On potentially $K_{r+1} - U$-graphical Sequences [*]

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Abstract

Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). We use the symbol Z_4 to denote $K_4 - P_2$. A sequence S is potentially $K_m - H$-graphical if it has a realization containing a $K_m - H$ as a subgraph. Let $\sigma(K_m - H, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_m - H, n)$ is potentially $K_m - H$-graphical. In this paper, we determine the values of $\sigma(K_{r+1} - U, n)$ for $n \geq 5r + 18, r + 1 \geq k \geq 7, j \geq 6$ where U is a graph on k vertices and j edges which contains a graph $K_3 \bigcup P_3$ but not contains a cycle on 4 vertices and not contains Z_4.

Key words: graph; degree sequence; potentially $K_{r+1} - U$-graphic sequence; potentially $K_{r+1} - K_3 \bigcup P_3$-graphic sequence

AMS Subject Classifications: 05C07, 05C35

1 Introduction

The set of all non-increasing nonnegative integers sequence $\pi = (d_1, d_2, ..., d_n)$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be graphic if it is the

degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H-graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. Let $\sigma(\pi) = d_1 + d_2 + ... + d_n$, and $[x]$ denote the largest integer less than or equal to x. If G and G_1 are graphs, then $G \cup G_1$ is the disjoint union of G and G_1. If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. We denote $G + H$ as the graph with $V(G + H) = V(G) \bigcup V(H)$ and $E(G + H) = E(G) \bigcup E(H) \bigcup \{xy : x \in V(G), y \in V(H)\}$. Let K_k, C_k, T_k, and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k + 1$ vertices, and a path on $k + 1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). Let F_k denote the friendship graph on $2k + 1$ vertices, that is, the graph of k triangles intersecting in a single vertex. For $0 \le r \le t$, denote the generalized friendship graph on $kt - kr + r$ vertices by $F_{t,r,k}$, where $F_{t,r,k}$ is the graph of k copies of K_t meeting in a common r set. We use the symbol Z_4 to denote $K_4 - P_2$. We use the symbol $G[v_1, v_2, ..., v_k]$ to denote the subgraph of G induced by vertex set $\{v_1, v_2, ..., v_k\}$. We use the symbol $\epsilon(G)$ to denote the number of edges in graph G.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. Mantel [21] proved that $ex(n, K_3) = [\frac{n^2}{4}]$. This was rediscovered by Turán [22] as a special case of his results on $ex(n, K_k)$. In terms of graphic sequences, the number $2ex(n, H)+2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \ge l$ is forcibly H-graphical. Here we consider the following variant: determine the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \ge l$ is potentially H-graphical. We denote this minimum l by $\sigma(H, n)$. Erdös, Jacobson and Lehel [3] showed that $\sigma(K_k, n) \ge (k - 2)(2n - k + 1) + 2$ and conjectured that the equality holds. They proved that if π does not contain zero terms, this conjecture is true for $k = 3$, $n \ge 6$. The conjecture is confirmed in [6],[15],[16],[17] and [18].

Gould, Jacobson and Lehel [6] also proved that $\sigma(pK_2, n) = (p-1)(2n-2) + 2$ for $p \ge 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \ge 4$. They also pointed out that it would be nice to see where in the range for $3n - 2$ to $4n - 4$, the value $\sigma(K_4 - e, n)$ lies. Luo [19] characterized the potentially C_k graphic sequence for $k = 3, 4, 5$. Luo and Warner [20] characterized the potentially K_4-graphic sequences. Yin and Yin [32] characterize the potentially

$(K_5 - e)$-positive graphic sequences and give two simple necessary and sufficient conditions for a positive graphic sequence π to be potentially K_5-graphic. Moreover, they also give a simple necessary and sufficient condition for a positive graphic sequence π to be potentially K_6-graphic. Ferrara, Gould and Schmitt [5] determined $\sigma(F_k, n)$ for n sufficiently large. Ferrara [4] determined $\sigma(F_{t,0,k}, n)$ for a sufficiently large choice of n and determined $\sigma(F_{t,t-2,k}, n)$ for a sufficiently large choice of n. Yin and Chen [23] determined $\sigma(F_{t,t-1,k}, n)$ for $n \geq 3t + 2k^2 + 3k - 6$. Yin, Chen and Schmitt [24] determined $\sigma(F_{t,r,k}, n)$ for $k \geq 2, t \geq 3, 1 \leq r \leq t - 2$ and n sufficiently large. Gould et al. [6] determined $\sigma(K_{2,2}, n)$ for $n \geq 4$. Yin et al. [27-30] determined $\sigma(K_{r,s}, n)$ for $s \geq r \geq 2$ and sufficiently large n. Lai [8] determined $\sigma(K_4 - e, n)$ for $n \geq 4$. Yin, Li and Mao[26] determined $\sigma(K_{r+1} - e, n)$ for $r \geq 3$, $r + 1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$. Yin and Li[25] gave a good method (Yin-Li method) of determining the values $\sigma(K_{r+1} - e, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$ (In fact, Yin and Li[25] also determining the values $\sigma(K_{r+1} - ke, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$). After reading[25], using Yin-Li method Yin [31] determined $\sigma(K_{r+1} - K_3, n)$ for $n \geq 3r + 5, r \geq 3$. Lai [9] determined $\sigma(K_5 - K_3, n)$, for $n \geq 5$. Lai [10,11] determined $\sigma(K_5 - C_4, n), \sigma(K_5 - P_3, n)$ and $\sigma(K_5 - P_4, n)$, for $n \geq 5$. Determining $\sigma(K_{r+1} - H, n)$, where H is a tree on 4 vertices is more useful than a cycle on 4 vertices (for example, $C_4 \not\subset C_i$, but $P_3 \subset C_i$ for $i \geq 5$). So, after reading[25] and [31], using Yin-Li method Lai and Hu[12] determined $\sigma(K_{r+1} - H, n)$ for $n \geq 4r + 10, r \geq 3, r + 1 \geq k \geq 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices and $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r + 8, r \geq 3$. Using Yin-Li method Lai and Sun[13] determined $\sigma(K_{r+1} - (kP_2 \bigcup tK_2), n)$ for $n \geq 4r + 10, r + 1 \geq 3k + 2t, k + t \geq 2, k \geq 1, t \geq 0$. To now, the problem of determining $\sigma(K_{r+1} - H, n)$ for H not containing a cycle on 3 vertices and sufficiently large n has been solved. Using Yin-Li method Lai[14] determine the values of $\sigma(K_{r+1} - Z, n)$ for $n \geq 5r + 19, r + 1 \geq k \geq 5$, $j \geq 5$ where Z is a graph on k vertices and j edges which contains a graph Z_4 but not contains a cycle on 4 vertices. Using Yin-Li method Lai[14] also determine the values of $\sigma(K_{r+1} - Z_4, n)$, $\sigma(K_{r+1} - (K_4 - e), n)$, $\sigma(K_{r+1} - K_4, n)$ for $n \geq 5r + 16, r \geq 4$. In this paper, using Yin-Li method we prove the following two theorems.

Theorem 1.1. If $r \geq 6$ and $n \geq 5r + 18$, then

$$\sigma(K_{r+1} - (K_3 \bigcup P_3), n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \quad \text{if } n - r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \quad \text{if } n - r \text{ is even} \end{cases}$$

Theorem 1.2. If $n \geq 5r + 18, r + 1 \geq k \geq 7$, and $j \geq 6$, then

$$\sigma(K_{r+1} - U, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \quad \text{if } n - r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \quad \text{if } n - r \text{ is even} \end{cases}$$

where U is a graph on k vertices and j edges which contains a graph $(K_3 \bigcup P_3)$ but not contains a cycle on 4 vertices and not contains Z_4.

There are a number of graphs on k vertices and j edges which contains a graph $(K_3 \bigcup P_3)$ but not contains a cycle on 4 vertices and not contains Z_4. (for example, $C_3 \bigcup C_{i_1} \bigcup C_{i_2} \bigcup \cdots \bigcup C_{i_p}$ $(i_j \neq 4, j = 2, 3, \cdots, p, i_1 \geq 5)$, $C_3 \bigcup P_{i_1} \bigcup P_{i_2} \bigcup \cdots \bigcup P_{i_p}$ $(i_1 \geq 3)$, $C_3 \bigcup P_{i_1} \bigcup C_{i_2} \bigcup \cdots \bigcup C_{i_p}$ $(i_j \neq 4, j = 2, 3, \cdots, p, i_1 \geq 3)$, etc)

2 Preparations

In order to prove our main result, we need the following notations and results.

Let $\pi = (d_1, \cdots, d_n) \in NS_n, 1 \leq k \leq n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), \\ \quad \text{if } d_k \geq k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), \\ \quad \text{if } d_k < k. \end{cases}$$

Denote $\pi_k' = (d_1', d_2', \cdots, d_{n-1}')$, where $d_1' \geq d_2' \geq \cdots \geq d_{n-1}'$ is a rearrangement of the $n - 1$ terms of π_k''. Then π_k' is called the residual sequence obtained by laying off d_k from π.

Theorem 2.1 [25] Let $n \geq r + 1$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r+1} \geq r$. If $d_i \geq 2r - i$ for $i = 1, 2, \cdots, r - 1$, then π is potentially A_{r+1}-graphic.

Theorem 2.2 [25] Let $n \geq 2r + 2$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r+1} \geq r$. If $d_{2r+2} \geq r - 1$, then π is potentially A_{r+1}-graphic.

Theorem 2.3 [25] Let $n \geq r + 1$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r+1} \geq r - 1$. If $d_i \geq 2r - i$ for $i = 1, 2, \cdots, r - 1$, then π is potentially $K_{r+1} - e$-graphic.

Theorem 2.4 [25] Let $n \geq 2r + 2$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r-1} \geq r$. If $d_{2r+2} \geq r - 1$, then π is potentially $K_{r+1} - e$ -graphic.

Theorem 2.5 [7] Let $\pi = (d_1, \cdots, d_n) \in NS_n$ and $1 \leq k \leq n$. Then $\pi \in GS_n$ if and only if $\pi_k' \in GS_{n-1}$.

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Theorem 2.6 [2] Let $\pi = (d_1, \cdots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any $t, 1 \leq t \leq n-1$,

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{j=t+1}^{n} min\{t, d_j\}.$$

Theorem 2.7 [6] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Lemma 2.1 [31] If $\pi = (d_1, d_2, \cdots, d_n) \in NS_n$ is potentially $K_{r+1} - e$-graphic, then there is a realization G of π containing $K_{r+1} - e$ with the $r+1$ vertices $v_1, \cdots, v_{r+1}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \cdots, r+1$ and $e = v_r v_{r+1}$.

Lemma 2.2 [14] Let $n \geq 2r$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r-1} \geq r$, $d_{r+1} \geq r-1$. If $d_i \geq 2r-i$ for $i = 1, 2, \cdots, r-2$, then π is potentially $K_{r+1} - e$-graphic.

Lemma 2.3 [14] Let $\pi = (d_1, \cdots, d_n) \in GS_n$ and G be a realization of π. If $\epsilon(G[v_1, v_2, ..., v_{r+1}]) \leq \epsilon(K_{r+1}) - 1$, then there is a realization H of π such that $d_H(v_i) = d_i$ for $i = 1, 2, \cdots, r+1$ and $v_r v_{r+1} \notin E(H)$.

Lemma 2.4 [14] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r-4} \geq r$,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \quad \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \quad \text{if } n-r \text{ is even} \end{cases}$$

If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - (P_2 \bigcup K_2)$-graphic.

Lemma 2.5 [14] Let $n \geq 2r$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r-2} \geq r+1$, $d_{r+1} \geq r$, $d_r - 1 \geq d_{d_{r+1}+2}$. If $d_i \geq 2r-i$ for $i = 1, 2, \cdots, r-3$, then π is potentially A_{r+1}-graphic.

3 Proof of Main Results

Lemma 3.1 Let $n \geq 2r+2, r \geq 4$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r-2} \geq r-1$ and $d_{r+1} \geq r-2$,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \quad \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \quad \text{if } n-r \text{ is even} \end{cases}$$

If $d_i \geq 2r-i$ for $i = 1, 2, \cdots, r-3$, then π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r - 1$.

Subcase 1.1: $d_{r-1} \geq r + 1$.

If $d_{r-2} \geq r + 2$, then π is potentially $K_{r+1} - e$-graphic by Theorem 2.3. Hence,π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic.

If $d_{r-2} = r + 1$.

If $d_{r+1} = r + 1$, then $d_{r-2} = d_{r-1} = d_r = d_{r+1} = r + 1$. Suppose π is not potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic. Let H be a realization of π, then $\epsilon(H[v_1, v_2, ..., v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$. Let $S = (d_1, d_2, \cdots, d_{r-3}, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, d_{r+2}, \cdots, d_n)$, then by Theorem 2.1, S is potentially A_{r+1}-graphic (Denote $S' = (d'_1, d'_2, \cdots, d'_n)$,where $d'_1 \geq d'_2 \geq \cdots \geq d'_n$ is a rearrangement of the n terms of S. Therefore $S' \in GS_n$ by Lemma 2.3. Then S' satisfies the conditions of Theorem 2.1). Therefore, there is a realization G of S with $v_1, v_2, \cdots, v_{r+1}$ $(d(v_i) = d_i, i = 1, 2, \cdots, r - 1, d(v_r) = d_r + 1, d(v_{r+1}) = d_{r+1} + 1)$, the $r + 1$ vertices of highest degree containing a K_{r+1} by Theorem 2.7. Hence, $G - v_{r+1}v_r$ is a realization of π. Thus, π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic, which is a contradiction.

If $d_{r+1} = r$ or $d_{r+1} = r - 1$, then $d_{r-1} - 1 \geq r \geq d_{r+2}$. The residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 = d_1 - 1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$, and $d'_{(r-1)+1} = d'_r \geq r - 2$. By Lemma 2.2, π'_{r+1} is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_{r-1} - 1\} \subseteq \{d'_1, \cdots, d'_r\}$ and Lemma 2.1.

Subcase 1.2: $d_{r-1} \leq r$.

If $d_{r-2} \geq r + 1$, then $d_{r-2} - 1 \geq d_{r-1}$. The residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 = d_1 - 1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$, and $d'_{(r-1)+1} = d'_r \geq r - 2$. By Lemma 2.2, π'_{r+1} is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1\} \subseteq \{d'_1, \cdots, d'_r\}$ and Lemma 2.1.

If $d_{r-2} = r$.

If $d_{r+1} = r$, then $d_{r-2} = d_{r-1} = d_r = d_{r+1} = r$. Suppose π is not potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic. Let H be a realization of π, then $\epsilon(H[v_1, v_2, ..., v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$. Let $S = (d_1, d_2, \cdots, d_{r-3}, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, d_{r+2}, \cdots, d_n)$. Denote $S' = (d'_1, d'_2, \cdots, d'_n)$,where $d'_1 \geq d'_2 \geq \cdots \geq d'_n$ is a rearrangement of the n terms of S. Therefore $S' \in GS_n$ by Lemma 2.3. The residual sequence $S''_{r+1} = (d''_1, \cdots, d''_{n-1})$ obtained by laying off $d'_{r+1} = d_{r-1} = r$ from S' satisfies: $d''_1 = d'_1 - 1 \geq 2(r-1) - 1, \cdots, d''_{(r-1)-2} = d''_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d''_{(r-1)-1} = d''_{r-2} \geq r$, and $d''_{(r-1)+1} = d''_r \geq r - 1$. By Theorem 2.1, S''_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic by

$\{d_1 - 1, \cdots, d_{r-3} - 1, d_r, d_{r+1}\} \subseteq \{d_1'', \cdots, d_r''\}$ and Theorem 2.7, which is a contradiction.

If $d_{r+1} = r - 1$ and $d_r = r$, then $d_{r-2} = d_{r-1} = d_r = r$. The residual sequence $\pi_{r+1}' = (d_1', \cdots, d_{n-1}')$ obtained by laying off d_{r+1} from π satisfies: $d_1' = d_1 - 1 \geq 2(r-1) - 1, \cdots, d_{(r-1)-2}' = d_{r-3}' \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d_{(r-1)-1}' = d_{r-2}' \geq r - 1$, and $d_{(r-1)+1}' = d_r' \geq r - 2$. By Lemma 2.2, π_{r+1}' is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1\} \subseteq \{d_1', \cdots, d_r'\}$, $d_{r-2}' = d_r$, $d_{r-1}' = d_{r-2} - 1$, $d_r' = d_{r-1} - 1$ and Lemma 2.1.

If $d_{r+1} = r - 1$ and $d_r = r - 1$, then $d_{r-2} - 1 \geq d_{r+2}$. The residual sequence $\pi_{r+1}' = (d_1', \cdots, d_{n-1}')$ obtained by laying off d_{r+1} from π satisfies: $d_1' = d_1 - 1 \geq 2(r-1) - 1, \cdots, d_{(r-1)-2}' = d_{r-3}' \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d_{(r-1)-1}' = d_{r-2}' \geq r - 1$, and $d_{(r-1)+1}' = d_r' \geq r - 2$. By Lemma 2.2, π_{r+1}' is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1\} \subseteq \{d_1', \cdots, d_r'\}$, $d_{r-2}' = d_{r-2} - 1$ and Lemma 2.1.

If $d_{r-2} = r - 1$, then $d_{r-2} = d_{r-1} = d_r = d_{r+1} = r - 1$. Suppose π is not potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic. Let H be a realization of π, then $\epsilon(H[v_1, v_2, ..., v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$. Let $S = (d_1, d_2, \cdots, d_{r-3}, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, d_{r+2}, \cdots, d_n)$, Denote $S' = (d_1', d_2', \cdots, d_n')$, where $d_1' \geq d_2' \geq \cdots \geq d_n'$ is a rearrangement of the n terms of S. Therefore $S' \in GS_n$ by Lemma 2.3. The residual sequence $S_{r+1}'' = (d_1'', \cdots, d_{n-1}'')$ obtained by laying off $d_{r+1}' = d_{r-1} = r - 1$ from S' satisfies: $d_1'' = d_1' - 1 \geq 2(r-1) - 1, \cdots, d_{(r-1)-2}'' = d_{r-3}'' \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d_{(r-1)-1}'' = d_{r-2}'' = r - 1$, and $d_{(r-1)+1}'' = d_r'' = r - 1$. By Lemma 2.2, S_{r+1}'' is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic by $\{d_1 - 1, \cdots, d_{r-3} - 1, d_r, d_{r+1}, d_{r-2}\} = \{d_1'', \cdots, d_r''\}$ and Lemma 2.1, which is a contradiction.

Case 2: $d_{r+1} = r - 2$.

Subcase 2.1: $d_{r-1} < d_{r-2}$.

If $d_{r-2} \geq r$, then the residual sequence $\pi_{r+1}' = (d_1', \cdots, d_{n-1}')$ obtained by laying off $d_{r+1} = r - 2$ from π satisfies: (1) $d_i' = d_i - 1$ for $i = 1, 2, \cdots, r - 2$,(2) $d_1' = d_1 - 1 \geq 2(r-1) - 1, \cdots, d_{(r-1)-2}' = d_{r-3}' \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d_{(r-1)-1}' = d_{r-2}' \geq r - 1$, and $d_{(r-1)+1}' = d_r' = d_r \geq r - 2$. By Lemma 2.2, π_{r+1}' is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_{r-1}, d_r\} = \{d_1', \cdots, d_r'\}$ and Lemma 2.1.

If $d_{r-2} = r - 1$, then

$$
\begin{aligned}
\sigma(\pi) &\leq (r-3)(n-1) + r - 1 + (r-2)(n-r+2) \\
&= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+2) \\
&= (r-1)(2n-r) - 3(n-r) - 2 \\
&< \sigma(\pi),
\end{aligned}
$$

which is a contradiction.

Subcase 2.2: $d_{r-1} = d_{r-2}$, then π'_{r+1} satisfies: $d'_1 \geq d_1 - 1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1)-2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$ and $d'_{(r-1)+1} = d'_r \geq r - 2$. By Lemma 2.2, π'_{r+1} is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic by $\{d_{r-1}, d_r, d_1 - 1 \cdots, d_{r-2} - 1\} = \{d'_1, \cdots, d'_r\}$ and Lemma 2.1.

Lemma 3.2. If $n \geq r + 1, r + 1 \geq k \geq 7$, then

$$
\sigma(K_{r+1} - U, n) \geq
\begin{cases}
(r-1)(2n-r) - 3(n-r) - 1, \\
\quad \text{if } n - r \text{ is odd} \\
(r-1)(2n-r) - 3(n-r), \\
\quad \text{if } n - r \text{ is even}
\end{cases}
$$

where U is a graph on k vertices and j edges which not contains a cycle on 4 vertices and not contains Z_4.

Proof. Let

$$
G =
\begin{cases}
K_{r-3} + (\frac{n-r-1}{2} K_2 \bigcup P_2 \bigcup K_1), \\
\quad \text{if } n - r \text{ is odd} \\
K_{r-3} + (\frac{n-r}{2} K_2 \bigcup P_2), \\
\quad \text{if } n - r \text{ is even}
\end{cases}
$$

Then G is a unique realization of

$$
\pi =
\begin{cases}
((n-1)^{r-3}, (r-1)^1, (r-2)^{n-r+1}, (r-3)^1) \\
\quad \text{if } n - r \text{ is odd} \\
((n-1)^{r-3}, (r-1)^1, (r-2)^{n-r+2}) \\
\quad \text{if } n - r \text{ is even}
\end{cases}
$$

and G clearly does not contain $K_{r+1} - U$, where the symbol x^y means x repeats y times in the sequence. Thus $\sigma(K_{r+1}-U, n) \geq \sigma(\pi)+2$. Therefore,

$$
\sigma(K_{r+1} - U, n) \geq
\begin{cases}
(r-1)(2n-r) - 3(n-r) - 1, \\
\quad \text{if } n - r \text{ is odd} \\
(r-1)(2n-r) - 3(n-r), \\
\quad \text{if } n - r \text{ is even}
\end{cases}
$$

The Proof of Theorem 1.1 According to Lemma 3.2, it is enough to verify that for $n \geq 5r + 18$,

$$\sigma(K_{r+1} - (K_3 \bigcup P_3), n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \quad \text{if } n - r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \quad \text{if } n - r \text{ is even} \end{cases}$$

We now prove that if $n \geq 5r + 18$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \quad \text{if } n - r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \quad \text{if } n - r \text{ is even} \end{cases}$$

then π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic.

If $d_{r-4} \leq r - 1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-5)(n-1) + (r-1)(n-r+5) \\ &= (r-1)(n-1) - 4(n-1) + (r-1)(n-r+5) \\ &= (r-1)(2n-r) - 4(n-r) \\ &< (r-1)(2n-r) - 3(n-r) - 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r-4} \geq r$.

If $d_{r-2} \leq r - 2$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-3)(n-1) + (r-2)(n-r+3) \\ &= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+3) \\ &= (r-1)(2n-r) - 3(n-r) - 3 \\ &< (r-1)(2n-r) - 3(n-r) - 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r-2} \geq r - 1$.

If $d_{r+1} \leq r - 3$, then

$$\begin{aligned} \sigma(\pi) &= \sum_{i=1}^{r} d_i + \sum_{i=r+1}^{n} d_i \\ &\leq (r-1)r + \sum_{i=r+1}^{n} min\{r, d_i\} + \sum_{i=r+1}^{n} d_i \\ &= (r-1)r + 2\sum_{i=r+1}^{n} d_i \\ &\leq (r-1)r + 2(n-r)(r-3) \\ &= (r-1)(2n-r) - 4(n-r) \\ &< (r-1)(2n-r) - 3(n-r) - 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r+1} \geq r - 2$.

If $d_i \geq 2r - i$ for $i = 1, 2, \cdots, r-3$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - (K_3 \bigcup P_3)$-graphic by Lemma 3.1 or Lemma 2.4 . If $d_{2r+2} \leq r - 2$

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and there exists an integer i, $1 \leq i \leq r - 3$ such that $d_i \leq 2r - i - 1$, then

$$
\begin{aligned}
\sigma(\pi) \;\leq\; & (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\
& + (r-2)(n+1-2r-2) \\
=\; & i^2 + i(n - 4r - 2) - (n-1) \\
& + (2r-1)(2r+2) + (r-2)(n-2r-1).
\end{aligned}
$$

Since $n \geq 5r + 18$, it is easy to see that $i^2 + i(n - 4r - 2)$, consider as a function of i, attains its maximum value when $i = r - 3$. Therefore,

$$
\begin{aligned}
\sigma(\pi) \;\leq\; & (r-3)^2 + (n - 4r - 2)(r-3) - (n-1) \\
& + (2r-1)(2r+2) + (r-2)(n-2r-1) \\
=\; & (r-1)(2n-r) - 3(n-r) - n + 5r + 16 \\
<\; & \sigma(\pi),
\end{aligned}
$$

which is a contradiction.

Thus,

$$
\sigma(K_{r+1} - (K_3 \bigcup P_3), n) \leq
\begin{cases}
(r-1)(2n-r) - 3(n-r) - 1, \\
\quad \text{if } n - r \text{ is odd} \\
(r-1)(2n-r) - 3(n-r), \\
\quad \text{if } n - r \text{ is even}
\end{cases}
$$

for $n \geq 5r + 18$.

The Proof of Theorem 1.2 By Lemma 3.2, for $n \geq 5r + 18, r + 1 \geq k \geq 7$, and $j \geq 6$,

$$
\sigma(K_{r+1} - U, n) \geq
\begin{cases}
(r-1)(2n-r) - 3(n-r) - 1, \\
\quad \text{if } n - r \text{ is odd} \\
(r-1)(2n-r) - 3(n-r), \\
\quad \text{if } n - r \text{ is even}
\end{cases}
$$

Obviously, $\sigma(K_{r+1} - U, n) \leq \sigma(K_{r+1} - (K_3 \bigcup P_3), n)$. By theorem 1.1,

$$
\sigma(K_{r+1} - (K_3 \bigcup P_3), n) =
\begin{cases}
(r-1)(2n-r) - 3(n-r) - 1, \\
\quad \text{if } n - r \text{ is odd} \\
(r-1)(2n-r) - 3(n-r), \\
\quad \text{if } n - r \text{ is even}
\end{cases}
$$

Then

$$
\sigma(K_{r+1} - U, n) =
\begin{cases}
(r-1)(2n-r) - 3(n-r) - 1, \\
\quad \text{if } n - r \text{ is odd} \\
(r-1)(2n-r) - 3(n-r), \\
\quad \text{if } n - r \text{ is even}
\end{cases}
$$

for $n \geq 5r + 18, r + 1 \geq k \geq 7$, and $j \geq 6$.

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