J. Math. Anal. Appl. 437 (2016) 720–735

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic analysis for time harmonic wave problems with small wave number

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ARTICLE INFO

Article history:
Received 25 August 2015
Available online 18 January 2016
Submitted by J. Xiao

Keywords:
Asymptotic analysis
Operator equation
Small wave number

ABSTRACT

We study the asymptotic behavior of the solution to some time harmonic wave problems when the wave number is taken as a small asymptotic parameter. Our basic strategy is to introduce suitable Lagrangian multipliers into the governing equations, and transforming them into saddle point problems. These saddle point problems are uniformly invertible with respect to the wave number $k \in [0, k_0]$, with k_0 being an arbitrary but fixed positive number. The asymptotic expansion is then derived by the standard regular perturbation technique.

1. Introduction

PDE problems with small asymptotic parameter are ubiquitous in the science and engineering applications. The study of asymptotic behavior for these problems is crucial at least for two reasons, understanding the possible new physics in the asymptotic regime and designing uniformly stable numerical schemes. Generally, if the limiting problem is well-posed, we call these PDE problems regularly perturbed. Otherwise, we call them singularly perturbed. The regularly perturbed problems are easier, since the solution admits a power series expansion which is valid at least when the asymptotic parameter is sufficiently small. It is the singularly perturbed problems which make the analysis more complicated. Even in the linear case, the asymptotic solutions behavior can be very different, strongly depending on the nature of these PDE problems. A correct solution ansatz with respect to the asymptotic parameter is the key ingredient for this kind of investigations. In this paper, we are interested in the time harmonic wave problems with small wave number.

The first problem we consider is the boundary value problem of the Helmholtz equation

$$-\Delta u - k^2 u = f,\ \forall x \in \Omega, \tag{1}$$

$$\partial_n u - iku = g,\ \forall x \in \Gamma, \tag{2}$$

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http://dx.doi.org/10.1016/j.jmaa.2016.01.008
0022-247X/

where $i = \sqrt{-1}$ denotes the imaginary unit, k is the wave number parameter, $\Omega \subset \mathrm{R}^n$ ($n = 2$ or 3) is a bounded connected Lipschitz domain with connected boundary Γ, n denotes the unit outward normal, $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$ are prescribed complex-valued functions. The weak formulation associated with the boundary value problem (1)–(2) is to find $u \in H^1(\Omega)$, such that for all $v \in H^1(\Omega)$ it holds that

$$(\nabla v, \nabla u) - ik < v, u > -k^2(v, u) = (v, f)+ < v, g > . \tag{3}$$

Here and hereafter, we define the volume and boundary duals as

$$(v, u) = \int_\Omega \bar{v}u dx, \quad < v, u >= \int_\Gamma \bar{v}u ds.$$

By the Riesz representation theorem, we can define three bounded linear operators $\{A_j\}_{j=0}^2$ from $H^1(\Omega)$ to $H^1(\Omega)$, and an element $b \in H^1(\Omega)$ as

$$(v, A_0u)_1 = (\nabla v, \nabla u), \tag{4}$$

$$(v, A_1u)_1 =< v, u >, \tag{5}$$

$$(v, A_2u)_1 = (v, u), \tag{6}$$

$$(v, b)_1 = (v, f)+ < v, g > . \tag{7}$$

In the above, $(\cdot, \cdot)_1$ stands for the standard H^1-inner product, i.e.,

$$(v, w)_1 = (\nabla v, \nabla w) + (v, w).$$

The variational problem (3) can then be written into an equivalent form of operator equation: find $u \in H^1(\Omega)$ such that

$$(A_0 - ikA_1 - k^2A_2)u = b. \tag{8}$$

The second problem we consider is the boundary value problem of Navier equation

$$-\nabla \cdot \sigma(u) - k^2u = f, \ \forall x \in \Omega, \tag{9}$$

$$n \cdot \sigma(u) - iku = g, \ \forall x \in \Gamma, \tag{10}$$

where $\Omega \subset \mathrm{R}^n$ ($n = 2$ or 3) is a bounded connected Lipschitz domain with connected boundary Γ, $\sigma(u)$ stands for the stress tensor of the displacement vector field u. For simplicity, we assume that the stress tensor $\sigma(u)$ relates to the strain tensor $\epsilon(u)$ through

$$\epsilon(u) = (\nabla u + (\nabla u)^\dagger)/2, \quad \sigma(u) = \lambda \mathrm{tr}\epsilon(u)I + 2\mu\epsilon(u).$$

In the above, λ and μ denote the Lame's constants. The weak formulation associated with the boundary value problem (9)–(10) is to find $u \in (H^1(\Omega))^n$, such that for all $v \in (H^1(\Omega))^n$ it holds that

$$\lambda(\mathrm{tr}\epsilon(v), \mathrm{tr}\epsilon(u)) + 2\mu(\epsilon(v), \epsilon(u)) - ik < v, u > -k^2(v, u) = (v, f)+ < v, g > . \tag{11}$$

By the Riesz representation theorem, we can define three bounded linear operators $\{A_j\}_{j=0}^2$ from $(H^1(\Omega))^n$ to $(H^1(\Omega))^n$, and an element $b \in (H^1(\Omega))^n$ as

$$(v, A_0 u)_1 = \lambda(\mathrm{tr}\epsilon(v), \mathrm{tr}\epsilon(u)) + 2\mu(\epsilon(v), \epsilon(u)), \tag{12}$$

$$(v, A_1 u)_1 =< v, u >, \tag{13}$$

$$(v, A_2 u)_1 = (v, u), \tag{14}$$

$$(v, b)_1 = (v, f)+ < v, g > . \tag{15}$$

The variational problem (11) can then be written into the equivalent form of operator equation: find $u \in (H^1(\Omega))^n$ such that

$$(A_0 - ikA_1 - k^2 A_2)u = b. \tag{16}$$

The last problem we consider is the boundary value problem of the Maxwell equation

$$\mathrm{curl}\,\mathrm{curl}\,u - k^2 u = f, \ \forall x \in \Omega, \tag{17}$$

$$n \times \mathrm{curl}\,u + iku_t = g, \ \forall x \in \Gamma, \tag{18}$$

where $\Omega \subset \mathrm{R}^3$ is a bounded connected Lipschitz domain with connected boundary Γ, $u_t = (n \times u) \times n$ on Γ, f is the source field, and g is a prescribed tangential vector field on Γ. Let us introduce the function space

$$H(\mathrm{imp}; \Omega) = \{v \in (L^2(\Omega))^3 |\, \mathrm{curl}\,v \in (L^2(\Omega))^3,\ v_t \in L_t^2(\Gamma)\},$$

where

$$L_t^2(\Gamma) = \{v \in (L(\Gamma))^3 | n \cdot v = 0\}.$$

The function space $H(\mathrm{imp}; \Omega)$ is a Hilbert space equipped with the inner product ([10])

$$(v, u)_{H(\mathrm{imp};\Omega)} = (\mathrm{curl}\,v, \mathrm{curl}\,u) + (v, u)+ < v_t, u_t > .$$

Given $f \in (L^2(\Omega))^3$ and $g \in L_t^2(\Gamma)$, the weak formulation associated with the boundary value problem (17)–(18) is to determine $u \in H(\mathrm{imp}; \Omega)$, such that for all $v \in H(\mathrm{imp}; \Omega)$ it holds that

$$(\mathrm{curl}\,v, \mathrm{curl}\,u) - k^2(v, u) - ik < v_t, u_t >= (v, f)- < v_t, g > . \tag{19}$$

By the Riesz representation theorem, we can define three bounded linear operators $\{A_j\}_{j=0}^2$ from $H(\mathrm{imp}; \Omega)$ to $H(\mathrm{imp}; \Omega)$, and an element $b \in H(\mathrm{imp}; \Omega)$ as

$$(v, A_0 u)_{H(\mathrm{imp};\Omega)} = (\mathrm{curl}\,v, \mathrm{curl}\,u), \tag{20}$$

$$(v, A_1 u)_{H(\mathrm{imp};\Omega)} =< v_t, u_t >, \tag{21}$$

$$(v, A_2 u)_{H(\mathrm{imp};\Omega)} = (v, u), \tag{22}$$

$$(v, b)_{H(\mathrm{imp};\Omega)} = (v, f)- < v, g > . \tag{23}$$

The variational problem (19) is then equivalent to the operator equation: find $u \in H(\mathrm{imp}; \Omega)$ such that

$$(A_0 - ikA_1 - k^2 A_2)u = b. \tag{24}$$

It is known that for all $k > 0$, the Helmholtz boundary value problem (1)–(2) (see [4]), the Navier boundary value problem (9)–(10) (see [5]), and the Maxwell boundary value problem (17)–(18) (see [10]) admit a unique solution in the corresponding function spaces which continuously depends on the data f and g. Equivalently speaking, the operator families $A_0 - ikA_1 - k^2A_2$ in (8), (16) and (24) are invertible for all $k > 0$. Since these operator families are analytic functions of k, the inverse operator families $(A_0 - ikA_1 - k^2A_2)^{-1}$ are also analytic for all $k > 0$. However, the analyticity property cannot be extended to $k = 0$, since the operator A_0 is not invertible. The implication of this observation is such that the operator equations (8), (16) and (24), thus the boundary value problems (1)–(2), (9)–(10), and (17)–(18) are singularly perturbed if we take the wave number k as a small asymptotic parameter.

The study of asymptotic behavior for the time harmonic wave problems with small wave number has a long history, and appeared in the literature as early as in [9,11,3]. Later on, Feng and Sheen considered this issue for the Helmholtz problem and the Navier problem in [6,5]. Hsiao and Wendland [7,8] performed the asymptotic analysis for the Helmholtz problem. They derived the first two asymptotic terms for both interior and exterior problems by the integral equation method. This paper aims mainly at an asymptotic analysis for the Helmholtz problem (1)–(2), the Navier problem (9)–(10), and the Maxwell problem (17)–(18). It turns out that we can make this analysis for all these problems in a uniform manner, since their equivalent operator equations (8), (16) and (24) have a similar form. By introducing suitable Lagrangian multipliers, we transform the governing equations into saddle point problems, which are uniformly invertible with respect to the wave number $k \in [0, k_0]$, with k_0 being an arbitrary but fixed positive number. The regular perturbation technique can then be straightforwardly applied to derive the asymptotic solution behavior. Different from the integral equation method used in [6,5,7,8], our method can be easily extended to problems with variable coefficients.

The rest of this paper is organized as follows. We first study the asymptotic solution behavior for this abstract operator equation in Section 2. Then taking the Helmholtz problem (1)–(2), the Navier problem (9)–(10), and the Maxwell problem (17)–(18) as three specific instances, we figure out the leading order terms of the corresponding asymptotic expansions in Section 3, Section 4 and Section 5, respectively.

2. An abstract operator equation

Let X be a complex Hilbert space with inner product $(\cdot,\cdot)_X$. The induced norm is denoted by $\|\cdot\|_X$. Given three bounded linear operators A_0, A_1 and A_2 from X to X, we consider the operator equation with real parameter $k > 0$: find $u \in X$ such that

$$(A_0 - ikA_1 - k^2A_2)u = b, \tag{25}$$

where b is a prescribed element in X. We hypothesize that:

H0: the bounded linear operators A_0, A_1 and A_2 are self-adjoint, namely,

$$(A_i v, u)_X = (v, A_i u)_X, \ \forall v, u \in X;$$

H1: the bounded linear operator family $A_0 - ikA_1 - k^2A_2$ is invertible for all $k > 0$.

Note that the inverse of an invertible bounded linear operator is also bounded by the closed graph theorem.

The operator equation (25) is a perturbation problem when $k \to 0$. If A_0 is invertible, this perturbation is regular, and the solution of (25) admits a power series expansion with respect to k. However, when A_0 is not invertible, this perturbation becomes singular. This section aims at an asymptotic expansion for the solution of (25) in the singularly perturbed case, namely, when the kernel space of A_0 is not trivial.

Let us put

$$X_0 = \ker A_0, \quad X_{01} = \ker A_0 \cap \ker A_1.$$

Obviously, X_0 is a closed subspace of X, and X_{01} is a closed subspace of X_0. Suppose that X_{02} is the orthogonal complement of X_{01} in X_0, i.e., $X_0 = X_{01} \oplus X_{02}$. Let P_1 and P_2 be the orthogonal projection operator from X onto X_{01} and X_{02}, I_1 and I_2 be the inclusion operators from X_{01} and X_{02} to X, respectively. Furthermore, let I_0 be the inclusion operator from X_0 to X. Note that P_1 is the conjugate operator of I_1, and P_2 is the conjugate operator of I_2, i.e.,

$$(I_1 v, u)_X = (v, P_1 u)_X, \ \forall v \in X_{01}, \ \forall u \in X,$$
$$(I_2 v, u)_X = (v, P_2 u)_X, \ \forall v \in X_{02}, \ \forall u \in X.$$

By the hypothesis H0, A_0 and A_1 are self-adjoint. It is straightforward to verify that

$$P_1 A_0 = 0, \quad P_1 A_1 = 0, \quad P_2 A_0 = 0. \tag{26}$$

With these notations introduced, we further hypothesize that

H2: the bounded linear operator $P_1 A_2 I_1$ from X_{01} to X_{01} is invertible;
H3: the bounded linear operator $P_2 A_1 I_2$ from X_{02} to X_{02} is invertible;
H4: confined and projected onto $\ker P_1 A_2 \cap \ker P_2 A_1$, the operator A_0 is invertible;
H5: the bounded linear operator $(P_1 A_2 I_0, P_2 A_1 I_0)$ from X_0 to $X_{01} \times X_{02}$ is invertible.

2.1. Equivalent saddle point problem

Applying the projection operators P_1 and P_2 onto both sides of (25) and recalling (26), we obtain

$$P_1 A_2 u = -k^{-2} P_1 b,$$
$$(P_2 A_1 - ik P_2 A_2) u = (-ik)^{-1} P_2 b.$$

These are two constraints on the solution u. By introducing two Lagrangian multipliers $p \in X_{01}$ and $t \in X_{02}$ into the operator equation (25), we derive the following saddle point problem: find $(u, p, t) \in X \times X_{01} \times X_{02}$ such that

$$(A_0 - ikA_1 - k^2 A_2) u + A_2 I_1 p + A_1 I_2 t = b, \tag{27}$$
$$P_1 A_2 u = -k^{-2} P_1 b, \tag{28}$$
$$(P_2 A_1 - ik P_2 A_2) u = (-ik)^{-1} P_2 b. \tag{29}$$

Proposition 2.1. *For all $k > 0$, if u is a solution to the operator equation (25), then $(u, 0, 0)$ is a solution to the saddle point problem (27)–(29). On the other hand, if (u, p, t) is a solution to the saddle point problem (27)–(29), then $p = 0$, $t = 0$, and u is a solution to the operator equation (25).*

Proof. It suffices to prove the latter statement. Applying P_1 onto both sides of (27) and recalling (26), we derive

$$-k^2 P_1 A_2 u + P_1 A_2 I_1 p = P_1 b.$$

Using (28) yields

$$P_1 A_2 I_1 p = 0,$$

which leads to $p = 0$, since $P_1 A_2 I_1$ is invertible on X_{01} by the hypothesis H2. Furthermore, applying P_2 onto both sides of (27) and recalling (26), we derive

$$(-ikP_2A_1 - k^2 P_2 A_2)u + P_2 A_1 I_2 t = P_2 b.$$

Using (29) yields

$$P_2 A_1 I_2 t = 0,$$

which leads to $t = 0$, since $P_2 A_1 I_2$ is invertible on X_{02} by the hypothesis H3. The equation (27) is then reduced to (25), which implies that u is a solution to (25). □

2.2. Uniform solvability of saddle point problem

The saddle point problem (27)–(29) is a specific instance of the following problem with general data: find $(u, p, t) \in X \times X_{01} \times X_{02}$ such that

$$(A_0 - ikA_1 - k^2 A_2)u + A_2 I_1 p + A_1 I_2 t = b, \tag{30}$$

$$P_1 A_2 u = n_1, \tag{31}$$

$$(P_2 A_1 - ikP_2 A_2)u = n_2, \tag{32}$$

where $b \in X$, $n_1 \in X_{01}$ and $n_2 \in X_{02}$.

Proposition 2.2. *$\forall k > 0$, there exists a constant $c_k > 0$, such that $\forall b \in X$, $\forall n_1 \in X_{01}$ and $\forall n_2 \in X_{02}$, the saddle point problem (30)–(32) admits a unique solution $(u, p, t) \in X \times X_{01} \times X_{02}$ which satisfies*

$$\|u\|_X + \|p\|_X + \|t\|_X \le c_k \left(\|b\|_X + \|n_1\|_X + \|n_2\|_X\right).$$

Proof. Applying P_1 and P_2 onto both sides of (30) and recalling (26), we derive

$$-k^2 P_1 A_2 u + P_1 A_2 I_1 p = P_1 b, \tag{33}$$

$$(-ikP_2 A_1 - k^2 P_2 A_2)u + P_2 A_2 I_1 p + P_2 A_1 I_2 t = P_2 b. \tag{34}$$

Using (31)–(32) yields

$$P_1 A_2 I_1 p = P_1 b + k^2 n_1, \tag{35}$$

$$P_2 A_2 I_1 p + P_2 A_1 I_2 t = P_2 b + ikn_2. \tag{36}$$

The derivation of (35)–(36) implies that if (u, p, t) is a solution of problem (30)–(32), then it is also a solution of operator equations (30), (35) and (36). On the other hand, if (u, p, t) is a solution of (30), (35) and (36), applying (33)–(34) we derive (31)–(32). These imply that the problem (30)–(32) has the same solution as the operator equations (30), (35) and (36). According to the hypotheses H2, H3 and H1, p, t and u can be successively solved from (35), (36) and (30). The proof thus finishes since all operators involved are bounded. □

Next let us consider the saddle point problem (30)–(32) when $k = 0$. In this case, the problem (30)–(32) expresses as

$$A_0 u + A_2 I_1 p + A_1 I_2 t = b, \tag{37}$$

$$P_1 A_2 u = n_1, \tag{38}$$

$$P_2 A_1 u = n_2. \tag{39}$$

Proposition 2.3. *There exists a constant $c_0 > 0$, such that $\forall\, b \in X$, $\forall\, n_1 \in X_{01}$, $\forall\, n_2 \in X_{02}$, the saddle point problem (37)–(39) admits a unique solution $(u, p, t) \in X \times X_{01} \times X_{02}$ which satisfies*

$$\|u\|_X + \|p\|_X + \|t\|_X \le c_0(\|b\|_X + \|n_1\|_X + \|n_2\|_X).$$

Proof. According to the hypothesis H5, let $v \in X_0$ be the unique solution of

$$P_1 A_2 I_0 v = n_1, \quad P_2 A_1 I_0 v = n_2.$$

Set $\tilde{u} = u - I_0 v$, then $\tilde{u}$ solves

$$A_0 \tilde{u} + A_2 I_1 p + A_1 I_2 t = b - A_0 I_0 v, \tag{40}$$

$$P_1 A_2 \tilde{u} = 0, \tag{41}$$

$$P_2 A_1 \tilde{u} = 0. \tag{42}$$

The above implies that $\tilde{u} \in \ker P_1 A_2 \cap \ker P_2 A_1$. According to the hypothesis H4, A_0 is invertible when confined and projected to $\ker P_1 A_2 \cap \ker P_2 A_1$. On the other hand, by the hypothesis H5, there exists a constant $c > 0$ such that for all $(0, 0) \neq (p, t) \in X_{01} \times X_{02}$, there exists $0 \neq w \in X_0$ satisfying

$$P_1 A_2 I_0 w = p, \quad P_2 A_1 I_0 w = t,$$

and

$$\|w\|_X \le c(\|p\|_X + \|t\|_X).$$

Since A_1 and A_2 are self-adjoint, we have

$$(I_0 w, A_2 I_1 p + A_1 I_2 t) = (P_1 A_2 I_0 w, p) + (P_2 A_1 I_0 w, t) = (p, p) + (t, t) = \|p\|_X^2 + \|t\|_X^2,$$

which leads to

$$\|A_2 I_1 p + A_1 I_2 t\| \ge (\|p\|_X^2 + \|t\|_X^2)/\|w\|_X \ge (\|p\|_X + \|t\|_X)/(2c).$$

The above implies that the mapping $A_2 I_1 p + A_1 I_2 t$ from $X_1 \times X_2$ to X satisfies the inf–sup stability condition. By the classical mixed variation theory, we know that there exists a constant $c_2 > 0$, such that $\forall\, b \in X$, $\forall\, n_1 \in X_{01}$, $\forall\, n_2 \in X_{02}$, the problem (40)–(42) admits a unique solution $(\tilde{u}, p, t) \in X \times X_{01} \times X_{02}$ which satisfies

$$\|\tilde{u}\|_X + \|p\|_X + \|t\|_X \le c_2(\|b\|_X + \|v\|_X).$$

The proof thus finishes since v is bounded by $\|n_1\|_X + \|n_2\|_X$. $\square$

Proposition 2.2 and Proposition 2.3 reveal that the left hand of problem (30)–(32) defines an invertible bounded operator family on the augmented space $X \times X_{01} \times X_{02}$ for all $k \ge 0$. Since this operator family is analytic with respect to k, we then derive the main result of this paper.

Theorem 2.1. *Given an arbitrary but fixed $k_0 > 0$, there exists a constant $c_3 = c_3(k_0)$, such that $\forall\, k \in [0, k_0]$, $\forall\, b \in X$, $\forall\, n_1 \in X_{01}$ and $\forall\, n_2 \in X_{02}$, the saddle point problem (30)–(32) admits a unique solution $(u, p, t) \in X \times X_{01} \times X_{02}$ which satisfies*

$$\|u\|_X + \|p\|_X + \|t\|_X \le c_3(\|b\|_X + \|n_1\|_X + \|n_2\|_X).$$

Thanks to Proposition 2.1 and Theorem 2.1, we then derive the stability estimate with respect to k for the operator equation (25).

Theorem 2.2. *Let u be the solution of (25). Given an arbitrary but fixed $k_0 > 0$, there exists a constant $c_4 = c_4(k_0)$, such that for all $k \in (0, k_0]$ it holds that*

$$\|u\|_X \le c_4 \left(\frac{\|P_1 b\|_X}{k^2} + \frac{\|P_2 b\|_X}{k} + \|b\|_X \right).$$

2.3. Asymptotic expansion

Now we can make an asymptotic expansion for the solution u of the operator equation (25). By Proposition 2.1 and Proposition 2.2, (u, p, t) with $p = 0$ and $t = 0$ is the unique solution to the saddle point problem (27)–(29). Let us make the formal asymptotic expansions

$$u = \sum_{m=-2}^{\infty} (-ik)^m u_m, \quad p = \sum_{m=-2}^{\infty} (-ik)^m p_m, \quad t = \sum_{m=-2}^{\infty} (-ik)^m t_m. \tag{43}$$

Substituting the above into (27)–(29) and equating the different powers of k, for all $m \ge -2$ we have

$$A_0 u_m + A_2 I_1 p_m + A_1 I_2 t_m = \delta_{m,0} b - A_1 u_{m-1} - A_2 u_{m-2}, \tag{44}$$

$$P_1 A_2 u_m = \delta_{m,-2} P_1 b, \tag{45}$$

$$P_2 A_1 u_m = \delta_{m,-1} P_2 b - P_2 A_2 u_{m-1}. \tag{46}$$

In the above, $\delta_{\cdot,\cdot}$ indicates the Kronecker symbol, which values 1 if two indices are equal, and 0 in the other cases. We have also made convention that $u_m = 0$ for all $m < -2$.

Proposition 2.4. *For all $m \ge -2$, the saddle point problem (44)–(46) admits a unique solution (u_m, p_m, t_m) with $p_m = 0$ and $t_m = 0$. Besides, there exists a constant $c_m > 0$ such that $\|u_m\| \le c_m \|b\|$.*

Proof. The existence and uniqueness follow by Proposition 2.3. Applying P_1 onto the both sides of (44) and using (45) yields

$$P_1 A_2 I_1 p_m = \delta_{m,0} P_1 b - P_1 A_2 u_{m-2} = \delta_{m,0} P_1 b - \delta_{m-2,-2} P_1 b = 0.$$

According to the hypothesis H2, we obtain $p_m = 0$. Furthermore, applying P_2 onto the both sides of (44) and using (46) yields

$$P_2 A_1 I_2 t_m = \delta_{m,0} P_2 b - P_2 A_1 u_{m-1} - P_2 A_2 u_{m-2} = 0.$$

According to the hypothesis H3, we obtain $t_m = 0$. The stability estimate follows by Proposition 2.3 and a sequential deduction. $\square$

Remark 2.1. When $m = -2$, the equations (44)–(46) express as

$$A_0 u_{-2} + A_2 I_1 p_{-2} + A_1 I_2 t_{-2} = 0,$$
$$P_1 A_2 u_{-2} = P_1 b,$$
$$P_2 A_1 u_{-2} = 0.$$

If $P_1 b = 0$, then by Proposition 2.4, we have $u_{-2} = 0$, $p_{-2} = 0$ and $t_{-2} = 0$.

Now for all $J \geq -1$, let us truncate the series terms in (43) and put

$$u^{(J)} = \sum_{m=-2}^{J} (-ik)^m u_m, \quad p^{(J)} = \sum_{m=-2}^{J} (-ik)^m p_m, \quad t^{(J)} = \sum_{m=-2}^{J} (-ik)^m t_m,$$
$$u_e^{(J)} = u^{(J)} - u, \quad p_e^{(J)} = p^{(J)} - p, \quad t_e^{(J)} = t^{(J)} - t.$$

A direct computation yields that

$$\begin{aligned}
&(A_0 - ikA_1 - k^2 A_2) u_e^{(J)} + A_2 I_1 p_e^{(J)} + A_1 I_2 t_e^{(J)} \\
&\qquad = (-ik)^{J+1} A_1 u_J + (-ik)^{J+1} A_2 u_{J-1} + (-ik)^{J+2} A_2 u_J, \\
&P_1 A_2 u_e^{(J)} = 0, \\
&(P_2 A_1 - ikP_2 A_2) u_e^{(J)} = (-ik)^{J+1} P_2 A_2 u_J.
\end{aligned}$$

Applying Proposition 2.4 and Theorem 2.1, we then derive the following asymptotic error estimate.

Theorem 2.3. *Let u be the solution of (25) and $(u_m, 0, 0)$ the unique solution of (44)–(46). Given an arbitrary but fixed $k_0 > 0$ and $J \geq -1$, there exists a constant $c_5 = c_5(J, k_0)$, such that for all $k \in (0, k_0]$, it holds that*

$$\left\| \sum_{m=-2}^{J} (-ik)^m u_m - u \right\|_X \leq c_5 k^{J+1} \|b\|_X.$$

3. The Helmholtz problem

The operator equation (8) is an instance of (25) with $X = H^1(\Omega)$ and $\{A_j\}_{j=0}^2$ defined by (4)–(6). It is easy to verify that

$$X_0 = \mathrm{C}, \quad X_{01} = \{0\}, \quad X_{02} = X_0 = \mathrm{C}.$$

Here and hereafter, the symbol C denotes the field of complex numbers. Besides, the hypotheses made on the operator equation (25) in Section 2 are all fulfilled for this instance:

H0: The self-adjoint property for $\{A_j\}_{j=0}^2$ is obvious;
H1: For all $k > 0$, $A_0 - ikA_1 - k^2 A_2$ is invertible (see [4]);
H2: The hypothesis H2 is trivial since $X_{01} = \{0\}$ and $P_1 = 0$;
H3: Given $t \in X_{02} = \mathrm{C}$, let us consider the operator equation: find $v \in X_{02} = \mathrm{C}$ such that

$$P_2 A_1 I_2 v = t.$$

This equation is equivalent to

$$(1,t)_1 = (1, P_2A_1I_2v)_1 = (1, A_1I_2v)_1 =< 1, I_2v >=< 1, v >,$$

which leads to

$$v = |\Omega|t/|\Gamma|,$$

where $|\Omega|$ and $|\Gamma|$ denote the volume and boundary measures, respectively. This certainly implies that the operator $P_2A_1I_2$ from X_{02} to X_{02} is invertible;

H4: Actually, since $P_1 = 0$, according to the definition of A_1 (see (5)), we have

$$\ker P_1A_2 \cap \ker P_2A_1 = \ker P_2A_1 = H^1_*(\Omega) \equiv \{v \in H^1(\Omega)| < 1, v >= 0\}.$$

It is known that confined to $H^1_*(\Omega)$, the semi-norm $|\cdot|_{1,\Omega}$ is a norm equivalent to the standard H^1-norm. This implies that the operator A_0 (see (4)) is invertible when confined and projected to $H^1_*(\Omega)$;

H5: For this instance, this hypothesis is the same as H3, since $X_{01} = \{0\}$, $P_1 = 0$ and $I_0 = I_2$.

Note that since $X_{01} = \{0\}$ and $P_1 = 0$, the saddle point problem (44)–(46) is reduced to

$$A_0u_m + A_1I_2t_m = \delta_{m,0}b - A_1u_{m-1} - A_2u_{m-2}, \tag{47}$$

$$P_2A_1u_m = \delta_{m,-1}P_2b - P_2A_2u_{m-1}. \tag{48}$$

According to Remark 2.1, we know that $u_{-2} = 0$. In the case that $m = -1$, the equations (47)–(48) express as

$$A_0u_{-1} + A_1I_2t_{-1} = 0, \tag{49}$$

$$P_2A_1u_{-1} = P_2b. \tag{50}$$

The corresponding variational form is to find $(u_{-1}, t_{-1}) \in H^1(\Omega) \times C$ such that for all $v \in H^1(\Omega)$ it holds that

$$(\nabla v, \nabla u_{-1})+ < v, t_{-1} >= 0,$$
$$< 1, u_{-1} >= (1, b)_1 = (1, f)+ < 1, g > .$$

Therefore, by Proposition 2.4 we have $t_{-1} = 0$ and

$$u_{-1} = [(1, f)+ < 1, g >]/|\Gamma|.$$

Applying Theorem 2.3 we derive

Theorem 3.1. *Let u be the solution of (3) and $(u_m, 0)$ the solution of (47)–(48). Given an arbitrary but fixed $k_0 > 0$ and $J \geq -1$, there exists a constant $c_6 = c_6(k_0, J)$, such that for all $k \in (0, k_0]$, it holds that*

$$\left\| \frac{(1,f)+ < 1, g >}{-ik|\Gamma|} + \sum_{m=0}^{J} (-ik)^m u_m - u \right\|_{1,\Omega} \leq c_6 k^{J+1}(\|f\|_{0,\Omega} + \|g\|_{\frac{1}{2},\Gamma}).$$

As a by-product, we derive the following stability estimate.

Theorem 3.2. *Let u be the solution of (3). Given an arbitrary but fixed $k_0 > 0$, there exists a constant $c_7 = c_7(k_0)$, such that for all $k \in (0, k_0]$, it holds that*

$$\|u\|_{1,\Omega} \leq c_7 \left(\frac{|(1,f)+ <1,g>|}{k} + \|f\|_{0,\Omega} + \|g\|_{\frac{1}{2},\Gamma} \right),$$
$$|u|_{2,\Omega} \leq c_7 \left(\|f\|_{0,\Omega} + \|g\|_{\frac{1}{2},\Gamma} \right).$$

4. The Navier problem

The operator equation (16) is an instance of (25) with $X = (H^1(\Omega))^n$ and $\{A_j\}_{j=0}^2$ defined by (12)–(14). For this instance, it is easy to verify that (see [2])

$$X_0 = \{\mathcal{M}x + r | \forall \mathcal{M} \in \mathrm{C}^{n\times n},\ \mathcal{M}^\dagger = -\mathcal{M},\ \forall r \in \mathrm{C}^n\}, \quad X_{01} = \{0\}, \quad X_{02} = X_0.$$

Note that X_0 is simply the rigid body displacement space. Besides, the hypotheses made on the operator equation (25) in Section 2 are all fulfilled for this instance:

H0: The self-adjoint property for $\{A_j\}_{j=0}^2$ is obvious;
H1: For all $k > 0$, $A_0 - ikA_1 - k^2A_2$ is invertible (see [5]);
H2: The hypothesis H2 is trivial since $X_{01} = \{0\}$ and $P_1 = 0$;
H3: Given $t \in X_{02}$, let us consider the operator equation: find $v \in X_{02}$ such that

$$P_2A_1I_2v = t.$$

This equation is equivalent to find $v \in X_{02}$ such that for all $w \in X_{02}$, it holds that

$$(w, t)_1 = (w, P_2A_1I_2v)_1.$$

Note that according to the definition of A_1 (see (13)), we have

$$(w, P_2A_1I_2v)_1 = (w, A_1I_2v)_1 =< w, v > .$$

Since X_{02} is of finite dimension, we know that the duals $(w, t)_1$ and $< w, v >$ define two equivalent inner products on X_{02}. This implies that the operator $P_2A_1I_2$ is invertible;
H4: Actually, since $P_1 = 0$ we have

$$\ker P_1A_2 \cap \ker P_2A_1 = \ker P_2A_1 = \mathbf{H}^1_*(\Omega) \equiv \{v \in (H^1(\Omega))^n | < w, v >= 0,\ \forall w \in X_0\}.$$

It is known that confined to $\mathbf{H}^1_*(\Omega)$, the semi-norm $|\cdot|_{1,\Omega}$ is a norm equivalent to the standard H^1-norm. Since $(\cdot, A_0\cdot)$ is an equivalent quadratic form as $(\nabla\cdot, \nabla\cdot)$ (see [2]), we know that the operator A_0 is invertible when confined and projected to $\mathbf{H}^1_*(\Omega)$;
H5: For this instance, this hypothesis is the same as H3, since $X_{01} = \{0\}$, $P_1 = 0$ and $I_0 = I_2$.

Note that since $X_{01} = \{0\}$ and $P_1 = 0$, the saddle point problem (44)–(46) is reduced to

$$A_0u_m + A_1I_2t_m = \delta_{m,0}b - A_1u_{m-1} - A_2u_{m-2}, \tag{51}$$
$$P_2A_1u_m = \delta_{m,-1}P_2b - P_2A_2u_{m-1}. \tag{52}$$

According to Remark 2.1, we have $u_{-2} = 0$. In the case that $m = -1$, the equations (51)–(52) express as

$$A_0 u_{-1} + A_1 I_2 t_{-1} = 0, \tag{53}$$

$$P_2 A_1 u_{-1} = P_2 b. \tag{54}$$

The corresponding variational form is to find $(u_{-1}, t_{-1}) \in (H^1(\Omega))^n \times X_{02}$ such that for all $(v, s) \in (H^1(\Omega))^n \times X_{02}$ it holds that

$$\lambda(\mathrm{tr}\epsilon(v), \mathrm{tr}\epsilon(u_{-1})) + 2\mu(\epsilon(v), \epsilon(u_{-1})) + < v, t_{-1} >= 0,$$

$$< s, u_{-1} >= (s, b)_1 = (s, f) + < s, g > .$$

Applying Theorem 2.3 we derive

Theorem 4.1. *Let u be the solution of (11) and $(u_m, 0)$ the solution of (51)–(52). Given an arbitrary but fixed $k_0 > 0$ and $J \geq -1$, there exists a constant $c_8 = c_8(k_0, J)$, such that for all $k \in (0, k_0]$, it holds that*

$$\left\| \frac{u_{-1}}{-ik} + \sum_{m=0}^{J} (-ik)^m u_m - u \right\|_{1,\Omega} \leq c_8 k^{J+1} (\|f\|_{0,\Omega} + \|g\|_{\frac{1}{2},\Gamma}),$$

where $u_{-1} \in X_0$ solves

$$< s, u_{-1} >= (s, b)_1 = (s, f) + < s, g >, \ \forall s \in X_0.$$

Considering u_{-1} is a linear function of x, as a by-product of the above theorem, we have

Theorem 4.2. *Let u be the solution of (11). Given an arbitrary but fixed $k_0 > 0$, there exists a constant $c_9 = c_9(k_0)$, such that for all $k \in (0, k_0]$, it holds that*

$$|u|_{2,\Omega} \leq c_9 \left(\|f\|_{0,\Omega} + \|g\|_{\frac{1}{2},\Gamma} \right).$$

5. The Maxwell problem

The operator equation (24) is an instance of (25) with $X = H(\mathrm{imp}; \Omega)$ and $\{A_j\}_{j=0}^{2}$ defined by (20)–(22). Let us put

$$Y(\Omega) = \{w \in H^1(\Omega) | w|_\Gamma \in H^1(\Gamma)\}.$$

For this instance, it is easy to verify that

$$X_0 = \nabla Y(\Omega), \quad X_{01} = \nabla H_0^1(\Omega), \quad X_{02} = \nabla e(H^1(\Gamma)).$$

In the above, e indicates the harmonic extension from $H^{\frac{1}{2}}(\Gamma)$ to $H^1(\Omega)$. We show that the hypotheses made on the operator equation (25) in Section 2 are all fulfilled:

- H0: The self-adjointness of A_0, A_1 and A_2 is obvious;
- H1: This hypothesis holds since $A_0 - ikA_1 - k^2 A_2$ is invertible ([10]);
- H2: This hypothesis holds since $P_1 A_2 I_1 = Id_{X_1}$;
- H3: Given $t' \in H^1(\Gamma)/\mathrm{C}$, i.e., $\nabla e(t') \in X_{02}$, let us determine $t \in H^1(\Gamma)/\mathrm{C}$, i.e., $\nabla e(t) \in X_{02}$, such that for all $s \in H^1(\Gamma)/\mathrm{C}$, it holds that

$$(\nabla e(s), P_2 A_1 I_2 \nabla e(t))_{H(\mathrm{imp};\Omega)} = (\nabla e(s), \nabla e(t'))_{H(\mathrm{imp};\Omega)}.$$

The above equation is equivalent to

$$< \nabla_\Gamma s, \nabla_\Gamma t >= (\nabla e(s), \nabla e(t')) + < \nabla_\Gamma s, \nabla_\Gamma t' >, \tag{55}$$

since

$$(\nabla e(s), P_2 A_1 I_2 \nabla e(t))_{H(\mathrm{imp};\Omega)} =< \nabla_\Gamma s, \nabla_\Gamma t >,$$

and

$$(\nabla e(s), \nabla e(t'))_{H(\mathrm{imp};\Omega)} = (\nabla e(s), \nabla e(t')) + < \nabla_\Gamma s, \nabla_\Gamma t' > .$$

Since $\|\nabla_\Gamma \cdot \|_{0,\Gamma}$ is an equivalent norm on $H^1(\Gamma)/\mathrm{C}$, from (55) we know that t is uniquely determined in $H^1(\Gamma)/\mathrm{C}$. Besides, we have

$$\|\nabla e(t)\|_{H(\mathrm{imp};\Omega)} \lesssim |t|_{1,\Gamma} \lesssim |t'|_{1,\Gamma} \lesssim \|\nabla e(t')\|_{H(\mathrm{imp};\Omega)}.$$

Here and hereafter, the symbol $\lesssim$ implies that the left quantity is bounded by the right quantity multiplied with a positive constant which depends only on the geometry of definition domain. The above implies that the operator $P_2 A_1 I_2$ from X_{02} to X_{02} is invertible. Therefore, the hypothesis H3 holds;

H4: It is straightforward to check that

$$\ker P_1 A_2 \cap \ker P_2 A_1 = H_0(\mathrm{imp};\Omega) \equiv \{v \in H(\mathrm{imp};\Omega) | \mathrm{div}\, v = 0,\ \mathrm{div}_\Gamma\, v_t = 0\}.$$

The reader is referred to [1,10] for the definition of boundary divergence operator div_Γ. By Theorem A.1 in the Appendix, the hypothesis H4 follows;

H5: Given $p \in H_0^1(\Omega)$ and $t \in H^1(\Gamma)/\mathrm{C}$, i.e., $\nabla p \in X_{01}$ and $\nabla e(t) \in X_{02}$, the system of operator equations: find $v \in X_0$ such that

$$P_1 A_2 I_0 v = \nabla p, \quad P_2 A_1 I_0 v = \nabla e(t),$$

is equivalent to seek $v = \nabla z$ with $z \in Y(\Omega)/\mathrm{C}$ such that for all $q \in H_0^1(\Omega)$ and all $s \in H^1(\Gamma)/\mathrm{C}$, it holds that

$$\begin{aligned} &(\nabla q, \nabla z) = (\nabla q, \nabla p), \\ &< \nabla_\Gamma s, \nabla_\Gamma z >= (\nabla e(s), \nabla e(t)) + < \nabla_\Gamma s, \nabla_\Gamma t > . \end{aligned}$$

The solution z is thus unique and existent, and it continuously depends on p and t.

Now we can apply Theorem 2.3 to derive the asymptotic expansion for the solution to the variational problem (19). We study the first two terms as follows:

- Term u_{-2}: In the case that $m = -2$, we have

$$\begin{aligned} &A_0 u_{-2} + A_2 I_1 p_{-2} + A_1 I_2 t_{-2} = 0, \\ &P_1 A_2 u_{-2} = P_1 b, \\ &P_2 A_1 u_{-2} = 0. \end{aligned}$$

The corresponding variational problem is to find $(u_{-2}, p_{-2}, t_{-2}) \in H(\mathrm{imp}; \Omega) \times H_0^1(\Omega) \times H^1(\Gamma)/\mathrm{C}$ such that for all $(v, q, s) \in H(\mathrm{imp}; \Omega) \times H_0^1(\Omega) \times H^1(\Gamma)/\mathrm{C}$ it holds that

$$\begin{aligned}
&(\operatorname{curl} v, \operatorname{curl} u_{-2}) + (v, \nabla p_{-2}) + < v_t, \nabla_\Gamma t_{-2} >= 0, \\
&(\nabla q, u_{-2}) = (\nabla q, f), \\
&< \nabla_\Gamma s, u_{-2,t} >= 0.
\end{aligned}$$

By Proposition 2.4, we have $p_{-2} = 0$ and $t_{-2} = 0$. Besides, $u_{-2} = \nabla\phi$ with $\phi \in H_0^1(\Omega)$ solving

$$(\nabla q, \nabla\phi) = (\nabla q, f), \ \forall q \in H_0^1(\Omega).$$

- Term u_{-1}: In the case that $m = -1$, we have

$$\begin{aligned}
&A_0 u_{-1} + A_2 I_1 p_{-1} + A_1 I_2 t_{-1} = 0, \\
&P_1 A_2 u_{-1} = 0, \\
&P_2 A_1 u_{-1} = P_2 b - P_2 A_2 u_{-2}.
\end{aligned}$$

The corresponding variational problem is to find $(u_{-1}, p_{-1}, t_{-1}) \in H(\mathrm{imp}; \Omega) \times H_0^1(\Omega) \times H^1(\Gamma)/\mathrm{C}$ such that for all $(v, q, s) \in H(\mathrm{imp}; \Omega) \times H_0^1(\Omega) \times H^1(\Gamma)/\mathrm{C}$ it holds that

$$\begin{aligned}
&(\operatorname{curl} v, \operatorname{curl} u_{-1}) + (v, \nabla p_{-1}) + < v_t, \nabla_\Gamma t_{-1} >= 0, \\
&(\nabla q, u_{-1}) = 0, \\
&< \nabla_\Gamma s, u_{-1,t} >= (e(s), f) - < s, g > -(e(s), \nabla\phi).
\end{aligned}$$

By Proposition 2.4, we have $p_{-1} = 0$ and $t_{-1} = 0$. In addition, $u_{-1} = \nabla e(\psi)$ with $\psi \in H^1(\Gamma)/\mathrm{C}$ solving

$$< \nabla_\Gamma s, \nabla_\Gamma \psi >= (e(s), f) - < s, g > -(e(s), \nabla\phi).$$

Applying Theorem 2.3 we derive

Theorem 5.1. *Let u be the solution of (19) and $(u_m, 0, 0)$ the solution of (44)–(46). Given an arbitrary but fixed $k_0 > 0$ and $J \geq -1$, there exists a constant $c_{10} = c_{10}(k_0, J)$, such that for all $k \in (0, k_0]$, it holds that*

$$\left\| (-ik)^{-2}\nabla\phi + (-ik)^{-1}\nabla e(\psi) + \sum_{m=0}^{J} (-ik)^m u_m - u \right\|_{\mathbf{H}(\mathrm{imp};\Omega)} \leq c_{10} k^{J+1} (\|f\|_{0,\Omega} + \|g\|_{\frac{1}{2},\Gamma}),$$

where $\phi \in H_0^1(\Omega)$ solves

$$(\nabla q, \nabla\phi) = (\nabla q, f), \ \forall q \in H_0^1(\Omega),$$

and $\psi \in H^1(\Gamma)/\mathrm{C}$ solves

$$< \nabla_\Gamma s, \nabla_\Gamma \psi >= (e(s), f) - < s, g > -(e(s), \nabla\phi), \ \forall s \in H^1(\Gamma)/\mathrm{C}.$$

6. Conclusion

We have performed asymptotic analysis for several time-harmonic wave problems with small wave number. These problems can be categorized into an operator equation frame, whose coefficient operator forms a quadratic analytic operator family. When the wave number parameter is set as zero, the coefficient operator is not invertible, which renders the operator equation singularly perturbed when the wave number is small. By introducing suitable Lagrangian multipliers, we transformed the operator equation into a saddle point problem. We proved that the saddle point problem is uniformly solvable. Based on this fact, we derived the asymptotic expansion for the solution of operator equation.

It turns out that $k = 0$ is an order-one pole of the analytic inverse operator family for the Helmholz and Navier problems, and a order-two pole of that for the Maxwell problem. The former is a recovered result built in [6] and [5]. However, to the authors' knowledge, the result for the Maxwell problem is new. Besides, different from the integral equation method employed in [6] and [5], our method can be easily extended to problems with variable coefficient problems.

Appendix A. On the equivalent norm in $\boldsymbol{H_0}(\mathbf{imp}; \boldsymbol{\Omega})$

Theorem A.1. *Confined to* $H_0(\mathrm{imp};\Omega)$, $\|\mathrm{curl}\cdot\|_{0,\Omega}$ *is a norm equivalent to* $\|\cdot\|_{H(\mathrm{imp};\Omega)}$.

Proof. It suffices to show that for all $u \in H_0(\mathrm{imp};\Omega)$, it holds that

$$\|u\|_{0,\Omega} + \|u_t\|_{0,\Gamma} \lesssim \|\mathrm{curl}\, u\|_{0,\Omega}.$$

Let us set $T = \mathrm{curl}\, u$. Since $\mathrm{div}\, T = 0$, there exists $W \in H^1(\Omega)$ such that

$$T = \mathrm{curl}\, W,\ \ \mathrm{div}\, W = 0,\ \ \|W\|_{1,\Omega} \lesssim \|\mathrm{curl}\, u\|_{0,\Omega}.$$

Set $Z = u - W$, then we have $\mathrm{curl}\, Z = 0$ and $\mathrm{div}\, Z = 0$. These imply that there exists a scalar potential $p \in H^1(\Omega)/\mathrm{C}$ such that $Z = \nabla p$ and $\Delta p = 0$. Since $u = W + \nabla p$ and $\mathrm{div}_\Gamma\, u_t = 0$, it holds that

$$\Delta_\Gamma p = -\mathrm{div}_\Gamma\, W_T.$$

This implies that

$$|p|_{1,\Gamma} \lesssim \|\mathrm{div}_\Gamma\, W_T\|_{-1,\Gamma} \lesssim \|W\|_{0,\Gamma} \lesssim \|W\|_{1,\Omega} \lesssim \|\mathrm{curl}\, u\|_{0,\Omega}.$$

By the standard regularity argument, we know that $p \in H^{\frac{3}{2}}(\Omega)/\mathrm{C}$ and

$$|p|_{\frac{3}{2},\Omega} \lesssim \|\mathrm{curl}\, u\|_{0,\Omega}.$$

We then derive

$$\|u\|_{\frac{1}{2},\Omega} \le \|W\|_{\frac{1}{2},\Omega} + \|\nabla p\|_{\frac{1}{2},\Omega} \lesssim \|W\|_{1,\Omega} + |p|_{\frac{3}{2},\Omega} \lesssim \|\mathrm{curl}\, u\|_{0,\Omega},$$

which leads to

$$\|u\|_{0,\Omega} \lesssim \|\mathrm{curl}\, u\|_{0,\Omega}. \tag{A.1}$$

Besides, since

$$u_t = W_T + \nabla_\Gamma p,$$

we have

$$\|u_t\|_{0,\Gamma} \leq \|W_T\|_{0,\Gamma} + \|\nabla_\Gamma p\|_{0,\Gamma} \lesssim \|W\|_{1,\Omega} + |p|_{1,\Gamma} \lesssim \|\operatorname{curl} u\|_{0,\Omega}. \tag{A.2}$$

Combining (A.1)–(A.2) we finish the proof. □

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