases} \mathcal{Q}_{\lambda, \mu} \setminus \{\hat{c}\} & \text{if } \mathcal{Q}_{\lambda, \mu} \neq \emptyset \text{ and } p \geq n-m, \\ \mathcal{Q}_{\lambda, \mu} & \text{otherwise.} \end{cases} \\ m' &:= 2n - \#\mathcal{L}_{\lambda, \mu} - \#\mathcal{Q}'_{\lambda, \mu}, \\ X' &:= Gr(m', 2n). \end{aligned}$$

Let $\nu(\lambda, \mu) := [1, 2n] \setminus \mathcal{L}_{\lambda, \mu}$. For each subset $\mathcal{I} \subset \mathcal{Q}'_{\lambda, \mu}$, we let

$$\nu_{\mathcal{I}}(\lambda, \mu) := \nu(\lambda, \mu) \setminus (\mathcal{I} \cup \{2n+1-c : c \in \mathcal{Q}'_{\lambda, \mu} \setminus \mathcal{I}\}).$$

We will simply denote these sets as ν and $\nu_{\mathcal{I}}$ whenever there is no confusion about λ, μ . Moreover, we shall naturally think of $\nu_{\mathcal{I}}$ as a Schubert symbol for the type A Grassmannian $X' = Gr(m', 2n)$, by noting that the set $\nu_{\mathcal{I}}$ always has cardinality m' for any \mathcal{I} . We therefore obtain projected Richardson varieties $Z_{\nu_{\mathcal{I}}, \nu_{\mathcal{I}}} \subset \mathbb{P}^{2n-1}$ as defined in the case of type A Grassmannians using (4.1).

The maximal torus $T \subset G = SO(2n, \mathbb{C})$ acts on $V = \mathbb{C}^{2n}$ by diagonal matrices $\text{diag}\{z_1, z_2, \dots, z_n, z_n^{-1}, \dots, z_2^{-1}, z_1^{-1}\}$. It is embedded in a larger torus $(\mathbb{C}^*)^{2n} \subset GL(2n, \mathbb{C})$ of diagonal matrices $\text{diag}\{\hat{z}_1, \dots, \hat{z}_{2n}\}$. This induces a ring homomorphism $F_D : \mathbb{Z}[\hat{t}_1, \dots, \hat{t}_{2n}] \rightarrow \mathbb{Z}[t_1, \dots, t_n]$ defined by

$$\hat{t}_i \mapsto \begin{cases} t_i & \text{if } i \leq n \\ -t_{2n+1-i} & \text{if } i \geq n+1. \end{cases}$$

The simple roots of G are given by $\alpha_n = t_{n-1} + t_n$ and $\alpha_i = t_i - t_{i+1}$ for $i = 1, \dots, n-1$. The map F_D sends positive roots of $GL(2n, \mathbb{C})$ to positive roots of G , with the sole exception of the root $\hat{\alpha}_n = \hat{t}_n - \hat{t}_{n+1}$, which is sent to $2t_n$. However, using Corollary A.3, we will ensure that specializations of the root $\hat{\alpha}_n$ never occur in our Pieri rule. We can now state the main result of this section.

Theorem 7.4 (Equivariant Pieri rule for $OG(m, 2n)$). *Given $\lambda, \mu \in \mathfrak{S}(X)$ and an integer $1 \leq p \leq 2n - m - 1$, we have $N_{\lambda, p}^{\mu} = 0$ unless $\lambda \rightarrow \mu$ and $|\mu| \leq |\lambda| + p$. When both hypotheses hold, we define $p' := |\lambda| + p - |\mu| \geq 0$. We then have*

$$(7.1) \quad N_{\lambda, p}^{\mu}(X) = \begin{cases} N_{\nu, p'}^{\nu}(OG(m', 2n)) & \text{if } \mathcal{Q}_{\lambda, \mu} = \emptyset \text{ and } p \geq n - m, \\ \sum_{\mathcal{I} \subset \mathcal{Q}'_{\lambda, \mu}} F_D \left(N_{\nu_{\mathcal{I}}, p'}^{\nu_{\mathcal{I}}}(X') \right) & \text{otherwise.} \end{cases}$$

Furthermore if $\mu \leq s_p$, then $N_{\lambda, p}^{\mu}(X) \neq 0$.

The following lemma holds for all isotropic Grassmannians. We will need it to prove Theorem 7.4 in the case $\mathcal{Q}_{\lambda, \mu} = \emptyset$ and $p \geq n - m$.

Lemma 7.5. *Let $IG_{\omega}(m, N)$ be a Grassmannian of Lie type D_n (respectively B_n or C_n), and suppose $\lambda \rightarrow \mu$ and $1 \leq p \leq 2n - m - 1$ (respectively $1 \leq p \leq 2n - m$). If $\mathcal{Q}_{\lambda, \mu} = \emptyset$, then we have $\#\nu \leq n$, where $\nu = \nu(\lambda, \mu) = [1, N] \setminus \mathcal{L}_{\lambda, \mu}$. Furthermore if $|\mu| \leq |\lambda| + p$, then*

$$N_{\lambda, p}^{\mu}(IG_{\omega}(m, N)) = N_{\nu, |\lambda|+p-|\mu|}^{\nu}(IG_{\omega}(\#\nu, N)).$$

Proof. If $\mathcal{Q}_{\lambda, \mu} = \emptyset$, then c is a cut in $D(\lambda, \mu)$ for any $c \in [1, N]$. We then claim that $\{c, N+1-c\} \cap \mathcal{L}_{\lambda, \mu} \neq \emptyset$ for any $c \in [1, N]$. To see why, we reproduce the argument from [41, Lemma 4.14]: If we are working in type B and $c = n+1$, then $n+1$ must be a zero column, and hence be in $\mathcal{L}_{\lambda, \mu}$. Otherwise, let us consider $c \leq N/2$. Since $c-1$ is a cut, we have $\lambda_i \leq c-1 < \mu_{i+1}$ (or $\lambda_i \leq N+1-c < \mu_{i+1}$) for some i . Then c (resp. $N+1-c$) is a zero column, in which case we are done; or we have $c = \mu_{i+1} \in \mu$ (resp. $N+1-c = \lambda_i \in \lambda$). It follows that $N+1-c \in \mathcal{L}_{\lambda, \mu}$ (resp. $c \in \mathcal{L}_{\lambda, \mu}$), proving our claim. We therefore have $\#\nu = N - \#\mathcal{L}_{\lambda, \mu} \leq n$.

As a consequence, ν can be treated as a Schubert symbol for $IG_{\omega}(\#\nu, N)$. We now consider $Z_{\nu, \nu} \subset \mathbb{P}^{N-1}$ to be the projected Richardson variety coming from the

Richardson variety $Y'_{\nu,\nu} \subset IG_\omega(\#\nu, N)$. As in Lemma 4.1, we then see that $Z_{\nu,\nu} = Z_{\lambda,\mu}$. Since $|\lambda| + p - |\mu| \geq 0$, we have $N_{\lambda,p}^\mu(IG_\omega(m, N)) = N_{\nu,|\lambda|+p-|\mu|}^\nu(IG_\omega(\#\nu, N))$ by Proposition 3.1. \square

The proof of our main theorem follows easily.

Proof of Theorem 7.4. We note that as in the proof of Theorem 6.2, we have $p = m + p - m' + 1$ if $\mathcal{Q}_{\lambda,\mu} \neq \emptyset$ and $p \geq n - m$, and $p' = m + p - m'$ otherwise. If $\mathcal{Q}_{\lambda,\mu} = \emptyset$ and $p \geq n - m$ then we are done by Lemma 7.5. Let $Q := OG(1, 2n)$ denote the $(2n - 2)$ dimensional quadric of isotropic lines, with inclusion $\iota : Q \hookrightarrow \mathbb{P}^{2n-1}$. Note that $[Q \cap \mathbb{P}(E_{n_p})]^T = \iota^*[\mathbb{P}(E_{n_p})]^T$ if and only if $p < n - m$. If $\mathcal{Q}_{\lambda,\mu} = \emptyset$ and $p < n - m$, it follows that $N_{\lambda,p}^\mu = \int_Q [Z_{\nu,\nu}]^T \cdot \iota^*[\mathbb{P}(E_{n_p})]^T = F_D(N_{\nu,p'}^\nu(X'))$, as in the proof of Case 1 of Theorem 6.2. Finally, if $\mathcal{Q}_{\lambda,\mu} \neq \emptyset$, the proof is identical to the proof of Case 2 of Theorem 6.2 (simply replace F_B with F_D , $2n$ with $2n - 1$, $p \leq n - m$ with $p < n - m$, and $p > n - m$ with $p \geq n - m$). \square

In conclusion, the combination of Theorem 7.4 with Lemma A.2 and Corollary A.3 yields a manifestly positive type D Pieri rule. To see that the hypotheses of Corollary A.3 are met, note that if $I_1 \cap \nu_{\mathcal{I}} = \{n\}$, then $\mu_1 = n$, since $\mu_1 \in \nu_{\mathcal{I}}$ for any $\mathcal{I} \subset \mathcal{Q}_{\lambda,\mu}$. Therefore c is a cut for every $c \in [1, n]$, and hence $\mathcal{Q}_{\lambda,\mu} = \emptyset$. Moreover, since $I_1 = [1, n]$, we have $2n + 1 - p' - m' = n$, so $p = n - m + 1 > n - m$. In this case, the Pieri rule does not make use of the specialization F_D , and positivity follows from any one of the known positive formulas for the restriction coefficient $N_{\nu,p'}^\nu(OG(m', 2n))$ (e.g. [27, 2, 5]).

Example 7.6. Let $X := OG(1, 8)$, and suppose we wish to calculate $N_{\{2\},4}^{\{1\}}(X)$. Note that $\#\mathcal{Q}_{\{2\},\{1\}} = \emptyset$, and $p = 4 > 3 = n - m$. It follows that

$$\begin{aligned} N_{\{2\},4}^{\{1\}}(X) &= N_{\{1,2\},3}^{\{1,2\}}(OG(2, 8)) \\ &= (-t_1 - t_2)((-t_4 - t_2)(t_4 - t_2) + (-t_2 - t_1)(-t_3 - t_1)), \end{aligned}$$

where the final polynomial is computed using Anders Buch's Equivariant Schubert Calculator [9]. Interestingly, this type D restriction coefficient is not the specialization of any type A restriction coefficient (see Remark 7.8).

Remark 7.7. For the even orthogonal Grassmannian $OG(m, 2n)$, there is an additional special Schubert variety \tilde{X}_{n-m} of codimension $n - m$ defined by $\tilde{X}_{n-m} := \{\Sigma \in X : \dim(\Sigma \cap \langle \mathbf{e}_1, \dots, \mathbf{e}_{n-1}, \mathbf{e}_{n+1} \rangle) \geq 1\}$ ⁵. The type D_n Dynkin diagram automorphism induces an involution of $OG(m, 2n)$ that interchanges Schubert varieties of types 1 and 2, and maps \tilde{X}_{n-m} to X_{n-m} . Multiplication with $[\tilde{X}_{n-m}]^T$ is equivalent to multiplication with $[X_{n-m}]^T$ in the following sense: We denote by $\tilde{N}_{\lambda,n-m}^\mu(X)$ the structure coefficients of $[X_\lambda]^T \cdot [\tilde{X}_{n-m}]^T$. Then, it follows from (7.1) and the involution of $OG(m, 2n)$ that

$$\tilde{N}_{\lambda,p}^\mu(X) = \begin{cases} \tilde{N}_{\nu,p'}^\nu(X') & \text{if } \mathcal{Q}_{\lambda,\mu} = \emptyset, \text{ and} \\ \sum_{\mathcal{I} \subset \mathcal{Q}'_{\tilde{\lambda},\tilde{\mu}}} \tilde{F}_D \left(N_{\tilde{\nu}_{\mathcal{I}},p'}^{\tilde{\nu}_{\mathcal{I}}}(X') \right) & \text{otherwise;} \end{cases}$$

⁵The definitions of \tilde{X}_{n-m} and X_{n-m} are reversed in [12] when n is even.

where X' and p' are defined as before, and

$$\tilde{F}_D(\hat{t}_i) := \begin{cases} F_D(\hat{t}_{2n+1-i}) & \text{if } i \in \{n, n+1\}, \text{ and} \\ F_D(\hat{t}_i) & \text{if } i \notin \{n, n+1\} \end{cases}$$

is induced by F_D and the involution. We note that \tilde{F}_D also sends all positive roots of type A_{2n-1} to positive roots of type D_n except for the simple root $\hat{t}_n - \hat{t}_{n+1}$. It follows from Corollary A.3 that the aforementioned formula for $\tilde{N}_{\lambda,p}^\mu(X)$ is manifestly positive.

Remark 7.8. Suppose $\mathcal{Q}_{\lambda,\mu} = \emptyset$ and $p \geq n - m$. The restriction coefficient $N_{\lambda,p}^\mu = N_{\nu,p'}^\nu(OG(m', 2n))$ may not be the specialization (via F_D or \tilde{F}_D) of a type A restriction coefficient from a Grassmannian $Gr(m'', 2n)$ for any $1 \leq m'' \leq 2n-1$, even up to a factor of 2 (in contrast to the type B case; see Remark 6.4). We have verified several such examples by computer, including Example 7.6.

APPENDIX A. FORMULA FOR RESTRICTIONS OF SPECIAL SCHUBERT CLASSES

In this appendix, we provide a manifestly positive formula for the restriction of a special Schubert class in $H_T^*(Gr(m, N))$ to an arbitrary T -fixed point, stated in terms of Schubert symbols. We derive this formula directly from the Atiyah-Bott-Berline-Verge integration formula, but it can also be deduced from other restriction formulas, such as [14, 5]. We thank Sushmita Venugopalan for her insight in proving the following key lemma.

Lemma A.1. Consider the polynomial ring $\mathbb{Z}[x_1, \dots, x_r; y_1, \dots, y_{p+r-1}]$. We have the following algebraic identity:

$$(A.1) \quad \sum_{j \in [1, r]} \frac{\prod_{i \in [1, p+r-1]} (y_i - x_j)}{\prod_{i \in [1, r] \setminus \{j\}} (x_i - x_j)} = \sum_{1 \leq c_1 < \dots < c_p \leq p+r-1} \prod_{i=1}^p (y_{c_i} - x_{c_i-i+1}).$$

Proof. We claim that both sides of (A.1) are equal to

$$\sum_{k=0}^p e_k(y_1, \dots, y_{p+r-1}) h_{p-k}(-x_1, \dots, -x_r),$$

where $e_k(y_1, \dots, y_{p+r-1})$ is an elementary symmetric polynomial of degree k , and $h_j(-x_1, \dots, -x_r)$ is a complete homogeneous symmetric polynomial of degree j for $j \geq 0$. We set $h_j(-x_1, \dots, -x_r) = 0$ for $j < 0$.

For any positive integer k , let $V(x_1, \dots, x_k)$ denote the Vandermonde determinant $\prod_{1 \leq i < j \leq k} (x_j - x_i)$. Combining fractions on the left-hand side of (A.1), we get

$$\sum_{j \in [1, r]} \frac{(-1)^{j-1} (y_1 - x_j) \cdots (y_{p+r-1} - x_j) V(x_1, \dots, \hat{x}_j, \dots, x_r)}{V(x_1, \dots, x_r)} = \sum_{\substack{\text{all sequences} \\ 1 \leq c_1 < \dots < c_k \leq p+r-1 \\ \text{for } 0 \leq k \leq p+r-1}} \left(\sum_{j \in [1, r]} \frac{(y_{c_1} \cdots y_{c_k}) (-1)^{j-1} (-x_j)^{p+r-1-k} V(x_1, \dots, \hat{x}_j, \dots, x_r)}{V(x_1, \dots, x_r)} \right).$$

The coefficient in $\mathbb{Z}[x_1, \dots, x_r]$ of the monomial $y_{c_1} \cdots y_{c_k}$ above is equal to

$$(-1)^{p-k} s_{(p-k)}(x_1, \dots, x_r).$$

Here,

$$s_{(p-k)}(x_1, \dots, x_r) := \frac{\det \begin{bmatrix} x_1^{p+r-1-k} & \dots & x_r^{p+r-1-k} \\ x_1^{r-2} & \dots & x_r^{r-2} \\ \vdots & \ddots & \vdots \\ x_1 & \dots & x_r \\ 1 & \dots & 1 \end{bmatrix}}{(-1)^{r-1} V(x_1, \dots, x_r)}$$

is a Schur polynomial. By the Jacobi-Trudi formula, we have $s_{(p-k)}(x_1, \dots, x_r) = h_{p-k}(x_1, \dots, x_r) = (-1)^{p-k} h_{p-k}(-x_1, \dots, -x_r)$ as desired.

To prove the equality from the right-hand side, we set some more notation. For any positive integers $b \leq a$, let $\left\{ \begin{smallmatrix} a \\ b \end{smallmatrix} \right\}$ denote the set of strictly increasing subsequences $\{c_i\}_{i=1}^b \subset [1, a]$. Fix an integer $1 \leq k \leq p-1$ and a sequence $\{f_i\}_{i=1}^k \in \left\{ \begin{smallmatrix} p+r-1 \\ k \end{smallmatrix} \right\}$. Let $\left\{ \begin{smallmatrix} p+r-1 \\ p \end{smallmatrix} \right\}'$ denote the elements of $\left\{ \begin{smallmatrix} p+r-1 \\ p \end{smallmatrix} \right\}$ containing $\{f_i\}_{i=1}^k$ as a subsequence. For any $\{c_i\}_{i=1}^p \in \left\{ \begin{smallmatrix} p+r-1 \\ p \end{smallmatrix} \right\}'$, let $\{\widehat{c}_j\}_{j=1}^{p-k}$ denote the *weakly* increasing sequence $\{c_i - i + 1 : c_i \neq f_j \text{ for any } j\}$. The coefficient in $\mathbb{Z}[x_1, \dots, x_r]$ of the monomial $y_{f_1} \dots y_{f_k}$ on the right-hand side of (A.1) is then

$$(A.2) \quad \sum_{\{c_i\} \in \left\{ \begin{smallmatrix} p+r-1 \\ p \end{smallmatrix} \right\}'} \left(\prod_{j=1}^{p-k} (-x_{\widehat{c}_j}) \right).$$

The set $\left\{ \begin{smallmatrix} p+r-1 \\ p \end{smallmatrix} \right\}'$ has cardinality $\binom{p+r-k-1}{p-k}$, exactly the number of degree $p-k$ monomials in $\mathbb{Z}[x_1, \dots, x_r]$, up to scalar multiples. We claim that no monomial occurs more than once in (A.2). Suppose on the contrary that $\{\widehat{c}_j\}_{j=1}^{p-k} = \{\widehat{d}_j\}_{j=1}^{p-k}$ for some pair $\{c_i\}_{i=1}^p \neq \{d_i\}_{i=1}^p \in \left\{ \begin{smallmatrix} p+r-1 \\ p \end{smallmatrix} \right\}'$. Then there exists a minimum integer h such that $f_h = c_i = d_j$ for some $i \neq j$. Assuming $i < j$, we have $d_i < d_j = c_i$. Since $d_i \neq f_l$ for any $l \in [1, k]$, we have $\widehat{c}_{i-h+1} \neq \widehat{d}_{i-h+1}$, a contradiction. It follows that (A.2) equals $h_{p-k}(-x_1, \dots, -x_r)$. \square

Now we consider the Grassmannian $X = Gr(m, N)$, and compute the structure coefficient $N_{\nu, p}^\nu$ in the equivariant product $[X_\nu]^T \cdot [X_p]^T$ of equivariant Schubert classes. Here $\nu = \{\nu_1 < \dots < \nu_m\}$ is a general Schubert symbol; $p \in \{1, \dots, N-m\}$, and $X_p = X_{s_p}$ is labeled by the special Schubert symbol $s_p = \{N+1-m-p, N+2-m, \dots, N\}$. We can further assume $\nu \leq s_p$, since $N_{\nu, p}^\nu$ would vanish otherwise (see e.g. [27]). Let $I_1 := [1, N-m-p+1]$, $I_2 := [N-m-p+2, N]$ and $r := \#(I_1 \cap \nu)$. It follows from $\nu \leq s_p$ that $r \geq 1$ and $I_2 \setminus \nu$ consists of $p+r-1$ elements. Write $I_1 \cap \nu = \{a_1 < \dots < a_r\}$ and $I_2 \setminus \nu = \{b_1 < \dots < b_{p+r-1}\}$. We have the following formula of $N_{\nu, p}^\nu$, which gives the restriction of the special Schubert class $[X_p]^T$ to the T -fixed point corresponding to ν .

Lemma A.2. *The restriction coefficient $N_{\nu, p}^\nu$ is given by*

$$(A.3) \quad N_{\nu, p}^\nu = \sum_{1 \leq c_1 < \dots < c_p \leq p+r-1} \prod_{i=1}^p (t_{b_{(c_i)}} - t_{a_{(c_i-i+1)}}).$$

Proof. Let $\mathcal{A} := [Z_{\nu, \nu}]^T \cdot [\mathbb{P}(E_{N-m-p+1})]^T \in H_T^*(\mathbb{P}^{N-1})$. By Proposition 3.1 we have $N_{\nu, p}^\nu = \int_{\mathbb{P}^{N-1}}^T \mathcal{A}$. For $1 \leq j \leq N$, let $\iota_j^* : H_T^*(\mathbb{P}^{N-1}) \rightarrow H_T^*(\mathbb{P}(\langle \mathbf{e}_j \rangle))$ denote the restriction to the T -fixed point of \mathbb{P}^{N-1} corresponding to the j -th basis vector.

Note that $t_j^* \mathcal{A} = 0$ unless $j \in I_1 \cap \nu$, in which case we have $t_j^* \mathcal{A} = \prod_{i \notin \nu} (t_i - t_j) \prod_{i \in I_2} (t_i - t_j)$. By the Atiyah-Bott-Berline-Vergne integration formula (see e.g. [1, §2.5]), we therefore have

$$\begin{aligned} \int_{\mathbb{P}^{N-1}} \mathcal{A} &= \sum_{j=1}^N \frac{t_j^* \mathcal{A}}{\prod_{i \in [1, N] \setminus \{j\}} (t_i - t_j)} \\ &= \sum_{j \in I_1 \cap \nu} \frac{\prod_{i \in I_1 \setminus \nu} (t_i - t_j) \prod_{i \in I_2 \setminus \nu} (t_i - t_j)^2 \prod_{i \in I_2 \cap \nu} (t_i - t_j)}{\prod_{i \in [1, N] \setminus \{j\}} (t_i - t_j)} \\ &= \sum_{j \in I_1 \cap \nu} \frac{\prod_{i \in I_2 \setminus \nu} (t_i - t_j)}{\prod_{i \in I_1 \cap \nu \setminus \{j\}} (t_i - t_j)} \\ &= \sum_{j \in [1, r]} \frac{\prod_{i \in [1, p+r-1]} (t_{b_i} - t_{a_j})}{\prod_{i \in [1, r] \setminus \{j\}} (t_{a_i} - t_{a_j})}. \end{aligned}$$

Lemma A.1 yields the result, after identifying y_i with t_{b_i} for $1 \leq i \leq p+r-1$, and x_i with t_{a_i} for $1 \leq i \leq r$. \square

We observe that Lemma A.2 is manifestly positive in the sense that the terms $(t_{b_{c_i}} - t_{a_{c_i-i+1}})$ are elements of $\mathbb{Z}_{\geq 0}[t_2 - t_1, \dots, t_N - t_{N-1}]$. Specializations of these terms yield manifestly positive Pieri rules in types C and B (Theorems 5.3 and 6.2). When $N = 2n$, the specialization F_D sends all positive roots $t_a - t_b$ (where $a < b$) to positive roots of type D_n except for the simple root $t_n - t_{n+1}$ of type A_{2n-1} . Nevertheless, the following corollary ensures the type D Pieri rule (Theorem 7.4) is manifestly positive as well.

Corollary A.3. *If $N = 2n$, then none of the terms $(t_{b_{c_i}} - t_{a_{c_i-i+1}})$ in the summation (A.3) are given by $(t_{n+1} - t_n)$ unless $I_1 = [1, n]$ and $I_1 \cap \nu = \{n\}$.*

Proof. If the term $(t_{n+1} - t_n)$ occurs, then there exists $1 \leq c_1 < \dots < c_p \leq p+r-1$ and $1 \leq i \leq p$ such that $t_{b_{(c_i)}} - t_{a_{(c_i-i+1)}} = t_{n+1} - t_n$. It follows that $n = a_{(c_i-i+1)} \leq a_r < b_1 \leq b_{(c_i)} = n+1$. Hence, we have $c_i = 1$, $r = 1$, $b_1 = n+1$, and $a_1 = n$. \square

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