A note on potentially $K_4 - e$ graphical sequences*

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Abstract

A sequence S is potentially $K_4 - e$ graphical if it has a realization containing a $K_4 - e$ as a subgraph. Let $\sigma(K_4 - e, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_4 - e, n)$ is potentially $K_4 - e$ graphical. Gould, Jacobson, Lehel raised the problem of determining the value of $\sigma(K_4 - e, n)$. In this paper, we prove that $\sigma(K_4 - e, n) = 2[(3n-1)/2]$ for $n \geq 7$ and $n = 4, 5$, and $\sigma(K_4 - e, 6) = 20$.

1. Introduction

If $S = (d_1, d_2, \ldots, d_n)$ is a sequence of non-negative integers, then it is called *graphical* if there is a simple graph G of order n, whose degree sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is precisely S. If G is such a graph then G is said to *realize* S or be a *realization* of S. A graphical sequence S is *potentially H graphical* if there is a realization of S containing H as a subgraph, while S is *forcibly H graphical* if every realization of S contains H as a subgraph. We define $\sigma(S) = d_1 + d_2 + \cdots + d_n$. If G and G_1 are graphs, then $G \cup G_1$ is the disjoint union of G and G_1. If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. Let K_k be a complete graph on k vertices, and C_k be a cycle of length k. We write $[x]$ for the largest integer less than or equal to x.

Given a graph H, what is $\text{ex}(n, H)$, the maximum number of edges of a graph with n vertices not containing H as a subgraph? This problem was proposed for $H = C_4$ by Erdös [2] in 1938 and in general by Turan [9]. In terms of graphical sequences, the number $2\text{ex}(n, H)+2$ is the minimum even integer m such that every n-term graphical sequence S with $\sigma(S) \geq m$ is forcibly H graphical. Here we consider the following variant: determine the minimum even integer m such that every n-term graphical sequence S with $\sigma(S) \geq m$ is potentially H graphical. We denote this minimum m by

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$\sigma(H,n)$. Erdös, Jacobson and Lehel [1] showed that $\sigma(K_k,n) \geq (k-2)(2n-k+1)+2$; and conjectured that $\sigma(K_k,n) = (k-2)(2n-k+1)+2$. They proved that if S does not contain zero terms, this conjecture is true for $k=3, n \geq 6$. Li and Song [6,7,8] proved that if S does not contain zero terms, this conjecture is true for $k=4, n \geq 8$ and $k=5, n \geq 10$, and $\sigma(K_k,n) \leq 2n(k-2)+2$ for $n \geq 2k-1$. Gould, Jacobson and Lehel [3] proved that this conjecture is true for $k=4, n \geq 9$; if $n=8$ and $\sigma(S) \geq 28$, then either there is a realization of S containing K_4 or $S=(4^7,0^1)$ (i.e. S consists of seven integers 4 and one integer 0); $\sigma(pK_2,n) = (p-1)(2n-2)+2$ for $p \geq 2; \sigma(C_4,n) = 2[(3n-1)/2]$ for $n \geq 4, \sigma(C_4,n) \leq \sigma(K_4-e,n) \leq \sigma(K_4,n)$; and they raised the problem of determining the value of $\sigma(K_4-e,n)$. Lai [4,5] proved that $\sigma(C_{2m+1},n) = m(2n-m-1)+2$, for $m \geq 2, n \geq 3m$; $\sigma(C_{2m+2},n) = m(2n-m-1)+4$, for $m \geq 2, n \geq 5m-2$.

In this paper, we determine the values of $\sigma(K_4-e,n)$.

2. $\sigma(K_4-e,n)$

Theorem 1. For $n=4,5$ and $n \geq 7$

$$\sigma(K_4-e,n) = \begin{cases} 3n-1 & \text{if } n \text{ is odd} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

For $n=6$, if S is a 6-term graphical sequence with $\sigma(S) \geq 16$, then either there is a realization of S containing K_4-e or $S=(3^6)$. (Thus $\sigma(K_4-e,6)=20$.)

Proof. By [3], for $n \geq 4$,

$$\sigma(K_4-e,n) \geq \sigma(C_4,n) = \begin{cases} 3n-1 & \text{if } n \text{ is odd} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

We need to show that if S is an n-term graphical sequence with $\sigma(S) \geq 3n-1$ if n is odd, or $\sigma(S) \geq 3n-2$ if n is even, then there is a realization of S containing a K_4-e (unless $S=(3^6)$).

For $n=4$, if a graph has size $q \geq 5$, then clearly it contains a K_4-e, so that $\sigma(K_4-e,n) \leq 3n-2$.

For $n=5$, we have $q \geq 7$. There are exactly four graphs of order 5 and size 7 and each contains a K_4-e. Thus $\sigma(K_4-e,n) \leq 3n-1$.

We proceed by induction on n. Take $n \geq 6$ and make the inductive assumption that for $5 \leq t < n$, whenever S_1 is a t-term graphical sequence such that

$$\sigma(S_1) \geq \begin{cases} 3t-1 & \text{if } t \text{ is odd} \\ 3t-2 & \text{if } t \text{ is even} \end{cases}$$

then either S_1 has a realization containing a K_4-e or $S_1=(3^6)$.

We first consider even n. Let S be an n-term graphical sequence with $\sigma(S) \geq 3n-2$. Let G be a realization of S. Assume $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$.

Case 1: Suppose $\sigma(S)=3n-2$. If $d_n \leq 1$, let S' be the degree sequence of $G-v_n$. Then $\sigma(S') \geq 3n-2-2 = 3(n-1)-1$. By induction, S' has a realization containing a $K_4 - e$. Therefore S has a realization containing a $K_4 - e$. Hence, we may assume that $d_n \geq 2$. Since $\sigma(S) = 3n - 2$, then $d_n = d_{n-1} = 2$. Let v_n be adjacent to x and y.

If $x = v_{n-1}$ or $y = v_{n-1}$, let S'' be the degree sequence of $G - v_n - v_{n-1}$. Then $\sigma(S'') = 3n - 2 - 6 = 3(n-2) - 2$. Clearly $S'' \neq (3^6)$. By induction, S'' has a realization containing a $K_4 - e$. Hence, S has a realization containing a $K_4 - e$.

Suppose $x \neq v_{n-1}$ and $y \neq v_{n-1}$, and v_{n-1} is adjacent to x and y. If x is adjacent to y, then G contains a $K_4 - e$. If x is not adjacent to y, then the edge interchange that removes the edges xv_{n-1} and yv_n and inserts the edges xy and v_nv_{n-1} produces a realization G' of S containing $v_{n-1}v_n$, and we are done as before.

Suppose $x \neq v_{n-1}$ and $y \neq v_{n-1}$, and v_{n-1} is not adjacent to x. Let v_{n-1} be adjacent to z_1 and z_2. We first consider the case that x is not adjacent to z_1. Then the edge interchange that removes the edges $v_{n-1}z_1$ and v_nx and inserts the edges xz_1 and $v_{n-1}v_n$ produces a realization G' of S containing $v_{n-1}v_n$. We have reduced this case to a graph G' as above. If x is not adjacent to z_2, as in the previous case, we can prove that S has a realization containing a $K_4 - e$. Finally, consider what happens if x is adjacent to both z_1 and z_2. If z_1 is adjacent to z_2, then G contains a $K_4 - e$. If z_1 is not adjacent to z_2, then the edge interchange that removes the edges $v_{n-1}z_1$, $v_{n-1}z_2$ and v_nx and inserts the edges $v_{n-1}v_n$, z_1z_2 and $v_{n-1}x$ produces a realization G' of S containing $v_{n-1}v_n$, and we are done as before.

The case that $x \neq v_{n-1}$ and $y \neq v_{n-1}$, and v_{n-1} is not adjacent to y is of course similar to the previous case.

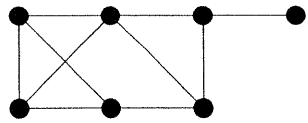
Case 2: Suppose $\sigma(S) = 3n$. If $d_n \leq 2$, let S' be the degree sequence of $G - v_n$. Then $\sigma(S') \geq 3n - 4 = 3(n-1) - 1$. By induction, S' has a realization containing a $K_4 - e$. Hence, S has a realization containing a $K_4 - e$. Thus, we may assume that $d_n \geq 3$, and so $S = (3^n)$. If $n = 6$ then $S = (3^6)$ (and clearly no realization of S contains a $K_4 - e$). Let G_1 be a realization of (3^6). If $n = 4p$ ($p \geq 2$), then pK_4 is a realization of $S = (3^n)$ which contains a $K_4 - e$. And if $n = 4p + 2$ ($p \geq 2$), then $G_1 \cup (p-1)K_4$ is a realization of $S = (3^n)$ which contains a $K_4 - e$.

Case 3: Suppose $3n + 2 \leq \sigma(S) \leq 4n - 2$. Then $d_n \leq 3$. Let S' be a degree sequence of $G - v_n$, then $\sigma(S') \geq 3n + 2 - 6 = 3(n-1) - 1$. By induction, S' has a realization containing a $K_4 - e$. Hence, S has a realization containing a $K_4 - e$.

Case 4: Suppose $\sigma(S) \geq 4n$. If $n \geq 8$, by Proposition 2 and Theorem 4 of [3], S has a realization containing a K_4. Now consider $n = 6$. If $4n \leq \sigma(S) \leq 5n - 2$, then $d_n \leq 4$. Let S' be a degree sequence of $G - v_n$, so $\sigma(S') \geq 4n - 8 = 16 = 3(n-1) + 1$. By induction, S' has a realization containing a $K_4 - e$. Hence, S has a realization containing a $K_4 - e$. If $\sigma(S) \geq 5n = 30$, then $\sigma(S) = 30$. The realization of S is K_6 which contains $K_4 - e$. This completes the discussion for even n.

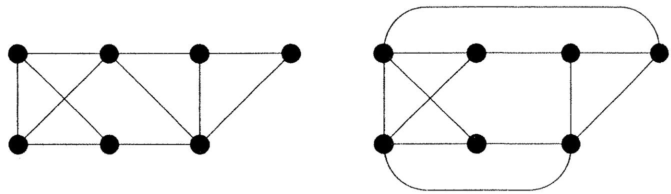
Now we consider odd n. Let S be an n-term graphical sequence with $\sigma(S) \geq 3n - 1$. Let G be a realization of S. Assume $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$.

Case 1: Suppose $\sigma(S) = 3n - 1$. Then $d_n \leq 2$. Let S' be the degree sequence of $G - v_n$, so $\sigma(S') \geq 3n - 1 - 4 = 3(n-1) - 2$. By induction, either S' has a realization containing a $K_4 - e$ or $S' = (3^6)$. Therefore either S has a realization containing a $K_4 - e$ or $S = (4^1, 3^5, 1^1)$. Clearly, $(4^1, 3^5, 1^1)$ has a realization containing a $K_4 - e$ (see Figure 1). In either event, S has a realization containing a $K_4 - e$.



$(4^1, 3^5, 1^1)$

Figure 1



$(4^2, 3^4, 2^1)$ $(4^3, 3^4)$

Figure 2

Case 2: Suppose $3n + 1 \leq \sigma(S) \leq 4n - 2$. Then $d_n \leq 3$. Let S' be the degree sequence of $G - v_n$, so $\sigma(S') \geq 3n+1-6 = 3(n-1)-2$. By induction, either S' has a realization containing a $K_4 - e$ or $S' = (3^6)$. Therefore either S has a realization containing a $K_4 - e$ or $S = (4^2, 3^4, 2^1), S = (4^3, 3^4)$. Clearly, both $(4^2, 3^4, 2^1)$ and $(4^3, 3^4)$ have a realization containing a $K_4 - e$ (see Figure 2). In any event, S has a realization containing a $K_4 - e$.

Case 3: Suppose $\sigma(S) \geq 4n$. If $n \geq 9$, then by Theorem 4 of [3], S has a realization containing a K_4. Next, if $n = 7$ and if $4n \leq \sigma(S) \leq 5n - 1$, then $d_n \leq 4$. Let S' be a degree sequence of $G - v_n$. Then $\sigma(S') \geq 4n - 8 = 3n - 1 = 3(n-1)+2$. Clearly $S' \neq (3^6)$, so by induction, S' has a realization containing $K_4 - e$. Thus S has a realization containing a $K_4 - e$. Finally, suppose that $\sigma(S) \geq 5n + 1 = 36$. Clearly, $(6^6, 0^1)$ is not graphical. Hence $d_7 \geq 1$ and by Theorem 2.2 of [6], S has a realization containing a K_4.

This completes the discussion for odd n, and so finishes the inductive step. The Theorem is proved.

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