

A LOWER BOUND FOR THE NUMBER OF EDGES IN A GRAPH CONTAINING NO TWO CYCLES OF THE SAME LENGTH

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph of n vertices in which any two cycles are of different lengths. In this paper, it is proved that

$$f(n) \geq n + 32t - 1$$

for $t = 27720r + 169$ ($r \geq 1$) and $n \geq \frac{6911}{16}t^2 + \frac{514441}{8}t - \frac{3309665}{16}$. Consequently,
 $\liminf_{n \rightarrow \infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2 + \frac{2562}{6911}}$.

1 Introduction

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see [1], p.247, Problem 11). Shi[2] proved that

$$f(n) \geq n + [(\sqrt{8n - 23} + 1)/2]$$

for $n \geq 3$. Lai[3,4,5,6] proved that for $n \geq (1381/9)t^2 + (26/45)t + 98/45$, $t = 360q + 7$,

$$f(n) \geq n + 19t - 1,$$

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and for $n \geq e^{2m}(2m + 3)/4$,

$$f(n) < n - 2 + \sqrt{nl \ln(4n/(2m + 3))} + 2n + \log_2(n + 6).$$

Boros, Caro, Füredi and Yuster[7] proved that

$$f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

Let $v(G)$ denote the number of vertices, and $\epsilon(G)$ denote the number of edges. In this paper, we construct a graph G having no two cycles with the same length which leads to the following result.

Theorem. Let $t = 27720r + 169$ ($r \geq 1$), then

$$f(n) \geq n + 32t - 1$$

for $n \geq \frac{6911}{16}t^2 + \frac{514441}{8}t - \frac{3309665}{16}$.

2 Proof of Theorem

Proof. Let $t = 27720r + 169$, $r \geq 1$, $n_t = \frac{6911}{16}t^2 + \frac{514441}{8}t - \frac{3309665}{16}$, $n \geq n_t$. We shall show that there exists a graph G on n vertices with $n + 32t - 1$ edges such that all cycles in G have distinct lengths.

Now we construct the graph G which consists of a number of subgraphs: B_i , ($0 \leq i \leq 21t + \frac{7t+1}{8} - 58$, $22t - 798 \leq i \leq 22t + 64$, $23t - 734 \leq i \leq 23t + 267$, $24t - 531 \leq i \leq 24t + 57$, $25t - 741 \leq i \leq 25t + 58$, $26t - 740 \leq i \leq 26t + 57$, $27t - 741 \leq i \leq 27t + 57$, $28t - 741 \leq i \leq 28t + 52$, $29t - 746 \leq i \leq 29t + 60$, $30t - 738 \leq i \leq 30t + 60$, and $31t - 738 \leq i \leq 31t + 799$).

Now we define these B_i 's. These subgraphs all have a common vertex x , otherwise their vertex sets are pairwise disjoint.

For $\frac{7t+1}{8} \leq i \leq t - 742$, let the subgraph $B_{19t+2i+1}$ consist of a cycle

$$C_{19t+2i+1} = xx_i^1x_i^2 \dots x_i^{144t+13i+1463}x$$

and eleven paths sharing a common vertex x , the other end vertices are on the cycle $C_{19t+2i+1}$:

$$\begin{aligned} & xx_{i,1}^1x_{i,1}^2 \dots x_{i,1}^{(11t-1)/2} x_i^{(31t-115)/2+i} \\ & xx_{i,2}^1x_{i,2}^2 \dots x_{i,2}^{(13t-1)/2} x_i^{(51t-103)/2+2i} \\ & xx_{i,3}^1x_{i,3}^2 \dots x_{i,3}^{(13t-1)/2} x_i^{(71t+315)/2+3i} \\ & xx_{i,4}^1x_{i,4}^2 \dots x_{i,4}^{(15t-1)/2} x_i^{(91t+313)/2+4i} \\ & xx_{i,5}^1x_{i,5}^2 \dots x_{i,5}^{(15t-1)/2} x_i^{(111t+313)/2+5i} \end{aligned}$$

$$\begin{aligned}
& xx_{i,6}^1 x_{i,6}^2 \dots x_{i,6}^{(17t-1)/2} x_i^{(131t+311)/2+6i} \\
& xx_{i,7}^1 x_{i,7}^2 \dots x_{i,7}^{(17t-1)/2} x_i^{(151t+309)/2+7i} \\
& xx_{i,8}^1 x_{i,8}^2 \dots x_{i,8}^{(19t-1)/2} x_i^{(171t+297)/2+8i} \\
& xx_{i,9}^1 x_{i,9}^2 \dots x_{i,9}^{(19t-1)/2} x_i^{(191t+301)/2+9i} \\
& xx_{i,10}^1 x_{i,10}^2 \dots x_{i,10}^{(21t-1)/2} x_i^{(211t+305)/2+10i} \\
& xx_{i,11}^1 x_{i,11}^2 \dots x_{i,11}^{(t-571)/2} x_i^{(251t+2357)/2+11i}.
\end{aligned}$$

From the construction, we notice that $B_{19t+2i+1}$ contains exactly seventy-eight cycles of lengths:

$21t + i - 57,$	$22t + i + 7,$	$23t + i + 210,$	$24t + i,$
$25t + i + 1,$	$26t + i,$	$27t + i,$	$28t + i - 5,$
$29t + i + 3,$	$30t + i + 3,$	$31t + i + 742,$	$19t + 2i + 1,$
$32t + 2i - 51,$	$32t + 2i + 216,$	$34t + 2i + 209,$	$34t + 2i,$
$36t + 2i,$	$36t + 2i - 1,$	$38t + 2i - 6,$	$38t + 2i - 3,$
$40t + 2i + 5,$	$40t + 2i + 744,$	$49t + 3i + 1312,$	$42t + 3i + 158,$
$43t + 3i + 215,$	$44t + 3i + 209,$	$45t + 3i - 1,$	$46t + 3i - 1,$
$47t + 3i - 7,$	$48t + 3i - 4,$	$49t + 3i - 1,$	$50t + 3i + 746,$
$58t + 4i + 1314,$	$53t + 4i + 157,$	$53t + 4i + 215,$	$55t + 4i + 208,$
$55t + 4i - 2,$	$57t + 4i - 7,$	$57t + 4i - 5,$	$59t + 4i - 2,$
$59t + 4i + 740,$	$68t + 5i + 1316,$	$63t + 5i + 157,$	$64t + 5i + 214,$
$65t + 5i + 207,$	$66t + 5i - 8,$	$67t + 5i - 5,$	$68t + 5i - 3,$
$69t + 5i + 739,$	$77t + 6i + 1310,$	$74t + 6i + 156,$	$74t + 6i + 213,$
$76t + 6i + 201,$	$76t + 6i - 6,$	$78t + 6i - 3,$	$78t + 6i + 738,$
$87t + 7i + 1309,$	$84t + 7i + 155,$	$85t + 7i + 207,$	$86t + 7i + 203,$
$87t + 7i - 4,$	$88t + 7i + 738,$	$96t + 8i + 1308,$	$95t + 8i + 149,$
$95t + 8i + 209,$	$97t + 8i + 205,$	$97t + 8i + 737,$	$106t + 9i + 1308,$
$105t + 9i + 151,$	$106t + 9i + 211,$	$107t + 9i + 946,$	$115t + 10i + 1307,$
$116t + 10i + 153,$	$116t + 10i + 952,$	$125t + 11i + 1516,$	$126t + 11i + 894,$
$134t + 12i + 1522,$	$144t + 13i + 1464.$		

Similarly, for $58 \leq i \leq \frac{7t-7}{8}$, let the subgraph $B_{21t+i-57}$ consist of a cycle

$$xy_i^1 y_i^2 \dots y_i^{126t+11i+893} x$$

and ten paths

$$\begin{aligned}
& xy_{i,1}^1 y_{i,1}^2 \dots y_{i,1}^{(11t-1)/2} y_i^{(31t-115)/2+i} \\
& xy_{i,2}^1 y_{i,2}^2 \dots y_{i,2}^{(13t-1)/2} y_i^{(51t-103)/2+2i} \\
& xy_{i,3}^1 y_{i,3}^2 \dots y_{i,3}^{(13t-1)/2} y_i^{(71t+315)/2+3i}
\end{aligned}$$

$$\begin{aligned}
& xy_{i,4}^1 y_{i,4}^2 \dots y_{i,4}^{(15t-1)/2} y_i^{(91t+313)/2+4i} \\
& xy_{i,5}^1 y_{i,5}^2 \dots y_{i,5}^{(15t-1)/2} y_i^{(111t+313)/2+5i} \\
& xy_{i,6}^1 y_{i,6}^2 \dots y_{i,6}^{(17t-1)/2} y_i^{(131t+311)/2+6i} \\
& xy_{i,7}^1 y_{i,7}^2 \dots y_{i,7}^{(17t-1)/2} y_i^{(151t+309)/2+7i} \\
& xy_{i,8}^1 y_{i,8}^2 \dots y_{i,8}^{(19t-1)/2} y_i^{(171t+297)/2+8i} \\
& xy_{i,9}^1 y_{i,9}^2 \dots y_{i,9}^{(19t-1)/2} y_i^{(191t+301)/2+9i} \\
& xy_{i,10}^1 y_{i,10}^2 \dots y_{i,10}^{(21t-1)/2} y_i^{(211t+305)/2+10i}.
\end{aligned}$$

Based on the construction, $B_{21t+i-57}$ contains exactly sixty-six cycles of lengths:

$21t + i - 57,$	$22t + i + 7,$	$23t + i + 210,$	$24t + i,$
$25t + i + 1,$	$26t + i,$	$27t + i,$	$28t + i - 5,$
$29t + i + 3,$	$30t + i + 3,$	$31t + i + 742,$	$32t + 2i - 51,$
$32t + 2i + 216,$	$34t + 2i + 209,$	$34t + 2i,$	$36t + 2i,$
$36t + 2i - 1,$	$38t + 2i - 6,$	$38t + 2i - 3,$	$40t + 2i + 5,$
$40t + 2i + 744,$	$42t + 3i + 158,$	$43t + 3i + 215,$	$44t + 3i + 209,$
$45t + 3i - 1,$	$46t + 3i - 1,$	$47t + 3i - 7,$	$48t + 3i - 4,$
$49t + 3i - 1,$	$50t + 3i + 746,$	$53t + 4i + 157,$	$53t + 4i + 215,$
$55t + 4i + 208,$	$55t + 4i - 2,$	$57t + 4i - 7,$	$57t + 4i - 5,$
$59t + 4i - 2,$	$59t + 4i + 740,$	$63t + 5i + 157,$	$64t + 5i + 214,$
$65t + 5i + 207,$	$66t + 5i - 8,$	$67t + 5i - 5,$	$68t + 5i - 3,$
$69t + 5i + 739,$	$74t + 6i + 156,$	$74t + 6i + 213,$	$76t + 6i + 201,$
$76t + 6i - 6,$	$78t + 6i - 3,$	$78t + 6i + 738,$	$84t + 7i + 155,$
$85t + 7i + 207,$	$86t + 7i + 203,$	$87t + 7i - 4,$	$88t + 7i + 738,$
$95t + 8i + 149,$	$95t + 8i + 209,$	$97t + 8i + 205,$	$97t + 8i + 737,$
$105t + 9i + 151,$	$106t + 9i + 211,$	$107t + 9i + 946,$	$116t + 10i + 153,$
$116t + 10i + 952,$	$126t + 11i + 894.$		

B_0 is a path with an end vertex x and length $n - n_t$. Other B_i is simply a cycle of length i .

It is easy to see that

$$\begin{aligned}
v(G) = & v(B_0) + \sum_{i=1}^{19t+\frac{7t+1}{4}} (v(B_i) - 1) + \sum_{i=\frac{7t+1}{8}}^{t-742} (v(B_{19t+2i+1}) - 1) \\
& + \sum_{i=\frac{7t+1}{8}}^{t-742} (v(B_{19t+2i+2}) - 1) + \sum_{i=21t-1481}^{21t} (v(B_i) - 1) \\
& + \sum_{i=58}^{\frac{7t-7}{8}} (v(B_{21t+i-57}) - 1) + \sum_{i=22t-798}^{22t+64} (v(B_i) - 1) + \sum_{i=23t-734}^{23t+267} (v(B_i) - 1) \\
& + \sum_{i=24t-531}^{24t+57} (v(B_i) - 1) + \sum_{i=25t-741}^{25t+58} (v(B_i) - 1) + \sum_{i=26t-740}^{26t+57} (v(B_i) - 1) \\
& + \sum_{i=27t-741}^{27t+57} (v(B_i) - 1) + \sum_{i=28t-741}^{28t+52} (v(B_i) - 1) + \sum_{i=29t-746}^{29t+60} (v(B_i) - 1) \\
& + \sum_{i=30t-738}^{30t+60} (v(B_i) - 1) + \sum_{i=31t-738}^{31t+799} (v(B_i) - 1)
\end{aligned}$$

$$\begin{aligned}
&= n - n_t + 1 + \sum_{i=1}^{19t+\frac{7t+1}{4}} (i-1) + \sum_{i=\frac{7t+1}{8}}^{t-742} (144t + 13i + 1463 \\
&\quad + \frac{11t-1}{2} + \frac{13t-1}{2} + \frac{13t-1}{2} + \frac{15t-1}{2} + \frac{15t-1}{2} + \frac{17t-1}{2} + \frac{17t-1}{2} \\
&\quad + \frac{19t-1}{2} + \frac{19t-1}{2} + \frac{21t-1}{2} + \frac{t-571}{2}) + \sum_{i=\frac{7t+1}{8}}^{t-742} (19t + 2i + 1) \\
&\quad + \sum_{i=21t-1481}^{21t} (i-1) + \sum_{i=\frac{7t-7}{8}}^{21t} (126t + 11i + 893 \\
&\quad + \frac{11t-1}{2} + \frac{13t-1}{2} + \frac{13t-1}{2} + \frac{15t-1}{2} + \frac{15t-1}{2} + \frac{17t-1}{2} + \frac{17t-1}{2} \\
&\quad + \frac{19t-1}{2} + \frac{19t-1}{2} + \frac{21t-1}{2}) + \sum_{i=22t-798}^{22t+64} (i-1) \\
&\quad + \sum_{i=23t+267}^{23t+267} (i-1) + \sum_{i=24t-531}^{24t+57} (i-1) + \sum_{i=25t-741}^{25t+58} (i-1) \\
&\quad + \sum_{i=26t-740}^{26t+57} (i-1) + \sum_{i=27t-741}^{27t+57} (i-1) + \sum_{i=28t-741}^{28t+52} (i-1) \\
&\quad + \sum_{i=29t-746}^{29t+60} (i-1) + \sum_{i=30t-738}^{30t+60} (i-1) + \sum_{i=31t-738}^{31t+799} (i-1) \\
&= n - n_t + \frac{1}{16}(-3309665 + 1028882t + 6911t^2) \\
&= n.
\end{aligned}$$

Now we compute the number of edges of G

$$\begin{aligned}
\epsilon(G) &= \epsilon(B_0) + \sum_{i=1}^{19t+\frac{7t+1}{4}} \epsilon(B_i) + \sum_{i=\frac{7t+1}{8}}^{t-742} \epsilon(B_{19t+2i+1}) \\
&\quad + \sum_{i=\frac{7t+1}{8}}^{t-742} \epsilon(B_{19t+2i+2}) + \sum_{i=21t-1481}^{21t} \epsilon(B_i) \\
&\quad + \sum_{i=\frac{7t-7}{8}}^{21t} \epsilon(B_{21t+i-57}) + \sum_{i=22t-798}^{22t+64} \epsilon(B_i) + \sum_{i=23t-734}^{23t+267} \epsilon(B_i) \\
&\quad + \sum_{i=24t-531}^{24t+57} \epsilon(B_i) + \sum_{i=25t-741}^{25t+58} \epsilon(B_i) + \sum_{i=26t-740}^{26t+57} \epsilon(B_i) \\
&\quad + \sum_{i=27t-741}^{27t+57} \epsilon(B_i) + \sum_{i=28t-741}^{28t+52} \epsilon(B_i) + \sum_{i=29t-746}^{29t+60} \epsilon(B_i) \\
&\quad + \sum_{i=30t-738}^{30t+60} \epsilon(B_i) + \sum_{i=31t-738}^{31t+799} \epsilon(B_i) \\
&= n - n_t + \sum_{i=1}^{19t+\frac{7t+1}{4}} i + \sum_{i=\frac{7t+1}{8}}^{t-742} (144t + 13i + 1464 \\
&\quad + \frac{11t+1}{2} + \frac{13t+1}{2} + \frac{13t+1}{2} + \frac{15t+1}{2} + \frac{15t+1}{2} + \frac{17t+1}{2} + \frac{17t+1}{2} \\
&\quad + \frac{19t+1}{2} + \frac{19t+1}{2} + \frac{21t+1}{2} + \frac{t-571+2}{2}) + \sum_{i=\frac{7t+1}{8}}^{t-742} (19t + 2i + 2) \\
&\quad + \sum_{i=21t-1481}^{21t} i + \sum_{i=\frac{7t-7}{8}}^{21t} (126t + 11i + 894 \\
&\quad + \frac{11t+1}{2} + \frac{13t+1}{2} + \frac{13t+1}{2} + \frac{15t+1}{2} + \frac{15t+1}{2} + \frac{17t+1}{2} + \frac{17t+1}{2} \\
&\quad + \frac{19t+1}{2} + \frac{19t+1}{2} + \frac{21t+1}{2}) + \sum_{i=22t-798}^{22t+64} i \\
&\quad + \sum_{i=23t-734}^{23t+267} i + \sum_{i=24t-531}^{24t+57} i + \sum_{i=25t-741}^{25t+58} i \\
&\quad + \sum_{i=26t-740}^{26t+57} i + \sum_{i=27t-741}^{27t+57} i + \sum_{i=28t-741}^{28t+52} i \\
&\quad + \sum_{i=29t-746}^{29t+60} i + \sum_{i=30t-738}^{30t+60} i + \sum_{i=31t-738}^{31t+799} i \\
&= n - n_t + \frac{1}{16}(-3309681 + 1029394t + 6911t^2) \\
&= n + 32t - 1.
\end{aligned}$$

Then $f(n) \geq n + 32t - 1$, for $n \geq n_t$. This completes the proof of the theorem.

From the above theorem, we have

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \sqrt{2 + \frac{2562}{6911}},$$

which is better than the previous bounds $\sqrt{2}$ (see [2]), $\sqrt{2 + \frac{487}{1381}}$ (see [6]).

Combining this with Boros, Caro, Füredi and Yuster's upper bound, we have

$$1.98 \geq \limsup_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq 1.5397.$$

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