

TROPICAL LAGRANGIAN MULTI-SECTIONS AND SMOOTHING OF LOCALLY FREE SHEAVES OVER DEGENERATE CALABI-YAU SURFACES

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ABSTRACT. We introduce the notion of tropical Lagrangian multi-sections over any 2-dimensional integral affine manifold B with singularities, and use them to study the reconstruction problem for higher rank locally free sheaves over Calabi-Yau surfaces. To certain tropical Lagrangian multi-sections \mathbb{L} over B , which are explicitly constructed by prescribing local models around the ramification points, we associate locally free sheaves $\mathcal{E}_0(\mathbb{L})$ over the singular projective scheme $X_0(B, \mathcal{P}, s)$ associated to B equipped with a polyhedral decomposition \mathcal{P} and a gluing data s . We find combinatorial conditions on such an \mathbb{L} under which the sheaf $\mathcal{E}_0(\mathbb{L})$ is simple. This produces explicit examples of smoothable pairs $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$ in dimension 2.

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1. INTRODUCTION

The spectacular Gross-Siebert program [13, 14, 15] is usually referred as an algebro-geometric approach to the famous Strominger-Yau-Zaslow (SYZ) conjecture [25]. It gives an algebro-geometric way to construct mirror pairs. A polarized Calabi-Yau manifold \check{X} near a large volume limit point should admit a toric degeneration $\check{p} : \check{X} \rightarrow S$ to a singular variety $\check{X}_0 := \check{p}^{-1}(0)$, which is a union of toric varieties. These toric components intersect along toric strata. By gluing the fans that correspond to these toric pieces, we obtain an integral affine manifold \check{B} with singularities together with a polyhedral decomposition $\check{\mathcal{P}}$. Then by choosing a strictly convex multi-valued piecewise

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linear function $\check{\varphi}$ on B , a mirror family $p : \mathcal{X} \rightarrow S$ is obtained by the following procedure (the fan construction):

- (1) Take the discrete Legendre transform $(B, \mathcal{P}, \varphi)$ of $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$.
- (2) Let X_v be the toric variety associated to the fan Σ_v , defined around a vertex $v \in B$. Glue these toric varieties together along toric divisors with some twisting data s to obtain a projective scheme $X_0(B, \mathcal{P}, s)$.
- (3) Smooth $X_0(B, \mathcal{P}, s)$ to obtain a family $\check{p} : \check{\mathcal{X}} \rightarrow S$.

This is usually referred as the *reconstruction problem* in mirror symmetry.

The most difficult step is (3), namely, to prove that $X_0(B, \mathcal{P}, s)$ is smoothable. In [13], Gross and Siebert showed that $X_0(B, \mathcal{P}, s)$ carries the structure of a *log scheme* and it is *log smooth* away from a subset $Z \subset X_0(B, \mathcal{P}, s)$ of codimension at least 2. The subset Z should be thought of as the singular locus of a Lagrangian torus fibration (SYZ fibration) on the original side and it is the main obstacle in the reconstruction problem. Inspired by Kontsevich-Soibelman's earlier work [19] and Fukaya's program [10], Kontsevich and Soibelman invented the notion of *scattering diagrams* in the innovative work [20] and applied it to solve the reconstruction problem in dimension 2 over non-Archimedean fields. This was extensively generalized by Gross and Siebert [15], in which they solved the reconstruction problem over \mathbb{C} in *any* dimension. Roughly speaking, they defined a notion called *structure*, which consists of combinatorial (or tropical) data called *slaps* and *walls*. Applying Kontsevich-Soibelman's scattering diagram technique, they were able to construct a remarkable *explicit* order-by-order smoothing of $X_0(B, \mathcal{P}, s)$. Recently, in [1] and [8], it was shown that purely algebraic techniques were enough to prove the existence of smoothing and this can be applied to a more general class of varieties (called toroidal crossings varieties).

In view of Kontsevich's homological mirror symmetry (HMS) conjecture [18], it is natural to ask if one can reconstruct coherent sheaves from combinatorial or tropical data as well, where the latter should arise as tropical limits of Lagrangian submanifolds on the mirror side. In the rank one case, this was essentially accomplished by the series of works [11, 12, 16], in which the Gross-Siebert program was extended and applied to reconstruct (generalized) theta functions, or sections of ample line bundles, on Calabi-Yau varieties, proving (a strong form of) Tyurin's conjecture [27]. This paper represents an initial attempt to tackle the reconstruction problem in the higher rank case. We will restrict our attention to the dimension 2 case as in [11].

In [26], the third author of this paper demonstrated that the tangent bundle $T_{\mathbb{P}^2}$ of the complex projective plane \mathbb{P}^2 can be reconstructed from some tropical data on the fan of \mathbb{P}^2 , which he called a *tropical Lagrangian multi-section*. However, the definition there is not general enough to cover many interesting cases which could arise in mirror symmetry. In Section 3, we will give a more general definition of tropical Lagrangian multi-sections over any 2-dimensional integral affine manifold B with singularities equipped with a polyhedral decomposition \mathcal{P} . We expect such an object to arise as a tropical limit of Lagrangian multi-sections in an SYZ fibration of the mirror. Roughly speaking, it consists of a topological (possibly branched) covering map $\pi : L \rightarrow B$, a polyhedral decomposition \mathcal{P}_π on L respecting \mathcal{P} and a multi-valued piecewise linear function φ' on L . A key difference from the usual polyhedral decomposition is that we require the ramification locus of $\pi : L \rightarrow B$ to be contained in the codimension 2 strata of (L, \mathcal{P}_π) . In particular, the pullback affine structure on L is also singular along the ramification locus. See Definition 3.7.

To have explicit examples of tropical Lagrangian multi-sections, we need good local models. In Section 4, we define local models of φ' around the ramification points. In [22], Payne used the equivariant structure of $T_{\mathbb{P}^2}$ on each affine chart to define a piecewise linear function $\varphi_{2,1}$ on a

2-fold covering of $|\Sigma_{\mathbb{P}^2}| \cong N_{\mathbb{R}}$. It takes the form

$$\varphi_{2,1} := \begin{cases} -\xi_1 & \text{on } \sigma_0^+, \\ -\xi_2 & \text{on } \sigma_0^-, \\ \xi_1 & \text{on } \sigma_1^+, \\ \xi_1 - \xi_2 & \text{on } \sigma_1^-, \\ -\xi_1 + \xi_2 & \text{on } \sigma_2^+, \\ \xi_2 & \text{on } \sigma_2^-, \end{cases}$$

where σ_i^\pm are two copies of the cone σ_i and (ξ_1, ξ_2) are affine coordinates on $N_{\mathbb{R}} \cong \mathbb{R}^2$. It is natural to consider modifications of the coefficients of this function, as shown in Figure 1, where m, n are integers with $m \neq n$, and we call the resulting function $\varphi_{m,n}$.

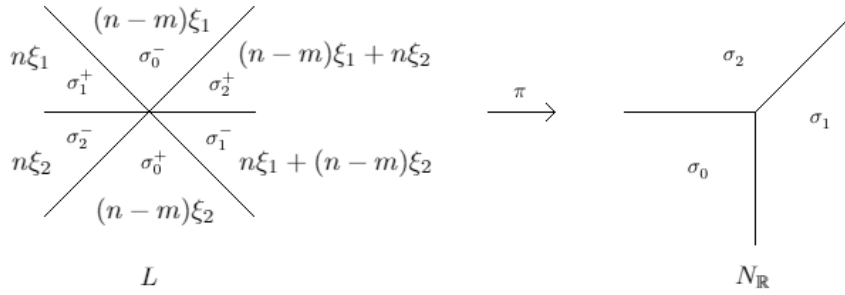


FIGURE 1. The tropical Lagrangian multi-section \mathbb{L}

Let B be a closed topological surface equipped with the structure of an integral affine manifold with singularities. We say a tropical Lagrangian multi-section is *of class \mathcal{S}* (denoted as $\mathbb{L} \in \mathcal{S}$) if φ' is locally modeled by $\varphi_{m,n}$ for some m, n around each ramification point. Furthermore, if the integers m, n are independent of the ramification points, we obtain a special subclass called $\mathcal{S}_{m,n} \subset \mathcal{S}$.

Section 5 is devoted to the construction of a locally free sheaf $\mathcal{E}_0(\mathbb{L})$ over the singular variety $X_0(B, \mathcal{P}, s)$ from a tropical Lagrangian multi-section \mathbb{L} of class \mathcal{S} . This is already nontrivial because there are obstructions to the gluing processes. First of all, taking the equivariant line bundle corresponding to φ' on each affine chart $U_i := \text{Spec}(\mathbb{C}[\check{v}_i \cap M])$ and direct sum, one obtains a rank r equivariant bundle on each U_i . If a vertex $v \in B$ is not a branched point, these local pieces glue to give a rank r bundle which splits. However, when v is a branched point, on each local piece U_i , there are two line bundles $\mathcal{L}_i^+, \mathcal{L}_i^-$ which *cannot* be glued due to nontrivial monodromy around the ramification points. In such a case, we follow [26] (which was in turn motivated by Fukaya's proposal for reconstructing bundles in [10]) and try to glue $\mathcal{L}_i^+ \oplus \mathcal{L}_i^-$'s equivariantly to obtain a set of naive transition functions:

$$\tau_{10}^{sf} := \begin{pmatrix} \frac{a_0}{(w_0^1)^m} & 0 \\ 0 & \frac{b_0}{(w_0^1)^n} \end{pmatrix}, \tau_{21}^{sf} := \begin{pmatrix} \frac{b_1}{(w_1^2)^n} & 0 \\ 0 & \frac{a_1}{(w_1^2)^m} \end{pmatrix}, \tau_{02}^{sf} := \begin{pmatrix} 0 & \frac{b_2}{(w_2^0)^n} \\ \frac{a_2}{(w_2^0)^m} & 0 \end{pmatrix}.$$

The problem is that these do *not* satisfy the cocycle condition. Thus we have to modify τ_{ij}^{sf} by multiplying by an invertible factor Θ_{ij} (the wall-crossing factors), namely, $\tau_{ij} := \tau_{ij}^{sf} \Theta_{ij}$. We then obtain the following

Proposition 1.1 (=Proposition 5.1). *If we impose the condition $\prod_i a_i b_i = -1$, then*

$$\tau_{02}\tau_{21}\tau_{10} = I.$$

Moreover, the equivariant structure defined by (2) can be extended.

From this we get an equivariant rank 2 holomorphic vector bundle $E_{m,n}$ on \mathbb{P}^2 . Denote the corresponding rank 2 bundle on a toric piece $X_v \cong \mathbb{P}^2$ by $\mathcal{E}(v)'$. Together with $r - 2$ line bundles $\mathcal{L}_v^{(k)}$, $k = 1, \dots, r - 2$, we obtain a rank r bundle $\mathcal{E}(v)$ on X_v even if v is a branched point. To glue $\{\mathcal{E}(v)\}$ together, a key observation is that the factors Θ_{ij} act *trivially* on the boundary divisors.¹ However, there is further obstruction to gluing these bundles together. This obstruction, which is given by a cohomology class $o_{\mathbb{L}}(s) \in H^2(L, \mathbb{C}^\times)$, is analogous to that in [13, Theorem 2.34] (that arises in gluing of the projective scheme $X_0(B, \mathcal{P}, s)$).

Theorem 1.2 (=Theorem 5.6). *If $o_{\mathbb{L}}(s) = 1$, then there exists a rank r locally free sheaf $\mathcal{E}_0(\mathbb{L})$ on $X_0(B, \mathcal{P}, s)$.*

We then proceed to study smoothability of the pair $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$ in Section 6. We will assume that the polyhedral decomposition \mathcal{P} is positive and simple (as in [13, 14, 15]) as well as *elementary*, meaning that every cell in \mathcal{P} is an elementary simplex.² From the Gross-Siebert program, we already know that $X_0(B, \mathcal{P}, s)$ can be smoothed to a formal polarized family $p : \mathcal{X} \rightarrow S$ of Calabi-Yau surfaces; here, we work over $S := \text{Spec}(\mathbb{C}[[t]])$. We will focus on the sheaves $\mathcal{E}_0(\mathbb{L})$ which correspond to tropical Lagrangian multi-sections $\mathbb{L} \in \mathcal{S}_{n+1,n}$.

To prove smoothability of the pair $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$ for $\mathbb{L} \in \mathcal{S}_{n+1,n}$, our strategy is to apply the main result [2, Corollary 4.7] in a previous work of the first and second authors, for which we need the condition that $H^2(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}_0(\mathbb{L}))) = 0$. In general, it is not easy to deal with higher cohomologies. Exploiting the fact that $X_0(B, \mathcal{P}, s)$ is a Calabi-Yau surface and Serre duality, we are reduced to showing that $H^0(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}_0(\mathbb{L}))) = 0$, or equivalently, that the locally free sheaf $\mathcal{E}_0(\mathbb{L})$ is *simple*.

Our main results give a combinatorial condition on the tropical Lagrangian multi-section \mathbb{L} which is equivalent to simplicity of $\mathcal{E}_0(\mathbb{L})$. The condition looks particularly appealing in the rank 2 case, so we will first discuss this case in Section 6.1. In order to state our result, we consider the embedded graph $\Gamma \subset B$ given by the union of the 1-cells in the polyhedral decomposition \mathcal{P} , and let $G(\mathbb{L})$ be the subgraph in Γ obtained by removing all the branched vertices. Then we have the following theorem

Theorem 1.3 (=Theorem 6.4). *Let $\mathbb{L} \in \mathcal{S}_{n+1,n}$. Then the locally free sheaf $\mathcal{E}_0(\mathbb{L})$ is simple if and only if $G(\mathbb{L})$ does not bound any 2-cell in \mathcal{P} .*

Because of this result, we say a tropical Lagrangian multi-section $\mathbb{L} \in \mathcal{S}_{n+1,n}$ is *simple* if $G(\mathbb{L})$ does not bound any 2-cell in \mathcal{P} (see Definition 6.3).

Corollary 1.4 (=Corollary 6.7). *If $\mathbb{L} \in \mathcal{S}_{n+1,n}$ is simple, then the pair $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$ is smoothable.*

¹This is because each irreducible component of the boundary divisor is isomorphic to \mathbb{P}^1 and hence any bundle splits into a direct sum of two line bundles; one should not expect such a nice property for $\dim(B) \geq 3$.

²In dimension 2, every polyhedral decomposition can actually be subdivided into elementary simplices (or equivalently, standard simplices).

The higher rank case, which is considerably more subtle, will be taken up in Section 6.2. The appropriate definition of simplicity of $\mathbb{L} \in \mathcal{S}_{n+1,n}$ and proof of the analogous theorem for $r \geq 3$ involves more complicated combinatorial data on the fiber product

$$P(\mathbb{L}) := L \times_{\pi} L,$$

which should be regarded as a certain fiberwise path space of \mathbb{L} . We will define an embedded graph $\tilde{G}(\mathbb{L}) \subset P(\mathbb{L})$ similar to $G(\mathbb{L})$ by lifting the polyhedral decomposition of B . To define simplicity for \mathbb{L} , we study how sections on each toric component are glued together along boundary divisors.

First of all, a non-trivial section of $End_0(\mathcal{E}_0(\mathbb{L}))$ restricts to non-trivial sections on some toric components. Since $m = n + 1$, these sections turn out to be either sections of $\mathcal{O}_{\mathbb{P}^2}(1)$ or sections of the rank 2 bundle $E_{n+1,n}(-n)$. Non-trivial sections of these two bundles have different vanishing loci along the boundary divisors which can be described combinatorially or tropically, as shown in Figure 5. We will introduce the notion of a *well-colored subgraph* (Definition 6.12) of $\tilde{G}(\mathbb{L})$ to classify these types of vanishing behavior. Roughly speaking, a well-colored graph is obtained by gluing the “dual” (see Figure 4) of the local pieces as shown in Figure 5 such that the coloring is respected. However, well-colored subgraphs alone are not enough to produce sections of $End_0(\mathcal{E}_0(\mathbb{L}))$ due to their tropical nature. To handle this, we introduce yet another piece of combinatorial information, called *compatible data* (Definition 6.18). Then for each well-colored subgraph $\tilde{G} \subset \tilde{G}(\mathbb{L})$ and compatible data D , we can define a rank 1 local system $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ on a natural subgraph \tilde{G}_0 of \tilde{G} and prove the following correspondence theorem:

Theorem 1.5 (=Theorem 6.20). *Let $\mathbb{L} \in \mathcal{S}_{n+1,n}$. Define*

$$\mathcal{WC}(\mathbb{L}) := \{\tilde{G} \subset \tilde{G}(\mathbb{L}) \mid \tilde{G} \text{ is well-colored}\},$$

and for each $\tilde{G} \in \mathcal{WC}(\mathbb{L})$, define

$$\mathcal{D}_{\tilde{G}} := \{D \mid \mathcal{L}_{\tilde{G}}(D, \mathbf{a}) \text{ is trivial}\}.$$

Then, for any $\tilde{G} \in \mathcal{WC}(\mathbb{L})$ and $D \in \mathcal{D}_{\tilde{G}}$, there is an injection

$$i_{\mathcal{L}_{\tilde{G}}(D, \mathbf{a})} : H^0(\tilde{G}_0, \mathcal{L}_{\tilde{G}}(D, \mathbf{a})) \hookrightarrow H^0(X_0(B, \mathcal{P}, s), End_0(\mathcal{E}_0(\mathbb{L})))$$

such that

$$H^0(X_0(B, \mathcal{P}, s), End_0(\mathcal{E}_0(\mathbb{L}))) = \bigcup_{\tilde{G} \in \mathcal{WC}(\mathbb{L})} \bigcup_{D \in \mathcal{D}_{\tilde{G}}} Im(i_{\mathcal{L}_{\tilde{G}}(D, \mathbf{a})}).$$

Theorem 6.20 gives a purely combinatorial description of sections of $End_0(\mathcal{E}_0(\mathbb{L}))$, which can be regarded as a tropical mirror symmetric statement. It also provides a natural definition for simplicity (Definition 6.23) of $\mathbb{L} \in \mathcal{S}_{n+1,n}$, which is explicit and checkable, though combinatorially much more complicated than the rank 2 case. Now we arrive at the following

Theorem 1.6 (=Theorem 6.26). *A tropical Lagrangian multi-section $\mathbb{L} \in \mathcal{S}_{n+1,n}$ is simple if and only if $\mathcal{E}_0(\mathbb{L})$ is simple.*

By applying Serre duality and the smoothing result in [2] again, Corollary 1.4 can be generalized to $r \geq 2$:

Corollary 1.7 (=Corollary 6.27). *If $\mathbb{L} \in \mathcal{S}_{n+1,n}$ is simple, then the pair $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$ is smoothable.*

We end this introduction with a short discussion on the relation between the locally free sheaf $\mathcal{E}_0(\mathbb{L})$ and a constructible sheaf \mathcal{F} on $P(\mathbb{L}) := L \times_{\pi} L$, which is defined as follows: Let $\pi_B : P(\mathbb{L}) \rightarrow B$ be the natural projection map. We define

$$\tilde{\mathcal{P}} := \{\sigma'_1 \times_{\pi} \sigma'_2 \mid \sigma'_1, \sigma'_2 \in \mathcal{P}_{\pi} \text{ such that } \pi(\sigma'_1) = \pi(\sigma'_2)\}.$$

Let $\{v\} \in \mathcal{P}$ be a vertex and $(v'_1, v'_2) := \tilde{v} \in \pi_B^{-1}(v)$. Each v'_i gives a vector bundle $\mathcal{E}(v'_i)$ on X_v , which is either a line bundle or a rank 2 bundle. For $\tilde{\sigma} \in \tilde{\mathcal{P}}$ with $\tilde{v} \in \tilde{\sigma}$, we define

$$\mathcal{F}(\tilde{\sigma}) := H^0(X_{\sigma}, (\mathcal{E}(v'_1)^* \otimes \mathcal{E}(v'_2))|_{X_{\sigma}}),$$

where $\sigma := \pi_B(\tilde{\sigma})$. If $\tilde{\tau} \subset \tilde{\sigma}$, we have the generalization map

$$g_{\tilde{\tau}\tilde{\sigma}} : \mathcal{F}(\tilde{\tau}) \rightarrow \mathcal{F}(\tilde{\sigma})$$

induced by the inclusion $\iota_{\sigma\tau} : X_{\sigma} \hookrightarrow X_{\tau}$. Clearly,

$$g_{\tilde{\rho}\tilde{\sigma}} = g_{\tilde{\tau}\tilde{\sigma}} \circ g_{\tilde{\rho}\tilde{\tau}},$$

whenever $\tilde{\rho} \subset \tilde{\tau} \subset \tilde{\sigma}$. Hence the data $(\{\mathcal{F}(\tilde{\sigma})\}, \{g_{\tilde{\tau}\tilde{\sigma}}\})$ defines a constructible sheaf \mathcal{F} on $P(\mathbb{L})$. Define

$$P_0(\mathbb{L}) := P(\mathbb{L}) \setminus \Delta_L,$$

where Δ_L denotes the diagonal. By restricting \mathcal{F} to $P_0(\mathbb{L})$, we get a sheaf \mathcal{F}_0 on $P_0(\mathbb{L})$. By construction, we have canonical identifications

$$(1) \quad \begin{aligned} H^0(X_0(B, \mathcal{P}, s), \text{End}(\mathcal{E}_0(\mathbb{L}))) &\cong H^0(P(\mathbb{L}), \mathcal{F}), \\ H^0(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}_0(\mathbb{L}))) &\cong H^0(P_0(\mathbb{L}), \mathcal{F}_0). \end{aligned}$$

The *coherent-constructible correspondence* of toric varieties was established by Fang-Liu-Treumann-Zaslow in [4, 5] and applied to prove HMS for toric varieties [6, 7]. As mentioned above, one should think of the fiber product $P(\mathbb{L})$ as a certain (fiberwise) path space of \mathbb{L} . For a non-singular SYZ fibration $p : \check{X} \rightarrow B$ and an honest Lagrangian multi-section $\mathbb{L} \subset \check{X}$, one can talk about the fiberwise geodesic path space as in [19, 21]. A point $(x'_1, x'_2) \in P(\mathbb{L})$ is regarded as the end points of an affine geodesic from \mathbb{L} to itself. The identifications (1) suggest that if one consider higher rank sheaves \mathcal{E} on X , the self-Hom space of \mathcal{E} should be computed by certain (possibly derived) constructible sheaf on the ‘‘path space’’ $P(\mathbb{L})$. We leave this for future research.

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2. THE GROSS-SIEBERT PROGRAM

In this section, we review some machinery in the Gross-Siebert program, mainly following [13].

2.1. Affine manifolds with singularities and their polyhedral decompositions.

Definition 2.1 ([13], Definition 1.15). *An integral affine manifold with singularities is a topological manifold B together with a closed subset $\Delta \subset B$, which is a finite union of locally closed submanifolds of codimension at least 2, such that $B_0 := B \setminus \Delta$ is an integral affine manifold (meaning that the transition functions are integral affine). Let B be an integral affine manifold with singularities and $U \subset B$ be an open subset. A continuous function $f : U \rightarrow \mathbb{R}$ is called integral affine if $f|_{U \cap B_0} : U \cap B_0 \rightarrow \mathbb{R}$ is an integral affine function. The sheaf of integral affine functions on B is denoted by $\mathcal{A}f(B, \mathbb{Z})$.*

Fix a rank n free abelian group $N \cong \mathbb{Z}^n$ and let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.2 ([13], Definition 1.21). *A polyhedral decomposition of a closed subset $R \subset N_{\mathbb{R}}$ is a locally finite covering \mathcal{P} of R by closed convex polytopes (called cells) with the property that*

- (1) *If $\sigma \in \mathcal{P}$ and $\tau \subset \sigma$ is a face, then $\tau \in \mathcal{P}$.*
- (2) *If $\sigma, \sigma' \in \mathcal{P}$, then $\sigma \cap \sigma'$ is a common face of σ, σ' .*

We say \mathcal{P} is integral if all vertices (0-dimensional cells) are contained in N .

For a polyhedral decomposition \mathcal{P} and a cell $\sigma \in \mathcal{P}$, we denote the relative interior of σ by

$$\text{Int}(\sigma) := \sigma \setminus \bigcup_{\tau \in \mathcal{P}, \tau \subsetneq \sigma} \tau.$$

Definition 2.3 ([13], Definition 1.22). *Let B be an integral affine manifold with singularities. A polyhedral decomposition of B is a collection \mathcal{P} of closed subsets of B (called cells) covering B which satisfies the following properties. If $\{v\} \in \mathcal{P}$ for some $v \in B$, then $v \notin \Delta$, and there exist a polyhedral decomposition \mathcal{P}_v of a closed subset $R_v \subset T_v B \cong \Lambda_v \otimes \mathbb{R}$, which is the closure of an open neighborhood of $0 \in T_v B$, and a continuous map $\exp_v : R_v \rightarrow B$ with $\exp_v(0) = v$ satisfying the following conditions:*

- (1) *\exp_v is a local homeomorphism onto its image, is injective on $\text{Int}(\tau)$ for all $\tau \in \mathcal{P}_v$, and is an integral affine map in some neighborhood of $0 \in R_v$.*
- (2) *For every top dimensional cell $\tilde{\sigma} \in \mathcal{P}_v$, $\exp_v(\text{Int}(\tilde{\sigma})) \cap \Delta = \emptyset$ and the restriction of \exp_v to $\text{Int}(\tilde{\sigma})$ is an integral affine map. Furthermore, $\exp_v(\tilde{\tau}) \in \mathcal{P}$ for all $\tilde{\tau} \in \mathcal{P}_v$.*
- (3) *$\sigma \in \mathcal{P}$ and $v \in \sigma$ if and only if $\sigma = \exp_v(\tilde{\sigma})$ for some $\tilde{\sigma} \in \mathcal{P}_v$ with $0 \in \tilde{\sigma}$.*
- (4) *Every $\sigma \in \mathcal{P}$ contains a point v with $\{v\} \in \mathcal{P}$.*

We say the polyhedral decomposition is toric if it satisfies the additional condition:

- (5) *For each $\sigma \in \mathcal{P}$, there is a neighborhood $U_\sigma \subset B$ of $\text{Int}(\sigma)$ and an integral affine submersion $S_\sigma : U_\sigma \rightarrow N'_{\mathbb{R}}$, where N' is a lattice of rank $\dim(B) - \dim(\sigma)$ and $S_\sigma(\sigma \cap U_\sigma) = \{0\}$.*

A polyhedral decomposition of B is called integral if all vertices are integral points of B .

The k -dimensional strata of (B, \mathcal{P}) is defined by

$$B^{(k)} := \bigcup_{\tau: \dim(\tau)=k} \tau.$$

If \mathcal{P} is a toric polyhedral decomposition \mathcal{P} , then for each $\tau \in \mathcal{P}$, one defines the fan Σ_τ as the collection of the cones

$$K_\sigma := \mathbb{R}_{\geq 0} \cdot S_\tau(\sigma),$$

where σ runs over all elements in \mathcal{P} such that $\tau \subset \sigma$ and $\text{Int}(\sigma) \cap U_\tau \neq \emptyset$. For a point $y \in \text{Int}(\tau) \setminus \Delta$, we put

$$\mathcal{Q}_\tau := \mathcal{Q}_{\tau,y} := \Lambda_y / \Lambda_{\tau,y},$$

which can be identified with the lattice N' in Condition (5) in Definition 2.3. These lattices define a sheaf $\mathcal{Q}_{\mathcal{P}}$ on B .

Definition 2.4 ([13], Definition 1.43). *Let B be an integral affine manifold with singularities and \mathcal{P} a polyhedral decomposition of B . Let $U \subset B$ be an open set. An integral piecewise linear function on U is a continuous function φ so that φ is integral affine on $U \cap \text{Int}(\sigma)$ for any top dimensional cell $\sigma \in \mathcal{P}$, and for any $y \in U \cap \text{Int}(\tau)$ (for some $\tau \in \mathcal{P}$), there exists a neighborhood V of y and $f \in \Gamma(V, \text{Aff}(B, \mathbb{Z}))$ such that $\varphi = f$ on $V \cap \text{Int}(\tau)$. We denote the sheaf of integral piecewise linear functions on B by $\mathcal{PL}_{\mathcal{P}}(B, \mathbb{Z})$.*

There is a natural inclusion $\text{Aff}(B, \mathbb{Z}) \hookrightarrow \mathcal{PL}_{\mathcal{P}}(B, \mathbb{Z})$, and we let $\mathcal{MPL}_{\mathcal{P}}$ be the quotient:

$$0 \rightarrow \text{Aff}(B, \mathbb{Z}) \rightarrow \mathcal{PL}_{\mathcal{P}}(B, \mathbb{Z}) \rightarrow \mathcal{MPL}_{\mathcal{P}} \rightarrow 0.$$

Locally, an element $\varphi \in \Gamma(B, \mathcal{MPL}_{\mathcal{P}})$ is a collection of piecewise linear functions $\{\varphi_U\}$ so that on each overlap $U \cap V$, the difference

$$\varphi_U|_{B_0} - \varphi_V|_{B_0}$$

is an integral affine function on $U \cap V \cap B_0$.

Definition 2.5 ([13], Definition 1.45). *The sheaf $\mathcal{MPL}_{\mathcal{P}}$ is called the sheaf of multi-valued piecewise linear functions of the pair (B, \mathcal{P}) .*

The sheaf $\mathcal{MPL}_{\mathcal{P}}$ also fits into the following exact sequence:

$$0 \rightarrow i_* \Lambda^* \rightarrow \mathcal{PL}_{\mathcal{P}}(B, \mathbb{Z}) / \mathbb{Z} \rightarrow \mathcal{MPL}_{\mathcal{P}} \rightarrow 0,$$

where $\Lambda \subset TB_0$ is the lattice inherited from the integral structure and $\Lambda^* \subset T^*B_0$ is the dual lattice.

Definition 2.6 ([13], Definition 1.46). *For each element $\varphi \in H^0(B, \mathcal{MPL}_{\mathcal{P}})$, its image under the connecting map $c_1 : H^0(B, \mathcal{MPL}_{\mathcal{P}}) \rightarrow H^1(B, i_* \Lambda^*)$ is called the first Chern class of φ .*

Definition 2.7 ([13], Definitions 1.47). *A section $\varphi \in H^0(B, \mathcal{MPL}_{\mathcal{P}})$ is said to be (strictly) convex if for any vertex $\{v\} \in \mathcal{P}$, there is a neighborhood $U \subset B$ of v such that there is a (strictly) convex representative φ_i .*

Definition 2.8 ([13], Definition 1.48). *A toric polyhedral decomposition \mathcal{P} is said to be regular if there exists a strictly convex multi-valued piecewise linear function $\varphi \in H^0(B, \mathcal{MPL}_{\mathcal{P}})$.*

Assumption 2.9. *All polyhedral decompositions in this paper are assumed to be regular; in particular, they are integral and toric.*

Given a regular polyhedral decomposition (\mathcal{P}, φ) , one can obtain another affine manifold with singularities \check{B} together with a regular polyhedral decomposition $(\check{\mathcal{P}}, \check{\varphi})$ by taking the dual cell of each cell in \mathcal{P} . We will not give the precise construction here but let us mention some facts about $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$. Topologically, \check{B} is same as B and their singular loci coincide. However, their affine structures and monodromies around the singular loci are dual to each other. See [13], Section 1.4 for the precise construction of $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$.

Definition 2.10 (cf. [13], Propositions 1.50 & 1.51). *The triple $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ is called the discrete Legendre transform of $(B, \mathcal{P}, \varphi)$.*

We will need the following notion later.

Definition 2.11. *A regular polyhedral decomposition \mathcal{P} is called elementary if for any cell $\sigma \in \mathcal{P}$, the dual cell $\check{\sigma}$ is an elementary simplex.*

2.2. Toric degenerations. In [13], Gross and Siebert defined a *toric degeneration* (of Calabi-Yau varieties) as a flat family $\check{p} : \check{\mathcal{X}} \rightarrow S$ such that generic fiber of \check{p} is smooth and the central fiber $\check{X}_0 := \check{p}^{-1}(0)$ is a union of toric varieties, intersecting along toric strata. By gluing the fan of each toric piece, they obtained an affine manifold with singularities \check{B} together with a polyhedral decomposition $\check{\mathcal{P}}$. When the family $\check{\mathcal{X}}$ is polarized, the resulting polyhedral decomposition is regular, so there is a strictly convex multi-valued piecewise linear function $\check{\varphi}$ on \check{B} , giving rise to the discrete Legendre transform $(B, \mathcal{P}, \varphi)$ of $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$. The important *reconstruction problem in mirror symmetry* is asking whether one can construct another toric degeneration $p : \mathcal{X} \rightarrow S$ (which acts as a mirror family) from $(B, \mathcal{P}, \varphi)$.

In this section, we review the fan construction of the algebraic spaces associated to (B, \mathcal{P}) in [13]; with a good choice of gluing data, such spaces serve as the central fibers of the toric degenerations $p : \mathcal{X} \rightarrow S$ and $\check{p} : \check{\mathcal{X}} \rightarrow S$.³

2.2.1. The fan construction. For each $\sigma \in \mathcal{P}$, the map $S_\sigma : U_\sigma \rightarrow \mathcal{Q}_\sigma$ defines a fan Σ_σ on \mathcal{Q}_σ . Let X_σ be the toric variety associated to Σ_σ . For $\tau \subset \sigma$, let

$$\Sigma_\tau(\sigma) := \{K \in \Sigma_\tau : K \supset K_\sigma\}.$$

There is a fan projection $p_{\tau\sigma} : \Sigma_\tau(\sigma) \rightarrow \Sigma_\sigma$, given by quotienting along the direction Λ_σ and an inclusion $j_{\sigma\tau} : \Sigma_\tau(\sigma) \rightarrow \Sigma_\tau$. There is a natural embedding $\iota_{\sigma\tau} : X_\sigma \rightarrow X(\Sigma_\tau(\sigma)) \subset X_\tau$ induced by the ring map

$$z^m \mapsto \begin{cases} z^m & \text{if } m \in K^\vee \cap K_\sigma^\perp \cap \mathcal{Q}_\sigma^* = ((K + \Lambda_{\sigma, \mathbb{R}})/\lambda_{\sigma, \mathbb{R}})^\vee \cap \mathcal{Q}_\sigma^*, \\ 0 & \text{otherwise.} \end{cases}$$

For $e \in \text{Hom}(\tau, \sigma)$, we define the functor $F_A : \text{Cat}(\mathcal{P}) \rightarrow \text{Sch}$ by

$$F(\tau) := X_\tau, \quad F(e) := \iota_{\tau\sigma}.$$

We can also twist the functor by certain gluing data. The barycentric subdivision $\text{Bar}(\mathcal{P})$ of \mathcal{P} defines an open covering $\mathcal{W} := \{W_\tau\}$ of B , where

$$W_\tau := \bigcup_{\substack{\sigma \in \text{Bar}(\mathcal{P}) \\ \sigma \cap \text{Int}(\tau) \neq \emptyset}} \text{Int}(\sigma).$$

For $e \in \text{Hom}(\tau, \sigma)$, we define $W_e := W_\tau \cap W_\sigma$.

Definition 2.12 ([13], Definition 2.10). *Let S be a scheme. A closed gluing data (for the fan picture) for \mathcal{P} over S is Čech 1-cocycle $s = (s_e)_{e \in \coprod_{\tau, \sigma \in \mathcal{P}} \text{Hom}(\tau, \sigma)}$ of the sheaf $\mathcal{Q}_{\mathcal{P}} \otimes_{\mathbb{Z}} \mathbb{G}_m(S)$ with respect to the cover \mathcal{W} of B . Here, $s_e \in \Gamma(W_e, \mathcal{Q}_{\mathcal{P}} \otimes_{\mathbb{Z}} \mathbb{G}_m(S)) = \mathcal{Q}_\sigma \otimes_{\mathbb{Z}} \mathbb{G}_m(S)$ for $e \in \text{Hom}(\tau, \sigma)$.*

The torus $\mathcal{Q}_\sigma \otimes_{\mathbb{Z}} \mathbb{G}_m(S)$ acts on $X_\sigma \times S$, so an element $s_e \in \mathcal{Q}_\sigma \otimes_{\mathbb{Z}} \mathbb{G}_m(S)$ gives an automorphism $s_e : X_\sigma \times S \rightarrow X_\sigma \times S$. We then obtain an s -twisted functor $F_{S,s} : \text{Cat}(\mathcal{P}) \rightarrow \text{Sch}_S$ by setting

$$F_{S,s}(\tau) := X_\tau \times S, \quad F_{S,s}(e) := (F(s) \times id_S) \circ s_e.$$

³For the cone construction, please refer to [13, Section 2.1].

We define

$$X_0(B, \mathcal{P}, s) := \varinjlim F_{S,s}.$$

In [13], Gross and Siebert introduced a special set of gluing data, which they called *open gluing data* (for the fan picture). We will not go into details here (readers are referred to [13, Definition 2.25] for the precise definition), but it is essential for $X_0(B, \mathcal{P}, s)$ to be the central fiber of some toric degeneration. Given such an open gluing data for (B, \mathcal{P}) , one can associate a closed gluing data s for the fan picture of (B, \mathcal{P}) (see [13, Proposition 2.32]). There is then an obstruction map $o : H^1(\mathcal{W}, \mathcal{Q}_{\mathcal{P}} \otimes_{\mathbb{Z}} \mathbb{C}^\times) \rightarrow H^2(B, \mathbb{C}^\times)$ such that when $o(s) = 1$, a projective scheme $X_0(B, \mathcal{P}, s)$ can be constructed. Throughout this paper, we assume that all closed gluing data are induced by open gluing data for the fan picture. We also assume that $A = \mathbb{C}$ and $S = \text{Spec}(\mathbb{C})$. One of the main result in [13] is the following

Theorem 2.13 ([13], Theorem 5.2). *Suppose (B, \mathcal{P}) is positive and simple and s satisfies the (LC) condition in [13, Proposition 4.25]. Then there exists a log structure on $X_0(B, \mathcal{P}, s)$ and a morphism $X_0(B, \mathcal{P}, s)^\dagger \rightarrow \text{Spec}(\mathbb{C})^\dagger$ which is log smooth away from a subset $Z \subset X_0(B, \mathcal{P}, s)$ of codimension 2.*

Remark 2.14. *When (B, \mathcal{P}) (and hence $(\check{B}, \check{\mathcal{P}})$) is positive and simple, $X_0(B, \mathcal{P}, s)$ and $\check{X}_0(\check{B}, \check{\mathcal{P}}, \check{s})$ are mirror to each other as log Calabi-Yau spaces in an appropriate sense; see [13, Section 5.3] for details.*

As aforementioned, the reconstruction problem in mirror symmetry is to construct a polarized toric degeneration $p : \mathcal{X} \rightarrow S$ whose central fiber is given by $X_0(B, \mathcal{P}, s)$ for some open gluing data s over $\text{Spec}(\mathbb{C})$. This was solved by Gross and Siebert in [15] using *logarithmic deformation theory* and a key combinatorial object called *scattering diagram* first introduced by Kontsevich and Soibelman in [20] (who first solved the reconstruction problem in dimension 2, but over non-Archimedean fields). When (B, \mathcal{P}) is *positive* and *simple*, Gross and Siebert proved that $X_0(B, \mathcal{P}, s)$ is always smoothable by writing down an explicit toric degeneration.

In [14], Gross and Siebert studied a specific type of logarithmic deformations, called *divisoral deformations*. Similar to the classical deformation theory of schemes, the first order divisoral deformations of $X_0(B, \mathcal{P}, s)$ are parametrized by a first cohomology group $H^1(X_0(B, \mathcal{P}, s), j_* \Theta_{X_0(B, \mathcal{P}, s)^\dagger / \mathbb{C}^\dagger})$, while the obstructions lie in the second cohomology group $H^2(X_0(B, \mathcal{P}, s), j_* \Theta_{X_0(B, \mathcal{P}, s)^\dagger / \mathbb{C}^\dagger})$ (see [14, Theorem 2.11]); here $\Theta_{X_0(B, \mathcal{P}, s)^\dagger / \mathbb{C}^\dagger}$ is the sheaf of *logarithmic tangent vectors* of the log scheme $X_0(B, \mathcal{P}, s)^\dagger$ and $j : X_0(B, \mathcal{P}, s) \setminus Z \hookrightarrow X_0(B, \mathcal{P}, s)$ is the inclusion map. However, they did not prove existence of smoothings along this line of thought.

Very recently, the first two authors of this paper and Leung [1], by using gluing of local differential graded Batalin-Vilkovisky (dgBV) algebras and partly motivated by [14], developed an algebraic framework to prove existence of formal smoothings for singular Calabi-Yau varieties with prescribed local models. This covers the log smooth case studied by Friedman [9] and Kawamata-Namikawa [17] as well as the maximally degenerate case studied by Kontsevich-Soibelman [20] and Gross-Siebert [15]. More importantly, this approach provides a singular version of the classical Bogomolov-Tian-Todorov theory and bypasses the complicated scattering diagrams.

Our theory was subsequently applied by Felten-Filip-Ruddat in [8] to produce smoothings of a very general class of varieties called toriodal crossing spaces.⁴ Such an algebraic framework should be applicable in a variety of settings, e.g., it was applied to smoothing of pairs in [2]. In this

⁴Applying results in Ruddat-Siebert [23], they were able to prove that the smoothings are actually analytic.

paper, which can be regarded as a sequel to [2], we show how this approach can lead to an explicit, combinatorial construction of smoothable pairs in dimension 2.

3. TROPICAL LAGRANGIAN MULTI-SECTIONS

In this section, we introduce the notion of a tropical Lagrangian multi-section when $\dim(B) = 2$.⁵ These tropical objects should be viewed as limiting versions of Lagrangian multi-sections of a Lagrangian torus fibration (or SYZ fibration). In [3], the first and third authors of this paper considered the case of a semi-flat Lagrangian torus fibration $X(B) \rightarrow B$, where a Lagrangian multi-section can be described by an unbranched covering map $\pi : L \rightarrow B$ together with a Lagrangian immersion into the symplectic manifold $X(B)$. However, in general (e.g., when B is simply connected), the covering map $\pi : L \rightarrow B$ would have non-empty branched locus. We begin by describing what kind of covering maps is allowed.

Definition 3.1. *Let B be a 2-dimensional integral affine manifold with singularities equipped with a polyhedral decomposition \mathcal{P} . Let L be a topological manifold. A r -fold topological covering map $\pi : L \rightarrow B$ with branched locus S is called admissible if*

- (1) $S \subset B^{(0)}$, and
- (2) $\pi^{-1}(B_0 \setminus S)$ is an integral affine manifold such that $\pi|_{\pi^{-1}(B_0 \setminus S)} : \pi^{-1}(B_0 \setminus S) \rightarrow B_0 \setminus S$ is an integral affine map.

An admissible covering map $\pi : L \rightarrow B$ is said to have simple branching if it satisfies the following extra condition:

- (3) For any $x \in B$, there exists a neighborhood $U \subset B$ of x such that the preimage $\pi^{-1}(U)$ can be written as

$$U' \amalg \prod_{i=1}^{r-2} U'_i$$

for some open subsets $U', U'_1, \dots, U'_{r-2} \subset L$ so that $\pi|_{U'} : U' \rightarrow U$ is a (possibly branched) 2-fold covering map.

Given an admissible covering map $\pi : L \rightarrow B$, the domain L is naturally an integral affine manifold with singularities and the singular locus is given by $S' \amalg \pi^{-1}(\Delta)$, where $S' \subset L$ is the ramification locus of $\pi : L \rightarrow B$. We need to distinguish these two singular sets combinatorially.

Definition 3.2. *Let B be a 2-dimensional integral affine manifold with singularities equipped with a polyhedral decomposition \mathcal{P} . Let $\pi : L \rightarrow B$ be an admissible covering map. A polyhedral decomposition of $\pi : L \rightarrow B$ is a collection \mathcal{P}_π of closed subsets of L (also called cells) covering L so that*

- (1) $\pi(\sigma') \in \mathcal{P}$ for all $\sigma' \in \mathcal{P}_\pi$;
- (2) for any $\sigma \in \mathcal{P}$, we have

$$\pi^{-1}(\sigma) = \bigcup_{\sigma' \in \mathcal{P}_\pi : \pi(\sigma') = \sigma} \sigma'$$

and if we define the relative interior of σ' to be

$$\text{Int}(\sigma') := \pi^{-1}(\text{Int}(\sigma)) \cap \sigma',$$

then $\pi|_{\text{Int}(\sigma')} : \text{Int}(\sigma') \rightarrow \text{Int}(\sigma)$ is a homeomorphism; and

- (3) if $\pi(\sigma'_1) = \pi(\sigma'_2)$ and $\sigma'_1 \cap \sigma'_2 \neq \emptyset$, then $\sigma'_1 \cap \sigma'_2 \subset S'$.

⁵From this point on, we will always assume that $\dim(B) = 2$.

We define

$$\dim(\sigma') := \dim(\sigma)$$

for all $\sigma' \in \mathcal{P}_\pi$ with $\pi(\sigma') = \sigma \in \mathcal{P}$.

Clearly, if \mathcal{P}_π is a polyhedral decomposition of an admissible covering map $\pi : L \rightarrow B$, then

$$\pi(\mathcal{P}_\pi) := \{\pi(\sigma') \mid \sigma' \in \mathcal{P}_\pi\} = \mathcal{P}.$$

We denote the k -dimensional strata of (L, \mathcal{P}_π) by

$$L^{(k)} := \bigcup_{\sigma' : \dim(\sigma')=k} \sigma'.$$

Remark 3.3. *The vertex of the polyhedral decomposition \mathcal{P}_π of $\pi : L \rightarrow B$ may lie in S' but never in $\pi^{-1}(\Delta)$. Moreover, if $\sigma \in \mathcal{P}$ is a cell of dimension at least 1, then Conditions (2) and (3) in Definition 3.2 imply that*

$$\pi^{-1}(\text{Int}(\sigma)) = \coprod_{\sigma' \in \mathcal{P}_\pi : \pi(\sigma')=\sigma} \text{Int}(\sigma')$$

because S' lies in $L^{(0)}$.

Example 3.4. *Let B be the boundary of a 3-simplex Ξ . It carries a polyhedral decomposition \mathcal{P} inherited from the simplicial structure of Ξ . Let L be a 2-torus. There is a branched 2-fold covering map $\pi : L \rightarrow B$, which branch over B along four points. Figure 2 shows a polyhedral decomposition \mathcal{P}_π of $\pi : L \rightarrow B$.*

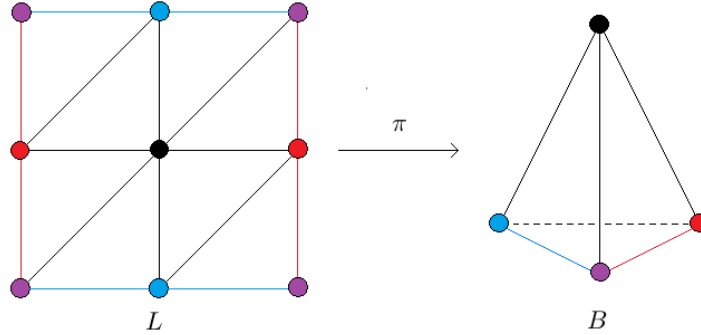


FIGURE 2

The largest square represents the 2-torus L and the colored vertices on B (resp. on L) are the branched (resp. ramification) points of the 2-fold covering map $\pi : L \rightarrow B$. The polyhedral decomposition \mathcal{P}_π is given by pulling back the cells in \mathcal{P} to L .

As the domain L is an integral affine manifold with singularities, we can therefore define the sheaf of integral affine functions $\mathcal{A}ff(L, \mathbb{Z})$ on L as before. However, since the singular locus S' lies in $L^{(0)}$, we need to clarify what it means by a piecewise linear function though it is similar to Definition 2.4:

Definition 3.5. Let B be an integral affine manifold with singularities equipped with a polyhedral decomposition \mathcal{P} . Let $\pi : L \rightarrow B$ be an admissible covering map equipped with a polyhedral decomposition \mathcal{P}_π . Let \tilde{U} be an open subset of L . A piecewise linear function on \tilde{U} is a continuous function φ' which is affine linear on $\text{Int}(\sigma')$ for any maximal cell $\sigma' \in \mathcal{P}_\pi$ and satisfies the following property: for any $y' \in \tilde{U}$ and $y' \in \text{Int}(\tau')$ (for some $\tau' \in \mathcal{P}_\pi$), there is a neighborhood \tilde{V} of y' and $f' \in \Gamma(\tilde{V}, \text{Aff}(L, \mathbb{Z}))$ such that

$$\varphi'|_{\tilde{V} \cap \tau'} = f'|_{\tilde{V} \cap \tau'}.$$

We denote the sheaf of piecewise linear functions on L by $\mathcal{PL}_{\mathcal{P}_\pi}(L, \mathbb{Z})$.

Definition 3.6. The sheaf of multi-valued piecewise linear functions $\mathcal{MPL}_{\mathcal{P}_\pi}$ on L is defined as the quotient

$$0 \rightarrow \text{Aff}(L, \mathbb{Z}) \rightarrow \mathcal{PL}_{\mathcal{P}_\pi}(L, \mathbb{Z}) \rightarrow \mathcal{MPL}_{\mathcal{P}_\pi} \rightarrow 0.$$

We are now ready to define the main object to be studied in this paper.

Definition 3.7. Let B be a 2-dimensional integral affine manifold with singularities equipped with a polyhedral decomposition \mathcal{P} . A tropical Lagrangian multi-section of rank r is a quadruple $\mathbb{L} := (L, \pi, \mathcal{P}_\pi, \varphi')$, where

- 1) L is a topological manifold and $\pi : L \rightarrow B$ is an admissible r -fold covering map,
- 2) \mathcal{P}_π is a polyhedral decomposition of $\pi : L \rightarrow B$, and
- 3) φ' is a multi-valued piecewise linear function on L .

4. A LOCAL MODEL AROUND THE RAMIFICATION LOCUS

In this section, we prescribe a local model for \mathbb{L} around each point in the ramification locus of $\pi : L \rightarrow B$; such a model is motivated by the previous work [26] of the third author. We also give an explicit construction of a certain class of rank 2 tropical Lagrangian multi-sections over B .

Definition 4.1. Let B be an integral affine manifold with singularities equipped with a polyhedral decomposition \mathcal{P} , and $\pi : L \rightarrow B$ an admissible covering map. A branched point $v \in S \subset B$ is called standard if there is an isomorphism $\Sigma_v \cong \Sigma_{\mathbb{P}^2}$ between the fan Σ_v and that of \mathbb{P}^2 .

For a standard vertex $v \in S$, we construct a fan $\Sigma_{v'}$, a 2-fold covering map $\pi_* : |\Sigma_{v'}| \rightarrow |\Sigma_v|$ with a unique branched point at 0 and a piecewise linear function $\varphi_{v'} : |\Sigma_{v'}| \rightarrow \mathbb{R}$ on $|\Sigma_{v'}|$ as follows.

Let $K_{\sigma_0}, K_{\sigma_1}, K_{\sigma_2} \in \Sigma_v$ be the top dimensional cones of Σ_v which correspond to the cones $\sigma_0, \sigma_1, \sigma_2 \in \Sigma_{\mathbb{P}^2}$ (see Figure 1) respectively. Let $K_{\sigma_i}^\pm$ be two copies of K_{σ_i} . Let ρ_j, ρ_k be the rays spanning K_{σ_i} and ρ_j^\pm, ρ_k^\pm be that for $K_{\sigma_i}^\pm$, for $i, j, k = 0, 1, 2$ being distinct. We glue $K_{\sigma_0}^\pm$ with $K_{\sigma_1}^\mp$ and $K_{\sigma_2}^\mp$ by identifying ρ_1^\pm with ρ_1^\mp and ρ_2^\pm with ρ_2^\mp respectively, and glue $K_{\sigma_1}^\pm$ with $K_{\sigma_2}^\mp$ by identifying ρ_0^\pm with ρ_0^\mp . Then the fan $\Sigma_{v'}$ is given by

$$\{K_{\sigma_i}^\pm, \rho_i^\pm, 0 \mid i = 0, 1, 2\}.$$

The projection $\pi_* : |\Sigma_{v'}| \rightarrow |\Sigma_v|$ is defined by $K_{\sigma_i}^\pm \rightarrow K_{\sigma_i}$.

On the fan $\Sigma_{v'}$, one can define a piecewise linear function $\varphi_{v'}$ by setting

$$\varphi_{v'} := \begin{cases} (n-m)\xi_1 & \text{on } K_{\sigma_0}^+, \\ (n-m)\xi_2 & \text{on } K_{\sigma_0}^-, \\ n\xi_1 & \text{on } K_{\sigma_1}^+, \\ n\xi_1 + (n-m)\xi_2 & \text{on } K_{\sigma_1}^-, \\ (n-m)\xi_1 + n\xi_2 & \text{on } K_{\sigma_2}^+, \\ n\xi_2 & \text{on } K_{\sigma_2}^-, \end{cases}$$

for some $m, n \in \mathbb{Z}$ with $m \neq n$. Here, ξ_1, ξ_2 are affine coordinates on $|\Sigma_v| \cong \mathbb{R}^2$. This gives a tropical Lagrangian multi-section $(|\Sigma_{v'}|, \pi_*, \Sigma_{v'}, \varphi_{v'})$ of the smooth affine manifold $|\Sigma_v|$ with polyhedral decomposition given by the fan Σ_v .

Recall that when π has simple branching, for any branched point $v \in S$, one can choose a neighborhood U of v such that

$$\pi^{-1}(U) = U' \amalg \prod_{i=1}^{r-2} U'_i,$$

for some open subsets $U', U'_1, \dots, U'_{r-2} \subset L$ and $\pi|_{U'} : U' \rightarrow U$ is a branched 2-fold covering map. Let \mathcal{P}_π be a polyhedral decomposition of $\pi : L \rightarrow B$. Denote by $\mathcal{P}_{v'}$ the collection of all cells $\sigma' \in \mathcal{P}_\pi$ such that $v' \in \sigma'$. Write

$$\pi^{-1}(\sigma \cap U) \cap U' = (\sigma^+ \cup \sigma^-) \cap U'$$

for $\sigma \in \mathcal{P}$ and $\sigma^\pm \in \mathcal{P}_{v'}$.

Definition 4.2. *Let B be a 2-dimensional integral affine manifold with singularities, \mathcal{P} a polyhedral decomposition of B . A tropical Lagrangian multi-section $\mathbb{L} := (L, \pi, \mathcal{P}_\pi, \varphi')$ is said to be of class \mathcal{S} , denoted as $\mathbb{L} \in \mathcal{S}$, if*

- (1) $\pi : L \rightarrow B$ has simple branching and every branched point is standard;
- (2) for any vertex $v \in S$, there exists a neighborhood $U \subset B$ of v and two integral affine embeddings $f' : U' \rightarrow |\Sigma_{v'}|$, $f : U \rightarrow |\Sigma_v|$ such that $f'(\sigma^\pm \cap U') \subset K_\sigma^\pm$ for all $\sigma \in \mathcal{P}$ with $v \in \sigma$ and $\pi_* \circ f' = f \circ \pi$; and
- (3) for each vertex $v \in S$, φ' can be represented by $\varphi_{v'}$ on U' , i.e., if $\varphi'_{U'}$ is a representative of φ' on U' , then

$$\varphi'_{U'} - \varphi_{v'}|_{f'(U')} \circ f' \in \mathcal{A}ff(U', \mathbb{Z}),$$

for some $m(v'), n(v') \in \mathbb{Z}$ defining $\varphi_{v'}$.

For each $m, n \in \mathbb{Z}$ with $m \neq n$, there is a subclass $\mathcal{S}_{m,n}$ of \mathcal{S} defined as

$$\mathcal{S}_{m,n} := \{\mathbb{L} \in \mathcal{S} : m(v'), n(v') \in \{m, n\} \text{ for all } v' \in S'\}.$$

This collection of tropical Lagrangian multi-sections will be our main object of study in Section 6.

Remark 4.3. *We may also use the local model given by*

$$\varphi'_{v'} = \begin{cases} 0 & \text{on } K_{\sigma_0}^\pm, \\ m\xi_1 & \text{on } K_{\sigma_1}^+, \\ n\xi_1 & \text{on } K_{\sigma_1}^-, \\ n\xi_2 & \text{on } K_{\sigma_2}^+, \\ m\xi_2 & \text{on } K_{\sigma_2}^-. \end{cases}$$

This local model is more related to metric structures while the previous one is more related to equivariant structures; see [26] for a discussion in the special case $m = 2, n = 1$. Nevertheless, we will see later that $\varphi'_{v'}$ and $\varphi_{v'}$ actually give rise to the same locally free sheaf on \mathbb{P}^2 .

5. CONSTRUCTION OF $\mathcal{E}_0(\mathbb{L})$ FOR $\mathbb{L} \in \mathcal{S}$

Let (B, \mathcal{P}) be a 2-dimensional integral affine manifold with singularities equipped with a regular polyhedral decomposition \mathcal{P} . Let \mathbb{L} be a rank r tropical Lagrangian multi-section of class \mathcal{S} as in Definition 4.2. The goal of this section is to construct a locally free sheaf $\mathcal{E}_0(\mathbb{L})$ of rank r on the scheme $X_0(B, \mathcal{P}, s)$ associated to \mathbb{L} .

First of all, for any vertex $\{v\} \in \mathcal{P}$, we need to construct a locally free sheaf $\mathcal{E}(\check{v})$ over the toric variety X_v . Let $\sigma_1, \dots, \sigma_l$ be the top dimensional cells that contain v and $U_{\sigma_i} \subset X_v$ be the toric affine chart corresponding to σ_i . Then $\{U_{\sigma_i}\}$ forms an open covering of X_v . Suppose the vertex $v \notin S$. Then the preimage of v consists of r distinct points v'_k , for $k = 1, \dots, r$. Hence we obtain r piecewise linear functions $\varphi_{v'_k} : |\Sigma_{v'_k}| \rightarrow \mathbb{R}$, which correspond to r equivariant line bundles $\mathcal{L}(v'_k)$ on X_v . In this case, we put

$$\mathcal{E}(v) := \bigoplus_{k=1}^r \mathcal{L}(v'_k).$$

Suppose $v \in S$. Then by our assumption, there are precisely three top dimensional cells $\sigma_0, \sigma_1, \sigma_2$ containing v . In this case, $X_v \cong \mathbb{P}^2$. Let $v' \in S'$ be the unique ramification point such that $\pi(v') = v$. In a neighborhood U of v , write

$$\pi^{-1}(\sigma_i \cap U) := ((\sigma_i^+ \cup \sigma_i^-) \cap U') \amalg \prod_{k=1}^{r-2} \sigma_i^{(k)} \cap U'_k$$

with $\sigma_i^+ \cap \sigma_i^- = \{v'\}$ being the ramification point. As $\varphi|_{U'}$ can be represented by $\varphi_{v'}$, by restricting to $\sigma_i^\pm, \sigma_i^{(k)}$, we obtain r integral affine functions. Hence they define r equivariant line bundles $\mathcal{L}_i^\pm, \mathcal{L}_i^{(k)}$, $k = 1, \dots, r-2$, on the affine chart \check{X}_i , where

$$\begin{aligned} \mathcal{L}_0^+ &= \mathcal{O}((n-m)D_1)|_{U_0}, & \mathcal{L}_0^- &= \mathcal{O}((n-m)D_2)|_{U_0}, \\ \mathcal{L}_1^+ &= \mathcal{O}(nD_0)|_{U_1}, & \mathcal{L}_1^- &= \mathcal{O}((m-n)D_2 + (2n-m)D_0)|_{U_2}, \\ \mathcal{L}_2^+ &= \mathcal{O}(nD_0)|_{U_2}, & \mathcal{L}_2^- &= \mathcal{O}((m-n)D_1 + (2n-m)D_0)|_{U_2}. \end{aligned}$$

Here, $U_i \subset \mathbb{P}^2$ are the affine charts corresponding to the cones K_{σ_i} and D_k is the divisor corresponding to the ray ρ_k . In this case, we set

$$\mathcal{E}_i := (\mathcal{L}_i^+ \oplus \mathcal{L}_i^-) \oplus \bigoplus_{k=1}^{r-2} \mathcal{L}_i^{(k)},$$

which is a rank r vector bundle on U_{σ_i} .

Next, we try to glue the bundles \mathcal{E}_i 's together. First of all, the line bundles $\{\mathcal{L}_i^{(k)}\}_{i=0,1,2}$ naturally glue together to form an equivariant line bundle $\mathcal{L}(v'_k)$ on X_v as before.

However, in the case $v \in S$, the rank 2 bundles $\{\mathcal{L}_i^+ \oplus \mathcal{L}_i^-\}_{i=0,1,2}$ cannot be glued equivariantly. This is because when we try to glue \mathcal{L}_i^\pm to \mathcal{L}_j^\mp equivariantly, the gluing data consists of two diagonal matrices and one off-diagonal matrix, and so the cocycle condition fails to hold. More precisely,

the equivariant structure on each $\mathcal{L}_i^+ \oplus \mathcal{L}_i^-$ is given by the minus of the slopes of $\varphi_{v'}$ on K_{σ_i} :

$$(2) \quad \begin{aligned} (\lambda_1, \lambda_2) \cdot (w_0^1, w_0^2, v_0^+, v_0^-) &= (\lambda_1 w_0^1, \lambda_2 w_0^2, \lambda_1^{m-n} v_0^+, \lambda_2^{m-n} v_0^-), \\ (\lambda_1, \lambda_2) \cdot (w_1^0, w_1^2, v_1^-, v_1^+) &= (\lambda_1^{-1} w_1^0, \lambda_1^{-1} \lambda_2 w_1^2, \lambda_1^{-n} v_1^-, \lambda_1^{-n} \lambda_2^{m-n} v_1^+), \\ (\lambda_1, \lambda_2) \cdot (w_2^0, w_2^1, v_2^+, v_2^-) &= (\lambda_2^{-1} w_2^0, \lambda_1 \lambda_2^{-1} w_2^1, \lambda_2^{-n} v_2^+, \lambda_1^{m-n} \lambda_2^{-n} v_2^-); \end{aligned}$$

here v_i^\pm are fiber coordinates of \mathcal{L}_i^\pm . According to the gluing of $|\Sigma_{v'}| \setminus \{v'\}$, one can write down the naive transition functions

$$\tau_{10}^{sf} := \begin{pmatrix} \frac{a_0}{(w_0^1)^m} & 0 \\ 0 & \frac{b_0}{(w_0^1)^n} \end{pmatrix}, \tau_{21}^{sf} := \begin{pmatrix} \frac{b_1}{(w_1^2)^n} & 0 \\ 0 & \frac{a_1}{(w_1^2)^m} \end{pmatrix}, \tau_{02}^{sf} := \begin{pmatrix} 0 & \frac{b_2}{(w_2^0)^n} \\ \frac{a_2}{(w_2^0)^m} & 0 \end{pmatrix},$$

where $w_i^j := \frac{\zeta^j}{\zeta^i}$ are inhomogeneous coordinates of a point $[\zeta^0 : \zeta^1 : \zeta^2] \in \mathbb{P}^2$ and a_i, b_i are arbitrary nonzero constants. It is clear that they do not compose to the identity.

To correct the gluing, we introduce three automorphisms. For $m > n$, we consider

$$\begin{aligned} \Theta_{10} &:= I + \begin{pmatrix} 0 & 0 \\ -a_0 b_1 a_2 \left(\frac{w_0^2}{w_0^1}\right)^{m-n} & 0 \end{pmatrix} \in \text{Aut}(\mathcal{O}|_{U_{10}} \oplus \mathcal{O}|_{U_{10}}), \\ \Theta_{21} &:= I + \begin{pmatrix} 0 & b_0^{-1} b_1^{-1} a_2^{-1} \left(\frac{w_1^0}{w_1^2}\right)^{m-n} \\ 0 & 0 \end{pmatrix} \in \text{Aut}(\mathcal{O}(mD_0)|_{U_{21}} \oplus \mathcal{O}(nD_0)|_{U_{21}}), \\ \Theta_{02} &:= I + \begin{pmatrix} 0 & 0 \\ a_0^{-1} b_1^{-1} b_2^{-1} \left(\frac{w_2^1}{w_2^0}\right)^{m-n} & 0 \end{pmatrix} \in \text{Aut}(\mathcal{O}(mD_0)|_{U_{02}} \oplus \mathcal{O}(nD_0)|_{U_{02}}), \end{aligned}$$

while for $m < n$, we consider

$$\begin{aligned} \Theta_{10} &:= I + \begin{pmatrix} 0 & -b_0 a_1 b_2 \left(\frac{w_0^2}{w_0^1}\right)^{n-m} \\ 0 & 0 \end{pmatrix} \in \text{Aut}(\mathcal{O}|_{U_{10}} \oplus \mathcal{O}|_{U_{10}}), \\ \Theta_{21} &:= I + \begin{pmatrix} 0 & 0 \\ a_0^{-1} a_1^{-1} b_2^{-1} \left(\frac{w_1^0}{w_1^2}\right)^{n-m} & 0 \end{pmatrix} \in \text{Aut}(\mathcal{O}(mD_0)|_{U_{21}} \oplus \mathcal{O}(nD_0)|_{U_{21}}), \\ \Theta_{02} &:= I + \begin{pmatrix} 0 & b_0^{-1} a_1^{-1} a_2^{-1} \left(\frac{w_2^1}{w_2^0}\right)^{n-m} \\ 0 & 0 \end{pmatrix} \in \text{Aut}(\mathcal{O}(mD_0)|_{U_{02}} \oplus \mathcal{O}(nD_0)|_{U_{02}}). \end{aligned}$$

The factors Θ_{ij} are written in terms of the frame of \mathcal{E}_j on U_j . Let

$$\tau_{ij} := \tau_{ij}^{sf} \Theta_{ij}.$$

Then a straightforward computation gives the following

Proposition 5.1. *If we impose the condition that $\prod_i a_i b_i = -1$, then*

$$\tau_{02} \tau_{21} \tau_{10} = I.$$

Moreover, the equivariant structure defined by (2) can be extended.

Hence we obtain an equivariant rank 2 bundle $E_{m,n}$ on $X_v \cong \mathbb{P}^2$ (for $v \in S$). The mysterious constants a_i, b_i 's are indeed irrelevant to the holomorphic structure.

Lemma 5.2. *The holomorphic structure of $E_{m,n}$ is independent of a_i, b_i 's as long as $\prod_i a_i b_i = -1$.*

Proof. We will only consider the case $m > n$; the proof for $n > m$ is similar. Let τ'_{ij} 's be the transition functions of $E_{m,n}$ with $a_i = -1, b_i = 1$, for all $i = 0, 1, 2$. We define f by

$$\begin{aligned} f|_{U_0} &:= f_0 := \begin{pmatrix} 1 & 0 \\ 0 & a_0 b_1 a_2 \end{pmatrix}, \\ f|_{U_1} &:= f_1 := \begin{pmatrix} -a_0 & 0 \\ 0 & -a_1^{-1} b_2^{-1} \end{pmatrix}, \\ f|_{U_2} &:= f_2 := \begin{pmatrix} a_0 b_1 & 0 \\ 0 & b_2^{-1} \end{pmatrix}. \end{aligned}$$

Using $\prod_i a_i b_i = -1$, one can check that

$$\tau_{02} f_2 = f_0 \tau'_{02}, \quad \tau_{21} f_1 = f_2 \tau'_{21}, \quad \tau_{10} f_0 = f_1 \tau'_{10}.$$

Hence f defines an isomorphism. \square

From now on, we will take $a_i = -1, b_i = 1$, for all $i = 0, 1, 2$. Then the corresponding transition functions are given by

$$(3) \quad \tau'_{10} := \begin{pmatrix} -\frac{1}{(w_0^1)^m} & 0 \\ -\frac{(w_0^2)^{m-n}}{(w_0^1)^m} & \frac{1}{(w_0^1)^n} \end{pmatrix}, \tau'_{21} := \begin{pmatrix} \frac{1}{(w_1^2)^n} & -\frac{(w_1^0)^{m-n}}{(w_1^2)^m} \\ 0 & -\frac{1}{(w_1^1)^m} \end{pmatrix}, \tau'_{02} := \begin{pmatrix} -\frac{(w_2^1)^{m-n}}{(w_2^0)^m} & \frac{1}{(w_2^0)^n} \\ -\frac{(w_2^1)^{m-n}}{(w_2^0)^m} & 0 \end{pmatrix},$$

for $m > n$ and

$$\tau'_{10} := \begin{pmatrix} -\frac{1}{(w_0^1)^m} & -\frac{(w_0^2)^{n-m}}{(w_0^1)^n} \\ 0 & \frac{1}{(w_0^1)^n} \end{pmatrix}, \tau'_{21} := \begin{pmatrix} -\frac{1}{(w_1^2)^n} & 0 \\ -\frac{(w_1^0)^{n-m}}{(w_1^1)^n} & \frac{1}{(w_1^1)^m} \end{pmatrix}, \tau'_{02} := \begin{pmatrix} 0 & \frac{1}{(w_2^0)^n} \\ \frac{1}{(w_2^0)^m} & -\frac{(w_2^1)^{m-n}}{(w_2^0)^m} \end{pmatrix},$$

for $n > m$.

Proposition 5.3. *For any $m, n \in \mathbb{Z}$ with $m \neq n$, we have $E_{m,n} \cong E_{n,m}$ and $E_{m,n}^* \cong E_{-m,-n}$.*

Proof. Let $\{\tau_{ij}^{m,n}\}$ be the transition functions of $E_{m,n}$. Put

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then we have

$$\tau_{10}^{m,n} J = J^{-1} \tau_{10}^{n,m}, \quad \tau_{21}^{m,n} J^{-1} = J \tau_{21}^{n,m}, \quad \tau_{02}^{m,n} J = J \tau_{02}^{n,m}.$$

Hence we have $E_{m,n} \cong E_{n,m}$. It is immediate that $(\tau_{ij}^{m,n})^{-1} = (\tau_{ij}^{-m,-n})^t$. Thus $E_{m,n}^* \cong E_{-m,-n}$. \square

Remark 5.4. *As aforementioned, one can as well use the local model defined in Remark 4.3. Then by applying the same technique of reconstruction as in [26], one can construct a rank 2 bundle $E'_{m,n}$ on $X_v \cong \mathbb{P}^2$. Using the homomorphism defined in [26, Theorem 3.9], we see that $E'_{m,n} \cong E_{m,n}$. See also [26] for the relation of Θ_{ij} 's and the wall-crossing phenomenon.*

Before constructing the sheaf $\mathcal{E}_0(\mathbb{L})$, we first prove the simplicity of the rank 2 bundle $E_{m,n}$. This will be crucial in the study of smoothability of the pair $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$ in Section 6. Recall that a locally free sheaf \mathcal{E} on a scheme X is called *simple* if $H^0(X, \text{End}(\mathcal{E})) = \mathbb{C}$, or equivalently, $H^0(X, \text{End}_0(\mathcal{E})) = 0$ where End_0 denotes the sheaf of traceless endomorphisms.

For this purpose, we compute the Chern classes of $E_{m,n}$. The piecewise linear function $\varphi_{m,n}$ allows us to obtain the equivariant Chern classes as piecewise polynomial functions on $|\Sigma_{\mathbb{P}^2}|$ (see [22, Section 3.2]), namely,

$$c_1^{(\mathbb{C}^\times)^2}(E_{m,n}) = \begin{cases} (n-m)(\xi_1 + \xi_2) & \text{on } \sigma_0, \\ 2n\xi_1 + (n-m)\xi_2 & \text{on } \sigma_1, \\ (n-m)\xi_1 + 2n\xi_2 & \text{on } \sigma_2. \end{cases}$$

$$c_2^{(\mathbb{C}^\times)^2}(E_{m,n}) = \begin{cases} (n-m)^2\xi_1\xi_2 & \text{on } \sigma_0, \\ n^2\xi_1^2 + n(n-m)\xi_1\xi_2 & \text{on } \sigma_1, \\ n(n-m)\xi_1\xi_2 + n^2\xi_2^2 & \text{on } \sigma_2. \end{cases}$$

Since the equivariant cohomology $H_{(\mathbb{C}^\times)^2}^\bullet(\mathbb{P}^2; \mathbb{Z})$ of \mathbb{P}^2 is given by $\mathbb{Z}[t_0, t_1, t_2]/(t_0t_1t_2)$, where t_i 's are piecewise linear functions on $|\Sigma_{\mathbb{P}^2}|$ so that $t_i(v_j) = \delta_{ij}$, and the forgetful map $H_{(\mathbb{C}^\times)^2}^\bullet(\mathbb{P}^2; \mathbb{Z}) \rightarrow H^\bullet(\mathbb{P}^2; \mathbb{Z}) \cong \mathbb{Z}[H]/(H^3)$ is given by mapping all the t_i 's to the hyperplane class H , we have

$$c(E_{m,n}) = 1 + (m+n)H + (m^2 + n^2 - mn)H^2.$$

Proposition 5.5. *For all $m, n \in \mathbb{Z}$ with $m \neq n$, the bundle $E_{m,n}$ is stable and hence simple.*

Proof. A straightforward calculation shows that we have the equality

$$c_2 - 4c_1^2 = -3(m-n)^2.$$

Hence $c_2 - 4c_1^2 < 0$ and $c_2 - 4c_1^2 \neq -4$. These conditions are equivalent to stability of rank 2 bundles on \mathbb{P}^2 . See [24]. \square

Let us go back to the construction of $\mathcal{E}_0(\mathbb{L})$. The fan structure around $v \in S$ defines a toric isomorphism $X_v \cong \mathbb{P}^2$. Denote the corresponding bundle on X_v by $\mathcal{E}(v')$ and put

$$\mathcal{E}(v) := \mathcal{E}(v') \oplus \bigoplus_{k=1}^{r-2} \mathcal{L}(v'_k),$$

which is a rank r bundle on \mathbb{P}^2 . In this way, the tropical Lagrangian multi-section \mathbb{L} defines a locally free sheaf $\mathcal{E}(\sigma)$ on X_σ for each $\sigma \in \mathcal{P}$. Now we fix a closed gluing data $s = (s_e)$ for the fan picture. For a morphism $e : \tau \rightarrow \sigma$, we write the corresponding scheme morphism $F_s(e) : X_\sigma \rightarrow X_\tau$ as $\iota_{\sigma\tau,s} : X_\sigma \rightarrow X_\tau$. In order to obtain a consistent gluing, we need to find a set of isomorphisms $\{g_{\sigma\tau,s} : \mathcal{E}(\sigma) \rightarrow \iota_{\sigma\tau,s}^* \mathcal{E}(\tau)\}_{\tau \subset \sigma}$ such that

$$(\iota_{\sigma\tau,s}^* g_{\tau\omega,s}) \circ g_{\sigma\tau,s} = g_{\sigma\omega,s}$$

for each triple $\omega \subset \tau \subset \sigma$.

To do this, note that by our definition of tropical Lagrangian multi-section, for $\diamond, \blacklozenge \in \{\omega, \tau, \sigma\}$ and lifts $\diamond', \blacklozenge' \in \mathcal{P}_\pi$ such that $\diamond' \subset \blacklozenge'$, there are piecewise linear functions $\varphi'_{\blacklozenge'}, \varphi'_{\diamond'}$ such that

$$f'_{\diamond', \blacklozenge'} := \varphi'_{\diamond'} - \varphi'_{\blacklozenge'}$$

is an affine function, whenever defined. Since the intersection of any two charts always avoids S , by composing with π^{-1} , we can regard $f'_{\diamond', \blacklozenge'}$ as a local affine function on B . We assume that for all $\diamond \in \{\omega, \tau, \sigma\}$, $\varphi'_{\diamond'}$ is induced from the pullback of some piecewise linear functions $\varphi_{\diamond'}$ on the fan $\Sigma_{\diamond'}$. Then we have

$$f'_{\diamond', \blacklozenge'} = S_{\diamond'}^*(\varphi_{\diamond'}) - S_{\blacklozenge'}^*(\varphi_{\blacklozenge'})$$

on $U_\diamond \cap U_\blacklozenge$ (recall that for any $\sigma \in \mathcal{P}$, $S_\sigma : U_\sigma \rightarrow N'_\mathbb{R}$ is an integral affine submersion assumed to exist in Condition (5) in Definition 2.3). Let

$$\Sigma_\diamond(\blacklozenge) := \{K \in \Sigma_\diamond : K \supset K_\blacklozenge\}.$$

Then there is a natural fan inclusion $j_{\blacklozenge\diamond} : \Sigma_\diamond(\blacklozenge) \rightarrow \Sigma_\diamond$ and a fan projection $p_{\blacklozenge\diamond} : \Sigma_\diamond(\blacklozenge) \rightarrow \Sigma_\blacklozenge$ given by quotienting $\Sigma_\diamond(\blacklozenge)$ along $\Lambda_{\blacklozenge, \mathbb{R}}$, so that

$$f_{\blacklozenge\diamond} := j_{\blacklozenge\diamond}^* \varphi_{\blacklozenge\diamond} - p_{\blacklozenge\diamond}^* \varphi_{\blacklozenge}$$

is an affine function on $|\Sigma_\diamond(\blacklozenge)|$.

We first consider the case $\omega \not\subset S$. Let $\sigma_i, \sigma_j \in \mathcal{P}$ be two cells containing σ . For $\diamond \in \{\omega, \tau, \sigma\}$, denote by $\mathcal{L}(\diamond')$ the line bundle on the toric stratum X_\diamond which corresponds to the piecewise linear function $\varphi_{\diamond'} : |\Sigma_\diamond| \rightarrow \mathbb{R}$. The transition function of $j_{\blacklozenge\diamond}^* \mathcal{L}(\diamond')$ on $V_{\sigma_i} \cap V_{\sigma_j} \subset X(\Sigma_\diamond(\blacklozenge))$ is given by the monomial

$$z^{j_{\blacklozenge\diamond}^*(m_i(\diamond') - m_j(\diamond'))},$$

where $m_i(\diamond'), m_j(\diamond') \in \mathcal{Q}_\diamond^*$ are the slopes of $\varphi_{\diamond'}$ on $K_{\sigma_i}, K_{\sigma_j} \in \Sigma_\diamond$. If $\diamond \subset \blacklozenge$, we have the projection $p_{\blacklozenge\diamond} : \Sigma_\diamond(\blacklozenge) \rightarrow \Sigma_\blacklozenge$. Then the transition function of the line bundle $p_{\blacklozenge\diamond}^* \mathcal{L}(\blacklozenge')$ on $V_{\sigma_i} \cap V_{\sigma_j}$ is given by the monomial

$$z^{p_{\blacklozenge\diamond}^*(m_i(\blacklozenge') - m_j(\blacklozenge'))}.$$

Since $f_{\blacklozenge\diamond}$ is affine, we have $p_{\blacklozenge\diamond}^* \mathcal{L}(\blacklozenge') \cong j_{\blacklozenge\diamond}^* \mathcal{L}(\diamond')$. Thus, the embedding $\iota_{\blacklozenge\diamond} : X_\blacklozenge \rightarrow X(\Sigma_\diamond(\blacklozenge)) \subset X_\diamond$ is given by

$$z^m \mapsto \begin{cases} z^m & \text{if } m \in K_{\sigma_i}^\vee \cap K_\blacklozenge^\perp \cap \mathcal{Q}_\blacklozenge^* = (K_{\sigma_i} + \Lambda_{\blacklozenge, \mathbb{R}}) / \Lambda_{\blacklozenge, \mathbb{R}} \cap \mathcal{Q}_\blacklozenge^*, \\ 0 & \text{otherwise} \end{cases}$$

on the affine chart V_{σ_i} . Twisting with $s_e \in \mathcal{Q}_\blacklozenge \otimes_{\mathbb{Z}} \mathbb{C}^\times$, the inclusion $\iota_{\blacklozenge\diamond, s} : X_\blacklozenge \rightarrow X(\Sigma_\diamond(\blacklozenge)) \subset X_\diamond$ is given by

$$z^m \mapsto \begin{cases} s_e(m) z^m & \text{if } m \in K_{\sigma_i}^\vee \cap K_\blacklozenge^\perp \cap \mathcal{Q}_\blacklozenge^* = (K_{\sigma_i} + \Lambda_{\blacklozenge, \mathbb{R}}) / \Lambda_{\blacklozenge, \mathbb{R}} \cap \mathcal{Q}_\blacklozenge^*, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\iota_{\blacklozenge\diamond, s}^* \mathcal{L}(\diamond')$ has transition function given by

$$\frac{s_e(m_i(\blacklozenge'))}{s_e(m_j(\blacklozenge'))} z^{m_i(\blacklozenge') - m_j(\blacklozenge')}.$$

So we can define $g_{\blacklozenge\diamond', s} : \mathcal{L}(\blacklozenge') \rightarrow \iota_{\blacklozenge\diamond, s}^* \mathcal{L}(\diamond')$ by

$$e_i(\blacklozenge') \mapsto s_e(m_i(\blacklozenge'))^{-1} \iota_{\blacklozenge\diamond, s}^* e_i(\diamond')$$

on $U_i(\blacklozenge)$, which gives an isomorphism $\mathcal{L}(\blacklozenge') \cong \iota_{\blacklozenge\diamond, s}^* \mathcal{L}(\diamond')$. Moreover, as $\iota_{\sigma\omega, s}^* \mathcal{L}(\omega') = \iota_{\sigma\tau, s}^* \iota_{\tau\omega, s}^* \mathcal{L}(\omega')$, we have

$$s_{\tau \rightarrow \sigma}(m_i(\sigma')) s_{\omega \rightarrow \tau}(m_i(\tau')) s_{\omega \rightarrow \sigma}(m_i(\sigma'))^{-1} = s_{\tau \rightarrow \sigma}(m_j(\sigma')) s_{\omega \rightarrow \tau}(m_j(\tau')) s_{\omega \rightarrow \sigma}(m_j(\sigma'))^{-1}.$$

Hence the quantity

$$s_{\omega'\tau'\sigma'} := s_{\tau \rightarrow \sigma}(m_i(\sigma')) s_{\omega \rightarrow \tau}(m_i(\tau')) s_{\omega \rightarrow \sigma}(m_i(\sigma'))^{-1}$$

is independent of i . Also, if we choose other local representatives of φ' , then

$$\hat{s}_{\omega'\tau'\sigma'} = s_{\omega'\tau'\sigma'} (s_{\tau \rightarrow \sigma}(a(\sigma')) s_{\omega \rightarrow \tau}(a(\tau')) s_{\omega \rightarrow \sigma}(a(\sigma'))^{-1}),$$

where $a(\diamond')$ is the slope some affine function.

Now, if $\omega = \{v\} \subset S$ and $\omega \subsetneq \tau$, observe that $\Theta_{ij}|_{D_k} \equiv I$ on the divisor $D_k \subset \mathbb{P}^2$, for $i, j, k = 0, 1, 2$ distinct, so $\mathcal{E}(v')$ splits into two line bundles $\mathcal{L}_\tau^+(v') \oplus \mathcal{L}_\tau^-(v')$ on the $(\mathbb{C}^\times)^2$ -orbit corresponding to $K_\tau \in \Sigma_v$. The affine functions $\varphi_{v'} - \varphi_{\tau^\pm}$ induce isomorphisms

$$g_{\tau^\pm v', s} : \mathcal{L}(\tau^\pm) \cong \iota_{\tau v, s}^* \mathcal{L}_\tau^\pm(v'),$$

as before, where τ^\pm are preimage cells of τ under $\pi : L \rightarrow B$ such that $\tau^+ \cap \tau^- = \{v'\}$. Similarly, we have

$$g_{\sigma^\pm v', s} : \mathcal{L}(\sigma^\pm) \cong \iota_{\sigma v, s}^* \mathcal{L}_\sigma^\pm(v').$$

Without lost of generality, we may assume $\tau^\pm \subset \sigma^\pm$, so that

$$(\iota_{\sigma v, s}^* g_{\tau^\pm v', s}) \circ g_{\sigma^\pm \tau^\pm, s} \circ g_{\sigma^\pm v', s}^{-1} = (s_{v \rightarrow \tau}(m_i(\tau^\pm)) s_{\tau \rightarrow \sigma}(m_i(\sigma^\pm)) s_{v \rightarrow \sigma}(m_i(\sigma^\pm))^{-1}) id_{\mathcal{L}(\sigma')}.$$

Then

$$s_{v' \tau^\pm \sigma^\pm} := s_{v \rightarrow \tau}(m_i(\tau^\pm)) s_{\tau \rightarrow \sigma}(m_i(\sigma^\pm)) s_{v \rightarrow \sigma}(m_i(\sigma^\pm))^{-1}$$

is independent of i . Moreover, choosing other local representatives of φ' leads to the equality

$$\hat{s}_{v' \tau^\pm \sigma^\pm} = s_{v' \tau^\pm \sigma^\pm} (s_{v \rightarrow \tau}(a(\tau^\pm)) s_{\tau \rightarrow \sigma}(a(\sigma^\pm)) s_{v \rightarrow \sigma}(a(\sigma^\pm))^{-1}).$$

Hence, for any $\omega', \tau', \sigma' \in \mathcal{P}_\pi$ with $\omega' \subset \tau' \subset \sigma'$, we obtain an element $s_{\omega' \tau' \sigma'}$, so that $(s_{\omega' \tau' \sigma'})_{\omega' \subset \tau' \subset \sigma'}$ gives a \mathbb{C}^\times -valued Čech 2-cocycle with respect to the simplicial structure on L induced by \mathcal{P}_π . This defines a cohomology class

$$o_{\mathbb{L}}(s) := [(s_{\omega' \tau' \sigma'})_{\omega' \subset \tau' \subset \sigma'}] \in H^2(L, \mathbb{C}^\times).$$

It is also clear that $o_{\mathbb{L}}(s)$ only depends on the cohomology class of s in $H^1(\mathcal{W}, \mathcal{Q}_{\mathcal{P}} \otimes \mathbb{C}^\times)$. As a result, we obtain the *obstruction map* as a group homomorphism

$$o_{\mathbb{L}} : H^1(\mathcal{W}, \mathcal{Q}_{\mathcal{P}} \otimes \mathbb{C}^\times) \rightarrow H^2(L, \mathbb{C}^\times).$$

Theorem 5.6. *If $o_{\mathbb{L}}(s) = 1$, then there exists a rank r locally free sheaf $\mathcal{E}_0(\mathbb{L})$ on $X_0(B, \mathcal{P}, s)$.*

Proof. If $o_{\mathbb{L}}(s) = 1$, then for $\omega' \subset \tau' \subset \sigma'$, there exists $h_{\diamond' \blacklozenge'} \in \mathbb{C}^\times$ for any pair $\diamond', \blacklozenge' \in \{\omega', \tau', \sigma'\}$ such that

$$s_{\omega' \tau' \sigma'} = h_{\tau' \sigma'} h_{\omega' \tau'} h_{\omega' \sigma'}^{-1}.$$

Set $\tilde{s}_{\diamond \blacklozenge} := s_{\diamond \blacklozenge} \cdot h_{\diamond \blacklozenge}^{-1}$, and define $g_{\blacklozenge \diamond', \tilde{s}} : \mathcal{L}(\blacklozenge') \rightarrow \iota_{\blacklozenge \diamond, s}^* \mathcal{L}(\diamond')$ by

$$e_i(\blacklozenge') \mapsto \tilde{s}_e(m_i(\blacklozenge'))^{-1} \iota_{\blacklozenge \diamond, s}^* e_i(\diamond') = (s_e(m_i(\blacklozenge'))^{-1} h_{\diamond \blacklozenge'}) \iota_{\blacklozenge \diamond, s}^* e_i(\diamond')$$

Then they satisfy

$$(\iota_{\sigma \tau, s}^* g_{\tau' \omega', \tilde{s}}) \circ g_{\sigma' \tau', \tilde{s}} = g_{\sigma' \omega', \tilde{s}}.$$

Now, suppose $\diamond, \blacklozenge \in \{\omega, \tau, \sigma\}$ have preimages cells $\diamond^{(\alpha)}, \blacklozenge^{(\beta)} \in \mathcal{P}_\pi$, $\alpha, \beta = 1, \dots, r$ (counted with multiplicities). Then we have an invertible matrix

$$g_{\blacklozenge \diamond, \tilde{s}} := (g_{\blacklozenge^{(\beta)} \diamond^{(\alpha)}, \tilde{s}})_{\alpha, \beta = 1, \dots, r},$$

where we put $g_{\blacklozenge^{(\beta)} \diamond^{(\alpha)}, \tilde{s}} = 0$ if $\diamond^{(\alpha)} \cap \blacklozenge^{(\beta)} = \emptyset$. As $\pi : L \rightarrow B$ is homeomorphic on cells, for any $\alpha \in \{1, \dots, r\}$, there exist unique $\beta(\alpha), \gamma(\alpha) \in \{1, \dots, r\}$ such that $\omega^{(\gamma)} \subset \tau^{(\beta)} \subset \sigma^{(\alpha)}$, so we have

$$\sum_{\beta, \gamma=1}^r g_{\omega^{(\gamma)} \sigma^{(\alpha)}, \tilde{s}} (\iota_{\sigma \tau, s}^* g_{\tau^{(\beta)} \omega^{(\gamma)}, \tilde{s}}) g_{\sigma^{(\alpha)} \tau^{(\beta)}, \tilde{s}} = g_{\omega^{(\gamma(\alpha))} \sigma^{(\alpha)}, \tilde{s}} (\iota_{\sigma \tau, s}^* g_{\tau^{(\beta(\alpha))} \omega^{(\gamma(\alpha))}, \tilde{s})) g_{\sigma^{(\alpha)} \tau^{(\beta(\alpha))}, \tilde{s}} = \delta_{\alpha \eta}.$$

Hence $\{g_{\sigma \tau, \tilde{s}} : \mathcal{E}(\sigma) \rightarrow \iota_{\sigma \tau, \tilde{s}}^* \mathcal{E}(\tau)\}_{\tau \subset \sigma}$ satisfies the desired cocycle condition. \square

Definition 5.7. *If $o_{\mathbb{L}}(s) = 1$, we define $\mathcal{E}_0(\mathbb{L}) := \varinjlim \mathcal{E}(\sigma)$.*

Remark 5.8. *The obstruction map $o_{\mathbb{L}}$ is a higher rank analog of the obstruction map defined in [13, Theorem 2.34] via open gluing data. Since the space of open gluing data is embedded into the space of closed gluing data (see [13, Proposition 2.32]), we can restrict $o_{\mathbb{L}}$ to the space of open gluing data.*

Remark 5.9. *Different choices of the constants $\{h_{\tau',\sigma'}\} \subset \mathbb{C}^\times$ in the proof of Theorem 5.6 may produce different locally free sheaves. When we write $\mathcal{E}_0(\mathbb{L})$, it is understood that we have fixed one such choice.*

6. SIMPLICITY AND SMOOTHABILITY

As before, we assume that $\dim(B) = 2$ and the polyhedral decomposition \mathcal{P} is elementary (See Definition 2.11). The tropical Lagrangian multi-section \mathbb{L} we consider is in class $\mathcal{S}_{m,n}$ (See Definition 4.2). We also assume that the domain L of \mathbb{L} is connected. These assumptions imply that, at an unramified vertex $v' \in L$, $\varphi_{v'}$ can be represented by the piecewise linear function

$$\varphi_k := \begin{cases} 0 & \text{on } K_{\sigma_0} \\ k\xi_1 & \text{on } K_{\sigma_1} \\ k\xi_2 & \text{on } K_{\sigma_2} \end{cases},$$

for $k \in \{m, n\}$, which corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^2}(k)$ on \mathbb{P}^2 .

We are interested in smoothability of the pair $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$, where $o_{\mathbb{L}}(s) = 1$.⁶ In order to apply the Gross-Siebert program, we assume that (B, \mathcal{P}) positive and simple. Then, with a suitable choice of gluing data s and a choice of strictly convex $\varphi \in H^0(B, \mathcal{MPL}_{\mathcal{P}})$, the main theorem of [15] (or alternatively, combining the results in [1] and [8]) says that $X_0(B, \mathcal{P}, s)$ is smoothable to a polarized toric degeneration $p: \mathcal{X} \rightarrow S$ over $S := \text{Spec}(\mathbb{C}[[t]])$.

To prove smoothability of the pair $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$, we plan to apply the main result (Corollary 4.7) in [2], for which we need to show that the pair $(X_0(B, \mathcal{P}, s), \det(\mathcal{E}))$ is smoothable and $H^2(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}_0(\mathbb{L}))) = 0$. The first condition is easy as $\det(\mathcal{E}_0(\mathbb{L})) = \mathcal{O}_{X_0(B, \mathcal{P}, s)}(K)$, for some $K \in \mathbb{Z}$ and the pair $(X_0(B, \mathcal{P}, s), \mathcal{O}_{X_0(B, \mathcal{P}, s)}(1))$ is smoothable by [15]. If $K = 0$, $\mathcal{E}_0(\mathbb{L}) = \mathcal{O}_{X_0(B, \mathcal{P}, s)}$ and it is clear that $(X_0(B, \mathcal{P}, s), \mathcal{O}_{X_0(B, \mathcal{P}, s)})$ is smoothable because $X_0(B, \mathcal{P}, s)$ is. If $K > 0$ (resp. $K < 0$), we see that the pair $(X_0(B, \mathcal{P}, s), \det(\mathcal{E}_0(\mathbb{L})))$ is also smoothable after tensoring $\mathcal{O}_{X_0(B, \mathcal{P}, s)}(1)$ (resp. $\mathcal{O}_{X_0(B, \mathcal{P}, s)}(-1)$) with itself $|K|$ -times. For the second condition, higher cohomologies are usually hard to compute. Fortunately, we work in dimension 2 and $X_0(B, \mathcal{P}, s)$ is Calabi-Yau, so it reduces to showing that $H^0(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}_0(\mathbb{L}))) = 0$, or equivalently, that the locally free sheaf $\mathcal{E}_0(\mathbb{L})$ is *simple*. Our main result provides a combinatorial condition on the tropical Lagrangian multi-section \mathbb{L} to assure such a situation, and the condition looks particularly neat in the rank 2 case. We will therefore separately handle the $r = 2$ case and the higher rank ($r \geq 3$) case.

6.1. Smoothing in rank 2. For any vertex $\{v\} \in \mathcal{P}$, we have the restriction maps

$$\begin{aligned} \Pi_v &: H^0(X_0(B, \mathcal{P}, s), \text{End}(\mathcal{E}_0(\mathbb{L}))) \rightarrow H^0(X_v, \text{End}(\mathcal{E}(v))), \\ \Pi_v^0 &: H^0(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}_0(\mathbb{L}))) \rightarrow H^0(X_v, \text{End}_0(\mathcal{E}(v))). \end{aligned}$$

We have already proven that $\Pi_v^0 \equiv 0$ for $v \in S$. So to prove simplicity of $\mathcal{E}_0(\mathbb{L})$, it remains to study Π_v^0 for $v \notin S$. To do this, we introduce a graph associated to \mathbb{L} :

⁶Recall that we always assume that all closed gluing data for both the fan and cone pictures are induced by open gluing data for the fan picture.

Definition 6.1. *The union of all 1-cells in \mathcal{P} defines an embedded graph $\Gamma := \bigcup_{\tau \in B^{(1)}} \tau$ in B . Then by removing all the branched points of $\pi : L \rightarrow B$ from Γ , we get the graph*

$$G(\mathbb{L}) := \bigcup_{\tau \in B^{(1)} : \tau \cap S = \emptyset} \tau,$$

as a subgraph of Γ .

Note that $V(G(\mathbb{L})) \amalg S = V(\Gamma)$, where $V(\cdot)$ denotes the vertex set of a graph, so $G(\mathbb{L})$ is the maximum subgraph of Γ whose vertices are unbranched points of $\pi : L \rightarrow B$. Hence if $G(\mathbb{L}) = \emptyset$, we have $\Pi_v^0 \equiv 0$ for all $v \notin S$. However, if $G(\mathbb{L}) \neq \emptyset$, the group $H^0(X_v, \text{End}_0(\mathcal{E}(v)))$ never vanishes for $v \notin S$ since $\mathcal{E}(v) \cong \mathcal{O}_{\mathbb{P}^2}(n) \oplus \mathcal{O}_{\mathbb{P}^2}(m)$.

Definition 6.2. *A cycle $\gamma \subset \Gamma$ is called a minimal cycle if there exists a 2-cell $\sigma \in \mathcal{P}$ such that $\partial\sigma = \gamma$.*

We now focus on the case $m = n + 1$.

Definition 6.3. *A tropical Lagrangian multi-section $\mathbb{L} \in \mathcal{S}_{n+1,n}$ is called simple if $G(\mathbb{L})$ has no minimal cycles.*

For example, when $G(\mathbb{L})$ is a disjoint union of trees, \mathbb{L} is always simple.

Theorem 6.4. *$\mathbb{L} \in \mathcal{S}_{n+1,n}$ is simple if and only if the locally free sheaf $\mathcal{E}_0(\mathbb{L})$ is simple.*

Proof. Let s be a section of $\text{End}_0(\mathcal{E}_0(\mathbb{L}))$ such that $\Pi_v^0(s) \neq 0$ for some $v \in G(\mathbb{L})$. Since $\Pi_v(s)$ is a non-zero section of $\mathcal{O}_{\mathbb{P}^2}(1)$, there exists at least one vertex $\check{\sigma} \in \check{\mathcal{P}}$ such that $\Pi_v(s)$ is non-zero at the torus fixed point $X_{\check{\sigma}} \subset X_v$. Consider the minimal cycle $\gamma := \partial\sigma$. If there exists some vertex $v' \in V(\gamma)$ such that $\Pi_{v'}(s) = 0$, then by continuity, $\Pi_{v'}(s)$ must vanish at the torus fixed point $X_{\check{\sigma}}$ because \check{v} and \check{v}' share the common vertex $\check{\sigma}$ (see Figure 3 below). Thus $\Pi_{v'}^0(s) \neq 0$ for all $v' \in V(\sigma)$. In particular, $\gamma \subset G(\mathbb{L})$.

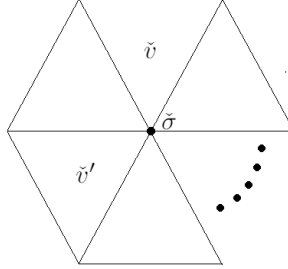


FIGURE 3. The dual of the minimal cycle γ .

Conversely, suppose $G(\mathbb{L})$ has a minimal cycle γ . Let

$$X_\gamma := \bigcup_{v \in V(\gamma)} X_v$$

and $\mathcal{E}_\gamma := \mathcal{E}_0(\mathbb{L})|_{X_\gamma}$. We want to construct a non-trivial section A of $\text{End}_0(\mathcal{E}_\gamma) \rightarrow X_\gamma$ by gluing non-trivial sections $\{s_v\}_{v \in V(\gamma)}$ of $\mathcal{O}(1) \rightarrow X_v$, $v \in V(\gamma)$, which has vanishing order 1 along the

boundary divisors of X_γ . To do this, we first define a cover $\{U_v\}_{v \in V(\gamma)}$ of γ by

$$U_v := \{v\} \cup \bigcup_{\tau \in E(\gamma): v \in \tau} \text{Int}(\tau).$$

Let $\{s_v\}_{v \in V(\gamma)}$ be sections of $\mathcal{O}(1)$ which vanish along the divisor correspond to the half-edges⁷ of γ . Then for any $\tau \in E(\gamma)$ with vertices $v, v' \in \tau$, $\iota_{\tau v, s}^* s_v$ and $\iota_{\tau v', s}^* s_{v'}$ can be regarded as non-trivial holomorphic sections of $\mathcal{O}_{X_\tau}(1) \rightarrow X_\tau$. Thus

$$\lambda_{vv'} := \frac{\iota_{\tau v, s}^* s_v}{\iota_{\tau v', s}^* s_{v'}}$$

is a meromorphic function on X_τ . But $\lambda_{vv'}$ has no zeros and poles, so $\lambda_{vv'} \in \mathbb{C}^\times$. Hence the data $\mathcal{L} := (\{U_v\}, \{\mathbb{C}\langle s_v \rangle\}, \{\lambda_{vv'}\})$ defines a rank 1 local system on $\gamma \cong S^1$. Let σ be such that $\partial\sigma = \gamma$. We extend \mathcal{L} as follows: Cover σ by

$$U_v^\sigma := U_v \cup \text{Int}(\sigma).$$

If $U_v^\sigma \cap U_{v'}^\sigma \neq \emptyset$, then

$$U_v^\sigma \cap U_{v'}^\sigma = \begin{cases} \text{Int}(\tau) \cup \text{Int}(\sigma) & \text{if } U_v \cap U_{v'} = \text{Int}(\tau), \\ \text{Int}(\sigma) & \text{otherwise.} \end{cases}$$

We define

$$\tilde{\lambda}_{vv'} := \begin{cases} \frac{\iota_{\tau v, s}^* s_v}{\iota_{\tau v', s}^* s_{v'}} & \text{if } U_v \cap U_{v'} = \text{Int}(\tau), \\ \frac{\iota_{\sigma v, s}^* s_v}{\iota_{\sigma v', s}^* s_{v'}} & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{\lambda}_{vv'} \in \mathbb{C}^\times$ and since $\iota_{\sigma\tau, s} \circ \iota_{\tau v, s} = \iota_{\sigma v, s}$, which is independent of τ , the cocycle condition is satisfied. Thus we get a local system $\tilde{\mathcal{L}} \rightarrow \sigma$ extending $\mathcal{L} \rightarrow \gamma$. Since σ is contractible, $\tilde{\mathcal{L}}$, and hence \mathcal{L} , are trivial as local systems. Therefore, we can find $\{c_v\} \subset \mathbb{C}^\times$ such that $\lambda_{vv'} = c_{v'}/c_v$. By definition, we have

$$\iota_{\tau v, s}^* (c_v s_v) = \iota_{\tilde{\tau} v', \tilde{s}}^* (c_{v'} s_{v'})$$

for all τ with vertices v, v' . Thus we obtain a section A of $\text{End}_0(\mathcal{E}_\gamma) \rightarrow X_\gamma$ which vanishes along the boundary divisor of X_γ . Extend A by zero to the other toric components, we see that $\text{End}_0(\mathcal{E}_0(\mathbb{L}))$ has a non-trivial section. \square

Remark 6.5. *The case when $m \geq n+2$ is much easier. By choosing sections of $\mathcal{O}_{\mathbb{P}^2}(m-n)$ which vanish only along boundary divisors, it is not hard to see that*

- when $m = n+2$, $\mathcal{E}_0(\mathbb{L})$ is simple if and only if $G(\mathbb{L})$ is a collection of vertices in $B \setminus S$;
- when $m \geq n+3$, $\mathcal{E}_0(\mathbb{L})$ is simple if and only if $G(\mathbb{L}) = \emptyset$.

Corollary 6.6. $\mathbb{L} \in \mathcal{S}_{n+1, n}$ is simple if and only if $H^2(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}_0(\mathbb{L}))) = 0$.

Proof. Because $X_0(B, \mathcal{P}, s)$ has Gorenstein singularities, Serre duality holds. The canonical sheaf is trivial by Calabi-Yau condition. \square

Because of Corollary 6.6, we can apply the smoothing result of [2] to obtain the following

Corollary 6.7. *If $\mathbb{L} \in \mathcal{S}_{n+1, n}$ is simple, then the pair $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$ is smoothable.*

⁷Let G be a graph and H be a subgraph of G . An edge $e \in E(G)$ is called a *half-edge* of H if $e \notin E(H)$ and $e \cap H \neq \emptyset$.

6.2. Smoothing in general rank. We now turn to the higher rank ($r \geq 3$) case. Let $(B, \mathcal{P}, \varphi)$ be as before and consider a tropical Lagrangian multi-section $\mathbb{L} = (L, \pi, \mathcal{P}_\pi, \varphi') \in \mathcal{S}_{n+1, n}$.

6.2.1. *The graph $\tilde{G}(\mathbb{L})$.* In order to characterize when \mathbb{L} gives rise to a simple $\mathcal{E}_0(\mathbb{L})$, we consider the fiber product

$$P(\mathbb{L}) := L \times_\pi L.$$

Since π has simple branching, the ramification locus S' sits inside the diagonal Δ_L as $S' \times_\pi S' \subset \Delta_L$. There is a natural projection $\pi_B : P(\mathbb{L}) \rightarrow B$. The preimage set $\pi_B^{-1}(x)$ has r^2 points if $x \notin S$ while it has $(r-1)^2$ points if $x \in S$. Note that the topological space $P(\mathbb{L})$ is connected but *not* a manifold, even topologically. Nevertheless,

$$P_0(\mathbb{L}) := P(\mathbb{L}) \setminus \Delta_L,$$

is a topological manifold but *not* necessarily connected.

Define

$$\tilde{\mathcal{P}} := \{\sigma'_1 \times_\pi \sigma'_2 \mid \sigma'_1, \sigma'_2 \in \mathcal{P}_\pi \text{ with } \pi(\sigma'_1) = \pi(\sigma'_2)\}.$$

For each $\sigma \in \mathcal{P}$, we have

$$\pi_B^{-1}(\sigma) = \bigcup_{\substack{\sigma'_1, \sigma'_2 \in \mathcal{P}_\pi \\ \pi(\sigma'_1) = \pi(\sigma'_2) = \sigma}} \sigma'_1 \times_\pi \sigma'_2$$

and $\pi_B|_{\sigma'_1 \times_\pi \sigma'_2} : \sigma'_1 \times_\pi \sigma'_2 \rightarrow \sigma$ is a homeomorphism. Denote by $\tilde{\mathcal{P}}^{(k)}$ the k -dimensional strata of $P(\mathbb{L})$:

$$\tilde{\mathcal{P}}^{(k)} := \{\sigma'_1 \times_\pi \sigma'_2 \mid \dim(\sigma'_1) = \dim(\sigma'_2) = k\}.$$

Let

$$\tilde{\Gamma} := \bigcup_{\sigma'_1 \times_\pi \sigma'_2 \in \tilde{\mathcal{P}}^{(0)} \cup \tilde{\mathcal{P}}^{(1)}} \sigma'_1 \times_\pi \sigma'_2$$

be the graph associated to $\tilde{\mathcal{P}}$, which satisfies $\pi_B(\tilde{\Gamma}) = \Gamma$. We denote by $V_j(\tilde{\Gamma})$ the set of all j -valent vertices in $\tilde{\Gamma}$, and write a vertex of $\tilde{\Gamma}$ as a pair $\tilde{v} := (v'_1, v'_2) \in V(\tilde{\Gamma})$. There are 3 scenarios:

- If $v'_i \notin S'$ for $i = 1, 2$, then $\tilde{v} \in V_3(\tilde{\Gamma})$.
- If $v'_i \in S'$ for only one i , then $\tilde{v} \in V_6(\tilde{\Gamma})$.
- Lastly, if $v'_i \in S'$ for all i , which means $v'_1 = v'_2 \in S'$, then $\tilde{v} \in V_{12}(\tilde{\Gamma})$ and six of the twelve edges are contained in the diagonal Δ_L .

Definition 6.8. Let $\tilde{G}(\mathbb{L}) \subset P_0(\mathbb{L})$ be the subgraph of $\tilde{\Gamma}$ whose vertex set $V(\tilde{G}(\mathbb{L}))$ and edge set $E(\tilde{G}(\mathbb{L}))$ are given as follows. A vertex $\tilde{v} = (v'_1, v'_2) \in V(\tilde{G}(\mathbb{L}))$ if and only if one of the following conditions is satisfied.

- (1) For any adjacent edge $\tilde{\tau} := \tau'_1 \times_\pi \tau'_2$ of $\tilde{v} = (v'_1, v'_2)$, put $\tau := \pi_B(\tilde{\tau})$. Then there exist neighborhoods $U_{\tau'_1}, U_\tau$ of $\text{Int}(\tau'_1), \text{Int}(\tau)$ respectively such that the piecewise linear function

$$\varphi_{v'_2}|_{U_{\tau'_2}} \circ \pi^{-1} - \varphi_{v'_1}|_{U_{\tau'_1}} \circ \pi^{-1}$$

is strictly convex on U_τ .

- (2) If $v'_i \notin S'$ for $i = 1, 2$, then

$$\varphi_{v'_2} \circ \pi^{-1} - \varphi_{v'_1} \circ \pi^{-1}$$

is strictly convex on $U_v \subset B$, where $v = \pi_B(\tilde{v})$.

A 1-cell $\tilde{e} \in E(\tilde{G}(\mathbb{L}))$ if and only if \tilde{e} connects two points in $V(\tilde{G}(\mathbb{L}))$.

In particular, $\tilde{G}(\mathbb{L})$ is at most 6-valent. For a vertex $\tilde{v} = (v'_1, v'_2) \in V(\tilde{G}(\mathbb{L})) \cap V_6(\tilde{\Gamma})$ with $v'_1 \in S'$, there are two edges $\tilde{\tau}_1^+, \tilde{\tau}_1^-$ containing \tilde{v} . We put

$$(4) \quad \tilde{\tau}^+ := \tau_1^+ \times_{\pi} \tau'_2, \quad \tilde{\tau}^- := \tau_1^- \times_{\pi} \tau'_2,$$

where \pm is determined by the conditions that

$$\begin{aligned} \varphi_{\tau'_2} \circ \pi^{-1} - \varphi_{\tau_1^+}|_{U_{\tau_1^+}} \circ \pi^{-1} &\text{ is straightly convex, and} \\ \varphi_{\tau'_2} \circ \pi^{-1} - \varphi_{\tau_1^-}|_{U_{\tau_1^-}} \circ \pi^{-1} &\text{ is an affine function.} \end{aligned}$$

Similar definitions apply for $v'_2 \in S'$.

6.2.2. *Well-colored subgraphs of $\tilde{G}(\mathbb{L})$.* To proceed, we make the following notation. For any subgraph $\tilde{G} \subset \tilde{\Gamma}$, define

$$X_{\tilde{G}} := \bigcup_{v \in V(\pi_B(\tilde{G}))} X_v$$

and let $\mathcal{E}nd_{\tilde{G}}$ be the sheaf on $X_{\tilde{G}}$ such that

$$\mathcal{E}nd_{\tilde{G}}|_{X_v} = \bigoplus_{\substack{\tilde{v} \in V(\tilde{G}) \\ \pi_B(\tilde{v})=v}} \mathcal{E}(\tilde{v}) := \bigoplus_{\substack{\tilde{v}=(v'_1, v'_2) \in V(\tilde{G}) \\ \pi_B(\tilde{v})=v}} \mathcal{E}(v'_1)^* \otimes \mathcal{E}(v'_2),$$

where $\mathcal{E}(v'_i)$ is the bundle which corresponds to the piecewise linear function $\varphi_{v'_i}$ defined in a neighborhood of $v'_i \in L$. Note that $\mathcal{E}nd_{\tilde{G}}$ may *not* be a vector bundle on $X_{\tilde{G}}$ because its rank can be non-constant.

To prove simplicity of $\mathcal{E}_0(\mathbb{L})$, we want to show that the cohomology $H^0(X_0(B, \mathcal{P}, s), \mathcal{E}nd_0(\mathcal{E}_0(\mathbb{L})))$ vanishes by showing that the maps

$$\Pi_{\tilde{v}}^0 : H^0(X_0(B, \mathcal{P}, s), \mathcal{E}nd_0(\mathcal{E}_0(\mathbb{L}))) \rightarrow H^0(X_v, \mathcal{E}(v'_1)^* \otimes \mathcal{E}(v'_2))$$

vanish for all $\tilde{v} \in V(\tilde{\Gamma})$. We already have $\Pi_{\tilde{v}}^0 \equiv 0$, for all $\tilde{v} \in \Delta_{S'}$. For $\tilde{v} \notin V(\tilde{G}(\mathbb{L}))$, we also have $\Pi_{\tilde{v}}^0 \equiv 0$ because $\mathcal{E}(v'_1)^* \otimes \mathcal{E}(v'_2) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. So we suppose $\tilde{v} \in V(\tilde{G}(\mathbb{L}))$. Then by definition of $\tilde{G}(\mathbb{L})$, we have $\mathcal{E}(v'_1)^* \otimes \mathcal{E}(v'_2) \cong \mathcal{O}_{\mathbb{P}^2}(1)$ if $\tilde{v}'_i \notin S'$ for $i = 1, 2$, and if $\tilde{v}'_i \in S'$ for some $i = 1, 2$, then $\mathcal{E}(v'_1)^* \otimes \mathcal{E}(v'_2) \cong E_{n+1, n}(-k)$ or $E_{n+1, n}^*(k)$ for some $k \in \{n+1, n\}$. Thus we also need to handle sections coming from $E_{n+1, n}(-k)$ and its dual. Let us first try to understand how sections of $E_{n+1, n}(-k)$ and its dual look like.

Proposition 6.9. *Let $k, n \in \mathbb{Z}$. We have the following:*

- (1) *If $k \geq n+1$, then $H^0(\mathbb{P}^2, E_{n+1, n}(-k)) = 0$.*
- (2) *If $k \leq n$, then $H^0(\mathbb{P}^2, E_{n+1, n}^*(k)) = 0$.*
- (3) *If $k = n$, then $H^0(\mathbb{P}^2, E_{n+1, n}(-n)) = \mathbb{C}\langle s_0, s_1, s_2 \rangle$ and $H^0(\mathbb{P}^2, E_{n+1, n}^*(n+1)) = \mathbb{C}\langle s'_0, s'_1, s'_2 \rangle$, where s_k, s'_k are sections which vanish at the torus fixed point $D_i \cap D_j$ for distinct i, j, k .*

Proof. By Proposition 5.3, we only need to take care of $E_{n+1, n}(-k)$. By using the transition functions given in (3), it is easy to see that $E_{n+1, n}(-k) \cong E_{2, 1}(n-k-1) \cong T_{\mathbb{P}^2}(n-k-1)$. The Euler sequence of \mathbb{P}^2 reads

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \rightarrow T_{\mathbb{P}^2}(-1).$$

Tensoring with $\mathcal{O}(n-k)$, we get

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(n-k-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}(n-k)^{\oplus 3} \rightarrow T_{\mathbb{P}^2}(n-k-1) \rightarrow 0.$$

The induced long exact sequence on cohomologies implies (1) and (2) as well as

$$\dim_{\mathbb{C}} H^0(\mathbb{P}^2, E_{n+1,n}(-n)) = \dim_{\mathbb{C}} H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(-1)) = 3.$$

One can check that for any $\lambda, \mu, \eta \in \mathbb{C}$, if we define s by

$$(5) \quad \begin{aligned} s|_{U_0} &= (\lambda - \mu w_0^1) e_0 + (\eta - \mu w_0^2) e'_0, \\ s|_{U_1} &= (\mu - \lambda w_1^0) e_1 + (\eta - \lambda w_1^2) e'_1, \\ s|_{U_2} &= (\mu - \eta w_2^0) e_2 + (\lambda - \eta w_2^1) e'_2, \end{aligned}$$

then s is a section of $E_{n+1,n}(-n)$. Hence any section of $E_{n+1,n}(-n)$ must of the above form. By setting two of them to be zero, we see that s vanishes at a torus fixed point. \square

Remark 6.10. *The role of Condition (1) in the definition of $\tilde{G}(\mathbb{L})$ is to rule out the vertices that correspond to the situations (1), (2) in Proposition 6.9.*

Remark 6.11. *Equation (5) gives an explicit description of sections of $E_{n+1,n}(-n)$. Note that if s is a non-trivial section of $E_{n+1,n}(-n)$, then on each divisor, it splits into the following form*

$$(c, t) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

and at least one of them is non-trivial. If $\mathcal{E}(\tilde{v}) \cong E_{n+1,n}(-n)$ via $X_v \cong \mathbb{P}^2$, the edges $\tilde{\tau}^+, \tilde{\tau}^-$ gives two projection maps $\Pi_{\tilde{\tau}^+}, \Pi_{\tilde{\tau}^-}$ defined on $H^0(X_{\tau}, \iota_{\tau v, s}^* \mathcal{E}(\tilde{v})) \cong V_- \oplus V_+$, which correspond to the projections $(c, t) \mapsto t, (c, t) \mapsto c$ respectively.

As in the proof of Theorem 6.4, we can produce minimal cycles in $\tilde{G}(\mathbb{L})$ if $\mathcal{E}_0(\mathbb{L})$ is not simple. However, unlike the rank 2 case, non-existence of minimal cycles is way too strong for simplicity to hold when $r \geq 3$. This is due to the fact that sections of $E_{n+1,n}(-n)$ can vanish at a single torus fixed point. For example, if $\tilde{G}(\mathbb{L})$ itself is a minimal cycle whose vertices are all in $V_6(\tilde{\Gamma})$, then any non-trivial section of $\mathcal{E}_{\tilde{G}(\mathbb{L})} \rightarrow X_{\tilde{G}(\mathbb{L})}$ cannot be extended. Thus we need a detailed classification of the graphs which arise as ‘‘supports’’ of sections of $End_0(\mathcal{E}_0(\mathbb{L}))$. This leads to the following rather complicated definition.

Definition 6.12. *Let \tilde{G} be a subgraph of $\tilde{G}(\mathbb{L})$.*

- (1) *Denote by $V(\tilde{G})$ the set of vertices of \tilde{G} , $E(\tilde{G})$ the set of edges of \tilde{G} , $H(\tilde{G})$ the set of half-edge of \tilde{G} and $F(\tilde{G})$ the set of 2-cells in $\tilde{\mathcal{P}}$ such that $\tilde{\sigma} \in F(\tilde{G})$ if and only if $\tilde{\sigma}$ contains an edge or a half edge of \tilde{G} .*
- (2) *A subgraph $\tilde{G} \subset \tilde{G}(\mathbb{L})$ is said to be well-colored if there are coloring maps $c_e : E(\tilde{G}) \amalg H(\tilde{G}) \rightarrow \{\text{red, blue, green, black}\}, c_f : F(\tilde{G}) \rightarrow \{\text{white, red}\}$ such that*
 - (a) *Let $\tilde{\sigma}, \tilde{\sigma}' \in F(\tilde{G})$ and $\tilde{\sigma} \cap \tilde{\sigma}' \in E(\tilde{G}) \amalg H(\tilde{G})$. Then $\tilde{\sigma}, \tilde{\sigma}'$ are red-colored if and only if $\tilde{\sigma} \cap \tilde{\sigma}'$ is red-colored.*
 - (b) *If \tilde{e} is blue- or green-colored, then the two adjacent faces of \tilde{e} are white-colored.*
 - (c) *If $\tilde{e} = \tilde{\tau}^+ \in E(\tilde{G})$, then \tilde{e} is blue-colored (See (4) for the notations $\tilde{\tau}^{\pm}$).*
 - (d) *If $\tilde{e} = \tilde{\tau}^- \in E(\tilde{G})$, then \tilde{e} is green-colored.*
 - (e) *If $\tilde{v} \in V(\tilde{G}) \cap V_3(\tilde{\Gamma})$, then we have the following cases:*
 - (i) *\tilde{v} has one red edge or half-edge and two black edges, two red faces. We call such vertex a Type I(a) vertex.*
 - (ii) *\tilde{v} has one red face, one blue edge and two black edges. We call such vertex a Type I(b) vertex.*
 - (iii) *\tilde{v} has three blue-colored edges. We call such vertex a Type I(c) vertex.*

- (f) If $\tilde{v} \in V(\tilde{G}) \cap V_6(\tilde{\Gamma})$, then we have the following cases:
- (i) \tilde{v} has one green edge, two black edges and three red edges or half-edges, four red faces. We call such vertex a Type II(a) vertex.
 - (ii) \tilde{v} has one red-colored edge or half-edge, one blue edge, two black edges, two red faces. We call such vertex a Type II(b) vertex.
 - (iii) \tilde{v} has three blue edges and three black edges. We call such vertex a Type II(c) vertex.
- These conditions are subordinate to the extra condition that if $\tilde{\tau}, \tilde{\tau}'$ are edges or half-edges such that $\pi_B(\tilde{\tau}) = \pi_B(\tilde{\tau}')$, then $\tilde{\tau}, \tilde{\tau}'$ are of different color.
- (g) All half-edges of \tilde{G} are red-colored.

Different types of vertices are shown in Figure 4.

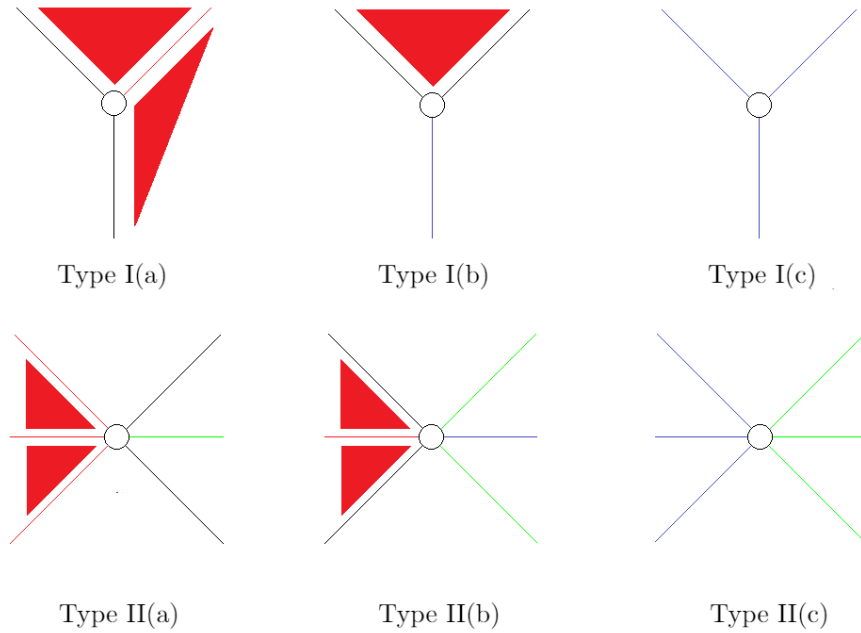


FIGURE 4

Example 6.13. If $\tilde{\gamma} \subset \tilde{G}(\mathbb{L})$ is a minimal cycle such that $V(\tilde{\gamma}) \subset V_3(\tilde{\Gamma})$, then $\tilde{\gamma}$ is well-colored. Explicitly, if $\partial\tilde{\sigma} = \tilde{\gamma}$, then we can color $\tilde{\sigma}$ white and all the outer faces red.

We shall give a more complex-geometric meaning of well-coloring.

Definition 6.14. Let s be a non-trivial section of $E_{n+1,n}(-n)$ and $\iota_i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ be an embedding mapping \mathbb{P}^1 to the boundary divisor D_i for $i = 0, 1, 2$. Write $\iota_i^*s = (c_i, t_i)$, where $c_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ and $t_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. The pseudo-zero locus of s along D_i is the set

$$PZ_i(s) := \{p \in \mathbb{P}^1 \mid c_i(p) = 0 \text{ or } t_i(p) = 0\}.$$

Then the pseudo-zero locus of s is defined as

$$PZ(s) := \bigcup_{i=0}^2 PZ_i(s).$$

We also define the pseudo-zero locus $PZ_i(s)$ of a section s of $\mathcal{O}_{\mathbb{P}^2}(1)$ along D_i to be the zero locus of $\iota_i^* s$

Remark 6.15. The red-colored faces and edges should be thought of as the zero locus or pseudo-zero locus of certain non-trivial sections of $\mathcal{O}_{\mathbb{P}^2}(1)$ or $E_{n+1,n}(-n)$ respectively. Different types of vertices correspond to different types of sections. The tropical images of the pseudo-zero loci of different types of sections are depicted in Figure 5. Each triangle is the dual cell of some vertex v such that $v = \pi_B(\tilde{v})$ for some $\tilde{v} \in V(\tilde{G})$. The “double triangle” should be thought of as a 2-fold covering of \tilde{v} , which record the pseudo-zero locus of a section of $E_{n+1,n}(-n)$ along toric divisors.

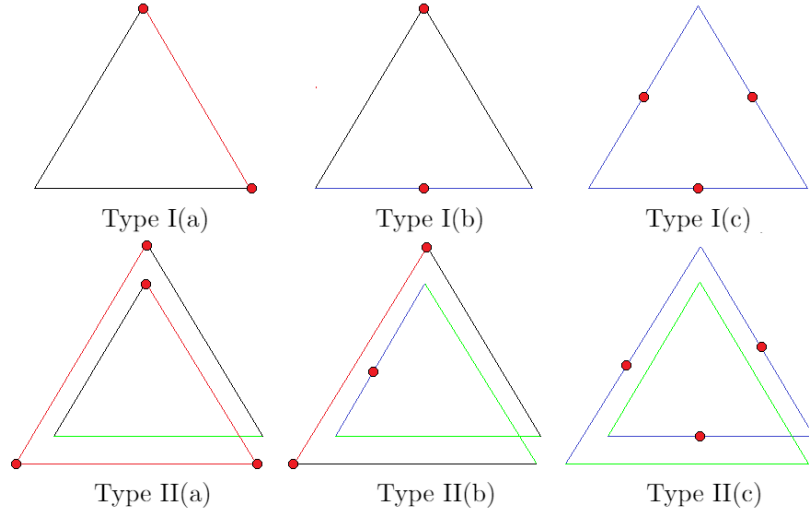


FIGURE 5. For Type I(a)-(c), the red-colored cells denote the tropical images of pseudo-zero loci of three types of sections of $\mathcal{O}_{\mathbb{P}^2}(1)$. For Type II(a)-(c), the red-colored cells denote the pseudo-zero locus of a non-trivial section of $E_{n+1,n}(-n)$.

6.2.3. *A rank 1 local system.* Let $\tilde{G} \subset \tilde{G}(\mathbb{L})$ be a well-colored subgraph. To study simplicity of $\mathcal{E}_0(\mathbb{L})$, we need one more ingredient, which is a rank 1 local system on the subgraph

$$\tilde{G}_0 := \tilde{G} \setminus \{Int(\tilde{\tau}) \mid \tilde{\tau} \text{ is a red edge}\} \subset \tilde{G}.$$

Note that $V(\tilde{G}_0) = V(\tilde{G})$. Fix $\tilde{v} \in V(\tilde{G}_0)$ and define the *open star* of \tilde{v} by

$$U_{\tilde{v}} := \bigcup_{\tilde{\sigma} \in V(\tilde{G}_0) \amalg E(\tilde{G}_0)} Int(\tilde{\sigma}) \subset \tilde{G}_0.$$

Then $\{U_{\tilde{v}}\}_{\tilde{v} \in V(\tilde{G}_0)}$ forms an open cover of \tilde{G}_0 . To define the local system, we choose a collection of sections $\mathcal{S} := \{s_{\tilde{v}}\}_{\tilde{v} \in V(\tilde{G})}$, where $s_{\tilde{v}}$ is a non-trivial section of $\mathcal{E}(\tilde{v})$. We assume that the pseudo-zero

locus of $s_{\tilde{v}}$'s respects the coloring of \tilde{G} , meaning that, for an adjacent cell $\tilde{\sigma}$ of \tilde{v} , $X_{\tilde{\sigma}}$ is contained in the pseudo-zero locus of $s_{\tilde{v}}$ if and only if $\tilde{\sigma}$ is red-colored. If $U_{\tilde{v}} \cap U_{\tilde{u}} \neq \emptyset$, then we have

$$U_{\tilde{v}} \cap U_{\tilde{u}} = \begin{cases} \text{Int}(\tilde{\tau}) \amalg \text{Int}(\tilde{\tau}') & \text{if } \tilde{v}, \tilde{u} \in V_6(\tilde{\Gamma}) \text{ such that } \tilde{v}, \tilde{u} \in \tilde{\tau} \cap \tilde{\tau}', \\ \text{Int}(\tilde{\tau}) & \text{otherwise} \end{cases}$$

for some $\tilde{\tau}, \tilde{\tau}' \in E(\tilde{G}_0)$ with $\pi_B(\tilde{\tau}) = \pi_B(\tilde{\tau}')$. In such a case, we define

$$\lambda_{\tilde{v}\tilde{u}}(\mathcal{S}) := \begin{cases} \frac{\iota_{\tau v, s}^* s_{\tilde{v}}}{\iota_{\tau v', s}^* s_{\tilde{u}}} & \text{if } \tilde{v}, \tilde{u} \in V_3(\tilde{\Gamma}), \\ \frac{t_{\tilde{\tau}\tilde{v}}}{\iota_{\tau v', s}^* s_{\tilde{u}}} & \text{if } \tilde{v} \in V_6(\tilde{\Gamma}), \tilde{u} \in V_3(\tilde{\Gamma}) \text{ and } \tilde{\tau} \text{ is a blue or black edge,} \\ \frac{c_{\tilde{\tau}\tilde{v}}}{c_{\tilde{\tau}\tilde{u}}} & \text{if } \tilde{v}, \tilde{u} \in V_6(\tilde{\Gamma}) \text{ and } \tilde{\tau} \text{ is a green edge,} \\ \frac{t_{\tilde{\tau}\tilde{v}}}{t_{\tilde{\tau}\tilde{u}}} & \text{if } \tilde{v}, \tilde{u} \in V_6(\tilde{\Gamma}) \text{ and } \tilde{\tau} \text{ is not a green edge.} \end{cases}$$

An explanation of the definition of $\lambda_{\tilde{v}\tilde{u}}(\mathcal{S})$ is in order:

- (1) Suppose $\tilde{\tau}$ is not a red-colored edge whose vertices \tilde{v}, \tilde{u} are in $V_3(\tilde{\Gamma})$. Then $\mathcal{E}(\tilde{v}), \mathcal{E}(\tilde{u})$ are line bundles and $\iota_{\tau v, s}^* s_{\tilde{v}}, \iota_{\tau v', s}^* s_{\tilde{u}}$ are non-trivial holomorphic sections of $\iota_{\tau v, s}^* \mathcal{E}(\tilde{v}), \iota_{\tau v', s}^* \mathcal{E}(\tilde{u})$, which can be identified with a line bundle $\mathcal{L}(\tilde{\tau})$ on $X_{\tilde{\tau}}$. In this case, the quotient $\lambda_{\tilde{v}\tilde{u}}(\mathcal{S})$ is a meromorphic function on $X_{\tilde{\tau}}$.
- (2) Suppose $\tilde{\tau}$ is not a red-colored edge and some of its vertices \tilde{v}, \tilde{u} are in $V_6(\tilde{\Gamma})$, say $\tilde{v} \in V_6(\tilde{\Gamma})$. Then $\mathcal{E}(\tilde{v})$ is a rank 2 bundle. In this case, $\iota_{\tau v, s}^* s_{\tilde{v}}$ splits into

$$(c_{\tilde{\tau}\tilde{v}}, t_{\tilde{\tau}\tilde{v}}) \in H^0(X_{\tilde{\tau}}, \iota_{\tau v, s}^* \mathcal{E}(\tilde{v})),$$

where $c_{\tilde{\tau}\tilde{v}} \in \mathbb{C}$ and $t_{\tilde{\tau}\tilde{v}}$ is a holomorphic section of a line bundle on $X_{\tilde{\tau}}$. Both of them are non-trivial since $\tilde{\tau}$ is not a red edge. The bundles $\iota_{\tau v, s}^* \mathcal{E}(\tilde{v}), \iota_{\tau v', s}^* \mathcal{E}(\tilde{v})$ are identified with a split rank 2 bundle $\mathcal{E}(\tilde{\tau})$ on $X_{\tilde{\tau}}$ and thus, $\lambda_{\tilde{v}\tilde{u}}(\mathcal{S})$ is also a meromorphic function on $X_{\tilde{\tau}}$.

Clearly $\lambda_{\tilde{v}\tilde{u}}(\mathcal{S})\lambda_{\tilde{u}\tilde{v}}(\mathcal{S}) = 1$ and since $U_{\tilde{v}} \cap U_{\tilde{u}} \cap U_{\tilde{u}'} = \emptyset$ for three distinct vertices, the cocycle condition is vacuous.

Definition 6.16. A collection of sections $\mathcal{S} = \{s_{\tilde{v}}\}_{\tilde{v} \in V(\tilde{G})}$, with $s_{\tilde{v}} \in H^0(X_{\tilde{v}}, \mathcal{E}(\tilde{v}))$, is called admissible if $\{\lambda_{\tilde{v}\tilde{u}}(\mathcal{S})\}_{\tilde{v}, \tilde{u} \in V(\tilde{G})} \subset \mathbb{C}^\times$.

Thus, if \mathcal{S} is a collection of admissible sections, the non-zero constants $\{\lambda_{\tilde{v}\tilde{u}}(\mathcal{S})\}_{\tilde{v}, \tilde{u} \in V(\tilde{G})}$ defines a rank 1 local system

$$\mathcal{L}_{\tilde{G}}(\mathcal{S})$$

on the support of \tilde{G}_0 . Clearly, $\mathcal{L}_{\tilde{G}}(\mathcal{S})$ only depends on the pseudo-zero loci of $s_{\tilde{v}}$'s, namely, if $\mathcal{S} = \{s_{\tilde{v}}\}_{\tilde{v} \in V(\tilde{G})}$ and $\mathcal{S}' = \{s'_{\tilde{v}}\}_{\tilde{v} \in V(\tilde{G})}$ are related by $s_{\tilde{v}} = c_{\tilde{v}} s'_{\tilde{v}}$ for some non-zero constants $\{c_{\tilde{v}}\}_{\tilde{v} \in V(\tilde{G})}$, then $\mathcal{L}_{\tilde{G}}(\mathcal{S}) \cong \mathcal{L}_{\tilde{G}}(\mathcal{S}')$ as local systems. It is also clear that a non-trivial global section of $\mathcal{E}nd_{\tilde{G}}$ exists if and only if $\mathcal{L}_{\tilde{G}}(\mathcal{S})$ is trivial as a local system. The key point is that admissible sections can be determined combinatorially.

Definition 6.17. Let \tilde{G} be a well-colored subgraph of $\tilde{G}(\mathbb{L})$. Let $\tilde{v} \in V(\tilde{G})$ and for an adjacent face $\tilde{\sigma}$, we define $\sigma := \pi_B(\tilde{\sigma})$. Let $\sigma_i, i = 0, 1, 2$, be the adjacent faces of $\pi_B(\tilde{v})$. A data D on \tilde{G} is a collection of triples $\{\delta(\tilde{v}) = (d_{\sigma_0}(\tilde{v}), d_{\sigma_1}(\tilde{v}), d_{\sigma_2}(\tilde{v}))\}_{\tilde{v} \in V(\tilde{G}_0)} \subset \mathbb{C}^3$ such that, if $\tilde{v} \in V_3(\tilde{G})$, then we have the following cases:

(Ia) If \tilde{v} is of Type I(a) with the unique white-colored adjacent face $\tilde{\sigma}_k$, then

$$d_{\sigma_k}(\tilde{v}) \neq 0$$

and $d_{\sigma_j}(\tilde{v}) = d_{\sigma_k}(\tilde{v}) = 0$ for distinct $i, j, k \in \{0, 1, 2\}$.

(Ib) If \tilde{v} is of Type I(a) with two white-colored adjacent faces $\tilde{\sigma}_i, \tilde{\sigma}_j$,

$$d_{\sigma_i}(\tilde{v}) \neq 0, \quad d_{\sigma_j}(\tilde{v}) \neq 0$$

and $d_{\sigma_k}(\tilde{v}) = 0$ for distinct $i, j, k \in \{0, 1, 2\}$.

(Ic) If \tilde{v} is of Type I(c), then

$$d_{\sigma_i}(\tilde{v}) \neq 0$$

for all $i \in \{0, 1, 2\}$;

while, if $\tilde{v} \in V_6(\tilde{G})$, then we have the following cases:

(IIa) If \tilde{v} is of Type II(a) with white-colored adjacent faces $\tilde{\sigma}_i, \tilde{\sigma}_j$ such that the edge $\tilde{\sigma}_i \cap \tilde{\sigma}_j$ is green-colored, then

$$d_{\sigma_i}(\tilde{v}) = d_{\sigma_j}(\tilde{v}) = 0$$

and $d_{\sigma_k}(\tilde{v}) \neq 0$ for distinct $i, j, k \in \{0, 1, 2\}$.

(IIb) If \tilde{v} is of Type II(b) with white-colored adjacent faces $\tilde{\sigma}_i, \tilde{\sigma}_j$ such that the edge $\tilde{\sigma}_i \cap \tilde{\sigma}_j$ is blue-colored then

$$d_{\sigma_i}(\tilde{v}) \neq 0, \quad d_{\sigma_j}(\tilde{v}) \neq 0$$

and $d_{\sigma_k}(\tilde{v}) = 0$ for distinct $i, j, k \in \{0, 1, 2\}$.

(IIc) If \tilde{v} is of Type II(c), then

$$d_{\sigma_i}(\tilde{v}) \neq 0$$

for all $i \in \{0, 1, 2\}$.

Definition 6.18. Let \tilde{G} be a well-colored subgraph of $\tilde{G}(\mathbb{L})$. Let $\tilde{\tau} := \tau'_1 \times_\pi \tau'_2 \in E(\tilde{G}_0)$ be an edge with vertices \tilde{v}, \tilde{u} and adjacent faces $\tilde{\sigma}, \tilde{\sigma}'$, and let $s_e, s_{e'}$ be the gluing data correspond to the morphisms $e : v \rightarrow \tau, e' : u \rightarrow \tau$. A data $D := \{\delta(\tilde{v})\}_{\tilde{v} \in V(\tilde{G})}$ is called compatible if the following conditions are satisfied:

(1) If $\tilde{\tau}$ has vertices $\tilde{v}, \tilde{u} \in V_3(\tilde{\Gamma})$ and two white-colored adjacent faces $\tilde{\sigma}, \tilde{\sigma}'$, then

$$\frac{s_e(m(\tilde{\sigma}))}{s_e(m(\tilde{\sigma}'))} \frac{d_\sigma(\tilde{v})}{d_{\sigma'}(\tilde{v})} = \frac{s_{e'}(m(\tilde{\sigma}))}{s_{e'}(m(\tilde{\sigma}'))} \frac{d_\sigma(\tilde{u})}{d_{\sigma'}(\tilde{u})},$$

where $m(\tilde{\sigma}), m(\tilde{\sigma}') \in \mathcal{Q}_\tau^*$ are the slopes of the piecewise linear function $\varphi_{\tau'_2} \circ \pi^{-1} - \varphi_{\tau'_1} \circ \pi^{-1}$ on the cones $K_\sigma, K_{\sigma'} \in \Sigma_\tau$ respectively.

(2) If $\tilde{\tau}$ is a blue edge with vertices $\tilde{v} \in V_3(\tilde{\Gamma}), \tilde{u} \in V_6(\tilde{\Gamma})$ and two white-colored adjacent faces $\tilde{\sigma}, \tilde{\sigma}'$, then

$$\frac{s_e(m(\tilde{\sigma}))}{s_e(m(\tilde{\sigma}'))} \frac{d_\sigma(\tilde{v})}{d_{\sigma'}(\tilde{v})} = - \frac{s_{e'}(m(\tilde{\sigma}))}{s_{e'}(m(\tilde{\sigma}'))} \frac{d_\sigma(\tilde{u})}{d_{\sigma'}(\tilde{u})}.$$

(3) If $\tilde{\tau}$ is a blue edge with vertices $\tilde{v}, \tilde{u} \in V_6(\tilde{\Gamma})$ and two white-colored adjacent faces $\tilde{\sigma}, \tilde{\sigma}'$, then

$$\frac{s_e(m(\tilde{\sigma}))}{s_e(m(\tilde{\sigma}'))} \frac{d_\sigma(\tilde{v})}{d_{\sigma'}(\tilde{v})} = \frac{s_{e'}(m(\tilde{\sigma}))}{s_{e'}(m(\tilde{\sigma}'))} \frac{d_\sigma(\tilde{u})}{d_{\sigma'}(\tilde{u})}.$$

Given a well-colored subgraph \tilde{G} and a compatible data D , we define

$$\mathcal{L}_{\tilde{G}}(D, \mathbf{a})|_{U_{\tilde{v}}} := \mathbb{C}\langle \tilde{v} \rangle,$$

and for $\tilde{v}, \tilde{u} \in V(\tilde{G})$ so that

$$U_{\tilde{v}} \cap U_{\tilde{u}} = \begin{cases} \text{Int}(\tilde{\tau}) \amalg \text{Int}(\tilde{\tau}') & \text{if } \tilde{v}, \tilde{u} \in V_6(\tilde{G}) \text{ such that } \tilde{v}, \tilde{u} \in \tilde{\tau} \cap \tilde{\tau}', \\ \text{Int}(\tilde{\tau}) & \text{otherwise} \end{cases}$$

for some $\tilde{\tau}, \tilde{\tau}' \in E(\tilde{G}_0)$ with $\pi_B(\tilde{\tau}) = \pi_B(\tilde{\tau}')$, we define

$$\lambda_{\tilde{v}\tilde{u}}(D, \mathbf{a}) := \lambda_{\tilde{v}\tilde{u}}(D, \sigma, \mathbf{a}) := \begin{cases} a_{\tilde{v}\tilde{u}} \frac{s_e(m(\tilde{\sigma}))}{s_{e'}(m(\tilde{\sigma}))} \frac{d_\sigma(\tilde{v})}{d_\sigma(\tilde{u})} & \text{if } \tilde{v}, \tilde{u} \in V_3(\tilde{\Gamma}), \\ a_{\tilde{v}\tilde{u}} \frac{s_e(m(\tilde{\sigma}))}{s_{e'}(m(\tilde{\sigma}))} \frac{d_\sigma(\tilde{v})}{d_\sigma(\tilde{u})} & \text{if } \tilde{v} \in V_3(\tilde{\Gamma}), \tilde{u} \in V_6(\tilde{\Gamma}), \\ a_{\tilde{v}\tilde{u}} \frac{d_\sigma(\tilde{u})}{d_\sigma(\tilde{v})} & \text{if } \tilde{v}, \tilde{u} \in V_6(\tilde{\Gamma}) \text{ and } \tilde{\tau} \text{ is a green edge,} \\ a_{\tilde{v}\tilde{u}} \frac{s_e(m(\tilde{\sigma}))}{s_{e'}(m(\tilde{\sigma}))} \frac{d_\sigma(\tilde{v})}{d_\sigma(\tilde{u})} & \text{if } \tilde{v}, \tilde{u} \in V_6(\tilde{\Gamma}) \text{ and } \tilde{\tau} \text{ is not a green edge,} \end{cases}$$

where $\tilde{\sigma}$ is any white-colored adjacent face of $\tilde{\tau}$ and $\mathbf{a} := \{a_{\tilde{v}\tilde{u}}\}$ is a set of non-zero constants that we will fix later. They are required to satisfy $a_{\tilde{v}\tilde{u}}a_{\tilde{u}\tilde{v}} = 1$. Clearly, $\lambda_{\tilde{v}\tilde{u}}(D, \mathbf{a})\lambda_{\tilde{u}\tilde{v}}(D, \mathbf{a}) = 1$. In this way we obtain a rank 1 local system

$$\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$$

on \tilde{G}_0 , which is independent of the white-colored cell we chose. Indeed, for each vertex $\tilde{v} \in V(\tilde{G})$, we define

$$f_{\tilde{v}} := \begin{cases} 1 & \text{if } \tilde{v} \in V_3(\tilde{G}), \\ -1 & \text{if } \tilde{v} \in V_6(\tilde{G}). \end{cases}$$

Then by Conditions (1), (2) and (3) in Definition 6.18, we have

$$\lambda_{\tilde{v}\tilde{u}}(D, \tilde{\sigma}, \mathbf{a})f_{\tilde{u}} = f_{\tilde{v}}\lambda_{\tilde{v}\tilde{u}}(D, \tilde{\sigma}', \mathbf{a}),$$

for all $\tilde{v}, \tilde{u} \in V(\tilde{G})$. Moreover, $\mathcal{L}_{\tilde{G}}(D, \mathbf{a}) \cong \mathcal{L}_{\tilde{G}}(D', \mathbf{a})$ as local systems if there exists a set of non-zero constants $\{c_{\tilde{v}}\}_{\tilde{v} \in V(\tilde{G})} \subset \mathbb{C}^\times$ such that $\delta(\tilde{v}) = c_{\tilde{v}}\delta'(\tilde{v})$ for all $\tilde{v} \in V(\tilde{G})$.

To relate $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ with $\mathcal{L}_{\tilde{G}}(\mathcal{S})$, for each $\tilde{v} \in V(\tilde{G})$, we associate a section $s_{\tilde{v}}$ of $\mathcal{E}(\tilde{v})$ such that the corresponding transition functions $\{\lambda_{\tilde{v}\tilde{u}}(\mathcal{S})\}$ are given by $\{\lambda_{\tilde{v}\tilde{u}}(D, \mathbf{a})\}$, for some choice of \mathbf{a} that only depends on \mathbb{L} and the gluing data $\{g_{\sigma\tau, s}\}_{\tau \subset \sigma}$ obtained in Theorem 5.6. Note that $X_{\tilde{v}} \cong \text{Proj}(\mathbb{C}[\tilde{P}_{\tilde{v}}])$, where \tilde{v} is the dual cell of $\{\tilde{v}\}$ and

$$\tilde{P}_{\tilde{v}} = \{(mr, r) \in \tilde{\Lambda}_{\tilde{v}} \oplus \mathbb{Z} \mid m \in \tilde{v}, r \geq 0\}.$$

When $\mathcal{E}(\tilde{v})$ is a line bundle, sections of $\mathcal{E}(\tilde{v})$ are degree 1 elements of $\mathbb{C}[P_{\tilde{v}}]$. Let $\tilde{\sigma}$ be a vertex of \tilde{v} . Put

$$\chi_{\tilde{\sigma}}(\tilde{v}) := z^{(\tilde{\sigma}, 1)},$$

which can be regarded as homogeneous coordinates of $X_{\tilde{v}}$. Let $\tilde{\tau}$ be the edge such that $\tilde{\sigma} \notin \tilde{\tau}$. Then $\chi_{\tilde{\sigma}}(\tilde{v})$ vanishes along the divisor corresponding to $\tilde{\tau}$. The assignment $\tilde{v} \mapsto s_{\tilde{v}}$ is as follows.

(Ia) If \tilde{v} is of Type I(a) and $\tilde{\sigma}$ is the unique white-colored adjacent face, we define

$$s_{\tilde{v}} := d_\sigma(\tilde{v})\chi_{\tilde{\sigma}}(\tilde{v}).$$

(Ib) If \tilde{v} is of Type I(b) such that $\tilde{\tau}_k$ has two white-colored adjacent faces $\tilde{\sigma}_i, \tilde{\sigma}_j$, for $i, j \neq k$, we define

$$s_{\tilde{v}} := d_{\sigma_i}(\tilde{v})\chi_{\tilde{\sigma}_i}(\tilde{v}) + d_{\sigma_j}(\tilde{v})\chi_{\tilde{\sigma}_j}(\tilde{v}).$$

(Ic) If \tilde{v} is of Type I(c), we define

$$s_{\tilde{v}} := d_{\sigma_0}(\tilde{v})\chi_{\tilde{\sigma}_0}(\tilde{v}) + d_{\sigma_1}(\tilde{v})\chi_{\tilde{\sigma}_1}(\tilde{v}) + d_{\sigma_2}(\tilde{v})\chi_{\tilde{\sigma}_2}(\tilde{v}).$$

(IIa) If \tilde{v} is of Type II(a) with two unique white-colored face $\tilde{\sigma}_1, \tilde{\sigma}_2$ so that the edge $\tilde{\sigma}_1 \cap \tilde{\sigma}_2$ is green, we define $s_{\tilde{v}}$ by

$$\begin{aligned} s_{\tilde{v}}|_{W_{\sigma_0}} &:= -\frac{1}{d_{\sigma_0}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_1}(\tilde{v})}{\chi_{\tilde{\sigma}_0}(\tilde{v})} e_0 - \frac{1}{d_{\sigma_0}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_2}(\tilde{v})}{\chi_{\tilde{\sigma}_0}(\tilde{v})} e'_0, \\ s_{\tilde{v}}|_{W_{\sigma_1}} &:= \frac{1}{d_{\sigma_0}(\tilde{v})} e'_1, \\ s_{\tilde{v}}|_{W_{\sigma_2}} &:= \frac{1}{d_{\sigma_0}(\tilde{v})} e'_2. \end{aligned}$$

Here, $W_{\sigma_i} := \text{Spec}(\mathbb{C}[\tilde{\sigma}_i \cap M]) = \{\chi_{\tilde{\sigma}_i}(\tilde{v}) \neq 0\}$ are the affine charts of X_v and $s_{\tilde{v}}$ is written in terms of the inhomogeneous coordinates $\chi_{\tilde{\sigma}_j}(\tilde{v})/\chi_{\tilde{\sigma}_i}(\tilde{v})$ of W_{σ_i} .

(IIb) If \tilde{v} is of Type II(b) with a pair of white-colored faces $\tilde{\sigma}_1, \tilde{\sigma}_2$ so that the edge $\tilde{\sigma}_1 \cap \tilde{\sigma}_2$ is blue-colored, we define $s_{\tilde{v}}$ by

$$\begin{aligned} s_{\tilde{v}}|_{W_{\sigma_0}} &:= \frac{1}{d_{\sigma_1}(\tilde{v})} e_0 + \frac{1}{d_{\sigma_2}(\tilde{v})} e'_0, \\ s_{\tilde{v}}|_{W_{\sigma_1}} &:= -\frac{1}{d_{\sigma_1}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_0}(\tilde{v})}{\chi_{\tilde{\sigma}_1}(\tilde{v})} e_1 + \left(\frac{1}{d_{\sigma_2}(\tilde{v})} - \frac{1}{d_{\sigma_1}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_2}(\tilde{v})}{\chi_{\tilde{\sigma}_1}(\tilde{v})} \right) e'_1, \\ s_{\tilde{v}}|_{W_{\sigma_2}} &:= -\frac{1}{d_{\sigma_2}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_0}(\tilde{v})}{\chi_{\tilde{\sigma}_2}(\tilde{v})} e_2 + \left(\frac{1}{d_{\sigma_1}(\tilde{v})} - \frac{1}{d_{\sigma_2}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_1}(\tilde{v})}{\chi_{\tilde{\sigma}_2}(\tilde{v})} \right) e'_2. \end{aligned}$$

(IIc) If \tilde{v} is of Type II(c), we define $s_{\tilde{v}}$ by

$$\begin{aligned} s_{\tilde{v}}|_{W_{\sigma_0}} &:= \left(\frac{1}{d_{\sigma_1}(\tilde{v})} - \frac{1}{d_{\sigma_0}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_1}(\tilde{v})}{\chi_{\tilde{\sigma}_0}(\tilde{v})} \right) e_0 + \left(\frac{1}{d_{\sigma_2}(\tilde{v})} - \frac{1}{d_{\sigma_0}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_2}(\tilde{v})}{\chi_{\tilde{\sigma}_0}(\tilde{v})} \right) e'_0, \\ s_{\tilde{v}}|_{W_{\sigma_1}} &:= \left(\frac{1}{d_{\sigma_0}(\tilde{v})} - \frac{1}{d_{\sigma_1}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_0}(\tilde{v})}{\chi_{\tilde{\sigma}_1}(\tilde{v})} \right) e_1 + \left(\frac{1}{d_{\sigma_2}(\tilde{v})} - \frac{1}{d_{\sigma_1}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_2}(\tilde{v})}{\chi_{\tilde{\sigma}_1}(\tilde{v})} \right) e'_1, \\ s_{\tilde{v}}|_{W_{\sigma_2}} &:= \left(\frac{1}{d_{\sigma_0}(\tilde{v})} - \frac{1}{d_{\sigma_2}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_0}(\tilde{v})}{\chi_{\tilde{\sigma}_2}(\tilde{v})} \right) e_2 + \left(\frac{1}{d_{\sigma_1}(\tilde{v})} - \frac{1}{d_{\sigma_2}(\tilde{v})} \frac{\chi_{\tilde{\sigma}_1}(\tilde{v})}{\chi_{\tilde{\sigma}_2}(\tilde{v})} \right) e'_2. \end{aligned}$$

The quotient of the characters are precisely transition functions of the line bundle $\mathcal{O}_{X_v}(1) \cong \mathcal{O}_{\mathbb{P}^2}(1)$, so we have

$$\iota_{\tau v, s}^* \left(\frac{\chi_{\tilde{\sigma}}(\tilde{v})}{\chi_{\tilde{\sigma}'}(\tilde{v})} \right) = \frac{s_e(m(\tilde{\sigma}))}{s_e(m(\tilde{\sigma}'))} \frac{\chi_{\tilde{\sigma}}(\tilde{\tau})}{\chi_{\tilde{\sigma}'}(\tilde{\tau})}$$

Then by using Conditions (1), (2) and (3) in Definition 6.18, one can see that $\iota_{\tau v, s}^* s_{\tilde{v}}, \iota_{\tau u, s}^* s_{\tilde{u}}$ share the same pseudo-zero locus on X_{τ} . For example, if \tilde{v} is of Type I(c) and \tilde{u} is of Type II(c), and $\tilde{\tau} \in E(\tilde{G})$ is a blue edge connecting \tilde{v}, \tilde{u} with white-colored adjacent faces $\tilde{\sigma}_1, \tilde{\sigma}_2$, then $X_{\tau} \cong \mathbb{P}^1$ is mapped into the divisors $\{\chi_{\tilde{\sigma}_0}(v) = 0\} \subset X_v$ and $\{\chi_{\tilde{\sigma}_0}(u) = 0\} \subset X_u$. We compute in terms of the inhomogeneous coordinate $\chi_{\tilde{\sigma}_2}(\tilde{\tau})/\chi_{\tilde{\sigma}_1}(\tilde{\tau})$ that

$$\begin{aligned} A_{\tilde{\tau}\tilde{v}, s}(\iota_{\tau v, s}^*(s_{\tilde{v}})) &= a_{\tilde{\tau}\tilde{v}} \left(d_{\sigma_1}(\tilde{v}) + d_{\sigma_2}(\tilde{v}) \frac{s_e(m(\tilde{\sigma}_2))}{s_e(m(\tilde{\sigma}_1))} \frac{\chi_{\tilde{\sigma}_2}(\tilde{\tau})}{\chi_{\tilde{\sigma}_1}(\tilde{\tau})} \right) \chi_{\tilde{\sigma}_1}(\tilde{\tau}), \\ A_{\tilde{\tau}\tilde{u}, s}(\iota_{\tau u, s}^*(s_{\tilde{u}})) &= \left(\frac{a_{\tilde{\tau}\tilde{u}}^-}{d_{\sigma_0}(\tilde{u})}, a_{\tilde{\tau}\tilde{u}}^+ \left(\frac{1}{d_{\sigma_2}(\tilde{u})} - \frac{1}{d_{\sigma_1}(\tilde{u})} \frac{s_e(m(\tilde{\sigma}_2))}{s_e(m(\tilde{\sigma}_1))} \frac{\chi_{\tilde{\sigma}_2}(\tilde{\tau})}{\chi_{\tilde{\sigma}_1}(\tilde{\tau})} \right) \chi_{\tilde{\sigma}_1}(\tilde{\tau}) \right), \end{aligned}$$

where

$$\begin{aligned} A_{\tilde{\tau}\tilde{v},s}^{-1} &:= g_{\tau'_1 v'_1, s}^* \otimes g_{\tau'_2 v'_2, s} : \mathcal{L}(\tau'_1)^* \otimes \mathcal{L}(\tau'_2) \rightarrow \iota_{\tau v, s}^* (\mathcal{L}(v'_1)^* \otimes \mathcal{L}(v'_2)), \\ A_{\tilde{\tau}\tilde{u},s}^{-1} &:= g_{\tau'_1 u'_1, s}^* \otimes g_{\tau'_2 u'_2, s} : \mathcal{L}(\tau'_1)^* \otimes \mathcal{L}(\tau'_2) \rightarrow \iota_{\tau u, s}^* (\mathcal{L}(u'_1)^* \otimes \mathcal{L}(u'_2)), \end{aligned}$$

and $g_{\tau'_i u'_i, s}, g_{\tau'_i v'_i, s}$ are components of the isomorphisms obtained in Theorem 5.6. As $a_{\tilde{\tau}\tilde{v}}, a_{\tilde{\tau}\tilde{u}}^+$ are non-zero, the sections $a_{\tilde{\tau}\tilde{v}} (\iota_{\tau v, s}^* (s_{\tilde{v}}))$ and $\Pi_{\tilde{\tau}+\tilde{u}} (a_{\tilde{\tau}\tilde{u}} (\iota_{\tau u, s}^* (s_{\tilde{u}})))$ (See Remark 6.11 for the notation $\Pi_{\tilde{\tau}\pm\tilde{u}}$) share the same zero locus along X_τ if and only if

$$d_{\sigma_1}(\tilde{v}) + d_{\sigma_2}(\tilde{v}) \frac{s_e(m(\tilde{\sigma}_2)) \chi_{\tilde{\sigma}_2}(\tilde{\tau})}{s_e(m(\tilde{\sigma}_1)) \chi_{\tilde{\sigma}_1}(\tilde{\tau})} \text{ and } \frac{1}{d_{\sigma_2}(\tilde{u})} - \frac{1}{d_{\sigma_1}(\tilde{u})} \frac{s_{e'}(m(\tilde{\sigma}_2)) \chi_{\tilde{\sigma}_2}(\tilde{\tau})}{s_{e'}(m(\tilde{\sigma}_1)) \chi_{\tilde{\sigma}_1}(\tilde{\tau})}$$

share the same zero locus on X_τ . Equivalently,

$$d_{\sigma_1}(\tilde{v}) \left(-\frac{1}{d_{\sigma_1}(\tilde{u})} \frac{s_{e'}(m(\tilde{\sigma}_2))}{s_{e'}(m(\tilde{\sigma}_1))} \right) = \frac{1}{d_{\sigma_2}(\tilde{u})} \left(d_{\sigma_2}(\tilde{v}) \frac{s_e(m(\tilde{\sigma}_2))}{s_e(m(\tilde{\sigma}_1))} \right)$$

which is Condition (2) in Definition 6.18 after reordering terms. Hence the corresponding $\{\lambda_{\tilde{v}\tilde{u}}(\mathcal{S})\}$ are exactly given by $\{\lambda_{\tilde{v}\tilde{u}}(D, \mathbf{a})\}$, with $\mathbf{a} = \{a_{\tilde{v}\tilde{u}}\}$ being given by the quotient of the constants $\{a_{\tilde{\tau}\tilde{v}}, a_{\tilde{\tau}\tilde{v}}^\pm\}$ and so $\mathcal{L}_{\tilde{G}}(\mathcal{S}) \cong \mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ as local systems. The converse is also true, namely, given a set of sections $\mathcal{S} = \{s_{\tilde{v}}\}_{\tilde{v} \in V(\tilde{G})}$ whose vanishing loci respect the coloring of \tilde{G} , and for any edge $\tilde{\tau}$ with vertices \tilde{v}, \tilde{u} , $s_{\tilde{v}}, s_{\tilde{u}}$ share the same pseudo-zero locus on X_τ , one can obtain a compatible data D by writing $s_{\tilde{v}}$ as a linear combination of characters in $X_{\tilde{v}}$ as above. We summarize by the following:

Proposition 6.19. *Given a well-colored subgraph $\tilde{G} \subset \tilde{G}(\mathbb{L})$. There is a 1-1 correspondence between the set of compatible data on \tilde{G} and the set of admissible sections on \tilde{G} . Moreover, the assignment $\tilde{v} \mapsto s_{\tilde{v}}$ above induces an isomorphism of local systems $\mathcal{L}_{\tilde{G}}(D, \mathbf{a}) \cong \mathcal{L}_{\tilde{G}}(\mathcal{S})$.*

6.2.4. *The main results.* We are now ready to prove the main result of this section, which gives a combinatorial description of sections of $End_0(\mathcal{E}(\mathbb{L}))$ for $\mathbb{L} \in \mathcal{S}_{n+1, n}$.

Theorem 6.20. *Let $\mathbb{L} \in \mathcal{S}_{n+1, n}$. Define*

$$\mathcal{WC}(\mathbb{L}) := \{\tilde{G} \subset \tilde{G}(\mathbb{L}) \mid \tilde{G} \text{ is well-colored}\},$$

and for each $\tilde{G} \in \mathcal{WC}(\mathbb{L})$, define

$$\mathcal{D}_{\tilde{G}} := \{D \mid \mathcal{L}_{\tilde{G}}(D, \mathbf{a}) \text{ is trivial}\}.$$

Then, for any $\tilde{G} \in \mathcal{WC}(\mathbb{L})$ and $D \in \mathcal{D}_{\tilde{G}}$, there is an injection

$$i_{\mathcal{L}_{\tilde{G}}(D, \mathbf{a})} : H^0(\tilde{G}_0, \mathcal{L}_{\tilde{G}}(D, \mathbf{a})) \hookrightarrow H^0(X_0(B, \mathcal{P}, s), End_0(\mathcal{E}_0(\mathbb{L})))$$

such that

$$H^0(X_0(B, \mathcal{P}, s), End_0(\mathcal{E}_0(\mathbb{L}))) = \bigcup_{\tilde{G} \in \mathcal{WC}(\mathbb{L})} \bigcup_{D \in \mathcal{D}_{\tilde{G}}} Im \left(i_{\mathcal{L}_{\tilde{G}}(D, \mathbf{a})} \right).$$

Proof. Suppose we have a well-colored subgraph $\tilde{G} \subset \tilde{G}(\mathbb{L})$ and a compatible data D such that the local system $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ is trivial. Let $\bar{s} \in H^0(\tilde{G}_0, \mathcal{L}_{\tilde{G}}(D, \mathbf{a}))$. Representing \bar{s} as a Čech 0-cocycle $\{c_{\tilde{v}}\}_{\tilde{v} \in V(\tilde{G})} \subset \mathbb{C}$, we have

$$c_{\tilde{u}} = c_{\tilde{v}} \lambda_{\tilde{v}\tilde{u}}.$$

Define

$$s|_{X_{\tilde{v}}} := c_{\tilde{v}} s_{\tilde{v}}.$$

This gives a section s of $\mathcal{E}nd_{\tilde{G}} \rightarrow X_{\tilde{G}}$, whose pseudo-zero locus is prescribed by the red-colored cells of \tilde{G} . In particular, s vanishes along the divisor $\bigcup_{\tilde{e} \in H(\tilde{G})} X_{\tilde{e}}$. Hence we can extend s by zero to obtain a non-trivial section of $H^0(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}_0(\mathbb{L})))$. Put

$$i_{\mathcal{L}_{\tilde{G}}(D, \mathbf{a})}(\bar{s}) := s.$$

This gives the injection

$$i_{\mathcal{L}_{\tilde{G}}(D, \mathbf{a})} : H^0(\tilde{G}_0, \mathcal{L}_{\tilde{G}}(D, \mathbf{a})) \hookrightarrow H^0(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}_0(\mathbb{L}))).$$

Now let $s \in H^0(X_0(B, \mathcal{P}, s), \text{End}_0(\mathcal{E}(\mathbb{L})))$. Consider the subgraph $\tilde{G} \subset \tilde{G}(\mathbb{L})$ such that $\tilde{v} \in V(\tilde{G})$ if and only if $\Pi_{\tilde{v}}(s) \neq 0$ and $\tilde{e} \in E(\tilde{G})$ if and only if \tilde{e} is connecting two vertices in $V(\tilde{G})$. Let $\tilde{\tau} \in E(\tilde{G}) \amalg H(\tilde{G})$ and $\tilde{\sigma} \in F(\tilde{G})$.

- Color $\tilde{\sigma}$ by red if $X_{\tilde{\sigma}}$ lies in the pseudo-locus of $\Pi_{\tilde{v}}(s)$ for some vertex $\tilde{v} \in \tilde{\sigma}$. Color the remaining 2-cells by white.
- Color $\tilde{\tau}$ by red if $\tilde{\tau}$ has two red-colored adjacent faces.
- Color $\tilde{\tau}$ by blue if $\tilde{v} \in \tilde{\tau}$ for some $\tilde{v} \in V(\tilde{G}) \cap V_3(\tilde{\Gamma})$ and has two white-colored adjacent faces or if $\tilde{\tau} = \tilde{\tau}^+$ (See (4) for the definition of $\tilde{\tau}^\pm$) for some $\tilde{v} \in V(\tilde{G}) \cap V_6(\tilde{\Gamma})$ such that $\Pi_{\tilde{\tau}+\tilde{v}}(\iota_{\tilde{\tau}, \tilde{v}}^* \Pi_{\tilde{v}}(s))$ (See Remark 6.11 for definition of $\Pi_{\tilde{\tau}+\tilde{v}}$) is non-trivial.
- Color $\tilde{\tau}$ by green if $\tilde{\tau} = \tilde{\tau}^-$ for some $\tilde{v} \in V(\tilde{G}) \cap V_6(\tilde{\Gamma})$ such that $\Pi_{\tilde{\tau}-\tilde{v}}(\iota_{\tilde{\tau}, \tilde{v}}^* \Pi_{\tilde{v}}(s))$ is non-trivial.
- Color the remaining edges by black.

By the geometry of pseudo-zero loci of sections of $\mathcal{O}_{\mathbb{P}^2}(1), E_{n+1, n}(-n) \cong E_{1,0} \cong E_{n+1, n}^*(n+1)$, Conditions (2a)-(2d), (2(e)i)-(2(e)iii), (2(f)i)-(2(f)iii) are all satisfied. See Figure 5 for the tropical image of different types of pseudo-zero loci and Figure 4 for their dual. For Condition (2g), let \tilde{e} be a half-edge of \tilde{G} that contains a vertex $\tilde{u} \in V(\tilde{G})$. By the definition of a half-edge, \tilde{e} must connect \tilde{u} to a point $\tilde{u}' \in V(\tilde{G}(\mathbb{L})) \setminus V(\tilde{G})$. Then by the definition of \tilde{G} , we have $\Pi_{\tilde{u}'}(s) \equiv 0$. In particular, the two adjacent faces of \tilde{e} must be red-colored. Hence we get Condition (2g). Clearly s induces a compatible data D such that the associated local system $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ is trivial. Therefore, s lies in the image of $i_{\mathcal{L}_{\tilde{G}}(D, \mathbf{a})}$. \square

Remark 6.21. *As we have mentioned in the introduction, the space $H^0(X_0(B, \mathcal{P}, s), \text{End}(\mathcal{E}_0(\mathbb{L})))$ can be identified with the 0-th cohomology of a constructible sheaf \mathcal{F} on $P(\mathbb{L})$. Via the inclusion $\tilde{G}_0 \subset P(\mathbb{L})$, the local system $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ can be regarded as a subsheaf of \mathcal{F} supported on $|\tilde{G}_0|$.*

Example 6.22. *Suppose the gluing data s is trivial and \tilde{G} has no Type II(c) vertices. Then we can set $d_\sigma(\tilde{v}) = 1$ whenever \tilde{v} is not of Type II(b). For a Type II(b) vertex \tilde{v} , we can set $d_\sigma(\tilde{v}) = 1, d_{\sigma'}(\tilde{v}) = -1$, whenever σ, σ' are images of white-colored adjacent faces of \tilde{v} under π_B . The data $\mathcal{D} := \{\delta(\tilde{v})\}$ is then compatible and hence defines a local system $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$. If $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ is trivial, sections of $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ correspond to those of $\text{End}_0(\mathcal{E}_0(\mathbb{L}))$ whose pseudo-zero loci have tropical images being red edges, red vertices and midpoints of blue edges (see Figure 5).*

Theorem 6.20 leads us to make the following

Definition 6.23. *A tropical Lagrangian multi-section $\mathbb{L} \in \mathcal{S}_{n+1, n}$ is called simple if for any well-colored subgraph $\tilde{G} \subset \tilde{G}(\mathbb{L})$ and compatible data D , the local system $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ is non-trivial.*

Remark 6.24. *The proof of Theorem 6.4 show that if $\tilde{\gamma} \subset \tilde{G}(\mathbb{L})$ is a minimal cycle whose vertices are all in $V_3(\tilde{\Gamma})$, then there exists a compatible data D such that $\mathcal{L}_{\tilde{\gamma}}(D, \mathbf{a})$ is trivial.*

Remark 6.25. *Definition 6.23 is equivalent to Definition 6.3 when $r = 2$. For $r = 2$, the projections $\pi_1, \pi_2 : P_0(\mathbb{L}) \rightarrow L \setminus S'$ define two homeomorphisms and hence $\pi_B|_{P_0(\mathbb{L})} : P_0(\mathbb{L}) \rightarrow B \setminus S$ can be identified with the unbranched covering map $\pi|_{L \setminus S'} : L \setminus S' \rightarrow B \setminus S$. Thus the graph $\tilde{G}(\mathbb{L})$ can be regarded as a graph in $L \setminus S'$ and it has at most trivalent vertices. By Condition (2) in Definition 6.8, we only take half of the vertices in defining $\tilde{G}(\mathbb{L})$, so $\tilde{G}(\mathbb{L})$ and $G(\mathbb{L})$ are actually homeomorphic via $\pi|_{L \setminus S'}$. Moreover, we have already seen that any minimal cycle $\gamma \subset G(\mathbb{L})$ with trivalent vertices is well-colored and always carry a compatible data D such that $\mathcal{L}_\gamma(D, \mathbf{a})$ is trivial. This explains why, when $r = 2$, we just need non-existence of minimal cycles to define simplicity of \mathbb{L} .*

Theorem 6.20 implies the following

Theorem 6.26. *A tropical Lagrangian multi-section $\mathbb{L} \in \mathcal{S}_{n+1, n}$ is simple if and only if $\mathcal{E}_0(\mathbb{L})$ is simple.*

Again, by applying Serre duality and Corollary 4.7 of [2], we obtain the following

Corollary 6.27. *If $\mathbb{L} \in \mathcal{S}_{n+1, n}$ is simple, then the pair $(X_0(B, \mathcal{P}, s), \mathcal{E}_0(\mathbb{L}))$ is smoothable.*

Finally, let us briefly discuss how one may define simplicity of $\mathbb{L} \in \mathcal{S}_{m, n}$ for $m \geq n + 2$. For $m \geq n + 2$, pseudo-zero loci of sections of $\mathcal{O}_{\mathbb{P}^2}(m - n)$ have more combinatorial types as m grows. For instance, there are 15 types of pseudo-zero loci when $m = n + 2$. One can use the transition functions given in (3) to study sections of $E_{m, n}(-n)$ and classify their pseudo-zero loci. Afterwards, one can modify the definition of well-colored subgraphs and compatible data according to the pseudo-zero loci of $\mathcal{O}_{\mathbb{P}^2}(m - n)$ and $E_{m, n}(-n)$ and also define a local system $\{\mathcal{L}_{\tilde{G}}(D, \mathbf{a})\}$ so that its triviality corresponds to a 1-parameter family of sections of $End_0(\mathcal{E}(\mathbb{L}))$ as in Theorem 6.20. In the case $m > n + 2$, one can easily deduce that $\mathcal{E}_0(\mathbb{L})$ is simple if and only if for any well-colored subgraph $\tilde{G} \subset \tilde{G}(\mathbb{L})$ which has only 6-valent vertices and any compatible data D on \tilde{G} , the local system $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ is non-trivial. For $m = n + 2$, $\mathcal{E}_0(\mathbb{L})$ is simple if and only if for any well-colored subgraph $\tilde{G} \subset \tilde{G}(\mathbb{L})$ whose edges have at least one 6-valent vertex and any compatible data D on \tilde{G} , the local system $\mathcal{L}_{\tilde{G}}(D, \mathbf{a})$ is non-trivial.

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