REGULAR DEPENDENCE OF THE PEIERLS BARRIERS ON PERTURBATIONS

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ABSTRACT. Let f be an exact area-preserving monotone twist diffeomorphism of the infinite cylinder and $P_{\omega,f}(\xi)$ be the associated Peierls barrier. In this paper, we give the Hölder regularity of $P_{\omega,f}(\xi)$ with respect to the parameter f. In fact, we prove that if the rotation symbol $\omega \in (\mathbb{R}\backslash\mathbb{Q})\bigcup(\mathbb{Q}+)\bigcup(\mathbb{Q}-)$, then $P_{\omega,f}(\xi)$ is 1/3-Hölder continuous in f, i.e.

$$|P_{\omega,f'}(\xi) - P_{\omega,f}(\xi)| \leq C\|f' - f\|_{C^1}^{1/3}, \ \forall \xi \in \mathbb{R}$$

where C is a constant. Similar results also hold for the Lagrangians with one and a half degrees of freedom. As application, we give an open and dense result about the breakup of invariant circles.

1. Introduction

The Peierls barrier $P_{\omega,f}(\xi)$ for the monotone twist diffeomorphism f is a function which can be thought of as a dislocation energy. It measures to which extent the stationary configuration $(x_i)_{i\in\mathbb{Z}}$ of rotation symbol ω, subject to the constraint $x_0 = \xi$, is not minimal. In [11], Mather established the modulus of continuity for $P_{\omega,f}$ with respect to the parameter ω, and by applying this property, he gave destruction results for invariant circles under arbitrary small perturbations ([12]). For further research, we need more information and properties about the Peierls barriers. In this paper, we prove the Hölder continuity of Peierls barriers with respect to the parameter f, which generalize J. Mather's results in [11][12]. This paper is organized as follows: In Section 2 and Section 3, we introduce the definition of Peierls barrier and some basic properties in Aubry-Mather theory. Our main results are Theorem 4.1 and Theorem 4.9 in Section 4. In Section 5, we give an open and dense result as an application example.

For convenience, we denote by ϑ (mod 1) the standard coordinate of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and x the corresponding coordinate of its universal cover $\mathbb{R}$. We will let (ϑ, y) denote the standard coordinates of $\mathbb{S}^1 \times \mathbb{R}$ and (x, y) the corresponding coordinates of the universal cover $\mathbb{R} \times \mathbb{R}$. The dynamical properties of exact area-preserving monotone twist diffeomorphisms of an infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$ have been studied by Mather ([9]–[13]) and by Bangert ([2]). In the following, we refer to these papers for the definitions and results that we'll need.

1.1. Monotone twist diffeomorphism.

Definition 1.1. We call f an exact area-preserving monotone twist diffeomorphism if $f : \mathbb{S}^1 \times \mathbb{R} \longrightarrow \mathbb{S}^1 \times \mathbb{R}$,

$$(\vartheta, y) \longmapsto (\vartheta', y')$$

is a diffeomorphism satisfying the following conditions:

(1) $f \in C^1(\mathbb{S}^1 \times \mathbb{R})$.
(2) The 1-form $y'd\vartheta' - yd\vartheta$ on $\mathbb{S}^1 \times \mathbb{R}$ is exact.

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(3) (positive monotone twist) $\frac{\partial\vartheta'(\vartheta,y)}{\partial y} > 0$, for all (ϑ, y).
(4) f twists the cylinder infinitely at either hand. To express this condition, we consider a lift $\bar{f}$ of f to the universal cover $\mathbb{R}\times\mathbb{R}$, $\bar{f}(x,y) = (x', y')$, the condition means that for fixed x,

$$x' \to +\infty \text{ as } y \to +\infty \text{ and } x' \to -\infty \text{ as } y \to -\infty.$$

The positive monotone twist condition has its geometrical meaning. Consider a point $P \in \mathbb{S}^1\times\mathbb{R}$ and denote by $v_P = (0, 1)$ the vertical vector at P. Let $\beta_f(P)$ denote the angle between v_P and $d_P f \cdot v_P$ (count in the clockwise direction). So the positive monotone twist condition means that

$$0 < \beta_f(P) < \pi$$

everywhere.

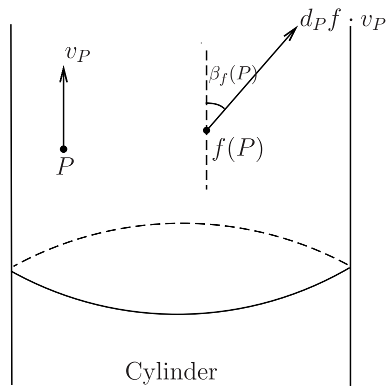

FIGURE 1. geometrical meaning of positive monotone twist condition

We denote by $\mathcal{J}$ the class of exact area-preserving monotone twist diffeomorphisms. Let $\mathcal{J}_\beta = \{f \in \mathcal{J} : \beta_f(P) \geq \beta, \text{ for all } P \in \mathbb{S}^1\times\mathbb{R}\}$. Although $\bigcup_{\beta>0} \mathcal{J}_\beta \subsetneqq \mathcal{J}$, most of our results can be generalized to $\mathcal{J}$ without any difficulty. This is because our main results concern what happens in a compact region K of $\mathbb{S}^1\times\mathbb{R}$. Thus, for all $f \in \mathcal{J}$, there exists $\beta > 0$ and $g \in \mathcal{J}_\beta$ such that $f|_K = g|_K$.

1.2. **The variational principle.** If $f \in \mathcal{J}$ and $\bar{f}$ is a lift of f to $\mathbb{R}\times\mathbb{R}$ such that $\bar{f}(x,y) = (x', y')$, then there exists a C^2 generating function $h(x, x') : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ such that $\partial_{12}h := \frac{\partial^2 h(x,x')}{\partial x \partial x'} < 0$ and

$$\begin{cases} y = -\partial_1 h(x, x') \\ y' = \partial_2 h(x, x') \end{cases} \tag{1.1}$$

where $\partial_1 h(x, x')$ and $\partial_2 h(x, x')$ denote the partial differential derivatives of h with respect to x and x'.

For $f \in \mathcal{J}$, the generating function satisfies the following conditions (**H**1) – (**H**4),
(**H**1) $h(x, x') = h(x + 1, x' + 1)$, for all $x, x' \in \mathbb{R}$.
(**H**2) $\lim_{|\xi|\to+\infty} h(x, x + \xi) = +\infty$ uniformly in x.
(**H**3) If $x < \xi, x' < \xi'$, then $h(x, x') + h(\xi, \xi') < h(x, \xi') + h(\xi, x')$.
(**H**4) If $(\bar{x}, x, x')$ and $(\bar{\xi}, x, \xi')$ are both minimal segments (see §2) and are distinct, then

$$(\bar{x} - \bar{\xi})(x' - \xi') < 0.$$

Moreover, we add two further conditions to h which was firstly introduced by Mather.
($\mathbf{H}5$) There exists a positive continuous function $\rho : \mathbb{R}^2 \to \mathbb{R}$ such that for $x < \xi$ and $x' < \xi'$, $h(x,\xi') + h(\xi,x') - h(x,x') - h(\xi,\xi') \geq \int_x^\xi \int_{x'}^{\xi'} \rho(s,s')dsds'$.
($\mathbf{H}6\theta$) There exists a positive number θ such that

$$x \mapsto \theta(x-x')^2/2 - h(x,x') \text{ is convex, for any } x',$$

$$x' \mapsto \theta(x'-x)^2/2 - h(x,x') \text{ is convex, for any } x.$$

Conditions ($\mathbf{H}1$) – ($\mathbf{H}4$) were firstly introduced by Bangert in [2], and conditions ($\mathbf{H}3$)($\mathbf{H}4$) can be implied by conditions ($\mathbf{H}5$)($\mathbf{H}6$) (see [11]). For $f \in \mathcal{J}_\beta$, it's not hard to observe that we can take $\rho = -\partial_{12}h$ in ($\mathbf{H}5$), the inequality "$\geq$" can be replaced by "$=$", and ($\mathbf{H}6\theta$) is satisfied with $\theta = \cot\beta$ $(0 < \beta < \pi/2)$.

Notice that if h is a generating function, $h + C$ is still a generating function. In next section, we'll introduce some basic results on the theory of minimal configurations, which was developed by Aubry and Le Daeron [1] and Mather. However, Bangert generalized this theory where h is not necessary differentiable and only satisfies ($\mathbf{H}1$) – ($\mathbf{H}4$) (see [2]).

2. Minimal configurations and Peierls barriers

2.1. **Minimal configurations.** In Bangert's set up, the variational principle h need not be differentiable, which is very useful for us in our proof of Theorem 4.1 and Theorem 4.9. Therefore, in this section, unless otherwise specified, we assume that h is only continuous and satisfies the conditions ($\mathbf{H}1$) – ($\mathbf{H}4$).

Definition 2.1. Let $\mathbb{R}^Z = \{x | x : \mathbb{Z} \to \mathbb{R}\}$ be bi-infinite sequences of real numbers with the product topology and let $x = (x_i)_{i\in\mathbb{Z}}$ be an element in $\mathbb{R}^Z$. We extend h to a finite segment $(x_j, ..., x_k), j < k$, $h(x_j, ..., x_k) := \sum_{i=j}^{k} h(x_i, x_{i+1})$. The segment $(x_j, ..., x_k)$ is called *minimal segment* with respect to h if

$$h(x_j, ..., x_k) \leq h(y_j, ..., y_k),$$

for all $(y_j, ..., y_k)$ with $y_j = x_j$ and $y_k = x_k$. $x = (x_i)_{i\in\mathbb{Z}}$ is called *minimal configuration* if every finite segment of x is minimal.

We denote by $\mathrm{M} := \mathrm{M}_h$ the set of all minimal configurations of h.

Definition 2.2. If $h \in C^2$, a segment $(x_j, ..., x_k)$ is called *stationary* if

$$\partial_2 h(x_{i-1}, x_i) + \partial_1 h(x_i, x_{i+1}) = 0, \text{ for all } j < i < k.$$

Remark. The stationary configurations $(..., x_i, ...)$ of h correspond to the orbits $(..., (x_i, y_i), ...)$ of $\bar{f}$, where $y_i = -\partial_1 h(x_i, x_{i+1}) = \partial_2 h(x_{i-1}, x_i)$.

Thus, the minimal configurations M_h of h correspond to a class of minimal orbits of the lift $\bar{f}$ of f, we denote by $\mathfrak{M} := \mathfrak{M}_{\bar{f}}$ the set of all such minimal orbits. In the following, we will give some dynamical properties for the minimal configurations for h, or equivalently, the minimal orbits of $\bar{f}$. For proofs, we refer to [11].

Given a configuration $x = (x_i)_{i\in\mathbb{Z}}$, we join (i, x_i) and $(i+1, x_{i+1})$ by a line segment in $\mathbb{R}^2$, the union of all such line segments is a piecewise linear curve in $\mathbb{R}^2$, which we call *the Aubry graph* of x. We say that configurations x and x^* *cross* if their Aubry graph cross. We say $x < x^*$ if $x_i < x_i^*$ for all i. Similarly, we can define $x > x^*$ and $x = x^*$. We say that x and x^* are *comparable* if $x < x^*$ or $x = x^*$ or $x > x^*$. By condition ($\mathbf{H}4$), we know that any two minimal configurations either cross or are comparable. Moreover, we say x and x^* are $\omega-asymptotic$ (resp. $\alpha-asymptotic$) if $\lim_{i\to+\infty}|x_i-x_i^*|=0$ (resp. $\lim_{i\to-\infty}|x_i-x_i^*|=0$).

Lemma 2.3 ([2], Aubry's Crossing Lemma)**.** *Let x and x^* be h-minimal configurations, then they cross at most once. If x and x^* coincide at some $i\in\mathbb{Z}$, i.e. $x_i=x_i^*$, then they cross at i.*

Notice that Aubry's Crossing Lemma can be proved by condition (**H**3). If x is a minimal configuration, then

$$\rho(x):=\lim_{n\to+\infty}x_n/n$$

exists. The number $\rho(x)$ is called *the rotation number* of x. In addition, the rotation function $\rho:\mathrm{M}_h\to\mathbb{R}$ is continuous and surjective, and $(pr_0,\rho):\mathrm{M}_h\to\mathbb{R}\times\mathbb{R}$ is proper, where $pr_0(x)=x_0$.

If $x\in\mathrm{M}_h$ and $\rho(x)=p/q$, where $q>0$ and p,q are relatively prime integers, then x must satisfy one of the three relations (see [11]):

(a) $x_{i+q}>x_i+p$, for all i.
(b) $x_{i+q}=x_i+p$, for all i.
(c) $x_{i+q}<x_i+p$, for all i.

This leads us to introduce the *symbol space* $S=(\mathbb{R}\setminus\mathbb{Q})\bigcup(\mathbb{Q}-)\bigcup(\mathbb{Q})\bigcup(\mathbb{Q}+)$, where $\mathbb{Q}+$ denotes the set of all symbols $\frac{p}{q}+$ and $\mathbb{Q}-$ is defined similarly. The symbol space has an obvious order so that $\frac{p}{q}-<\frac{p}{q}<\frac{p}{q}+$. We provide S with the order topology, i.e. the set of intervals (s_1,s_2)=$\{x:s_1<x<s_2\}$ is a basis for this topology. We also define the projection map $\pi:S\to\mathbb{R}$,

$$\pi(\omega)\triangleq\begin{cases}\omega, & \omega\in(\mathbb{R}\setminus\mathbb{Q})\\ \frac{p}{q}, & \omega=\frac{p}{q}\pm\ \text{ or }\ \frac{p}{q}\end{cases}\tag{2.1}$$

Obviously, the map π is weakly order preserving. For more details, see ([11], §3).

From now on, if $\omega\in\mathbb{R}\setminus\mathbb{Q}$, we denote by $\mathrm{M}_\omega:=\mathrm{M}_{\omega,h}$ the set of $x\in\mathrm{M}_h$ with $\rho(x)=\omega$. Let $\mathrm{M}_{p/q}:=\mathrm{M}_{p/q,h}$ denotes the set of $x\in\mathrm{M}_h$ with $\rho(x)=p/q$ and (b) holds. Let $\mathrm{M}_{p/q^-}:=\mathrm{M}_{p/q^-,h}$ denotes the set of $x\in\mathrm{M}_h$ with $\rho(x)=p/q$ and (b) or (c) hold. Similarly, we also denote by $\mathrm{M}_{p/q^+}:=\mathrm{M}_{p/q^+,h}$ the set of $x\in\mathrm{M}_h$ with $\rho(x)=p/q$ and (a) or (b) hold. Notice that for each $\omega\in S$, $\mathrm{M}_{\omega,h}$ is a non-empty closed set and totally ordered.

Equivalently, for the rotation symbol $\omega\in S$, we can denote by $\mathfrak{M}_\omega:=\mathfrak{M}_{\omega,\bar{f}}$ the set of all ω-minimal orbits of $f\in\mathcal{J}$.

Proposition 2.4. ([2]) *Let $x=(x_i)_{i\in\mathbb{Z}}\in\mathrm{M}_{\omega,h}$ be a minimal configuration, we have that*

$$|x_{i+j}-x_i-j\pi(\omega)|<1$$

for all $i<i+j\in\mathbb{Z}$.

Proposition 2.5. ([3]) *Let $(y_0,...,y_n)$ be a minimal segment, there exists $\alpha\in\mathbb{R}$ so that*

$$|y_{i+j}-y_i-j\alpha|<2$$

for all $0\le i\le i+j\le n$.

2.2. **Peierls barriers.** By the totally ordered property, the projection $pr_0:\mathrm{M}_{\omega,h}\to\mathbb{R}$, $pr_0(x)=x_0$ is a homeomorphism of $\mathrm{M}_{\omega,h}$ onto its image. We denote by $\mathrm{A}_{\omega,h}=pr_0(\mathrm{M}_{\omega,h})$, then it's a closed subset of $\mathbb{R}$ and invariant under the translation $x\mapsto x+1$ (see [11]). Now, we begin to introduce the definition of the Peierls barrier function $P_\omega(\xi):=P_{\omega,h}(\xi)$, it measures to which extent the stationary configuration $(y_i)_{i\in\mathbb{Z}}$, subject to the condition $y_0=\xi$, is not minimal.

Besides Bangert's conditions ($\mathbf{H}1$) – ($\mathbf{H}4$), we will assume h satisfies ($\mathbf{H}5$) and ($\mathbf{H}6\theta$).

Since $\mathbb{R} \setminus \mathrm{A}_{\omega,h}$ is an open set, it is a union of open intervals (J_-, J_+), where $J_-, J_+ \in \mathrm{A}_{\omega,h}$. *The Peierls barrier* is defined as follows. Let $\omega \in S$ and $\xi \in \mathbb{R}$, in the case $\xi \in \mathrm{A}_{\omega,h}$, we define $P_\omega(\xi) = 0$.

In the case $\xi \notin \mathrm{A}_{\omega,h}$, ξ belongs to a complementary interval (J_-, J_+) to $\mathrm{A}_{\omega,h}$. Denote by $x^{\pm}$ the minimal configurations of rotation symbol ω satisfying $x_0^{\pm} = J_{\pm}$ and let

$$I = \begin{cases} \mathbb{Z} & \omega \in (\mathbb{R} \setminus \mathbb{Q}) \bigcup (\mathbb{Q}+) \bigcup (\mathbb{Q}-) \\ 0, ..., q-1 & \omega = p/q, \end{cases}$$

we define

$$P_\omega(\xi) = \min\{\sum_I h(y_i, y_{i+1}) - h(x_i^-, x_{i+1}^-) | y_0 = \xi\}, \tag{2.2}$$

where the minimum is taken over the set of all configurations satisfying $x_i^- \le y_i \le x_i^+, \forall i \in \mathbb{Z}$ and in the case $\omega = p/q$, the minimum is taken under the additional periodicity constraint $y_{i+q} = y_i + p$.

Notice that $P_\omega(\xi)$ is well defined and finite, since h is Lipschitz and

$$0 \le \sum_{i \in I} y_i - x_i^- \le \sum_{i \in I} x_i^+ - x_i^- \le 1$$

(see [2]). It is worth pointing out that $P_\omega(\xi)$ is non-negative and 1-periodic, i.e.

$$P_\omega(\xi + 1) = P_\omega(\xi).$$

Now, let's recall some important properties of the Peierls barrier which are important for our main results in Section 4.

Proposition 2.6. ([11],[12]) *Let h be a continuous real valued function satisfying* ($\mathbf{H}1$) – ($\mathbf{H}5$) *and* ($\mathbf{H}6\theta$)*, then*

(1) $|P_{\frac{p}{q}}(\xi) - P_\omega(\xi)| \le 1200\theta(\frac{1}{q} + |\pi(\omega)q - p|)$,

(2) $|P_{\frac{p}{q}+}(\xi) - P_\omega(\xi)| \le 4800\theta(|\pi(\omega)q - p|)$ *in the case* $\omega \ge \frac{p}{q}+$ *and*
$|P_{\frac{p}{q}-}(\xi) - P_\omega(\xi)| \le 4800\theta(|\pi(\omega)q - p|)$ *in the case* $\omega \le \frac{p}{q}-$,

where π is (2.1).

The proof strongly relies on Aubry's Crossing Lemma. Proposition 2.6 (1) was firstly proved by Mather in ([11], Theorem 7.1). The proof of (2) could be found in [12], Theorem 2.2, where the author proved that if $\omega \ge \frac{p}{q}+$, then

$$|P_{\frac{p}{q}+}(\xi) - P_\omega(\xi)| \le 4C\theta(|\pi(\omega)q - p|),$$

where the constant $C = 1200$. Similarly, if $\omega \le \frac{p}{q}-$, then

$$|P_{\frac{p}{q}-}(\xi) - P_\omega(\xi)| \le 4C\theta(|\pi(\omega)q - p|),$$

where $C = 1200$.

Corollary 2.7. ([11]) *The map $\omega \mapsto P_{\omega,h}(\xi)$ is continuous at any irrational number ω, uniformly in ξ.*

Remark. In general, $\omega \mapsto P_{\omega,h}$ is not continuous at rational symbol $\omega = p/q$.

3. The conjunction operation

The reason for not restricting our attention to C^2 function h is that we consider not only the generating functions with respect to $f \in \mathcal{J}$, but also the class of functions generated by the conjunction operation. Let h_1 and $h_2 : \mathbb{R}^2 \to \mathbb{R}$ be two continuous functions satisfying the conditions $(\mathbf{H}1) - (\mathbf{H}5)$ and $(\mathbf{H}6\theta)$, *the conjunction* is defined in the following way:

$$h_1 * h_2(x, x') = \min_y h_1(x, y) + h_2(y, x').$$

Notice that even when both h_1 and h_2 are smooth, the conjunction $h_1 * h_2$ need not be smooth. However, we still have the following useful property.

Proposition 3.1. ([11]) *If h_1 and $h_2 : \mathbb{R}^2 \to \mathbb{R}$ are two continuous functions satisfying the conditions $(\mathbf{H}1) - (\mathbf{H}5)$ and $(\mathbf{H}6\theta)$, then $h_1 * h_2$ satisfies $(\mathbf{H}1) - (\mathbf{H}5)$ and $(\mathbf{H}6\theta)$ with the same θ.*

Therefore, one can still define the minimal configurations and Peierls barriers with respect to $h_1 * h_2$.

Given a function h as descried above and a rational number p/q (in lowest terms), we define the following conjunction

$$H_{(q,p)}(x, x') := h^{*q}(x, x' + p) = h * \cdots * h(x, x' + p), \tag{3.1}$$

where h^{*q} denotes the q-fold conjunction of h with itself. It is easy to verify the following equivalent relations about the minimal configurations and the Peierls barrier functions.

Proposition 3.2. ([7], [12]) *If h satisfies $(\mathbf{H}1) - (\mathbf{H}5)$ and $(\mathbf{H}6\theta)$, and $H_{(q,p)}(x, x')$ is the conjunction (3.1), then*

$$\mathrm{A}_{\omega,h} = \mathrm{A}_{q\omega - p, H_{(q,p)}} \text{ and } P_{\omega,h}(\xi) = P_{q\omega - p, H_{(q,p)}}(\xi),$$

where the rotation symbol $\omega \in (\mathbb{R} \setminus \mathbb{Q})$ or $\omega = \frac{p_1}{q_1} \pm, \frac{p_1}{q_1}$ whose denominator q_1 is divisible by q.

4. Main results

4.1. **Monotone twist diffeomorphism.** Let f be an exact area-preserving monotone twist diffeomorphism and let $\bar{f}$ be a lift of f which satisfies $\bar{f}(x + 1, y) = \bar{f}(x, y) + (1, 0)$, and we set $\bar{f}(x, y) = (\bar{f}_1(x, y), \bar{f}_2(x, y))$. For the rotation interval $[\omega_0, \omega_1]$, there is a compact annulus $\mathbf{A}_K := \mathbb{S}^1 \times [-K, K]$ with sufficiently large $K = K(\omega_0, \omega_1, f)$ so that the minimal orbits of f satisfies

$$\mathfrak{M}_{\omega,\bar{f}} \subseteq \mathbf{A}_{K-2} \subseteq \mathbf{A}_{K-1} \subseteq \mathbf{A}_K, \ \forall \omega \in [\omega_0, \omega_1]. \tag{4.1}$$

The space $C^1(\mathbf{A}_K) = C^1(\mathbf{A}_K, \mathbb{R})$ is provided with the norm:

$$\|f\|_{C^1(\mathbf{A}_K)} = \sup_{0 \le j \le 1} \max_{\mathbf{A}_K} |D^j f|.$$

Theorem 4.1. *Let $f \in \mathcal{J}$, $[\omega_0, \omega_1]$ and $\mathbf{A}_K$ be as shown above. There exist positive numbers $\delta_0 = \delta_0(\omega_0, \omega_1, f) \ll 1$ and $C_0 = C_0(\omega_0, \omega_1, f)$ such that for any $f' \in \mathcal{J}$, if $\|f' - f\|_{C^1(\mathbf{A}_K)} \le \delta_0$, then the corresponding Peierls barrier functions satisfy*

$$|P_{\omega,f'}(\xi) - P_{\omega,f}(\xi)| \le C_0 \|f' - f\|^{\frac{1}{3}}_{C^1(\mathbf{A}_K)}, \ \forall \xi \in \mathbb{R}$$

where $\omega \in (\mathbb{R} \setminus \mathbb{Q}) \bigcup (Q-) \bigcup Q+$, with $\pi(\omega) \in [\omega_0, \omega_1]$.

Remark. (1) The projection π is defined in (2.1). The conclusion of Theorem 4.1 also holds for C^r perturbations.

(2) The novelty here is that the constant C_0 does not depend on ξ and the 1/3-Hölder regularity is uniform in ξ, which could be derived from our proof below, see (4.27).

(3) For rational symbol $\omega = p/q$, the conclusion of Theorem 4.1 doesn't hold in general since the map $\omega \mapsto P_\omega$ is not continuous at rational symbol.

In order to prove Theorem 4.1, we need the following lemmas.

Lemma 4.2. *Under the same assumptions of Theorem 4.1, let h, h' be generating functions of f and f' respectively, satisfying the condition $h(0,0) = h'(0,0) = 0$. There exist positive numbers $\delta_1 = \delta_1(K, f) \ll 1$ and $C_1 = C_1(K, f)$, such that if $\|f' - f\|_{C^1(\mathbf{A}_K)} \leq \delta_1$, then*

$$\|h' - h\|_{C^0(\mathbf{B}_{K-1})} \leq C_1 \|f' - f\|_{C^1(\mathbf{A}_K)}$$

where $\mathbf{B}_{K-1} := \{(x, x') \in \mathbb{R}^2 | \bar{f}_1(x, -(K-1)) \leq x' \leq \bar{f}_1(x, K-1)\}$.

Proof. Take δ_1 small enough such that $\|f' - f\|_{C^1(\mathbf{A}_K)} \leq \delta_1 \ll 1$, then the lifts

$$\|\bar{f}' - \bar{f}\|_{C^1([0,1]\times[-K,K])} = \|f' - f\|_{C^1(\mathbf{A}_K)}.$$

Since for all $k \in \mathbb{Z}$, $\bar{f}'(x+k, y) = \bar{f}'(x, y) + (k, 0)$ and $\bar{f}(x+k, y) = \bar{f}(x, y) + (k, 0)$, one can deduce that

$$\|\bar{f}' - \bar{f}\|_{C^1(\mathbb{R}\times[-K,K])} = \|\bar{f}' - \bar{f}\|_{C^1([0,1]\times[-K,K])} = \|f' - f\|_{C^1(\mathbf{A}_K)}. \tag{4.2}$$

By adding a constant we can assume $h'(0,0) = h(0,0) = 0$. By choosing suitably large K, we can assume $(0,0) \in \mathbf{B}_{K-2}$. From the assumption (4.1), for all ω with $\pi(\omega) \in [\omega_0, \omega_1]$ and $x = (x_i)_{i\in\mathbb{Z}} \in \mathrm{M}_{\omega,h}$, we have $(x_i, x_{i+1}) \in \mathbf{B}_{K-2}$, i.e.

$$\bar{f}_1(x_i, -(K-2)) \leq x_{i+1} \leq \bar{f}_1(x_i, K-2).$$

Since $\bar{f}'$ and $\bar{f}$ are sufficiently close, then for all $x' = (x'_i)_{i\in\mathbb{Z}} \in \mathrm{M}_{\omega,h'}$, we have $(x'_i, x'_{i+1}) \in \mathbf{B}_{K-1}$, i.e.

$$\bar{f}'_1(x'_i, -(K-1)) \leq x'_{i+1} \leq \bar{f}'_1(x'_i, K-1).$$

From now on, we only consider h, h' restricted on the set $\mathbf{B}_{K-1}$ (See Figure 2).

In the following, we set the bounded region $\mathbf{D} := \mathbf{B}_{K-1} \bigcap ([0,1] \times \mathbb{R})$. By the condition ($\mathbf{H}1$), for all $(x, x') \in \mathbf{B}_{K-1}$,

$$\|h' - h\|_{C^0(\mathbf{B}_{K-1})} = \|h' - h\|_{C^0(\mathbf{D})} \tag{4.3}$$

Because h, h are C^2, so for $(x, x') \in \mathbf{D}$,

$$h(x, x') = h(0,0) + \int_0^x \partial_1 h(t, 0)dt + \int_0^{x'} \partial_2 h(x, t)dt$$
$$h'(x, x') = h'(0,0) + \int_0^x \partial_1 h'(t, 0)dt + \int_0^{x'} \partial_2 h'(x, t)dt$$

and

$$\begin{aligned} |h'(x, x') - h(x, x')| &\leq \int_0^x |\partial_1 h'(t, 0) - \partial_1 h(t, 0)|dt + \int_0^{x'} |\partial_2 h'(x, t) - \partial_2 h(x, t)|dt \\ &\triangleq I_1 + I_2 \end{aligned} \tag{4.4}$$

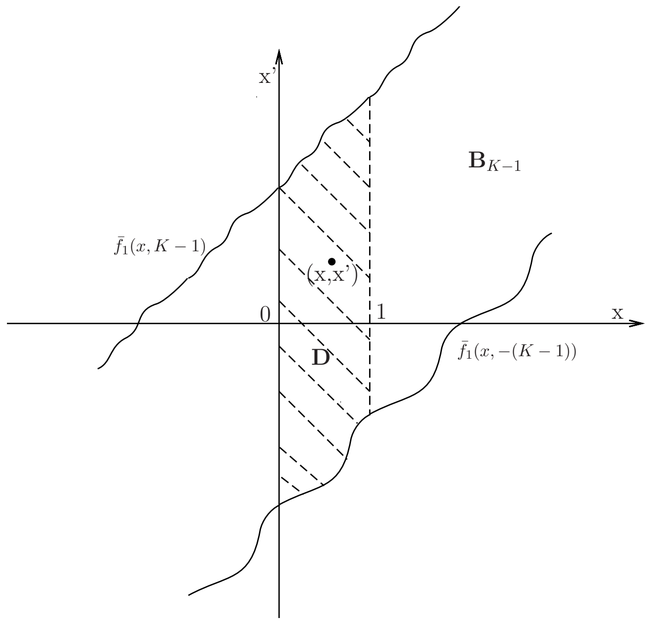

FIGURE 2.

Step 1. Firstly, let's estimate I_1. By (1.1), we have

$$\begin{cases} 0 = \bar{f}_1(t, -\partial_1 h(t,0)), \\ 0 = \bar{f}'_1(t, -\partial_1 h'(t,0)), \end{cases} \quad 0 \le t \le 1 \tag{4.5}$$

and by (4.5),

$$\begin{aligned} 0 &= \bar{f}_1(t, -\partial_1 h(t,0)) - \bar{f}'_1(t, -\partial_1 h'(t,0)) \\ &= \bar{f}_1(t, -\partial_1 h(t,0)) - \bar{f}_1(t, -\partial_1 h'(t,0)) + \bar{f}_1(t, -\partial_1 h'(t,0)) - \bar{f}'_1(t, -\partial_1 h'(t,0)) \end{aligned} \tag{4.6}$$

If we set $a := \min\limits_{(x,y)\in\mathbb{R}\times[-K,K]} \frac{\partial \bar{f}_1}{\partial y}(x,y)$, then $a > 0$ since $\frac{\partial \bar{f}_1}{\partial y} > 0$ and $\bar{f}_1(x+k,y) = \bar{f}_1(x,y) + (k,0)$.

Because $\bar{f}, \bar{f}'$ are sufficiently close, so it's easy to compute that

$$|-\partial_1 h(t,0))| \le K-1, \ |-\partial_1 h'(t,0))| \le K, \ \forall 0 \le t \le 1 \tag{4.7}$$

By (4.6) and (4.7),

$|\bar{f}_1(t, -\partial_1 h'(t,0)) - \bar{f}'_1(t, -\partial_1 h'(t,0))| = |\bar{f}_1(t, -\partial_1 h(t,0)) - \bar{f}_1(t, -\partial_1 h'(t,0))| \ge a|\partial_1 h'(t,0) - \partial_1 h(t,0)|$.

Therefore,

$$\|\bar{f}'_1 - \bar{f}_1\|_{C^1(\mathbb{R}\times[-K,K])} \ge a|\partial_1 h'(t,0) - \partial_1 h(t,0)|$$

and by (4.2),

$$I_1 \le \int_0^x \frac{1}{a}\|\bar{f}'_1 - \bar{f}_1\|_{C^1(\mathbb{R}\times[-K,K])} dt \le \int_0^x \frac{1}{a}\|f' - f\|_{C^1(\mathbf{A}_K)} dt \le \frac{1}{a}\|f' - f\|_{C^1(\mathbf{A}_K)}, \tag{4.8}$$

since $0 \le x \le 1$.

Step 2. Secondly, let's estimate I_2. For each fixed x, let

$$\phi_x(y) := \bar{f}_1(x,y) \text{ and } \phi'_x(y) := \bar{f}'_1(x,y).$$

Since $\frac{d\phi_x(y)}{dy} > 0$, we could choose a positive number $0 < b < 1$ such that

$$b \leq \frac{d\phi_x(y)}{dy} \leq \frac{1}{b} \text{ for all } (x,y) \in [0,1] \times [-K,K]. \tag{4.9}$$

Because $\bar{f}, \bar{f}'$ are sufficiently close, so it's easy to compute that for $(x,t) \in \mathbf{D}$,

$$|\phi_x^{-1}(t)| \leq K-1, \ |(\phi'_x)^{-1}(t)| \leq K \tag{4.10}$$

where $\phi_x^{-1}, (\phi'_x)^{-1}$ are the inverse function of ϕ_x, ϕ'_x respectively.

Then, by (4.2), (4.9), (4.10), we can conclude that for $(x,t) \in \mathbf{D}$,

$$\begin{aligned}
|(\phi'_x)^{-1}(t) - (\phi_x)^{-1}(t)| &= |(\phi_x)^{-1} \circ \phi_x \circ (\phi'_x)^{-1}(t) - (\phi_x)^{-1}(t)| \\
&\leq \max_{(x,s)\in\mathbf{D}} |\frac{d(\phi_x)^{-1}(s)}{ds}| \ |\phi_x \circ (\phi'_x)^{-1}(t) - t| \\
&\leq \frac{1}{b}|(\phi_x - \phi'_x + \phi'_x) \circ (\phi'_x)^{-1}(t) - t| \\
&= \frac{1}{b}|(\phi_x - \phi'_x) \circ (\phi'_x)^{-1}(t)| \\
&= \frac{1}{b}|\bar{f}_1(x,(\phi'_x)^{-1}(t)) - \bar{f}'_1(x,(\phi'_x)^{-1}(t))| \\
&\leq \frac{1}{b}\|\bar{f}' - \bar{f}\|_{C^1([0,1]\times[-K,K])} = \frac{1}{b}\|f' - f\|_{C^1(\mathbf{A}_K)}.
\end{aligned} \tag{4.11}$$

By (1.1), we get

$$\begin{cases} \partial_2 h(x,t) = \bar{f}_2(x, \phi_x^{-1}(t)), \\ \partial_2 h'(x,t) = \bar{f}'_2(x, (\phi'_x)^{-1}(t)), \end{cases} \tag{4.12}$$

and by (4.11) and (4.12), one can deduce that

$$\begin{aligned}
|\partial_2 h'(x,t) - \partial_2 h(x,t)| &\leq |\bar{f}'_2(x,(\phi'_x)^{-1}(t)) - \bar{f}_2(x,(\phi'_x)^{-1}(t))| + |\bar{f}_2(x,(\phi'_x)^{-1}(t)) - \bar{f}_2(x,\phi_x^{-1}(t))| \\
&\leq \|f' - f\|_{C^1(\mathbf{A}_K)} + \|\frac{\partial \bar{f}_2}{\partial y}\|_{C^0([0,1]\times[-K,K])} \ |(\phi'_x)^{-1}(t) - (\phi_x)^{-1}(t)| \\
&\leq (1 + \frac{L}{b})\|f' - f\|_{C^1(\mathbf{A}_K)}
\end{aligned} \tag{4.13}$$

where $L = \|\frac{\partial \bar{f}_2}{\partial y}\|_{C^0([0,1]\times[-K,K])}$.

Furthermore,

$$\begin{aligned}
I_2 &\leq \int_0^{x'} (1 + \frac{L}{b})\|f' - f\|_{C^1(\mathbf{A}_K)} dt \leq \max_{x\in[0,1]} (\ |\bar{f}_1(x,K)| + |\bar{f}_1(x,-K)| \ )(1 + \frac{L}{b})\|f' - f\|_{C^1(\mathbf{A}_K)} \\
&\leq 2\|\bar{f}\|_{C^1([0,1]\times[-K,K])}(1 + \frac{L}{b})\|f' - f\|_{C^1(\mathbf{A}_K)}
\end{aligned} \tag{4.14}$$

Finally, by (4.3), (4.4), (4.8) and (4.14), we get that

$$\|h' - h\|_{C^0(\mathbf{B}_{K-1})} \leq C_1 \|f' - f\|_{C^1(\mathbf{A}_K)}$$

with the constant C_1 only depending on K and f. □

The following lemma give an equivalent definition of the Peierls barrier for the rational rotation symbol.

Lemma 4.3. *Let h be a continuous real valued function satisfying* $(\mathbf{H}1)-(\mathbf{H}5)$ *and* $(\mathbf{H}6\theta)$*, then for rational rotation symbol $\frac{p}{q}\in\mathbb{Q}$(in lowest terms), the Peierls barrier has an equivalent defintion:*

$$P_{\frac{p}{q},h}(\xi)=\min_{\substack{y_0=\xi\\ y_q=y_0+p}}\sum_{i=0}^{q-1}h(y_i,y_{i+1})-\min_{x_q=x_0+p}\sum_{i=0}^{q-1}h(x_i,x_{i+1}).$$

Proof. Take $\xi\in(J_-,J_+)$, where (J_-,J_+) is a complementary interval to $\mathrm{A}_{\frac{p}{q},h}$. Let $x^\pm=(x_i^\pm)_{i\in\mathbb{Z}}$ satisfying $x^\pm_{i+q}=x^\pm_i+p, x^\pm_0=J_\pm$ be the periodic minimal configurations in $\mathrm{M}_{\frac{p}{q},h}$.

Comparing it with the definition (2.2), we only need to prove that the minimal segment $(\xi=y_0,y_1,...,y_{q-1},\, y_q=y_0+p)$ which achieves the minimum in the definition (2.2) satisfies the constraint

$$x_i^-\le y_i\le x_i^+.$$

In fact, we claim that the Aubry graphs of x^-,y,x^+ do not cross. Since $x^\pm$ are minimal configurations, $(y_i)_{i=0}^q$ is a minimal segment, the claim is an easy consequence according to Aubry's crossing lemma (Lemma 2.3). □

Lemma 4.4. *Let h be a continuous real valued function satisfying* $(\mathbf{H}1)-(\mathbf{H}5)$ *and* $(\mathbf{H}6\theta)$*, then the Peierls barriers have the following properties:*

(1) If $\frac{1}{q}>0$, then

$$|P_{\frac{1}{q}}(\xi)-P_{0+}(\xi)|\le\frac{16\theta}{q}.$$

(2) If $\frac{1}{q}>0$, then

$$|P_{-\frac{1}{q}}(\xi)-P_{0-}(\xi)|\le\frac{16\theta}{q}.$$

Proof. Notice that this lemma is not a direct conclusion of Proposition 2.6. Lemma 4.4 (1) was firstly claimed and proved by Mather in ([12], (4.1), (4.4a), (4.4b)), Lemma 4.4 (2) can be proved by the same approach. □

For symbol simplicity, we denote by $\mathbf{E}=\{(x,x')\in\mathbb{R}^2|\ |x'-x|\le 5\}$.

Lemma 4.5. *Assume that h,h' are the generating functions described in Lemma 4.2. Given $\frac{p}{q}\in[\omega_0,\omega_1]$, $q>0$ and p,q are relatively prime, and let $H_{(q,p)}$ (resp. $H'_{(q,p)}$) be the conjunction (3.1) of h (resp. h'), we have the following estimates:*

(1) $\|H'_{(q,p)}-H_{(q,p)}\|_{C^0(\mathbf{E})}\le q\|h'-h\|_{C^0(\mathbf{B}_{K-1})}$.

(2) For $0<m\in\mathbb{Z}$,

$$|P_{\frac{1}{m},H'_{(q,p)}}(\xi)-P_{\frac{1}{m},H_{(q,p)}}(\xi)|\le 2mq\|h'-h\|_{C^0(\mathbf{B}_{K-1})}.$$

Proof. (1). In fact, take $(x,x')\in\mathbf{E}$ and let $(x=x_0,x_1,...,x_{q-1},x_q=x'+p)$ be the minimal segment which achieves the minimum of $H_{(q,p)}(x,x')$, i.e.

$$H_{(q,p)}(x,x')=h(x_0,x_1)+\cdots+h(x_{q-1},x_q).$$

Then,

$$H'_{(q,p)}(x,x') \le \sum_{i=0}^{q-1} h'(x_i,x_{i+1}) = H_{(q,p)}(x,x') + \sum_{i=0}^{q-1}(h'(x_i,x_{i+1}) - h(x_i,x_{i+1})). \tag{4.15}$$

Next, we need to show that

$$(x_i,x_{i+1}) \in \mathbf{B}_{K-1}, \quad \forall 0 \le i \le q-1 \tag{4.16}$$

Recall that, by Proposition 2.5, there exists $\alpha \in \mathbb{R}$ such that $|x_q - x_0 - q\alpha| < 2$. Then, we obtain

$$|\alpha| < \frac{2+p+|x'-x|}{q} \le \frac{7}{q} + \frac{p}{q},$$

and by Proposition 2.5 again,

$$|x_{i+1} - x_i| \le 2 + |\alpha| \le 9 + \frac{p}{q} \le 10 + \max\{|\omega_0|, |\omega_1|\}.$$

Notice that we have assumed that the constant K is large enough, thus (4.16) holds.

By (4.15) and (4.16), we deduce that

$$H'_{(q,p)}(x,x') \le H_{(q,p)}(x,x') + q\|h'-h\|_{C^0(\mathbf{B}_{K-1})},\ \forall (x,x') \in \mathbf{E}.$$

Similarly, we can prove that

$$H_{(q,p)}(x,x') \le H'_{(q,p)}(x,x') + q\|h'-h\|_{C^0(\mathbf{B}_{K-1})},\ \forall (x,x') \in \mathbf{E}.$$

This completes the proof of (1).

(2). Firstly, we claim that the definition of $P_{\frac{1}{m},H_{(q,p)}}$ only depends on $H_{(q,p)}(x,x')$ restricted on $\mathbf{E}$.

Take $\xi \in (J_-,J_+)$, where (J_-,J_+) is a complementary interval to $\mathrm{A}_{\frac{1}{m},h}$. Let $x^\pm = (x_i^\pm)_{i\in\mathbb{Z}}$ satisfying $x_0^\pm = J_\pm$ be the $\frac{1}{m}$-minimal configurations of $\mathrm{M}_{\frac{1}{m},h}$,

$$P_{\frac{1}{m},H_{(q,p)}}(\xi) = \min\{\sum_{i=0}^{m-1} H_{(q,p)}(y_i,y_{i+1}) - H_{(q,p)}(x_i^-,x_{i+1}^-)|y_0 = \xi\}$$

where the minimum is taken over the set of all configurations satisfying $x_i^- \le y_i \le x_i^+$ and $y_{i+m} = y_i + 1$. Since $x_i^- \le y_i \le x_i^+, 0 \le i \le m-1$, by the totally ordered property of $x^\pm$, we get

$$\begin{aligned}|y_{i+1} - y_i| &\le |x_{i+1}^+ - x_i^-| \le |x_{i+1}^+ - x_{i+1}^-| + |x_{i+1}^- - x_i^-| \le 1 + |x_{i+1}^- - x_i^-| \\ &\le 2 + \pi(\omega) \qquad \text{(By Proposition 2.4)} \\ &\le 3\end{aligned} \tag{4.17}$$

Thus,

$$(y_i,y_{i+1}) \in \mathbf{E}, \quad \forall 0 \le i \le m-1,$$

which proves our claim. Similarly, we can prove this for $P_{\frac{1}{m},H'_{(q,p)}}$.

On the other hand, by Lemma 4.3, we have

$$P_{\frac{1}{m},H_{(q,p)}}(\xi) = \min_{\substack{y_0=\xi \\ y_m=y_0+1}} \sum_{i=0}^{m-1} H_{(q,p)}(y_i,y_{i+1}) - \min_{x_m=x_0+1} \sum_{i=0}^{m-1} H_{(q,p)}(x_i,x_{i+1})$$

$$P_{\frac{1}{m},H'_{(q,p)}}(\xi) = \min_{\substack{y_0=\xi \\ y_m=y_0+1}} \sum_{i=0}^{m-1} H'_{(q,p)}(y_i,y_{i+1}) - \min_{x_m=x_0+1} \sum_{i=0}^{m-1} H'_{(q,p)}(x_i,x_{i+1})$$

It follows that

$$|P_{\frac{1}{m},H'_{(q,p)}}(\xi) - P_{\frac{1}{m},H_{(q,p)}}(\xi)| \leq |I_1| + |I_2|. \tag{4.18}$$

where

$$I_1 = \min_{\substack{y_0=\xi \\ y_m=y_0+1}} \sum_{i=0}^{m-1} H'_{(q,p)}(y_i, y_{i+1}) - \min_{\substack{y_0=\xi \\ y_m=y_0+1}} \sum_{i=0}^{m-1} H_{(q,p)}(y_i, y_{i+1})$$

and

$$I_2 = \min_{x_m=x_0+1} \sum_{i=0}^{m-1} H'_{(q,p)}(x_i, x_{i+1}) - \min_{x_m=x_0+1} \sum_{i=0}^{m-1} H_{(q,p)}(x_i, x_{i+1})$$

Firstly, let's estimate $|I_1|$. Assume that $(\xi = a_0, a_1, ..., a_m = a_0+1)$ is the minimal segment satisfying

$$\sum_{i=0}^{m-1} H_{(q,p)}(a_i, a_{i+1}) = \min_{\substack{y_0=\xi \\ y_m=y_0+1}} \sum_{i=0}^{m-1} H_{(q,p)}(y_i, y_{i+1}).$$

We derive from Proposition 2.5 that there exists $\alpha \in \mathbb{R}$ such that

$$|a_m - a_0 - m\alpha| \leq 2$$

and since $a_m = a_0 + 1$, we obtain

$$|\alpha| \leq \frac{3}{m} \leq 3 \text{ and } |a_{i+1} - a_i| \leq 2 + |\alpha| \leq 5.$$

Consequently, $(a_i, a_{i+1}) \in \mathbf{E}, \quad \forall\ 0 \leq i \leq m-1$. Then,

$$\begin{aligned}
\min_{\substack{y_0=\xi \\ y_m=y_0+1}} \sum_{i=0}^{m-1} H'_{(q,p)}(y_i, y_{i+1}) &\leq \sum_{i=0}^{m-1} H'_{(q,p)}(a_i, a_{i+1}) \leq \sum_{i=0}^{m-1} H_{(q,p)}(a_i, a_{i+1}) \\
&\quad + \sum_{i=0}^{m-1} (H'_{(q,p)}(a_i, a_{i+1}) - H_{(q,p)}(a_i, a_{i+1})) \\
&\leq \min_{\substack{y_0=\xi \\ y_m=y_0+1}} \sum_{i=0}^{m-1} H_{(q,p)}(y_i, y_{i+1}) + m\kappa,
\end{aligned} \tag{4.19}$$

where $\kappa = \|H'_{(q,p)} - H_{(q,p)}\|_{C^0(\mathbf{E})}$. Similarly, we can prove

$$\min_{\substack{y_0=\xi \\ y_m=y_0+1}} \sum_{i=0}^{m-1} H_{(q,p)}(y_i, y_{i+1}) \leq \min_{\substack{y_0=\xi \\ y_m=y_0+1}} \sum_{i=0}^{m-1} H'_{(q,p)}(y_i, y_{i+1}) + m\kappa. \tag{4.20}$$

We conclude that, by (4.19) and (4.20),

$$|I_1| \leq m\|H'_{(q,p)} - H_{(q,p)}\|_{C^0(\mathbf{E})} \tag{4.21}$$

Secondly, let's estimate $|I_2|$. In fact, it can be similarly estimated as $|I_1|$,

$$|I_2| \leq m\|H'_{(q,p)} - H_{(q,p)}\|_{C^0(\mathbf{E})} \tag{4.22}$$

Therefore, it follows from (4.21), (4.22) and Lemma 4.5 (1) that

$$(4.18) \leq 2m\|H'_{(q,p)} - H_{(q,p)}\|_{C^0(\mathbf{E})} \leq 2mq\|h' - h\|_{C^0(\mathbf{B}_{K-1})},$$

which completes the proof of (2). $\square$

Because our proof of Theorem 4.1 needs the technique of rational approximation, so we introduce the following lemma.

Lemma 4.6 (Dirichlet approximation). *Given $\omega \in \mathbb{R}$ and $0 < n \in \mathbb{Z}$, then there exists a rational number $\frac{p}{q}$, $q > 0$ and p, q are relatively prime, such that*

$$0 < q \leq n,\ |\omega - \frac{p}{q}| \leq \frac{1}{q(n+1)}.$$

Proof. It can be easily proved by pigeon hole principle. Indeed, we firstly assume that $\omega \in \mathbb{R} \setminus \mathbb{Q}$. For every $0 < k \leq n$, we can find a integer h_k such that

$$a_k := k\omega + h_k \in (0, 1).$$

Then $a_1, ..., a_n$ are n distinct points in the interval $(0, 1)$ since $\omega \in \mathbb{R} \setminus \mathbb{Q}$. By pigeon hole principle, there exist two points $a_i, a_j (i < j)$ such that $|a_i - a_j| \leq \frac{1}{n+1}$, i.e.

$$|(i - j)\omega - (h_j - h_i)| \leq \frac{1}{n+1}$$

Thus,

$$|\omega - \frac{h_j - h_i}{i - j}| \leq \frac{1}{(j - i)(n + 1)},$$

which completes the proof for all $\omega \in \mathbb{R} \setminus \mathbb{Q}$.

For $\omega \in \mathbb{Q}$, it can be proved similarly. □

Proof of Theorem 4.1. To simplify notations, let's set

$$\delta := \|f' - f\|_{C^1(\mathbf{A}_K)}$$

Because we only concern what happens in the compact region $\mathbf{A}_K$, so we can assume that the generating function h of f satisfies $(\mathbf{H}1) - (\mathbf{H}5)$ and $(\mathbf{H}6\theta)$, with some $\theta = \theta(K, f)$ depending only on K and f.

In view of the geometrical meaning of θ, we know that, restricted on the compact region $\mathbf{A}_K$, $f|_{\mathbf{A}_K}$ turns every vertical vector to the right by an angle by at least β ($\cot\beta = \theta, 0 < \beta < \pi/2$). Thus there exists $0 < \delta_2 = \delta_2(K, f)$ such that for the perturbation f' of f, if $\delta \leq \delta_2$, the diffeomorphism $f'|_{\mathbf{A}_K}$ turns every vertical vector to the right by an angle by at least β' where $\beta' < \beta$ and $\cot\beta' = 2\theta$.

From now on, we set $\delta_0 = \min\{\delta_1, \delta_2\} \ll 1$. So if $\delta \leq \delta_0$, the generating function h' of f' satisfying $(\mathbf{H}1) - (\mathbf{H}5)$ and $(\mathbf{H}6\theta')$ with

$$\theta' = 2\theta.$$

Meanwhile, by Lemma 4.2, we choose the generating functions such that $h(0, 0) = h'(0, 0) = 0$ and

$$\|h' - h\|_{C^0(\mathbf{B}_{K-1})} \leq C_1\delta. \tag{4.23}$$

According to the proof of Lemma 4.2, for all rotation symbol

$$\omega \in (\mathbb{R} \setminus \mathbb{Q}) \bigcup (\mathbb{Q}+) \bigcup (\mathbb{Q}-)$$

with $\pi(\omega) \in [\omega_0, \omega_1]$, the minimal configurations $\mathrm{M}_{\omega,h}, \mathrm{M}_{\omega,h'} \subseteq \mathbf{B}_{K-1}$. Thus the definition of Peierls barrier functions $P_{\omega,h}, P_{\omega,h'}$ only depend on $h|_{B_{K-1}}, h'|_{B_{K-1}}$.

Let's begin our proof. Firstly, we approximate ω by rational number. In fact, by Lemma 4.6, for $n = [\delta^{-\frac{1}{3}}]$, we could find a rational number $\frac{p}{q}$ (in lowest terms) such that

$$0 < q \leq n \leq \delta^{-\frac{1}{3}} + 1, \quad |q\pi(\omega) - p| \leq \frac{1}{n+1} \leq \delta^{\frac{1}{3}}. \tag{4.24}$$

Assume that $\omega \geq \frac{p}{q}+$ (the case $\omega \leq \frac{p}{q}-$ is similar). By Proposition 2.6, Proposition 3.2 and the estimate (4.24), we have

$$\begin{aligned}\|P_{\omega,h'} - P_{\omega,h}\| &\leq \|P_{\omega,h'} - P_{\frac{p}{q}+,h'}\| + \|P_{\frac{p}{q}+,h'} - P_{\frac{p}{q}+,h}\| + \|P_{\frac{p}{q}+,h} - P_{\omega,h}\| \\ &\leq 4800\theta'|q\pi(\omega) - p| + \|P_{0+,H'_{(q,p)}} - P_{0+,H_{(q,p)}}\| + 4800\theta|q\pi(\omega) - p| \\ &\leq 14400\theta\delta^{\frac{1}{3}} + \|P_{0+,H'_{(q,p)}} - P_{0+,H_{(q,p)}}\|\end{aligned} \tag{4.25}$$

Next, we only need to give the estimate of $\|P_{0+,H'_{(q,p)}} - P_{0+,H_{(q,p)}}\|$. In fact, we take $m = [\delta^{-\frac{1}{3}}]$, and by Lemma 4.4, Lemma 4.5 (2), we obtain

$$\begin{aligned}\|P_{0+,H'_{(q,p)}} - P_{0+,H_{(q,p)}}\| &\leq \|P_{0+,H'_{(q,p)}} - P_{\frac{1}{m},H'_{(q,p)}}\| + \|P_{\frac{1}{m},H'_{(q,p)}} - P_{\frac{1}{m},H_{(q,p)}}\| + \|P_{\frac{1}{m},H_{(q,p)}} - P_{0+,H_{(q,p)}}\| \\ &\leq \frac{16}{m}\theta' + 2qm\|h' - h\|_{C^0(\mathbf{B}_{K-1})} + \frac{16}{m}\theta \\ &\leq \frac{48}{m}\theta + 2qmC_1\delta \quad (\text{ By (4.23) })\end{aligned}$$

We derive from (4.24) and $m = [\delta^{-\frac{1}{3}}]$ that

$$\|P_{0+,H'_{(q,p)}} - P_{0+,H_{(q,p)}}\| \leq 49\theta\delta^{\frac{1}{3}} + 3C_1\delta^{\frac{1}{3}}. \tag{4.26}$$

Finally, combining (4.25) with (4.26),

$$\|P_{\omega,h'}(\xi) - P_{\omega,h}(\xi)\| \leq (14449\theta + 3C_1)\delta^{\frac{1}{3}} = C_0\|f' - f\|^{\frac{1}{3}}_{C^1(\mathbf{A}_K)} \tag{4.27}$$

where $C_0 = 14449\theta + 3C_1$ which only depends on K, f. We know that K depends on ω_0, ω_1, f, so $C_0 = C_0(\omega_0, \omega_1, f)$. This completes the proof. □

4.2. **Lagrangians with one and a half degrees of freedom.** In [16], Moser showed that any monotone twist diffeomorphism on the cylinder $\mathbb{S}^1 \times \mathbb{R}$ can be regarded as the time-1 map of a periodic Lagrangian system. In what follows, we specialize to the Lagrangians with one and a half degrees of freedom. Firstly, let's briefly recall some basic notions and results of Mather theory. For proofs and details, we refer to [14], [15].

Let $L : T\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ be a $C^r (r \geq 2)$ Lagrangian satisfying Tonelli conditions:

(L1) Convexity: For each $(x, t) \in \mathbb{S}^1 \times \mathbb{S}^1$, L is strictly convex in v coordinate.

(L2) Superlinearity:

$$\lim_{\|v\| \to +\infty} \frac{L(x, v, t)}{\|v\|} = +\infty, \quad \text{uniformly on } (x, t).$$

(L3) Completeness: All solutions of the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(x, \dot{x}, t)\right) = \frac{\partial L}{\partial x}(x, \dot{x}, t)$$

are well defined for the whole $t \in \mathbb{R}$.

Let $I = [a, b]$ be an interval, and $\gamma : I \to \mathbb{S}^1$ be an absolutely continuous curve, we denote by

$$A(\gamma) := \int_a^b L(d\gamma(t), t)dt$$

the action of γ. An absolutely curve $\gamma : I \to \mathbb{S}^1$ is called a *minimizer* or *action minimizing curve* if

$$A(\gamma) = \min_{\substack{\xi(a)=\gamma(a),\xi(b)=\gamma(b)\\ \xi\in C^{ac}(I,\mathbb{S}^1)}} \int_a^b L(d\xi(t), t)dt.$$

We call $\gamma : (-\infty, +\infty) \to \mathbb{S}^1$ a *globally minimizing curve* if for all $a < b, \gamma$ is a minimizer on $[a, b]$. Notice that the minimizer satisfies the Euler-Lagrange equation.

Let $\mathcal{M}_L$ be the space of Euler-Lagrangian flow invariant probability measures on $T\mathbb{S}^1 \times \mathbb{S}^1$. To each $\mu \in \mathcal{M}_L$, note that $\int \lambda d\mu$=0 for each exact 1-form λ. Therefore, given $c \in H^1(\mathbb{S}^1, \mathbb{R})$ and a closed 1-form $\eta_c \in c = [\eta_c]$, we can define *Mather's α function*

$$\alpha(c) := -\inf_{\mu\in\mathcal{M}_L} A_c(\mu) = -\inf_{\mu\in\mathcal{M}_L} \int_{TM\times\mathbb{S}^1} L - \eta_c d\mu.$$

It's easy to be checked that $\alpha(c)$ is finite everywhere, convex and superlinear.

We associate to $\mu \in \mathcal{M}_L$ its rotation vector $\rho(\mu) \in H_1(\mathbb{S}^1, \mathbb{R})$ in the following sense:

$$\langle\rho(\mu), [\eta_c]\rangle = \int_{TM\times\mathbb{S}^1} \eta_c d\mu, \quad \forall c \in H^1(\mathbb{S}^1, \mathbb{R}).$$

So we can define *Mather's β function*:

$$\beta(\omega) := \inf_{\mu\in\mathcal{M}_L,\rho(\mu)=\omega} \int Ld\mu, \quad \forall\omega \in H_1(\mathbb{S}^1, \mathbb{R}).$$

β is finite, convex, superlinear and β is the Legendre-Fenchel dual of α .

Proposition 4.7. ([13]) *If $L : T\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ is a Tonelli Lagrangian, then*

(1) the function $\beta : H_1(\mathbb{S}^1, \mathbb{R}) \equiv \mathbb{R} \to \mathbb{R}$ is strictly convex and differentiable at all $\omega \in \mathbb{R} \setminus \mathbb{Q}$.
(2) the function $\alpha : H^1(\mathbb{S}^1, \mathbb{R}) \equiv \mathbb{R} \to \mathbb{R}$ is differentiable everywhere.

Next, we introduce the generalization of Peierls barrier to several degrees of freedom. For each $c \in H^1(\mathbb{S}^1, \mathbb{R})$ and $n \in \mathbb{Z}^+$, we define a function A_c^n,

$$A_c^n : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$$

$$A_c^n(x, x') := \inf_{\substack{\gamma(0)=x,\gamma(n)=x'\\ \gamma\in C^{ac}([0,n],\mathbb{S}^1)}} \int_0^n (L - \eta_c)(d\gamma(s), s)ds.$$

Then, following Mather, we introduce the barrier function on $\mathbb{S}^1 \times \mathbb{S}^1$

$$h_c^\infty(x, x') := \liminf_{n\to+\infty} A_c^n(x, x') + n\alpha(c). \tag{4.28}$$

This function is useful in the construction of connecting orbits (see [15]).

By Proposition 4.7, $\alpha'(c)$ exists for every $c \in H^1(\mathbb{S}^1, \mathbb{R})$, and the flat piece of graph α has rational slope.

Proposition 4.8. ([15], Proposition 7.1 and 7.2) *Let L be a Tonelli Lagrangian whose time-1 map is an area-preserving twist diffeomorphism, then*

(1) For $\omega \in \mathbb{R} \setminus \mathbb{Q}$, there exists a unique $c = c(\omega)$ such that $\alpha'(c) = \omega$, and

$$h_c^\infty(x, x) = P_\omega(x).$$

(2) For rational number $\frac{p}{q}$ (in lowest terms), let $c_+ := \max\{c : \alpha'(c) = \frac{p}{q}\}$, $c_- := \min\{c : \alpha'(c) = \frac{p}{q}\}$, then

$$h^\infty_{c_+}(x,x) = P_{\frac{p}{q}+}(x), \quad h^\infty_{c_-}(x,x) = P_{\frac{p}{q}-}(x)$$

Remark. $[c_-, c_+]$ corresponds to the flat of graph α with rational slope $\frac{p}{q}$.

- Now, let $L : T\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ be a Tonelli Lagrangian whose time-1 map Φ_1 is an exact area-preserving monotone twist diffeomorphism. For the rotation interval $[\omega_0, \omega_1]$, there is a compact annulus $\mathbf{A}_K := \mathbb{S}^1 \times [-K, K]$ with sufficiently large $K = K(\omega_0, \omega_1, L)$ so that the minimal orbits of Φ_1 satisfies

$$\mathfrak{M}_{\omega,\Phi} \subseteq \mathbf{A}_{K-2} \subseteq \mathbf{A}_{K-1} \subseteq \mathbf{A}_K, \ \forall \omega \in [\omega_0, \omega_1].$$

The space $C^2(\mathbf{A}_{2K}) = C^2(\mathbf{A}_{2K}, \mathbb{R})$ is provided with the norm:

$$\|f\|_{C^2(\mathbf{A}_{2K})} = \sup_{0 \leq j \leq 2} \max_{\mathbf{A}_{2K}} |D^j f|.$$

- $H^1(\mathbb{S}^1, \mathbb{R}) \equiv \mathbb{R}$, by abuse of notation, we use the same symbol c to denote the real number in $\mathbb{R}$ or the closed 1-form $cd\vartheta$ of $\mathbb{S}^1$.
- Let $h^\infty_{L_c}$ denote the barrier function (4.28) associated to the Lagrangian $L_c := L - c$.
- By Proposition 4.7 and 4.8, for irrational number $\omega \in H_1(\mathbb{S}^1, \mathbb{R}) \equiv \mathbb{R}$, we denote by $c(\omega) \in H^1(\mathbb{S}^1, \mathbb{R}) \equiv \mathbb{R}$ the unique number satisfying $\alpha'(c) = \omega$. In addition, we can define $c_+(\omega), c_-(\omega)$ for rational numbers as Proposition 4.8 (2). Similarly, let the Lagrangian L' be a perturbation of L, one can also define $c'(\omega), c'_+(\omega), c'_-(\omega)$ in the same way.

Theorem 4.9. *Let L, $[\omega_0, \omega_1]$, $\mathbf{A}_{2K}$ be as shown above. There exist constants $\delta_0 = \delta_0(\omega_0, \omega_1, L) \ll 1$ and $C_0 = C_0(\omega_0, \omega_1, L)$ such that for any Lagrangian L', if $\|L' - L\|_{C^2(\mathbf{A}_{2K} \times \mathbb{S}^1)} \leq \delta_0$ and $\omega \in [\omega_0, \omega_1]$, we have,*

(1) for $\omega \in \mathbb{R} \setminus \mathbb{Q}$, $|h^\infty_{L'_{c'(\omega)}}(x,x) - h^\infty_{L_{c(\omega)}}(x,x)| \leq C\|L' - L\|^{\frac{1}{3}}_{C^2(\mathbf{A}_{2K} \times \mathbb{S}^1)}$

(2) for rational number ω, $|h^\infty_{L'_{c'+(\omega)}}(x,x) - h^\infty_{L_{c_+(\omega)}}(x,x)| \leq C\|L' - L\|^{\frac{1}{3}}_{C^2(\mathbf{A}_{2K} \times \mathbb{S}^1)}$ and

$$|h^\infty_{L'_{c'-(\omega)}}(x,x) - h^\infty_{L_{c_-(\omega)}}(x,x)| \leq C\|L' - L\|^{\frac{1}{3}}_{C^2(\mathbf{A}_{2K} \times \mathbb{S}^1)}$$

Remark. For the Lagrangian of many degrees of freedom, the barrier function $h^\infty_c(x, x')$ could also be defined, one may ask whether there are similar results in this case. In general, it is not true and there're counterexamples which could be found in [4]. However, under additional assumptions, such as nearly-integrable Lagrangians of arbitrary degrees of freedom, we can also obtain some similar results, see [4] for details.

Proof of Theorem 4.9. The time-1 map Φ_1 of the Lagrangian L is an exact area-preserving monotone twist diffeomorphism. Then, there exists $\delta_0 = \delta_0(\omega_0, \omega_1, L)$ such that if $\|L' - L\|_{C^2(\mathbf{A}_{2K} \times \mathbb{S}^1)} \leq \delta_0$, the time-1 map Φ'_1 of L' is also a monotone twist diffeomorphism satisfying

$$\|\Phi'_1 - \Phi_1\|_{C^1(\mathbf{A}_K)} \leq D\|L' - L\|_{C^2(\mathbf{A}_{2K} \times \mathbb{S}^1)} \tag{4.29}$$

with the constant D. Therefore, by Proposition 4.8, it's not hard to observe that Theorem 4.9 is an easy consequence of (4.29) and Theorem 4.1. □

5. Application

The destruction of invariant circles or Lagrangian tori (Converse KAM theory) is an interesting and important topic in dynamic systems (see for example, [5] [6] [8] [12]). In this section, by applying Theorem 4.1, we give an open and dense property about the destruction of invariant circles. A real number $\omega \in \mathbb{R} \setminus \mathbb{Q}$ is *Diophantine* if there exist constants $C > 0$ and $\tau > 1$ such that

$$|q\omega - p| \geq \frac{C}{|q|^\tau} \quad \text{for all } p, q \in \mathbb{Z}, q \neq 0.$$

A real number is *Liouville* if it's not Diophantine .

Theorem 5.1. *Let $\mathcal{J}^r (r \geq 1)$ be the set of all C^r exact area-preserving monotone twist diffeomorphisms and let ω be a Liouville number. Then there exists a set $\mathcal{O}$ which is open and dense in $\mathcal{J}^r$ in the C^r topology, such that for all $f \in \mathcal{O}$, there is no homotopically non-trivial $f-$invariant circle with rotation number ω.*

Proof. **Case I**: ω is irrational.

Denote by $\mathcal{O}$ the set of all C^r exact area-preserving monotone twist diffeomorphisms that don't admit any homotopically non-trivial invariant circles with rotation number ω. We only need to prove that the set $\mathcal{O}$ is open and dense in $\mathcal{J}^r$.

Given an exact area-preserving monotone twist diffeomorphism $f \in J^r$, we assume that f admits a homotopically non-trivial $f-$invariant circle with rotation number ω. Then by Theorem 2.1 in [12], for any neighbourhood $\mathcal{U}_f$ of f in $\mathcal{J}^r$, we could find $g \in \mathcal{U}_f$ which does not admit any homotopically non-trivial $g-$invariant circle of rotation number ω. So $g \in \mathcal{O} \bigcap \mathcal{U}_f$, which proves that $\mathcal{O}$ is a dense set in $\mathcal{J}^r$.

On the other hand, we know that the Peierls barrier $P_{\omega,f}(\xi) \equiv 0$ if and only if there exists a homotopically non-trivial $f-$invariant circle of rotation number ω (see [11]). Take $f \in \mathcal{O}$, then there exists a point $\xi_0 \in \mathbb{R}$ such that $P_{\omega,f}(\xi_0) = a_0 > 0$. By Theorem 4.1, we have

$$|P_{\omega,f'}(\xi) - P_{\omega,f}(\xi)| \leq C_0 \|f' - f\|_{C^1}^{\frac{1}{3}} \leq C_0 \|f' - f\|_{C^r}^{\frac{1}{3}}.$$

Thus, there exists a small neighbourhood $\mathcal{V}_f$ of f in $\mathcal{J}^r$ such that for all $f' \in \mathcal{V}_f$, we have

$$P_{\omega,f'}(\xi_0) \geq a_0/2 > 0.$$

Then $\mathcal{V}_f \subseteq \mathcal{O}$, which proves that $\mathcal{O}$ is an open set.

Case II: $\omega = \frac{p}{q}$ is rational.

There is no homotopically non-trivial $f-$invariant circle of rotation number $\frac{p}{q}$ if and only if the Peierls barrier $P_{\frac{p}{q}+,f}(\xi) \not\equiv 0$ and $P_{\frac{p}{q}-,f}(\xi) \not\equiv 0$. Denote by $\mathcal{O}$ the set of all C^r exact area-preserving monotone twist diffeomorphisms that don't admit any homotopically non-trivial invariant circles with rotation number $\frac{p}{q}$. We need to prove that $\mathcal{O}$ is open and dense. In fact, the denseness of $\mathcal{O}$ was proved by Mather in [10] or [13], and the proof of openness is similar with Case I by Theorem 4.1.

□

Remark. Similar results also hold for analytic topology. Notice that in analytic situations, we need an additional condition on the irrational number ω, i.e.,

$$\limsup_{n \to +\infty} \frac{\log \log q_{n+1}}{\log q_n} > 0,$$

where $(p_n/q_n)_{n \in \mathbb{Z}}$ is the sequence of approximants given by the continued fraction expansion of ω (see[6]).

A set is called *residual* if it is a countable intersection of open and dense sets.

Corollary 5.2. *There exists a residual set $\mathcal{R} \subseteq \mathcal{J}^r$ in the C^r topology which satisfies: for each $f \in \mathcal{R}$, there is an open and dense set $\mathcal{H}(f) \subseteq \mathbb{R}$ such that for all $\omega \in \mathcal{H}(f)$, f does not admit any homotopically non-trivial invariant circle with rotation number ω.*

Proof. Let $\{r_n\}_{n\in\mathbb{Z}}$ denotes the set of all rational numbers in $\mathbb{R}$. By Theorem 5.1, we obtain an open and dense set $\mathcal{O}_n$ such that for all $f \in \mathcal{O}_n$, there is no homotopically non-trivial invariant circle with rotation number r_n. We set $\mathcal{R} = \bigcap_{n\in\mathbb{Z}} \mathcal{O}_n$, it is a residual set in $\mathcal{J}^r$ in the C^r topology.

Take $g \in \mathcal{R}$, then we obtain that the Peierls barriers $P_{r_n+,g}(\xi) \not\equiv 0$ and $P_{r_n-,g}(\xi) \not\equiv 0$ for all n. We deduce from Proposition 2.6 that $\omega \mapsto P_{\omega,g}$ is right-continuous at the rotation symbol $\omega = r_n+$ and left-continuous at the rotation symbol $\omega = r_n-$. Thus, there exists an open interval $(a_n, b_n) \ni r_n$ such that

$$P_{\omega,g}(\xi) \not\equiv 0, \text{ for all symbol } \omega \text{ with } \pi(\omega) \in (a_n, b_n).$$

If we set $\mathcal{H}(g) = \bigcup_{n\in\mathbb{Z}} (a_n, b_n)$, then it's open and dense since $\{r_n\}_{n\in\mathbb{Z}}$ is dense in $\mathbb{R}$, and

$$P_{\omega,g}(\xi) \not\equiv 0, \text{ for all symbol } \omega \text{ with } \pi(\omega) \in \mathcal{H}(g),$$

which means that there is no homotopically non-trivial g-invariant circle with rotation number $\omega \in \mathcal{H}(g)$. This completes our proof. □

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