LAGRANGIAN SECTIONS ON MIRRORS OF TORIC CALABI–YAU 3-FOLDS

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Abstract. We construct Lagrangian sections of a Lagrangian torus fibration on a 3-dimensional conic bundle, which are SYZ dual to holomorphic line bundles over the mirror toric Calabi–Yau 3-fold. We then demonstrate a ring isomorphism between the wrapped Floer cohomology of the zero-section and the regular functions on the mirror toric Calabi–Yau 3-fold.

1. Introduction

The goal of this paper is to provide evidence for and connect homological mirror symmetry [Kon95] and the Strominger–Yau–Zaslow (SYZ) conjecture [SYZ96] in the context of *3-dimensional conic bundles* of the form

$$Y = \left\{ (w_1, w_2, u, v) \in (\mathbb{C}^\times)^2 \times \mathbb{C}^2 \;\middle|\; h(w_1, w_2) = uv \right\}. \tag{1.0.1}$$

Mirror symmetry for such varieties goes back at least to [HIV]. The mirror partner $\check{Y}$ is an open subvariety of a 3-dimensional toric variety whose fan is the cone over a triangulation of the Newton polytope of the Laurent polynomial $h(w_1, w_2)$. These examples have proven to be a fertile testing ground for mathematical thinking on mirror symmetry as well, a pioneering example being [KS02, ST01].

These conic bundles have the distinguishing feature that they are among the few examples of Calabi–Yau threefolds which admit explicit Lagrangian fibrations. An intriguing feature of these examples is that the Lagrangian fibrations are only piecewise smooth and have codimension one discriminant locus and thus exhibit important features of the general case. There have therefore been a number of recent papers focusing on these examples through the lens of the SYZ conjecture. The essential idea for constructing such fibrations appears in [Gol01, Gro01a, Gro01b], and the construction has been carried out in [CBM05, CBM09, AAK]. The details of this construction with some modifications are presented in Section 2.

The manifolds Y are affine varieties and therefore have natural symplectic forms and Liouville structures. On the other hand, in algebro-geometric terms, Y is the *Kaliman modification* of $(\mathbb{C}^\times)^2$ along the hypersurface $Z \subset (\mathbb{C}^\times)^2$ defined by $h(w_1, w_2) = 0$ (see e.g. [Zai]). When viewed from this perspective, there is an alternative symplectic form used in [AAK], which arises by viewing Y as an open submanifold of the symplectic blow-up $\overline{Y}$ of $(\mathbb{C}^\times)^2 \times \mathbb{C}$ along the codimension two subvariety $Z \times 0$. This form is convenient for the purposes of the SYZ conjecture because one can construct a Lagrangian torus fibration whose behavior is easy to understand. We modify their construction by degenerating the hypersurface Z to a certain tropical localization introduced in [Abo06]. This modification plays an important role throughout the rest of the paper as we explain below.

The starting point for our work is the observation that the SYZ fibration comes equipped with certain natural *base-admissible Lagrangian sections*. A related construction of Lagrangian sections has been carried out independently by Gross and Matessi in

[GM], though we place an emphasis on the asymptotics of these sections since these are important for Floer cohomology. In Section 3, we prove the following theorem:

Theorem 1.1. *For every holomorphic line bundle $\mathcal{F}$ on the SYZ mirror $\check{Y}$, there is a base-admissible Lagrangian section $L_{\mathcal{F}}$ in Y whose SYZ transform is $\mathcal{F}$. Furthermore, such Lagrangian submanifolds are unique up to Hamiltonian isotopy.*

For the purposes of mirror symmetry for non-compact manifolds such as Y, it is essential to consider the *wrapped Floer cohomology* of non-compact Lagrangian submanifolds. The foundational theory of wrapped Fukaya categories on Liouville domains has been developed in [AS10]. However, wrapped Floer cohomology is very difficult to calculate directly and there are only a few computations in the literature. One of the main objectives of this paper is to prove the following theorem:

Theorem 1.2. *Let L_0 denote the zero-section of the fibration which is SYZ dual to the structure sheaf $\mathcal{O}_{\check{Y}}$. Then there exists an isomorphism*

$$\mathrm{HW}^*(L_0) \cong H^0(\mathcal{O}_{\check{Y}}) \tag{1.2.1}$$

of $\mathbb{C}$-algebras.

The wrapped Floer cohomology ring on the left hand side of (1.2.1) comes with a natural basis given by Hamiltonian chords. As suggested by Tyurin and emphasized by Gross, Hacking, Keel and Siebert [GHK11, GS12], the images of these Hamiltonian chords on the right hand side of (1.2.1) give generalizations of theta functions on Abelian varieties. The proof of Theorem 1.2, which is a direct consequence of Theorem 1.3 and Theorem 1.4 below, is a higher dimensional analogue of [Pas14]. However, the approach taken in this paper is slightly different. While Pascaleff's argument exploits a certain TQFT structure for Lefschetz fibrations due to Seidel, our argument is based upon a study of how Floer theory behaves under Kaliman modification. Though this birational geometry viewpoint governs our approach, we do not study this problem in maximal generality, but limit ourselves to this series of examples.

The first step in our argument is to construct a suitable family of Hamiltonians which are well behaved with respect to the conic bundle structure. The essential technical ingredient for wrapped Floer theory is the existence of suitable C^0-estimates for solutions to Floer's equation. However, base-admissible Lagrangian sections do not naturally fit into the setup of [AS10] and, for this reason, we adapt their theory of wrapped Floer cohomology to the above choice of symplectic forms and base-admissible Lagrangian sections.

In our setup, there are now two directions in which Floer curves can escape to infinity, namely, in the "base direction" and in the "fiber direction" of the conic fibration. We show that, outside of a compact set in the base, the Hamiltonian flow projects nicely to the Hamiltonian flow of a degree one Hamiltonian on the base. It follows that there is a maximum principle for solutions to Floer's equation with boundary on admissible Lagrangians which prevents curves from escaping to infinity in the "base direction". On the other hand, it is more difficult to prevent curves from escaping to infinity in the "fiber direction" of the conic fibration. In fact, curves can indeed escape to infinity; however, we show in Lemma 5.20 that they must break along *divisor chords*, which are Hamiltonian chords living completely inside the divisor at infinity. There is a natural auxiliary grading *relative* to the exceptional divisor E. The key observation is that, with respect to this grading, the grading of the divisor Hamiltonian chords becomes arbitrarily large as m gets larger. By restricting to generators for the Floer complex of H_m whose relative grading lies in a certain range, we are therefore able to exclude this breaking and obtain the necessary compactness needed to define wrapped Floer groups.

We now turn to the computational aspect of our paper. The zero-section L_0 is a base-admissible Lagrangian and so it makes sense to consider its adapted wrapped Floer ring $\mathrm{HW}^*_{\mathrm{ad}}(L_0)$. In Section 6, we prove the following theorem:

Theorem 1.3. *There exists an isomorphism*

$$\mathrm{HW}^*_{\mathrm{ad}}(L_0) \cong H^0(\mathcal{O}_{\check{Y}}) \tag{1.3.1}$$

of $\mathbb{C}$*-algebras.*

One case where wrapped Floer cohomology can sometimes be directly computed is in manifolds which are products of lower-dimensional manifolds and where the Lagrangians and Hamiltonians split according to the product structures. An important advantage of base-admissible Lagrangians is that they live away from the exceptional locus and hence can be regarded as Lagrangians in either $(\mathbb{C}^\times)^2 \times \mathbb{C}^\times$ or Y. When viewed as living in $(\mathbb{C}^\times)^2 \times \mathbb{C}^\times$, these Lagrangians respect the product structure. Moreover, the admissible Hamiltonians we use interplay nicely with this product structure and hence the Floer theory of base-admissible Lagrangians in $(\mathbb{C}^\times)^2 \times \mathbb{C}^\times$ is amenable to direct calculation.

In the case of L_0, we find that all Hamiltonian chords lie away from the exceptional locus and moreover that as a vector space, the Floer cohomologies agree when regarded as living in $(\mathbb{C}^\times)^2 \times \mathbb{C}^\times$ or Y. However, the product structure on Floer cohomology is deformed. This deformation can be formalized in terms of the *relative Fukaya category* of Seidel and Sheridan [Sei02, She11]. One slightly novel feature is that in typical situations, one works relative to a compactifying divisor at infinity, while here we work relative to the exceptional divisor E. In this case, one computes the deformed ring structure directly by exploiting the correspondence between holomorphic curves in $(\mathbb{C}^\times)^2 \times \mathbb{C}$ with incidence conditions relative to the submanifold $Z \times 0$ and holomorphic curves in Y (one direction of this correspondence is given by projection and the other direction is given by proper transform). The relevant enumerative calculation is then done in Lemma 6.15 based upon a simple degeneration of domain argument.

It is not difficult to see that the Lagrangian L_0 is also admissible in the sense of [AS10] when Y is equipped with its natural Stein structure. The final result in this paper is the following comparison theorem between the two types of Floer theories:

Theorem 1.4. *There exists an isomorphism*

$$\mathrm{HW}^*(L_0) \cong \mathrm{HW}_{\mathrm{ad}}(L_0) \tag{1.4.1}$$

of $\mathbb{C}$*-algebras.*

The proof of this theorem is a modification of [McL09, Theorem 5.5], which applies to Lefschetz fibrations. While his arguments do not generalize to general symplectic fibrations over higher dimensional bases, certain simplifying features of our situation allow us to adapt his arguments in a straightforward way.

We should emphasize that we do not consider any higher A_∞-operations or prove any version of homological mirror symmetry in this paper. However, in the simplest case when our polynomial $h(w_1, w_2)$ is $1+w_1+w_2$ (this corresponds to the case when the toric Calabi–Yau mirror is an affine space), it is possible to show that the Lagrangian L_0 generates the wrapped Fukaya category. A simple argument using the McKay correspondence then allows one to extend this generation result to toric Calabi–Yau varieties which are related by a variation of GIT quotients to orbifold quotients $\mathbb{C}^3/G$. We will describe this and some consequences of these generation results for symplectic topology and spaces of stability conditions in a separate paper.

Returning to the general case of a toric Calabi–Yau manifold, there are two natural ways in which one might try to extend our results to a homological mirror symmetry statement. The first is to establish a version of homological mirror symmetry between a suitable Fukaya categories generated by base-admissible Lagrangians and categories of coherent sheaves on the mirror $\check{Y}$. Base admissible Lagrangians L_i naturally fiber over Lagrangians $\underline{L}_i$ in $(\mathbb{C}^\times)^2$. The calculation of A_∞-operations can likely be reduced using the approach of Section 6 to the enumeration of Floer polygons with boundary on $\underline{L}_i$. It seems reasonable to hope that these latter curves can be put in bijection with tropical curves similar to [Abo09].

Let $\mathbf{D}$ be the union of the toric divisors in $\check{Y}$. A different direction of research begins with the observation that we have a Bousfield localization sequence

$$0 \to \mathrm{QC}_{\mathbf{D}}(\check{Y}) \to \mathrm{QC}(\check{Y}) \to \mathrm{QC}(\check{Y} \setminus \mathbf{D}) \to 0, \tag{1.4.2}$$

where $\mathrm{QC}(\check{Y})$ is the unbounded derived category of quasi-coherent sheaves on $\check{Y}$ and $\mathrm{QC}_{\mathbf{D}}(\check{Y})$ is the full subcategory consisting of objects whose cohomologies are supported on $\mathbf{D}$. Since $D^b \operatorname{coh}_{\mathbf{D}} \check{Y}$ generates $\mathrm{QC}_{\mathbf{D}}(\check{Y})$ and $\check{Y} \setminus \mathbf{D}$ is affine, the bounded derived category $D^b \operatorname{coh} \check{Y}$ is split-generated by $\mathcal{O}_{\check{Y}}$ and $D^b \operatorname{coh}_{\mathbf{D}} \check{Y}$.

This paper describes the subcategory of the wrapped Fukaya category generated by L_0. Furthermore, SYZ mirror symmetry predicts that there is a full-subcategory of this wrapped Fukaya category consisting of Liouville admissible Lagrangians which is equivalent to $D^b \operatorname{coh}_{\mathbf{D}}(\check{Y})$. These Lagrangians are again constructed using ideas from [Abo09] and a similar construction has already appeared in [GM].

For example, when $\check{Y}$ is the total space of the canonical line bundle over a toric Fano variety, we expect to have a collection of objects consisting of L_0 and compact Lagrangian spheres which split-generate the wrapped Fukaya category. We expect that the results in [Abo09] together with an analysis similar to that in [Sei10] would give a direct approach to studying the Floer cohomology of these Lagrangian spheres. Combining this with the results of this paper then gives an approach to studying the wrapped Fukaya categories for general $h(w_1, w_2)$.

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2. Lagrangian torus fibrations

2.1. **Tropical hypersurface.** Let $N = \mathbb{Z}^2$ be a free abelian group of rank 2, and $M = \operatorname{Hom}(N, \mathbb{Z})$ be the dual group. A *convex lattice polygon* in $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is the convex hull of a finite subset of M. Let Δ be a convex lattice polygon in $M_{\mathbb{R}}$ and $A = \Delta \cap M$ be the set of lattice points in Δ. A function $\nu \colon A \to \mathbb{R}$ defines a piecewise-linear function $\overline{\nu} \colon \Delta \to \mathbb{R}$ by the condition that

$$\operatorname{Conv}\{(\boldsymbol{\alpha}, u) \in A \times \mathbb{R} \mid u \geq \nu(\boldsymbol{\alpha})\} = \{(\boldsymbol{m}, u) \in \Delta \times \mathbb{R} \mid u \geq \overline{\nu}(\boldsymbol{\alpha})\}. \tag{2.1.1}$$

The set $\mathcal{P}_\nu$ consisting of maximal domains of linearity of $\overline{\nu}$ and their faces is a polyhedral decomposition of Δ. A polyhedral decomposition $\mathcal{P}$ of Δ is *coherent* (or *regular*) if there is a function $\nu\colon A \to \mathbb{Z}$ such that $\mathcal{P} = \mathcal{P}_\nu$. It is a *triangulation* if all the maximal-dimensional faces are triangles. A coherent triangulation is *unimodular* if every triangle can be mapped to the standard simplex (i.e., the convex hull of $\{(0,0),(1,0),(0,1)\}$) by the action of $\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

Let $\mathcal{P}$ be a unimodular coherent triangulation of Δ. A function $\nu\colon A \to \mathbb{R}$ is *adapted* to $\mathcal{P}$ if $\mathcal{P} = \mathcal{P}_\nu$. Given a function $\nu\colon A \to \mathbb{R}$ and an element $(c_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in A} \in \mathbb{C}^A$, a *patchworking polynomial* is defined by

$$h_t(\boldsymbol{w}) = \sum_{\boldsymbol{\alpha}\in A} c_{\boldsymbol{\alpha}} t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} \in \mathbb{C}[t^{\pm 1}][M]. \tag{2.1.2}$$

For a positive real number $t \in \mathbb{R}^{>0}$, a hypersurface Z_t of $N_{\mathbb{C}^\times} := N_{\mathbb{R}}/N \cong \operatorname{Spec}\mathbb{C}[M]$ is defined by

$$Z_t = \{\boldsymbol{w} \in N_{\mathbb{C}^\times} \mid h_t(\boldsymbol{w}) = 0\}. \tag{2.1.3}$$

The image

$$\Pi_t := \mathrm{Log}(Z_t) \tag{2.1.4}$$

of Z_t by the logarithmic map

$$\mathrm{Log}\colon N_{\mathbb{C}^\times} \to N_{\mathbb{R}}, \quad \boldsymbol{w} = (w_1, w_2) \mapsto \frac{1}{\log|t|}\left(\log|w_1|, \log|w_2|\right) \tag{2.1.5}$$

is called the *amoeba* of Z_t. The *tropical polynomial* associated with $h_t(\boldsymbol{w})$ is the piecewise-linear map defined by

$$L_\nu\colon N_{\mathbb{R}} \to \mathbb{R}, \quad \boldsymbol{n} \mapsto \max\left\{\boldsymbol{\alpha}(\boldsymbol{n}) - \nu(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in A\right\}. \tag{2.1.6}$$

The *tropical hypersurface* (or the *tropical curve*) associated with L_ν is defined as the locus $\Pi_\infty \subset N_{\mathbb{R}}$ where L_ν is not differentiable. The polyhedral decomposition of $N_{\mathbb{R}}$ defined by the tropical hypersurface Π_∞ is dual to the triangulation $\mathcal{P}$ of Δ. In particular, the connected components of the complement of Π_∞ can naturally be labeled by $\mathcal{P}^{(0)} = A$;

$$N_{\mathbb{R}} \setminus \Pi_\infty = \coprod_{\boldsymbol{\alpha}\in A} C_{\boldsymbol{\alpha},\infty}. \tag{2.1.7}$$

The amoeba Π_t converges to Π_∞ in the Hausdorff topology as t goes to infinity [Mik04, Rul].

We next introduce some terminology and notation which will be used throughout the rest of this paper. A *leg* is a one-dimensional polyhedral subcomplex of the tropical hypersurface Π_∞ which is non-compact. In what follows, we will use the variables r_i to denote $\log|w_i|$. Let $\{\Pi_i\}_{i=1}^{\ell}$ be the set of legs of the tropical curve Π_∞. For each leg Π_i, we write the endpoints of the edge in $\mathcal{P}$ dual to Π_i as $\boldsymbol{\alpha}_i = (\boldsymbol{\alpha}_{i,1}, \boldsymbol{\alpha}_{i,2}), \boldsymbol{\beta}_i = (\boldsymbol{\beta}_{i,1}, \boldsymbol{\beta}_{i,2}) \in A$, so that Π_i is defined by

$$\begin{cases} \boldsymbol{r}_{\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i} := (\alpha_{i,1} - \beta_{i,1}) r_1 + (\alpha_{i,2} - \beta_{i,2}) r_2 = \nu(\boldsymbol{\alpha}_i) - \nu(\boldsymbol{\beta}_i), \\ \boldsymbol{r}_{(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp} := (\beta_{i,2} - \alpha_{i,2}) r_1 + (\alpha_{i,1} - \beta_{i,1}) r_2 \geq a_i, \end{cases} \tag{2.1.8}$$

for some $a_i \in \mathbb{R}$. The maximal polyhedral subcomplex of Π_∞ which is compact will be denoted by $\Pi_{\infty,c}$.

Example 2.2. A prototypical example is

$$h_t(\boldsymbol{w}) = w_1 + w_2 + \frac{1}{w_1 w_2} + t^{\varepsilon} \tag{2.2.1}$$

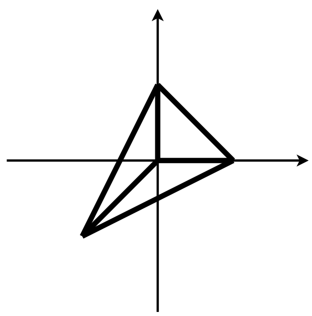

FIGURE 2.1. A triangulation

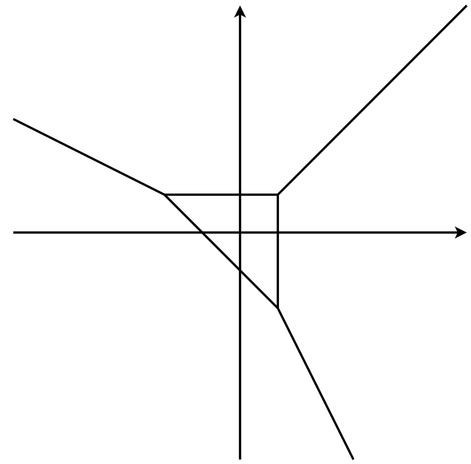

FIGURE 2.2. A tropical curve

for a small positive real number ε. The corresponding tropical polynomial is given by

$$L(\boldsymbol{n}) = \max\{n_1, n_2, -n_1 - n_2, \varepsilon\}, \tag{2.2.2}$$

and the tropical curve Π_∞ is shown in Figure 2.2. The set

$$A = \{(0,0), (1,0), (0,1), (-1,-1)\} \tag{2.2.3}$$

consists of four elements, corresponding to four connected components of $N_\mathbb{R} \setminus \Pi_\infty$. The tropical curve Π_∞ has three legs.

2.3. **Tropical localization.** Following [Abo06, Section 4], we set $C_{\boldsymbol{\alpha},t} := (\log t) \cdot C_{\boldsymbol{\alpha}}$ for each $\boldsymbol{\alpha} \in A$, and choose $\phi_{\boldsymbol{\alpha}} \colon N_\mathbb{R} \to \mathbb{R}$ such that

- $d(\boldsymbol{n}, C_{\boldsymbol{\alpha},t}) := \min_{\boldsymbol{n}' \in C_{\boldsymbol{\alpha},t}} \|\boldsymbol{n} - \boldsymbol{n}'\| \le (\varepsilon \log t)/2$ if and only if $\phi_{\boldsymbol{\alpha}}(\boldsymbol{n}) = 0$,
- $d(\boldsymbol{n}, C_{\boldsymbol{\alpha},t}) \ge \varepsilon \log t$ if and only if $\phi_{\boldsymbol{\alpha}}(\boldsymbol{n}) = 1$, and
- $\left|\dfrac{\partial \phi_{\boldsymbol{\alpha}}}{\partial n_1}\right| + \left|\dfrac{\partial \phi_{\boldsymbol{\alpha}}}{\partial n_2}\right| < \dfrac{4}{\varepsilon \log t}$

for a small positive real number ε. For an element $(c_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in A} \in \mathbb{C}^A$, define a family $\{h_{t,s}\}$ of maps $N_{\mathbb{C}^\times} \to \mathbb{C}$ by

$$h_{t,s}(\boldsymbol{w}) = \sum_{\boldsymbol{\alpha} \in A} c_{\boldsymbol{\alpha}} t^{-\nu(\boldsymbol{\alpha})} (1 - s\phi_{\boldsymbol{\alpha}}(\boldsymbol{w})) \boldsymbol{w}^{\boldsymbol{\alpha}}, \tag{2.3.1}$$

where we write $\phi_{\boldsymbol{\alpha}}(\boldsymbol{w}) := \phi_{\boldsymbol{\alpha}}(\operatorname{Log}_t(\boldsymbol{w}))$ by abuse of notation. For a face $\tau \in \mathcal{P}$, define

$$O_\tau = \{\boldsymbol{n} \in N_\mathbb{R} \mid (\phi_{\boldsymbol{\alpha}}(\boldsymbol{n}) \ne 1 \text{ for } \forall \boldsymbol{\alpha} \in \tau) \text{ and } (\phi_{\boldsymbol{\alpha}}(\boldsymbol{n}) = 1 \text{ for } \forall \boldsymbol{\alpha} \notin \tau)\}. \tag{2.3.2}$$

Then O_τ is contained in an ε-neighborhood of the face of Π_∞ dual to τ, and one has

$$N_\mathbb{R} = \coprod_{\tau \in \mathcal{P}} O_\tau. \tag{2.3.3}$$

The set

$$Z_{t,s} = \{\boldsymbol{w} \in N_{\mathbb{C}^\times} \mid h_{t,s}(\boldsymbol{w}) = 0\} \tag{2.3.4}$$

is a symplectic hypersurface in $N_{\mathbb{C}^\times}$ for a sufficiently large t [Abo06, Proposition 4.2], and the pairs $(N_{\mathbb{C}^\times}, Z_{t,s})$ for all $s \in [0,1]$ are symplectomorphic to each other [Abo06, Proposition 4.9].

The *tropical localization* of Z_t is defined by

$$Z \coloneqq Z_{t,1}. \tag{2.3.5}$$

We set $(c_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in A} = \mathbf{1} \coloneqq (1,\dots,1) \in \mathbb{C}^A$ and

$$h(\boldsymbol{w}) \coloneqq h_{t,s}(\boldsymbol{w})\big|_{s=1,(c_{\boldsymbol{\alpha}_i})_{\boldsymbol{\alpha}_i\in A}=\mathbf{1}} = \sum_{\boldsymbol{\alpha}_i\in A} t^{-\nu(\boldsymbol{\alpha}_i)}(1-\phi_{\boldsymbol{\alpha}_i}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\alpha}_i}. \tag{2.3.6}$$

The hypersurface Z is localized in the following sense:

- Over O_τ where τ is dual to an edge of Π_∞ (i.e., when $\tau \in \mathcal{P}^{(1)}$), all but two terms of h vanish, and hence Z is defined by

$$h(\boldsymbol{w}) = t^{-\nu(\boldsymbol{\alpha})}\left(1-\phi_{\boldsymbol{\alpha}}(\boldsymbol{w})\right)\boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})}\left(1-\phi_{\boldsymbol{\beta}}(\boldsymbol{w})\right)\boldsymbol{w}^{\boldsymbol{\beta}} = 0, \tag{2.3.7}$$

  where $\boldsymbol{\alpha},\boldsymbol{\beta}\in A$ are the endpoints of τ.
- Over O_σ where σ is dual to a vertex of Π_∞ (i.e., when $\sigma \in \mathcal{P}^{(2)}$), all but three terms of h vanish, whence Z is defined by

$$\begin{aligned} h(\boldsymbol{w}) = t^{-\nu(\boldsymbol{\alpha}_0)}\left(1-\phi_{\boldsymbol{\alpha}_0}(\boldsymbol{w})\right)\boldsymbol{w}^{\boldsymbol{\alpha}_0} &+ t^{-\nu(\boldsymbol{\alpha}_1)}\left(1-\phi_{\boldsymbol{\alpha}_1}(\boldsymbol{w})\right)\boldsymbol{w}^{\boldsymbol{\alpha}_1} \\ &+ t^{-\nu(\boldsymbol{\alpha}_1)}\left(1-\phi_{\boldsymbol{\alpha}_1}(\boldsymbol{w})\right)\boldsymbol{w}^{\boldsymbol{\alpha}_2} \quad (2.3.8) \\ = 0, & \quad (2.3.9) \end{aligned}$$

  where $\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\in A$ are the vertices of σ.

It follows that the amoeba $\Pi = \operatorname{Log}(Z)$ of the tropical localization Z agrees with the tropical hypersurface Π_∞ away from the ε-neighborhood of the 0-skeleton $\Pi_\infty^{(0)}$. An important property for us is that each connected component of the complement of a compact set in Z is defined by a single *algebraic* equation

$$t^{-\nu(\boldsymbol{\alpha})}\boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})}\boldsymbol{w}^{\boldsymbol{\beta}} = 0, \tag{2.3.10}$$

and fibers over a subset of a leg of the tropical hypersurface Π_∞. In a slight abuse of terminology, we will refer to this portion of the curve Z as a *leg* of Z.

For a sufficiently large number R_0 and a relatively small number $\varepsilon_n << R_0$, we may assume that the union

$$U_\Pi \coloneqq U_{\Pi_c} \cup \bigcup_{i=1}^{\ell} U_{\Pi_i} \tag{2.3.11}$$

of

$$U_{\Pi_c} \coloneqq \{\boldsymbol{r}\in N_{\mathbb{R}} \mid |\boldsymbol{r}| < R_0\} \tag{2.3.12}$$

and

$$U_{\Pi_i} = \left\{\boldsymbol{r}\in N_{\mathbb{R}} \;\middle|\; \left|\boldsymbol{r}_{\boldsymbol{\alpha}_i-\boldsymbol{\beta}_i}\right| < \varepsilon_n \text{ and } \boldsymbol{r}_{(\boldsymbol{\alpha}_i-\boldsymbol{\beta}_i)^\perp} \geq a_i + \varepsilon_n\right\} \tag{2.3.13}$$

is a neighborhood of Π. We assume that ε_n is large enough so that it contains the line $\boldsymbol{r}_{\boldsymbol{\alpha}_i-\boldsymbol{\beta}_i} - (\nu(\boldsymbol{\alpha}_i)-\nu(\boldsymbol{\beta}_i)) = 0$, and that if we have non-compact parallel legs Π_i, we will assume that all of the corresponding neighborhoods U_{Π_i} are the same. We next choose a neighborhood U_Z of Z in $N_{\mathbb{C}^\times}$ such that

- $\operatorname{Log}(U_Z) \subset U_\Pi$,
- $|h(\boldsymbol{w})| < \varepsilon_h$ for some small constant $\varepsilon_h > 0$ and any $\boldsymbol{w}\in U_Z$,
- the set $\{\boldsymbol{\alpha}\in A \mid \phi_{\boldsymbol{\alpha}}(\boldsymbol{w}) \neq 1\}$ consists of either two or three elements for any $\boldsymbol{w}\in U_Z$,
- $h = t^{-\nu(\boldsymbol{\alpha})}\boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})}\boldsymbol{w}^{\boldsymbol{\beta}}$ for any $\boldsymbol{w}\in U_Z\cap \operatorname{Log}^{-1}(U_{\Pi_i})$, and
- U_Z does not intersect $N_{\mathbb{R}^{>0}} \coloneqq \{(w_1,w_2)\in(\mathbb{R}^{>0})^2\} \subset N_{\mathbb{C}^\times}$.

We set

$$U_{Z_i} \coloneqq U_Z \cap \operatorname{Log}^{-1}(U_{\Pi_i}), \quad U_{Z_c} \coloneqq U_Z \cap \operatorname{Log}^{-1}(U_{\Pi_c}). \tag{2.3.14}$$

We write the natural projections as

$$\mathrm{pr}_1 \colon N_{\mathbb{C}^\times} \times \mathbb{C} \to N_{\mathbb{C}^\times}, \quad \mathrm{pr}_2 \colon N_{\mathbb{C}^\times} \times \mathbb{C} \to \mathbb{C}. \tag{2.3.15}$$

We choose a tubular neighborhood $U_{Z\times 0}$ of $Z \times 0$ in $N_{\mathbb{C}^\times} \times \mathbb{C}$ such that

- $\mathrm{pr}_1(U_{Z\times 0}) \subset U_Z$ and
- $|u|^2 + \left|t^{-\nu(\boldsymbol{\alpha})}\boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})}\boldsymbol{w}^{\boldsymbol{\beta}}\right|^2 < \varepsilon_h^2$ if $(\boldsymbol{w}, u) \in U_{Z\times 0}$ and $\operatorname{Log}(\boldsymbol{w}) \in U_{\Pi_i}$.

We set

$$U_i \coloneqq \{(\boldsymbol{w}, u) \in U_{Z\times 0} \mid \operatorname{Log}(\boldsymbol{w}) \in U_{\Pi_i}\}. \tag{2.3.16}$$

We fix an almost complex structure $J_{N_{\mathbb{C}^\times}}$ on $N_{\mathbb{C}^\times}$, which is adapted to Z in the following sense:

Definition 2.4. An $\omega_{N_{\mathbb{C}^\times}}$ compatible almost complex structure $J_{N_{\mathbb{C}^\times}}$ on $N_{\mathbb{C}^\times}$ is said to be *adapted to Z* if

- $J_{N_{\mathbb{C}^\times}}$ agrees with the standard complex structure $J_{N_{\mathbb{C}^\times},\mathrm{std}}$ of $N_{\mathbb{C}^\times}$ outside the inverse image by Log of a small neighborhood of the origin in $N_{\mathbb{R}}$, and
- the function $h(\boldsymbol{w})$ is $J_{N_{\mathbb{C}^\times}}$-holomorphic in U_Z.

2.5. **Symplectic blow-up.** We follow [AAK] in this subsection; see also [CBM09] for a closely related construction. As a smooth manifold, the blow-up

$$p \colon \overline{Y} \coloneqq \mathrm{Bl}_{Z\times 0}(N_{\mathbb{C}^\times} \times \mathbb{C}) \to N_{\mathbb{C}^\times} \times \mathbb{C} \tag{2.5.1}$$

of $N_{\mathbb{C}^\times} \times \mathbb{C}$ along the symplectic submanifold $Z \times 0$ is given by

$$\overline{Y} \coloneqq \left\{(\boldsymbol{w}, u, [v_0 : v_1]) \in N_{\mathbb{C}^\times} \times \mathbb{C} \times \mathbb{P}^1 \;\middle|\; uv_1 = h(\boldsymbol{w})v_0\right\}. \tag{2.5.2}$$

The compositions of the structure morphism (2.5.1) and the projections (2.3.15) will be denoted by

$$\overline{\pi}_{N_{\mathbb{C}^\times}} \coloneqq \mathrm{pr}_1 \circ p \colon \overline{Y} \to N_{\mathbb{C}^\times}, \quad \overline{\pi}_{\mathbb{C}} \coloneqq \mathrm{pr}_2 \circ p \colon \overline{Y} \to \mathbb{C}. \tag{2.5.3}$$

The exceptional set is given by

$$E \coloneqq p^{-1}(Z \times 0) = \left\{(\boldsymbol{w}, u, [v_0 : v_1]) \in N_{\mathbb{C}^\times} \times \mathbb{C} \times \mathbb{P}^1 \;\middle|\; h(\boldsymbol{w}) = u = 0\right\}, \tag{2.5.4}$$

which forms a $\mathbb{P}^1$-bundle $p|_E \colon E \to Z \times 0$ over $Z \times 0$. The total transform of the divisor $Z \times \mathbb{C} \subset N_{\mathbb{C}^\times} \times \mathbb{C}$ is given by

$$\overline{E} \coloneqq \overline{\pi}_{N_{\mathbb{C}^\times}}^{-1}(Z) = E \cup F \subset \overline{Y}, \tag{2.5.5}$$

where F is the strict transform of $Z \times \mathbb{C}$. There is an $\mathbb{S}^1$-action on $\overline{Y}$ defined by

$$(\boldsymbol{w}, u, [v_0 : v_1]) \mapsto \left(\boldsymbol{w}, e^{\sqrt{-1}\theta}u, \left[v_0 : e^{-\sqrt{-1}\theta}v_1\right]\right). \tag{2.5.6}$$

Let

$$D \coloneqq \left\{(\boldsymbol{w}, u, [v_0 : v_1]) \in \overline{Y} \;\middle|\; v_0 = 0\right\} \cong N_{\mathbb{C}^\times} \tag{2.5.7}$$

be the strict transform of the divisor

$$\overline{D} \coloneqq \{(\boldsymbol{w}, u) \in N_{\mathbb{C}^\times} \times \mathbb{C} \mid u = 0\}, \tag{2.5.8}$$

and write its complement as

$$Y \coloneqq \overline{Y} \setminus D \cong \left\{(\boldsymbol{w}, u, v) \in N_{\mathbb{C}^\times} \times \mathbb{C}^2 \;\middle|\; h(\boldsymbol{w}) = uv\right\}, \tag{2.5.9}$$

where $v = v_1/v_0$. The restrictions of (2.5.3) to Y will be denoted by

(2.5.10) $$\pi_{N_{\mathbb{C}^\times}} := \overline{\pi}_{N_{\mathbb{C}^\times}}|_Y : Y \to N_{\mathbb{C}^\times}, \quad \pi_{\mathbb{C}} := \overline{\pi}_{\mathbb{C}}|_Y : Y \to \mathbb{C}.$$

We also write

(2.5.11) $$\overline{\pi}_{N_{\mathbb{R}}} := \operatorname{Log} \circ \overline{\pi}_{N_{\mathbb{C}^\times}} : \overline{Y} \to N_{\mathbb{R}}, \quad \pi_{N_{\mathbb{R}}} := \operatorname{Log} \circ \pi_{N_{\mathbb{C}^\times}} : Y \to N_{\mathbb{R}}.$$

A tubular neighborhood of D in $\overline{Y}$ is given by

(2.5.12) $$U_D = \left\{ (\boldsymbol{w}, u, [v_0 : 1]) \in N_{\mathbb{C}^\times} \times \mathbb{C} \times \mathbb{P}^1 \;\middle|\; u = h(\boldsymbol{w}) v_0,\ |v_0| < \delta \right\}$$

for a small positive number δ. We identify U_D with $N_{\mathbb{C}^\times} \times \mathbb{D}_\delta$ by the map

(2.5.13) $$\begin{array}{ccc} U_D & \to & N_{\mathbb{C}^\times} \times \mathbb{D}_\delta \\ \cup & & \cup \\ (\boldsymbol{w}, u, [v_0 : 1]) & \mapsto & (\boldsymbol{w}, v_0) \end{array}$$

where $\mathbb{D}_\delta := \{ v_0 \in \mathbb{C} \mid |v_0| < \delta \}$ is an open disk of radius δ. The projection will be denoted by

(2.5.14) $$\pi_{\mathbb{D}_\delta} : U_D \cong N_{\mathbb{C}^\times} \times \mathbb{D}_\delta \to \mathbb{D}_\delta.$$

We consider a two-form

(2.5.15) $$\omega_\epsilon := p^* \left(\omega_{N_{\mathbb{C}^\times} \times \mathbb{C}} + \frac{\sqrt{-1}\epsilon}{2\pi} \partial \overline{\partial} \left(\chi(\boldsymbol{w}, u) \log \left(|u|^2 + |h(\boldsymbol{w})|^2 \right) \right) \right)$$

on $\overline{Y} \setminus p^*(Z \times 0)$ for a sufficiently small ϵ, where

(2.5.16) $$\omega_{N_{\mathbb{C}^\times} \times \mathbb{C}} = \frac{\sqrt{-1}}{2} \left(du \wedge d\bar{u} + \frac{dw_1}{w_1} \wedge \frac{d\bar{w}_1}{\bar{w}_1} + \frac{dw_2}{w_2} \wedge \frac{d\bar{w}_2}{\bar{w}_2} \right)$$

is the standard symplectic form on $N_{\mathbb{C}^\times} \times \mathbb{C}$, and the function $\chi : N_{\mathbb{C}^\times} \times \mathbb{C} \to [0,1]$ is a smooth S^1-invariant cut-off function supported on the tubular neighborhood $U_{Z \times 0}$ of $Z \times 0$ and satisfying $\chi \equiv 1$ in a smaller tubular neighborhood $U'_{Z \times 0}$ of $Z \times 0$. We require that

(2.5.17) $$\chi|_{U_i} = \chi_i \circ G_i$$

for a function $\chi_i : \mathbb{R}^{\geq 0} \to [0,1]$, where

(2.5.18) $$G_i : U_i \to \mathbb{R}^{\geq 0}, \quad (\boldsymbol{w}, u) \mapsto |u|^2 + \left| t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta}} \right|^2.$$

Also, for clarity, we emphasize that in (2.5.15) the operators ∂ and $\overline{\partial}$ are defined with respect to the almost complex structure $J_{N_{\mathbb{C}^\times}}$ adapted to Z in the sense of Definition 2.4. A crucial feature for us is that in the neighborhoods U_i, this form is actually invariant under the $\mathbb{T}^2$-action which preserves the monomial $\boldsymbol{w}^{\boldsymbol{\alpha} - \boldsymbol{\beta}}$. This 2-form extends naturally to a 2-form on $\overline{Y}$, which we write as ω_ϵ again by abuse of notation, since we may rewrite (2.5.15) as

(2.5.19) $$\omega_\epsilon = p^* \omega_{N_{\mathbb{C}^\times} \times \mathbb{C}} + \frac{\sqrt{-1}\epsilon}{2\pi} \partial \overline{\partial} \left(\log \left(|v_0|^2 + |v_1|^2 \right) \right)$$

when $\chi(\boldsymbol{w}, u) = 1$.

Proposition 2.6. *The two-form ω_ϵ is a symplectic form for sufficiently small ϵ.*

Proof. When $\chi = 1$, the symplectic form is the restriction of the form

$$\omega_\epsilon = p^*\omega_{N_{\mathbb{C}^\times} \times \mathbb{C}} + \frac{\sqrt{-1}\epsilon}{2\pi}\partial\overline{\partial}\left(\log\left(|v_0|^2 + |v_1|^2\right)\right) \tag{2.6.1}$$

on $N_{\mathbb{C}^\times} \times \mathbb{C} \times \mathbb{P}^1$ to $\overline{Y}$. It follows that, whenever $\chi = 1$, ω_ϵ is the restriction of a compatible symplectic form (2.6.1) to an almost complex submanifold, and hence symplectic in this region. When $\chi(\boldsymbol{w}, u) \neq 1$, the first term in (2.5.15) is a symplectic form, and the expression $\chi(\boldsymbol{w}, u)\log(|u|^2 + |h(\boldsymbol{w})|^2)$ and its derivatives are bounded from above, so ω_ϵ is a symplectic form for sufficiently small ϵ. □

We fix a convenient choice of a primitive θ_ϵ for the restriction of ω_ϵ to Y, which we write ω_ϵ by abuse of notation. For a function f on an almost complex manifold, we set

$$d^c f := df \circ J, \tag{2.6.2}$$

so that $-dd^c f = 2\sqrt{-1}\partial\overline{\partial} f$. The form

$$\theta_{\mathrm{vc}} := -\frac{\epsilon}{4\pi} d^c\left(\chi(\boldsymbol{w}, u)\log\left(|u|^2 + |h(\boldsymbol{w})|^2\right) - \log(|u|^2)\right) \tag{2.6.3}$$

is well-defined on the subset $Y \subset \overline{Y}$ and gives a primitive for the form

$$\frac{\sqrt{-1}\epsilon}{2\pi}\partial\overline{\partial}\left(\chi(\boldsymbol{w}, u)\log\left(|u|^2 + |h(\boldsymbol{w})|^2\right)\right). \tag{2.6.4}$$

Now we define

$$\theta_\epsilon := p^*\theta_{N_{\mathbb{C}^\times} \times \mathbb{C}} + \theta_{\mathrm{vc}}, \tag{2.6.5}$$

where $\theta_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ is the standard primitive of $\omega_{N_{\mathbb{C}^\times} \times \mathbb{C}}$, so that

$$d\theta_\epsilon = \omega_\epsilon. \tag{2.6.6}$$

The $\mathbb{S}^1$-action (2.5.6) is Hamiltonian with respect to the symplectic form (2.5.15) with the moment map

$$\mu = \pi|u|^2 + \frac{\epsilon}{2}|u|\frac{\partial}{\partial|u|}\left(\chi(\boldsymbol{w}, u)\log\left(|h(\boldsymbol{w})|^2 + |u|^2\right)\right), \tag{2.6.7}$$

Our conventions for the moment map follow those of [AAK] (in particular it differs from the more standard convention by a factor of 2π). This formula specializes to:

$$\mu = \begin{cases} \pi|u|^2 + \dfrac{\epsilon}{2}\dfrac{|u|^2}{|h(\boldsymbol{w})|^2 + |u|^2} & \text{where } \chi \equiv 1 \text{ (near } E), \\ \pi|u|^2 & \text{where } \chi \equiv 0 \text{ (away from } E). \end{cases} \tag{2.6.8}$$

The level set $\mu^{-1}(\lambda)$ is smooth unless $\lambda = \epsilon$, where it is singular along the fixed locus

$$\widetilde{Z} = \{(\boldsymbol{w}, u, [v_0 : v_1]) \in \overline{Y} \mid h(\boldsymbol{w}) = u = v_1 = 0\}. \tag{2.6.9}$$

The level set $\mu^{-1}(0)$ is the divisor D defined in (2.5.7). For points $\boldsymbol{w} \notin Z$, the fibers $\pi_{N_{\mathbb{C}^\times}}^{-1}(\boldsymbol{w})$ are smooth conics. Let $d\theta$ denote the natural $\mathbb{S}^1$-invariant angular one-form on these smooth fibers $\pi_{N_{\mathbb{C}^\times}}^{-1}(\boldsymbol{w})$. We have that the primitive θ_ϵ restricted to any fiber is given by

$$\theta_\epsilon|_{\pi_{N_{\mathbb{C}^\times}}^{-1}(\boldsymbol{w})} = |u|\frac{\partial}{\partial|u|}\left(\frac{1}{4}|u|^2 + \frac{\epsilon}{4\pi}\chi(\boldsymbol{w}, u)\log\left(|\boldsymbol{w}|^2 + |u|^2\right) - \frac{\epsilon}{4\pi}\log\left(|u|^2\right)\right)d\theta.$$

In view of (2.6.7) and (2.6.3), we may rewrite this in the much simpler form:

$$\theta_\epsilon|_{\pi_{N_{\mathbb{C}^\times}}^{-1}(\boldsymbol{w})} = \frac{1}{2\pi}(\mu - \epsilon)d\theta. \tag{2.6.10}$$

The same formula holds when $\boldsymbol{w} \in Z$, away from the singular points of $\pi_{N_{\mathbb{C}^\times}}^{-1}(\boldsymbol{w})$. For $\lambda \in \mathbb{R}^{>0} \setminus \{\epsilon\}$, the map $\pi_{N_{\mathbb{C}^\times}} : Y \to N_{\mathbb{C}^\times}$ induces a natural identification

$$Y_{\mathrm{red},\lambda} := \mu^{-1}(\lambda)/\mathbb{S}^1 \cong N_{\mathbb{C}^\times} \tag{2.6.11}$$

of the reduced space and $N_{\mathbb{C}^\times}$. The resulting reduced symplectic form $\omega_{\mathrm{red},\lambda}$ on $N_{\mathbb{C}^\times}$ can be averaged by the action of the torus $N_{\mathbb{S}^1} := N_{\mathbb{R}}/N$ to obtain a torus-invariant symplectic form $\omega'_{N_{\mathbb{C}^\times},\lambda}$. [AAK, Lemma 4.1] states that there exists a family $(\phi_\lambda)_{\lambda \in \mathbb{R}^{>0}}$ of diffeomorphisms of $N_{\mathbb{C}^\times}$ such that

- $\phi_\lambda^* \omega'_{N_{\mathbb{C}^\times},\lambda} = \omega_{\mathrm{red},\lambda}$,
- $\phi_\lambda = \mathrm{id}$ at every point whose $N_{\mathbb{S}^1}$-orbit is disjoint from the support of χ.
- ϕ_λ depends on λ in a piecewise smooth manner.

We set $\pi_\lambda := \mathrm{Log} \circ \phi_\lambda \colon N_{\mathbb{C}^\times} \to N_{\mathbb{R}}$ and define a continuous, piecewise smooth map by

$$\pi_B \colon Y \to B := N_{\mathbb{R}} \times \mathbb{R}^{>0}, \quad x \in \mu^{-1}(\lambda) \mapsto (\pi_\lambda([x]), \lambda)\,. \tag{2.6.12}$$

One can easily see as in [AAK, Section 4.2] that fibers of π_B^{-1} are smooth Lagrangian tori outside of the *discriminant locus* $\mathrm{Log} \circ \phi_\epsilon(Z) \times \{\epsilon\}$.

2.7. **SYZ mirror construction.** We continue to follow [AAK] in this subsection; see also [Aur07, Aur09, CLL12] for closely related constructions. The critical locus of the SYZ fibration $\pi_B : Y \to B$ is given by $Z \times \{(0,0)\} \subset N_{\mathbb{C}^\times} \times \mathbb{C}^2$, which is the fixed locus of the S^1-action. Hence the discriminant locus of π_B is given by $\Gamma = \Pi' \times \{\epsilon\} \subset B$, where $\Pi' := \pi_\epsilon(Z) \subset N_{\mathbb{R}}$ is essentially the amoeba of Z, except that the map π_ϵ differs from the logarithm map Log by ϕ_ϵ. The complement of the discriminant locus will be denoted by $B^{\mathrm{sm}} := B \setminus \Gamma$. The SYZ fibration induces an integral affine structure on B^{sm}. The corresponding local integral affine coordinates $\{x_j\}_{j=1}^3$ give local systems $T_{\mathbb{Z}} B^{\mathrm{sm}}$ and $T^*_{\mathbb{Z}} B^{\mathrm{sm}}$, generated by $\{\partial/\partial x_j\}_{j=1}^3$ and $\{dx_j\}_{j=1}^3$ respectively.

A choice of a section of π_B induces a symplectomorphism

$$\pi^{-1}(B^{\mathrm{sm}}) \cong T^* B^{\mathrm{sm}} / T^*_{\mathbb{Z}} B^{\mathrm{sm}} \tag{2.7.1}$$

given by the action-angle coordinates [Dui80]. The *semi-flat mirror* of Y is defined by

$$\check{Y}^{\mathrm{sf}} := T B^{\mathrm{sm}} / T_{\mathbb{Z}} B^{\mathrm{sm}}, \tag{2.7.2}$$

equipped with the natural complex structure $J_{\check{Y}^{\mathrm{sf}}}$ such that the holomorphic coordinates are given by $\left\{ z_j = \exp 2\pi \left(x_j + \sqrt{-1} y_j \right) \right\}_{j=1}^3$. Here $\{y_j\}_{j=1}^3$ are the coordinates on the fiber corresponding to $\{x_j\}_{j=1}^3$. To obtain the SYZ mirror $\check{Y}$, one first correct the semi-flat complex structure by contributions of the holomorphic disks bounded by Lagrangian torus fibers, and then add fibers over $\Gamma = B \setminus B^{\mathrm{sm}}$.

Instead of correcting complex structures of the semi-flat mirror, [AAK] considers the subset $B^{\mathrm{reg}} \subset B$ obtained by removing $\pi(p^{-1}(U_Z \times \mathbb{C}))$ from B. Here $U_Z \subset N_{\mathbb{C}^\times}$ is a sufficiently small neighborhood of Z containing the support of χ. The connected components of B^{reg} are in one-to-one correspondence with elements of A, and all fibers over B^{reg} are *tautologically unobstructed* (i.e., they do not bound any non-constant holomorphic disks). Let $U_{\boldsymbol{\alpha}}$ denote the connected component of B^{reg} corresponding to $\boldsymbol{\alpha} \in A$. The semi-flat mirror $TU_{\boldsymbol{\alpha}} / T_{\mathbb{Z}} U_{\boldsymbol{\alpha}}$ with coordinates $(z_{\boldsymbol{\alpha},1}, z_{\boldsymbol{\alpha},2}, z_{\boldsymbol{\alpha},3})$ can be completed to a torus $\widetilde{U}_{\boldsymbol{\alpha}} := \operatorname{Spec} \mathbb{C}[z_{\boldsymbol{\alpha},1}^{\pm 1}, z_{\boldsymbol{\alpha},2}^{\pm 1}, z_{\boldsymbol{\alpha},3}^{\pm 1}]$. Motivated by the counting of Maslov index two disks in a partial compactification of Y, [AAK] glues $\widetilde{U}_{\boldsymbol{\alpha}}$ and $\widetilde{U}_{\boldsymbol{\beta}}$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A$ together by

$$\begin{aligned} z_{\boldsymbol{\alpha},1} &= (1+z_{\boldsymbol{\beta},3})^{\beta_1-\alpha_1} z_{\boldsymbol{\beta},1}, \\ z_{\boldsymbol{\alpha},2} &= (1+z_{\boldsymbol{\beta},3})^{\beta_2-\alpha_2} z_{\boldsymbol{\beta},2}, \\ z_{\boldsymbol{\alpha},3} &= z_{\boldsymbol{\beta},3}. \end{aligned} \tag{2.7.3}$$

These local coordinates are related to coordinates (w_1, w_2, w_3) of the dense torus by

$$\begin{aligned} z_{\boldsymbol{\alpha},1} &= w_1 w_3^{-\alpha_1}, \\ z_{\boldsymbol{\alpha},2} &= w_2 w_3^{-\alpha_2}, \\ z_{\boldsymbol{\alpha},3} &= w_3 - 1. \end{aligned} \tag{2.7.4}$$

Let Σ be the fan in $M_{\mathbb{R}} \oplus \mathbb{R}$ associated with the coherent unimodular triangulation $\mathcal{P}$, and X_Σ be the associated toric variety. Let further K be an anticanonical divisor in X_Σ defined by the function $p := \chi_{\mathbf{0},1} - 1$, where $\chi_{\boldsymbol{n},k} : X_\Sigma \to \mathbb{C}$ is the function associated with the character $(\boldsymbol{n}, k) \in N \oplus \mathbb{Z}$ of the dense torus of X_Σ. By adding torus-invariant curves to $\bigcup_{\boldsymbol{\alpha} \in A} \widetilde{U}_{\boldsymbol{\alpha}}$, one obtains the complement $\check{Y} := X_\Sigma \setminus K$ of the anti-canonical divisor [AAK, Theorem 1.7].

2.8. **Coordinate ring of the mirror manifold.** One has $H^0(\mathcal{O}_{X_\Sigma}) = \bigoplus_{(\boldsymbol{n},k)\in\mathcal{C}} \chi_{\boldsymbol{n},k}$ where

$$\mathcal{C} := \{(\boldsymbol{n}, \ell) \in N \oplus \mathbb{Z} \mid \boldsymbol{n}(\boldsymbol{m}) + \ell \geq 0 \text{ for any } \boldsymbol{m} \in A\}. \tag{2.8.1}$$

If we define a function $\ell_1 \colon N \to \mathbb{Z}$ by

$$\ell_1(\boldsymbol{n}) := \min \{\ell \in \mathbb{Z} \mid (-\boldsymbol{n}, \ell) \in \mathcal{C}\}, \tag{2.8.2}$$

then the set $\left\{p^i \chi_{-\boldsymbol{n},\ell_1(\boldsymbol{n})}\right\}_{(\boldsymbol{n},i)\in N\times\mathbb{Z}}$ forms a basis of the algebra $H^0(\mathcal{O}_{\check{Y}}) = H^0(\mathcal{O}_{X_\Sigma})[p^{-1}]$. The product structure is given by

$$p^i \chi_{-\boldsymbol{n},\ell_1(\boldsymbol{n})} \cdot p^{i'} \chi_{-\boldsymbol{n}',\ell_1(\boldsymbol{n}')} = p^{i+i'} \chi_{-\boldsymbol{n}-\boldsymbol{n}',\ell_1(\boldsymbol{n})+\ell_1(\boldsymbol{n}')} \tag{2.8.3}$$

$$= p^{i+i'} \chi_{\mathbf{0},\ell_2(\boldsymbol{n},\boldsymbol{n}')} \cdot \chi_{-\boldsymbol{n}-\boldsymbol{n}',\ell_1(\boldsymbol{n}+\boldsymbol{n}')} \tag{2.8.4}$$

$$= p^{i+i'} (1+p)^{\ell_2(\boldsymbol{n},\boldsymbol{n}')} \cdot \chi_{-\boldsymbol{n}-\boldsymbol{n}',\ell_1(\boldsymbol{n}+\boldsymbol{n}')} \tag{2.8.5}$$

$$= \sum_{j=0}^{\ell_2(\boldsymbol{n},\boldsymbol{n}')} \binom{\ell_2(\boldsymbol{n},\boldsymbol{n}')}{j} p^{i+i'+j} \chi_{-\boldsymbol{n}-\boldsymbol{n}',\ell_1(\boldsymbol{n}+\boldsymbol{n}')}, \tag{2.8.6}$$

where the function $\ell_2 \colon N \times N \to \mathbb{Z}$ is defined by

$$\ell_2(\boldsymbol{n}, \boldsymbol{n}') = \ell_1(\boldsymbol{n}) + \ell_1(\boldsymbol{n}') - \ell_1(\boldsymbol{n}+\boldsymbol{n}'). \tag{2.8.7}$$

3. Base-admissible Lagrangian sections

3.1. **Liouville domains.** A pair $(X^{\mathrm{in}}, \theta)$ of a compact manifold X^{in} with boundary and a one-form θ on X^{in} is called a *Liouville domain* if

- $\omega := d\theta$ is a symplectic form on X^{in},
- the Liouville vector field V_θ, determined uniquely by the condition $\iota_{V_\theta}\omega = \theta$, points strictly outward along ∂X^{in}.

The one-form θ is called the *Liouville one-form.* The manifold $X := X^{\mathrm{in}} \cup_{\partial X^{\mathrm{in}}} [1, \infty) \times \partial X^{\mathrm{in}}$ obtained by gluing the positive symplectization of the contact manifold $\left(\partial X^{\mathrm{in}}, \theta|_{X^{\mathrm{in}}}\right)$ to X^{in} along ∂X^{in} is called the *Liouville completion* of $(X^{\mathrm{in}}, \theta)$. An exact symplectic manifold obtained as the Liouville completion of a Liouville domain will be called a *Liouville manifold.* The extension of the one-form θ to X will be denoted by θ again by abuse of notation. The coordinate on the symplectization end $[1,\infty)\times\partial X^{\mathrm{in}}$ corresponding to $[1,\infty)$ is called the *Liouville coordinate*.

If (X,J) is a Stein manifold with an exhaustive plurisubharmonic function $S\colon X\to\mathbb{R}$ whose critical values are less than $K\in\mathbb{R}$, then the manifold $X^{\mathrm{in}}=S^{-1}((-\infty,K])$ is a Liouville domain with a Liouville one-form $\theta:=-d^cS$ Under the additional assumption that the gradient flow of S is complete, the Liouville completion may be identified with X.

3.2. **Base-admissible Lagrangian sections.** Let S be an exhaustive plurisubharmonic function on $N_{\mathbb{C}^\times}$ defined by

$$S(\boldsymbol{w})=\frac{1}{2}|\boldsymbol{r}|^2=\frac{1}{2}(r_1^2+r_2^2) \tag{3.2.1}$$

in the logarithmic coordinates $\boldsymbol{w}=(w_1,w_2)=\left(e^{r_1+\sqrt{-1}\theta_1},e^{r_2+\sqrt{-1}\theta_2}\right)$. One has

$$dS=r_1dr_1+r_2dr_2 \tag{3.2.2}$$

and

$$\theta_{N_{\mathbb{C}^\times}}:=-d^cS=r_1d\theta_1+r_2d\theta_2. \tag{3.2.3}$$

Let $L_0=N_{\mathbb{R}^{>0}}\times\mathbb{R}^{>0}$ be the positive real locus of $N_{\mathbb{C}^\times}\times\mathbb{C}^\times=(N_{\mathbb{C}^\times}\times\mathbb{C})\setminus\overline{D}$, which is a Lagrangian submanifold diffeomorphic to $\mathbb{R}^3$. Since L_0 is disjoint from the tubular neighborhood $U_{Z\times 0}\subset N_{\mathbb{C}^\times}\times\mathbb{C}$ of the center $Z\times 0$ of the blow-up, it lifts to a Lagrangian submanifold of Y. By abuse of notation, we write the lifted Lagrangian again as L_0. More generally, we consider the following type of Lagrangians:

Definition 3.3. An exact Lagrangian section L of the SYZ fibration (2.6.12) of Y is *base-admissible* if the following conditions are satisfied:

- L is fibered over a Lagrangian submanifold $\underline{L}$ in $N_{\mathbb{C}^\times}\setminus U_Z$;

$$L=\underline{L}\times\mathbb{R}^{>0}\subset(N_{\mathbb{C}^\times}\setminus U_Z)\times\mathbb{C}\subset Y. \tag{3.3.1}$$

- $\underline{L}$ is Legendrian at infinity, i.e., $\theta_{N_{\mathbb{C}^\times}}\big|_{\underline{L}}=0$ outside of a compact set.

It is clear from Definition 3.3 that base-admissible Lagrangian sections of $\pi_B\colon Y\to B$ are in one-to-one correspondence with Lagrangian sections $\underline{L}$ of $\mathrm{Log}\colon N_{\mathbb{C}^\times}\to N_{\mathbb{R}}$ which are disjoint from U_Z and satisfying the Legendrian condition at infinity.

3.4. **Framed Lagrangian sections.** For each lattice point $\boldsymbol{\alpha}\in A$, consider the polynomial

$$h^{\boldsymbol{\alpha}}(\boldsymbol{w})=-t^{-\nu(\boldsymbol{\alpha})}(1-\phi_{\boldsymbol{\alpha}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\alpha}}+\sum_{\boldsymbol{\beta}\in A\setminus\{\boldsymbol{\alpha}\}}t^{-\nu(\boldsymbol{\beta})}(1-\phi_{\boldsymbol{\beta}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\beta}} \tag{3.4.1}$$

obtained by flipping the sign of one term in (2.3.6). The corresponding hypersurface will be denoted by

$$Z^{\boldsymbol{\alpha}}:=\{\boldsymbol{w}\in N_{\mathbb{C}^\times}\mid h^{\boldsymbol{\alpha}}_{t,1}(\boldsymbol{w})=0\}. \tag{3.4.2}$$

Lemma 3.5. *The amoeba of $Z^{\boldsymbol{\alpha}}$ coincides with that of Z.*

Proof. If $\mathrm{Log}(\boldsymbol{w})\in O_\tau$ for $\tau\in\mathcal{P}^{(1)}$ such that $\partial\tau=\{\boldsymbol{\alpha},\boldsymbol{\beta}\}$, then one has

$$\begin{aligned} h(\boldsymbol{w})&=t^{-\nu(\boldsymbol{\alpha})}(1-\phi_{\boldsymbol{\alpha}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\alpha}}+t^{-\nu(\boldsymbol{\beta})}(1-\phi_{\boldsymbol{\beta}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\beta}},\\ h^{\boldsymbol{\alpha}}(\boldsymbol{w})&=-t^{-\nu(\boldsymbol{\alpha})}(1-\phi_{\boldsymbol{\alpha}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\alpha}}+t^{-\nu(\boldsymbol{\beta})}(1-\phi_{\boldsymbol{\beta}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\beta}}.\end{aligned} \tag{3.5.1}$$

If $\operatorname{Log}(\boldsymbol{w}) \in O_\sigma$ where $\sigma \in \mathcal{P}^{(2)}$ is the simplex whose vertices are $\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma \in A$, then one has

(3.5.2)
$$h(\boldsymbol{w}) = t^{-\nu(\boldsymbol{\alpha})}(1-\phi_{\boldsymbol{\alpha}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})}(1-\phi_{\boldsymbol{\beta}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\beta}} + t^{-\nu(\gamma)}(1-\phi_{\gamma}(\boldsymbol{w}))\boldsymbol{w}^{\gamma},$$
$$h^{\boldsymbol{\alpha}}(\boldsymbol{w}) = -t^{-\nu(\boldsymbol{\alpha})}(1-\phi_{\boldsymbol{\alpha}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})}(1-\phi_{\boldsymbol{\beta}}(\boldsymbol{w}))\boldsymbol{w}^{\boldsymbol{\beta}} + t^{-\nu(\gamma)}(1-\phi_{\gamma}(\boldsymbol{w}))\boldsymbol{w}^{\gamma}.$$

By choosing a coordinate of M in such a way that $\boldsymbol{\beta} - \boldsymbol{\alpha} = (1,0)$ and $\gamma - \boldsymbol{\alpha} = (0,1)$, one can easily show that the amoebas are identical in both cases. □

Choose a sufficiently large t so that the connected components of the complement of $\Pi \coloneqq \operatorname{Log}(Z)$ are labeled by A as

$$N_{\mathbb{R}} \setminus \Pi = \coprod_{\boldsymbol{\alpha} \in A} \mathcal{Q}_{\boldsymbol{\alpha}} \tag{3.5.3}$$

just as in (2.1.7). For an interior lattice point $\boldsymbol{\alpha} \in A \cap \operatorname{Int} \Delta$, a *tropical Lagrangian section* is an exact Lagrangian section of the restriction of $\operatorname{Log}\colon N_{\mathbb{C}^\times} \to N_{\mathbb{R}}$ to the inverse image of $\mathcal{Q}_{\boldsymbol{\alpha}}$ with boundary in $Z^{\boldsymbol{\alpha}}$ which agrees with the parallel transport of ∂L along a segment in $\mathbb{C}$ in a small neighborhood of ∂L [Abo09, Definitions 3.7 and 3.16]. The prototypical example of a tropical Lagrangian section is the restriction of the positive real Lagrangian

$$\underline{L}_0 \coloneqq N_{\mathbb{R}^{>0}} = \{\boldsymbol{w} = (w_1, w_2) \in N_{\mathbb{C}^\times} \mid w_1, w_2 \in \mathbb{R}^{>0}\} \tag{3.5.4}$$

to the fibers over $\mathcal{Q}_{\boldsymbol{\alpha}}$.

Lemma 3.6. *A tropical Lagrangian section does not intersect a sufficiently small tubular neighborhood U_Z of Z.*

Proof. Since a tropical Lagrangian section is compact and Z is closed, it suffices to show that a tropical Lagrangian section does not intersect Z.

If $\operatorname{Log}(\boldsymbol{w}) \in O_\tau$ for $\tau \in \mathcal{P}^{(1)}$ such that $\partial\tau = \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$, then it follows from (3.5.1) that $\boldsymbol{w}^{\boldsymbol{\beta}-\boldsymbol{\alpha}}$ is in $\mathbb{R}^{>0}$ for $\boldsymbol{w} \in Z$ and $\mathbb{R}^{<0}$ for $\boldsymbol{w} \in Z^{\boldsymbol{\alpha}}$. It follows that a tropical Lagrangian section does not intersect Z in $\operatorname{Log}^{-1}(O_\tau)$ for $\tau \in \mathcal{P}^{(1)}$.

If $\operatorname{Log}(\boldsymbol{w}) \in O_\sigma$ for $\sigma \in \mathcal{P}^{(2)}$, then a tropical Lagrangian section agrees with the positive real Lagrangian $\underline{L}_0$ in the neighborhood $\partial\mathcal{Q} \cap O_\sigma$ of the vertex of Π_∞ dual to τ [Abo09, Lemma 3.18]. The positive real Lagrangian is clearly disjoint from Z, and Lemma 3.6 is proved. □

One can use the complex structure of $N_{\mathbb{C}^\times}$ to view $N_{\mathbb{C}^\times}$ as the trivial $N_{\mathbb{R}}/N$-bundle $TN_{\mathbb{R}}/T_{\mathbb{Z}}N_{\mathbb{R}}$ over $N_{\mathbb{R}}$, whose universal cover is the trivial $N_{\mathbb{R}}$-bundle $TN_{\mathbb{R}}$. A section of $TN_{\mathbb{R}}$ can be identified with a function on $N_{\mathbb{R}}$ with values in $N_{\mathbb{R}}$. Since the tropical Lagrangian section agrees with the positive real Lagrangian near $\mathcal{Q} \cap O_\tau$ for $\tau \in \mathcal{P}^{(2)}$, a lift $\mathcal{Q} \to TN_{\mathbb{R}}$ of a tropical Lagrangian section $\mathcal{Q} \to N_{\mathbb{C}^\times}$ to the universal cover $TN_{\mathbb{R}} \to N_{\mathbb{C}^\times} \cong TN_{\mathbb{R}}/T_{\mathbb{Z}}N_{\mathbb{R}}$ takes values in N near $\mathcal{Q} \cap O_\tau$. The Hamiltonian isotopy class of a tropical Lagrangian section is determined by the values $(n_\tau)_{\tau \in \mathcal{P}^{(2)}} \in N^{\mathcal{P}^{(2)}}$ of its lifts near the vertices of Π [Abo09, Proposition 3.20]. Two lifts come from the same section if and only if they are related by an overall shift by N.

For an edge $\tau \in \mathcal{P}^{(1)}$ in the interior of Δ, choose a coordinate of M in such a way that τ is the line segment between $\boldsymbol{\alpha} = (0,0)$ and $\boldsymbol{\beta} = (1,0)$. Then one has

$$h^{\boldsymbol{\alpha}}(\boldsymbol{w}) = -t^{-\nu(\boldsymbol{\alpha})} + t^{-\nu(\boldsymbol{\beta})} w_1 \tag{3.6.1}$$

for $\boldsymbol{w} \in O_\tau$, so that

$$\Pi \cap O_\tau = \{(r_1, r_2) \in N_\mathbb{R} \mid r_1 = -\nu(\boldsymbol{\alpha}) + \nu(\boldsymbol{\beta})\}. \tag{3.6.2}$$

It follows that a tropical Lagrangian section is constant in the w_1-variable above O_τ. A Lagrangian section $N_\mathbb{R} \to N_{\mathbb{C}^\times}$ is said to be *framed* if its restriction to $\mathcal{Q}_{\boldsymbol{\alpha}}$ is bounded by $Z^{\boldsymbol{\alpha}}$ for any interior lattice point $\boldsymbol{\alpha} \in A \cap \operatorname{Int} \Delta$.

If σ and σ' are elements of $\mathcal{P}^{(2)}$ adjacent to an edge $\tau \in \mathcal{P}^{(1)}$ in the interior of Δ, then the condition that the boundary of a Lagrangian section lies in Z implies that

$$\langle n_\sigma - n_{\sigma'}, \boldsymbol{\alpha} - \boldsymbol{\beta} \rangle = 0. \tag{3.6.3}$$

For a collection $(n_\sigma)_{\sigma \in \mathcal{P}^{(2)}} \in N^{\mathcal{P}^{(2)}}$ of elements of N, there exists a framed Lagrangian section whose lift takes the value n_σ on O_σ if and only if (3.6.3) is satisfied for any edge $\tau \in \mathcal{P}^{(1)}$.

3.7. **Legendrian condition at infinity.** Recall from Definition 3.3 that a Lagrangian submanifold $\underline{L} \subset N_{\mathbb{C}^\times}$ is Legendrian at infinity if $d^c S|_{\underline{L}} = 0$ outside of a compact set. A direct calculation shows that the graph

$$\Gamma_{df} := \left\{ (r_1, \theta_1, r_2, \theta_2) \in N_{\mathbb{C}^\times} \;\middle|\; \theta_1 = \frac{\partial f}{\partial r_1},\ \theta_2 = \frac{\partial f}{\partial r_2} \right\} \tag{3.7.1}$$

of the differential of a function $f \colon N_\mathbb{R} \to \mathbb{R}$ satisfies the Legendrian condition $d^c S|_{\Gamma_{df}} = 0$ if and only if

$$\left(r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2} \right) \frac{\partial f}{\partial r_1} = \left(r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2} \right) \frac{\partial f}{\partial r_2} = 0. \tag{3.7.2}$$

This happens if f homogeneous of degree one:

$$\left(r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2} \right) f = f. \tag{3.7.3}$$

Proposition 3.8. *Any framed Lagrangian section can be made Legendrian at infinity by a Hamiltonian isotopy.*

Proof. We can choose a framed Lagrangian in such a way that it coincides with the positive real Lagrangian in the neighborhood of each leg of Π outside of a compact set. Then the potential f of the Lagrangian is linear in that neighborhood. Now one can choose arbitrary homogeneous function of degree one which coincides with f in the compact set and in the neighborhood of each leg, and the Lagrangian generated by this function has the desired property. □

3.9. **SYZ transformation.** Let L be a base-admissible Lagrangian section of $\pi_B \colon Y \to B$ associated with a framed Lagrangian section $\underline{L}$ of $\operatorname{Log} \colon N_{\mathbb{C}^\times} \to N_\mathbb{R}$. Let further $\tau \in \mathcal{P}^{(1)}$ be an edge in the interior of Δ and $\ell \subset \check{Y}$ be the corresponding torus-invariant curve. We use the same coordinates as in Section 2.7.

For each interior lattice point $\boldsymbol{\alpha} \in A \cap \operatorname{Int} \Delta$, a framed Lagrangian section $\underline{L}$ restricts to a tropical Lagrangian section $\underline{L}_{\boldsymbol{\alpha}}$ over $\mathcal{Q}_{\boldsymbol{\alpha}}$. The fiberwise universal cover of the restriction of $\operatorname{Log} \colon N_{\mathbb{C}^\times} \to N_\mathbb{R}$ to $\mathcal{Q}_{\boldsymbol{\alpha}}$ can be identified with $T\mathcal{Q}_{\boldsymbol{\alpha}}$, with the positive real Lagrangian as the zero-section. We write the lift of $\underline{L}_{\boldsymbol{\alpha}}$ as the graph of a one-form

$$\omega = \xi_1 dy_1 + \xi_2 dy_2, \tag{3.9.1}$$

where ξ_1 and ξ_2 are functions on $N_\mathbb{R}$ satisfying

$$\frac{\partial \xi_1}{\partial y_2} - \frac{\partial \xi_2}{\partial y_1} = 0. \tag{3.9.2}$$

The *semi-flat SYZ transform* of $\underline{L}_{\boldsymbol{\alpha}}$ is the trivial bundle on $T\mathcal{Q}_{\boldsymbol{\alpha}}$, equipped with the connection

$$\nabla_{\boldsymbol{\alpha}} := d + 2\pi\sqrt{-1}\omega = d + 2\pi\sqrt{-1}(\xi_1 dy_1 + \xi_2 dy_2). \tag{3.9.3}$$

In cases without quantum corrections, this gives a holomorphic line bundle mirror to the given Lagrangian section [LYZ00]. In general, however, quantum corrections have to be taken into account [Cha13, CU13, CPU]. In our case, due to the nontrivial gluing formulas (2.7.3), the semi-flat SYZ transforms of $\underline{L}_{\boldsymbol{\alpha}}$ and $\underline{L}_{\boldsymbol{\beta}}$ do not coincide over the intersection

$$\widetilde{U}_{\boldsymbol{\alpha}} \cap \widetilde{U}_{\boldsymbol{\beta}} = T\ell / T_{\mathbb{Z}}\ell,$$

where $\ell \subset \Pi_\infty$ is the edge of the intersection of the connected components $\mathcal{Q}_{\boldsymbol{\alpha}}, \mathcal{Q}_{\boldsymbol{\beta}} \subset N_{\mathbb{R}} \setminus \Pi$ which is dual to $\tau \in \mathcal{P}^{(1)}$, but are related by

$$\nabla_{\boldsymbol{\alpha}} = \nabla_{\boldsymbol{\beta}} + \sqrt{-1}\langle df, \boldsymbol{\beta} - \boldsymbol{\alpha}\rangle d\arg(1+z_3), \tag{3.9.4}$$

where f is a primitive of $\underline{L}$, i.e. $\xi_i = \partial f/\partial x_i$ for $i = 1, 2$.

Since $\underline{L}_{\boldsymbol{\alpha}}$ and $\underline{L}_{\boldsymbol{\beta}}$ share the same boundary in Z over ℓ and the defining equation for Z is given as in (2.3.10), we have $k_{\boldsymbol{\alpha}\boldsymbol{\beta}} := \langle df, \boldsymbol{\beta} - \boldsymbol{\alpha}\rangle \in \mathbb{Z}$, so we may modify $\nabla_{\boldsymbol{\beta}}$ to the *gauge equivalent* connection

$$\nabla'_{\boldsymbol{\beta}} := \nabla_{\boldsymbol{\beta}} + \sqrt{-1}k_{\boldsymbol{\alpha}\boldsymbol{\beta}} d\arg(1+z_3). \tag{3.9.5}$$

Now $\nabla_{\boldsymbol{\alpha}}$ and $\nabla'_{\boldsymbol{\beta}}$ glue to give a connection $\nabla_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ on the chart $\widetilde{U}_{\boldsymbol{\alpha}} \cup \widetilde{U}_{\boldsymbol{\beta}} \subset \check{Y}$. It is clear that the cocycle condition is satisfied, so the connections $\{\nabla_{\boldsymbol{\alpha}\boldsymbol{\beta}}\}$ define a global $U(1)$-connection over $\check{Y}$ whose curvature has trivial $(0,2)$-part since L is Lagrangian. This produces a holomorphic line bundle $\mathcal{F}(L)$ over $\check{Y}$, called the *SYZ transform* of L.

To determine the isomorphism class of $\mathcal{F}(L)$, let $\ell \subset \Pi_\infty$ be an edge on the boundary of a connected component $C_{\boldsymbol{\alpha},\infty} \subset N_{\mathbb{R}} \setminus \Pi_\infty$ of the complement of the tropical curve Π_∞. We can choose a coordinate on M in such a way that the endpoints of the edge τ is given by $\boldsymbol{\alpha} = (0,0)$ and $\boldsymbol{\beta} = (1,0)$. A subset of the torus-invariant curve in $\check{Y}$ associated with the edge $\tau \in \mathcal{P}^{(1)}$ dual to ℓ can naturally be identified with $T\ell/T_{\mathbb{Z}}\ell$. Let $\sigma, \sigma' \in \mathcal{P}^{(2)}$ be the faces adjacent to τ, then the degree of the restriction of $\mathcal{F}(L)$ to $T\ell/T_{\mathbb{Z}}\ell$ is given by

$$\frac{\sqrt{-1}}{2\pi}\int_{T\ell/T_{\mathbb{Z}}\ell} F^{1,1}_{\nabla'_{\boldsymbol{\alpha}}} = -\int_{T\ell/T_{\mathbb{Z}}\ell} \frac{\partial \xi_2}{\partial y'_2} dx'_2 \wedge dy'_2 = -\int_\ell \frac{\partial \xi_2}{\partial y'_2} dy'_2 = \xi_2(s_\sigma) - \xi_2(s_{\sigma'}), \tag{3.9.6}$$

where $s_\sigma, s_{\sigma'} \in N_{\mathbb{R}}$ are the endpoints of ℓ dual to $\sigma, \sigma' \in \mathcal{P}^{(2)}$. More generally, it can be shown that the degree of the restriction of $\mathcal{F}(L)$ to $T\ell/T_{\mathbb{Z}}\ell$ is given by $\langle df, (\boldsymbol{\beta} - \boldsymbol{\alpha})^\perp\rangle|_{s_{\sigma'}}^{s_\sigma}$. This shows that the isomorphism class of $\mathcal{F}(L)$ depends only on the Hamiltonian isotopy class of L, hence proving Theorem 1.1.

4. Standard wrapped Floer theory

4.1. Let (X, θ) be a Liouville manifold. The induced contact structure on ∂X^{in} will be denoted by $\xi := \ker(\theta|_{\partial X^{\mathrm{in}}})$, and the Liouville coordinate on the symplectization end $[1,\infty) \times \partial X^{\mathrm{in}}$ will be denoted by r.

4.2. A Lagrangian submanifold $L \subset X$ is *Liouville-admissible* if it is the completion $L^{\mathrm{in}} \cup [1,\infty) \times \partial L^{\mathrm{in}}$ of a Lagrangian submanifold $L^{\mathrm{in}} \subset X^{\mathrm{in}}$ such that $\theta|L^{\mathrm{in}} \in \Omega^1(L^{\mathrm{in}})$ vanishes to infinite order along the boundary ∂L^{in}.

4.3. A *Liouville-admissible Hamiltonian* is a positive function $H\colon X \to \mathbb{R}^{>0}$ which is λr outside of a compact set for a positive real number λ called the *slope* of H. The set of Liouville-admissible Hamiltonians of slope λ is denoted by $\mathscr{H}_{\mathrm{La}}(X)_\lambda$ and we set $\mathscr{H}_{\mathrm{La}}(X) \coloneqq \bigcup_{\lambda\in\mathbb{R}^{>0}} \mathscr{H}_{\mathrm{La}}(X)_\lambda$.

4.4. A *Liouville-admissible almost complex structure* is a compatible almost complex whose restriction to the symplectization end is the direct sum of an almost complex structure on ξ and the standard complex structure on the rank 2 bundle spanned by the Liouville vector field and the Reeb vector field. The set of Liouville-admissible complex structures is denoted by $\mathscr{J}_{\mathrm{La}}(X)$.

4.5. Let $H\colon X \to \mathbb{R}$ be a function on a symplectic manifold X and L be a Lagrangian submanifold. A *Hamiltonian chord* is a trajectory $x\colon [0,1] \to X$ of the Hamiltonian flow such that $x(0) \in L$ and $x(1) \in L$. The set of Hamiltonian chords will be denoted by $\mathscr{X}(L, X; H)$. A Hamiltonian chord x is *non-degenerate* if the image $\varphi_1(L)$ of L by the time-one Hamiltonian flow $\varphi_1\colon X \to X$ intersects L transversally at the intersection point corresponding to x.

4.6. Let Σ be a closed disc with $d+1$ boundary punctures $\boldsymbol{\zeta} = \{\zeta_0, \dots, \zeta_d\}$, which are called the *points at infinity*. We denote by $\overline{\Sigma} \coloneqq \Sigma \cup \boldsymbol{\zeta}$ the closed disk obtained by filling in the punctures. The connected component of the boundary of Σ between ζ_i and ζ_{i+1}, which is homeomorphic to an open interval, will be denoted by $\partial_i\Sigma$. We also write $\partial\Sigma \coloneqq \bigcup_{i=0}^d \partial_i\Sigma$. The moduli space of such discs and its stable compactification will be denoted by $\mathcal{R}^d$ and $\overline{\mathcal{R}}^d$ respectively.

Remark 4.7. We will be exclusively interested in the cases when $d \leq 3$ in this paper.

4.8. A *strip-like end* around a point ζ_i is a holomorphic embedding

$$\begin{cases} \epsilon\colon \mathbb{R}^{\leq 0} \times [0,1] \to \Sigma, \\ \epsilon^{-1}(\partial\Sigma) = \mathbb{R}^{\leq 0} \times \{0,1\}, \\ \lim\limits_{s\to-\infty} \epsilon(s,-) = \zeta_i \end{cases} \tag{4.8.1}$$

if $i = 0$, and

$$\begin{cases} \epsilon\colon \mathbb{R}^{\geq 0} \times [0,1] \to \Sigma, \\ \epsilon^{-1}(\partial\Sigma) = \mathbb{R}^{\geq 0} \times \{0,1\}, \\ \lim\limits_{s\to\infty} \epsilon(s,-) = \zeta_i \end{cases} \tag{4.8.2}$$

otherwise.

4.9. A *Liouville-admissible Floer data* is a pair

$$(H, J) \in C^\infty([0,1], \mathscr{H}_{\mathrm{La}}(X)) \times C^\infty([0,1], \mathscr{J}_{\mathrm{La}}(X)) \tag{4.9.1}$$

of families of Liouville-admissible Hamiltonians and Liouville-admissible almost complex structures.

4.10. Let Σ be a closed disk with $d+1$ boundary punctures. A *Liouville-admissible perturbation data* (K, J) consists of

(1) a 1-form $K \in \Omega^1(\Sigma, \mathscr{H}_{\mathrm{La}}(X))$ on Σ with values in Liouville-admissible Hamiltonians satisfying
 - $K|_{\partial\Sigma} = 0$, and
 - $X_K = X_H \otimes \beta$ outside of a compact set, where H is of slope one and β is sub-closed (i.e., $d\beta \le 0$), and

(2) a family $J \in C^\infty(\Sigma, \mathscr{J}_{\mathrm{La}}(X))$ of Liouville-admissible almost complex structures on X parametrized by Σ.

It is *compatible* with a sequence $(\boldsymbol{H}, \boldsymbol{J}) = (H_j, J_j)_{j=0}^d$ of Liouville-admissible Floer data if

$$\epsilon_j^* K = H_j(t) dt \quad \text{and} \quad J(\epsilon_j(s,t)) = J_j(t) \tag{4.10.1}$$

for any $j \in \{0, \dots, d\}$ and any $t \in [0,1]$.

4.11. A sequence $\boldsymbol{x} \coloneqq (x_k \in \mathscr{X}(L; H_k))_{k=0}^d$ of Hamiltonian chords and a perturbation data (K, J) allow us to define *Floer's equation*

$$\begin{cases} y \colon \Sigma \to \overline{Y}, \\ y(\partial\Sigma) \subset L, \\ \lim\limits_{s \to \pm\infty} y(\epsilon_k(s, -)) = x_k, \quad k = 0, \dots, d, \\ (dy - X_K)^{0,1} = 0, \end{cases} \tag{4.11.1}$$

where X_K is the one-form with values in Hamiltonian vector fields on $\overline{Y}$ associated with K.

4.12. The *Floer complex* is defined by

$$\mathrm{CF}^*(L, X; H) \coloneqq \bigoplus_{x \in \mathscr{X}(L,X;H)} \mathfrak{o}_x, \tag{4.12.1}$$

where $\mathfrak{o}_x$ is the one-dimensional $\mathbb{C}$-normalized orientation space associated to x. For a pair $\mathbf{x} = (x_0, x_1)$ of Hamiltonian chords, the matrix element of the *Floer differential* $\mathfrak{m}_1$ is defined by counting the number of solutions to Floer's equation (4.11.1) on the strip $\Sigma \coloneqq \mathbb{R} \times [0,1]$. The cohomology of the Floer complex (4.12.1) is denoted by $\mathrm{HF}^*(L, X; H)$, and called the *Floer cohomology*.

4.13. A *Liouville-admissible sequence of Hamiltonians* is a sequence $(H_m)_{m=1}^\infty$ of Liouville-admissible Hamiltonians satisfying the following conditions:

(1) For each $m \in \mathbb{Z}^{>0}$, the set $\mathscr{X}(L, X; H_m)$ is finite and consists only of non-degenerate chords.
(2) The slopes λ_m of H_m satisfy $\lambda_m < \lambda_{m+1}$ and $\lambda_m + \lambda_{m'} \le \lambda_{m+m'}$ for any $m, m' \in \mathbb{Z}^{>0}$.

Note that (2) implies $\lim_{m\to\infty} \lambda_m = \infty$.

4.14. We fix a Liouville-admissible sequence $(H_m)_{m=1}^\infty$ of Hamiltonians and a sequence $(J_m)_{m=1}^\infty$ of Liouville-admissible almost complex structures. In addition, for each $m \in \mathbb{Z}^{>0}$, we fix a Liouville-admissible perturbation data $K(m, m+1)$ on the strip $\mathbb{R} \times [0,1]$, which is compatible with the pair $((H_m, J_m), (H_{m+1}, J_{m+1}))$ of Floer data. For any $n > m$, by gluing $(K(i, i+1))_{i=m}^{n-1}$, we obtain a perturbation data $K(m, n)$ on the strip $\mathbb{R} \times [0,1]$, which is compatible with the pair $((H_m, J_m), (H_n, J_n))$ of Floer data. By counting numbers of solutions to Floer's equation (4.11.1) on the strip with respect to this perturbation data, we obtain the *continuation map*

$$\mathrm{CF}^*(L,X;H_m) \to \mathrm{CF}^*(L,X;H_n) \tag{4.14.1}$$

on the Floer cochain complex. A standard argument in Floer theory shows that the continuation map commutes with the Floer differential, and induces the continuation map

$$\mathrm{HF}^*(L,X;H_m) \to \mathrm{HF}^*(L,X;H_n) \tag{4.14.2}$$

on the Floer cohomology. The colimit

$$\mathrm{HW}(L,X) \coloneqq \varinjlim_m \mathrm{HF}(L,X;H_m) \tag{4.14.3}$$

with respect to the continuation map (4.14.2) is called the *wrapped Floer cohomology.*

4.15. For any $m,n \in \mathbb{Z}^{>0}$, we fix a Liouville-admissible perturbation data $K(m,n,m+n)$ on a disk with three punctures, which is compatible with the triple

$$((H_m,J_m),(H_n,J_n),(H_{m+n},J_{m+n}))$$

of Floer data. This allows us to define a linear map

$$\mathfrak{m}_2\colon\ \mathrm{CF}^*(L,X;H_n)\otimes \mathrm{CF}^*(L,X;H_m) \to \mathrm{CF}^*(L,X;H_{m+n}) \tag{4.15.1}$$

by counting numbers of solutions to Floer's equation. A standard arguments in Floer theory shows that $\mathfrak{m}_2$ satisfies the Leibniz rule with respect to $\mathfrak{m}_1$, and hence induces a map on the Floer cohomology.

4.16. For any $m_1,m_2,m_3 \in \mathbb{Z}^{>0}$ with $m_1 < m_2$, we fix a one-parameter homotopy through Liouville-admissible perturbation data between the gluing of $K(m_1,m_3,m_1+m_3)$ with $K(m_1+m_3,m_2+m_3)$ and the gluing of $K(m_1,m_2)$ with $K(m_2,m_3,m_2+m_3)$. We also fix the analogous data for the case when the continuation map occur along the other positive strip-like end. A standard argument using this homotopy shows that the product is well-defined on the direct limit.

4.17. For any $m_1,m_2,m_3 \in \mathbb{Z}^{>0}$, we fix an $n \geq 0$, and set $\mathbf{m} = \sum_i m_i + n$. We then choose a family $K(m_1,m_2,m_3)$ of Liouville-admissible perturbation data on the universal family of disks with 4 punctures over the moduli space $\overline{\mathcal{R}}^3$ which is compatible with $(H_\mathbf{m},J_\mathbf{m})$ along the negative end and (H_{m_i},J_{m_i}) along the three positive ends. We assume that near one end of the boundary, the family $K(m_1,m_2,m_3)$ restricts to the gluing of a domain in $K(m_1,m_2,m_1+m_2+n)$ (for some domain in the interior of the above homotopy) with $K(m_1+m_2+n,m_3,\mathbf{m})$, and near the other end of the boundary, it restricts to the gluing of a domain in $K(m_2,m_3,m_2+m_3+n)$ (for some domain in the interior of the above homotopy) with $K(m_1,m_2+m_3+n,\mathbf{m})$. A standard argument in Floer theory using the moduli space of solutions to Floer's equation with respect to this perturbation data shows that the product on the Floer cohomology is associative.

5. Adapted wrapped Floer theory

5.1. Let $S\colon N_{\mathbb{C}^\times} \to \mathbb{R}$ be the standard exhaustive plurisubharmonic function defined in (3.2.1). A function $H : N_{\mathbb{C}^\times} \to \mathbb{R}$ is *homogeneous of degree one* with respect to S if

(5.1.1) $\quad -d^cS(X_H) = H$, and

(5.1.2) $\quad dS(X_H) = 0.$

5.2. A positive function $H_{\mathrm{b}}\colon N_{\mathbb{C}^\times} \to \mathbb{R}^{>0}$ is an *admissible base Hamiltonian* if it is homogeneous of degree one outside of a compact set.

5.3. For a function $H\colon N_{\mathbb{R}} \to \mathbb{R}$, the composition $H \circ \mathrm{Log}\colon N_{\mathbb{C}^\times} \to \mathbb{R}$ will also be denoted by H by abuse of notation. The Hamiltonian vector field is given by

$$X_H = \frac{\partial H}{\partial r_1}\frac{\partial}{\partial \theta_1} + \frac{\partial H}{\partial r_2}\frac{\partial}{\partial \theta_2}. \tag{5.3.1}$$

One has

$$-d^c S(X_H) = r_1 \frac{\partial H}{\partial r_1} + r_2 \frac{\partial H}{\partial r_2}, \tag{5.3.2}$$

so that $-d^c S(X_H) = H$ if and only if H is homogeneous of degree one in the usual sense.

5.4. A positive function $H_{\mathrm{ba}}\colon \overline{Y} \to \mathbb{R}^{>0}$ is a *base-admissible Hamiltonian* if there exists an admissible base Hamiltonian $H_{\mathrm{b}}\colon N_{\mathbb{C}^\times} \to \mathbb{R}^{>0}$ and a compact set $K \subset N_{\mathbb{C}^\times}$ such that

- for any $y \in \overline{Y} \setminus \overline{\pi}_{N_{\mathbb{C}^\times}}^{-1}(K)$, one has

$$(\overline{\pi}_{N_{\mathbb{C}^\times}})_*(X_{H_{\mathrm{ba}}}(y)) = X_{H_{\mathrm{b}}}(\overline{\pi}_{N_{\mathbb{C}^\times}}(y)). \tag{5.4.1}$$

- Outside of $\overline{\pi}_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$, H_{ba} is a C^2 small perturbation of $\overline{\pi}_{N_{\mathbb{C}^\times}}^{-1}(H_b)$

The set of base-admissible Hamiltonians on $\overline{Y}$ is denoted by $\mathscr{H}_{\mathrm{ba}}(\overline{Y})$.

Proposition 5.5. *There exists a base-admissible Hamiltonian.*

Proof. Recall from (2.5.18) that $\chi|_{U_i} = \chi_i \circ G_i$. The symplectic form on U_i is invariant under the $\mathbb{S}^1$-action on $N_{\mathbb{C}^\times}$ which preserves $\boldsymbol{w}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}$. The essential idea is to produce the base Hamiltonian by gluing together local moment maps for these actions. In more detail, define $\overline{\chi}_i(x) = \chi_i(x)\log(x)$. Then we have

$$\omega = \overline{\pi}_{N_{\mathbb{C}^\times}}^* \omega_{N_{\mathbb{C}^\times}} + \overline{\pi}_{\mathbb{C}_u}^* \omega_{\mathbb{C}_u} - \epsilon dd^c(\overline{\chi}_i(G_i)). \tag{5.5.1}$$

Note that

$$-dd^c(\overline{\chi}_i(G_i)) = -d(\overline{\chi}_i'(G_i) d^c(G_i)) \tag{5.5.2}$$

$$= -d\overline{\chi}_i'(G_i) \wedge d^c(G_i) - \overline{\chi}_i'(G_i) \wedge dd^c(G_i). \tag{5.5.3}$$

Let X be the vector field on $\overline{Y}$, which is $\partial_{\theta_{(\boldsymbol{\alpha}_i-\boldsymbol{\beta}_i)^\perp}}$ with $(u, \boldsymbol{w})$ as coordinates; it is characterized by

$$\iota_X du = \iota_X d\boldsymbol{w}^{\boldsymbol{\alpha}_i-\boldsymbol{\beta}_i} = \iota_X dr_{(\boldsymbol{\alpha}_i-\boldsymbol{\beta}_i)^\perp} = 0, \quad \iota_X d\theta_{(\boldsymbol{\alpha}_i-\boldsymbol{\beta}_i)^\perp} = 1. \tag{5.5.4}$$

Then we have

$$\iota_X (\overline{\pi}_{N_{\mathbb{C}^\times}})^* \omega_{N_{\mathbb{C}^\times}} = -dr_{(\boldsymbol{\alpha}_i-\boldsymbol{\beta}_i)^\perp} \tag{5.5.5}$$

By invariance of the function G_i under the local circle action generated by X, we have that

$$\iota_X dG_i = \iota_X d\left(|u|^2 + \left|t^{-\nu(\boldsymbol{\alpha}_i)}\boldsymbol{w}_i^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta}_i)}\boldsymbol{w}_i^{\boldsymbol{\beta}}\right|^2\right) = 0. \tag{5.5.6}$$

Define $c_i, c_i', c_i'' \in \mathbb{Q}$ by

$$c_i = \frac{1}{|\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i|}, \quad c_i' = \frac{\boldsymbol{\alpha}_i \cdot (\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)}{|\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i|^2}, \quad c_i^\perp = \frac{\boldsymbol{\alpha}_i \cdot (\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp}{|\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i|^2}, \tag{5.5.7}$$

so that

$$|\boldsymbol{r}|^2 \coloneqq r_1^2 + r_2^2 = c_i^2 \left(\left(\boldsymbol{r}_{\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i} \right)^2 + \left(\boldsymbol{r}_{(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp} \right)^2 \right), \tag{5.5.8}$$

$$\boldsymbol{\alpha}_i = c_i'(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i) + c_i''(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp. \tag{5.5.9}$$

Then we have $|\boldsymbol{w}^{\boldsymbol{\alpha}_i}| = |\boldsymbol{w}^{\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i}|^{c_i'} |\boldsymbol{w}^{(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp}|^{c_i''}$ and $d\theta_{\boldsymbol{\alpha}_i} = c_i' d\theta_{\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i} + c_i'' d\theta_{(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp}$, so that

$$\iota_X d^c |t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta}}|^2 = \iota_X d^c (|\boldsymbol{w}^{\boldsymbol{\alpha}}|^2 |t^{-\nu(\boldsymbol{\alpha})} - t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta} - \boldsymbol{\alpha}}|^2) \tag{5.5.10}$$

$$= (\iota_X d^c (|\boldsymbol{w}^{\alpha}|^2)) \cdot |t^{-\nu(\boldsymbol{\alpha})} - t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta} - \boldsymbol{\alpha}}|^2 \tag{5.5.11}$$

$$= -2c_i'' |t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta}}|^2. \tag{5.5.12}$$

Similarly, we have

$$\iota_X dd^c |t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta}}|^2 = (\mathscr{L}_X - d\iota_X) d^c |t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta}}|^2 \tag{5.5.13}$$

$$= -d \left(\iota_X d^c |t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta}}|^2 \right) \tag{5.5.14}$$

$$= d \left(2c_i'' |t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta}}|^2 \right). \tag{5.5.15}$$

Hence we have

$$-\iota_X \omega = d \left(r_{(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp} - 2c_i'' \epsilon \overline{\chi}'(G_i) |t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta}}|^2 \right). \tag{5.5.16}$$

If we define $\rho_i \colon U_i \to \mathbb{R}$ by

$$\rho_i \coloneqq \boldsymbol{r}_{(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp} - 2c_i'' \epsilon \overline{\chi}_i'(G_i) \left| t^{-\nu(\boldsymbol{\alpha})} \boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})} \boldsymbol{w}^{\boldsymbol{\beta}} \right|^2, \tag{5.5.17}$$

then $\rho_i - \boldsymbol{r}_{(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp}$ is a bounded function whose support is contained in the support of χ. Let R be a positive number satisfying $R \gg R_0$. Let H_{b} be a positive function on $N_{\mathbb{R}}$ (which is also considered as a function on $N_{\mathbb{C}^\times}$ by composing with $\mathrm{Log} \colon N_{\mathbb{C}^\times} \to N_{\mathbb{R}}$) such that

(5.5.18) H_{b} is homogeneous of degree one outside of a compact set,

(5.5.19) $H_{\mathrm{b}}|_{U_{\Pi_i}} = \frac{c_i \boldsymbol{r}^\perp_{\boldsymbol{\alpha} - \boldsymbol{\beta}}}{R}$, and

(5.5.20) $H_{\mathrm{b}}(\boldsymbol{r}) = \frac{|\boldsymbol{r}|}{R}$ outside of a neighborhood of U_Π.

Since the function $\frac{c_i \rho_i}{R}$ agrees with $\overline{\pi}_{N_{\mathbb{C}^\times}}^* H_{\mathrm{b}}$ outside the support of χ, we may glue $\frac{c_i \rho_i}{R}$ for $i = 1, \ldots, \ell$ and $\overline{\pi}_{N_{\mathbb{C}^\times}}^* H_{\mathrm{b}}$ together to obtain a positive function ρ defined on the complement of $\overline{\pi}_{N_{\mathbb{C}^\times}}^{-1}(K)$ for a compact subset K of $N_{\mathbb{C}^\times}$. We may extend this function to $\overline{Y}$ arbitrarily to obtain a function which satisfies the necessary axioms. □

We fix a function ρ appearing in the proof of Proposition 5.5 throughout the rest of this paper. We say that a base-admissible Hamiltonian H_{ba} has a *slope* $\lambda \in \mathbb{R}^{>0}$ if it is a C^2-small perturbation of $\lambda\rho$ which coincides with $\lambda\rho$ outside of the inverse image by $\overline{\pi}_{N_{\mathbb{C}^\times}}$ of a compact set in $N_{\mathbb{C}^\times}$.

5.6. An ω-compatible almost complex structure $J_{\overline{Y}}$ on $\overline{Y}$ is said to be *base-admissible* if the map $\overline{\pi}_{N_{\mathbb{C}^\times}} : \overline{Y} \to N_{\mathbb{C}^\times}$ is $(J_{\overline{Y}}, J_{N_{\mathbb{C}^\times}})$-holomorphic outside of a compact set. The set of base-admissible almost complex structures on $\overline{Y}$ will be denoted by $\mathscr{J}_{\mathrm{ba}}(\overline{Y})$.

5.7. A *base-admissible Floer data* is a pair

$$(H, J) \in C^\infty([0,1], \mathscr{H}_{\mathrm{ba}}(\overline{Y})) \times C^\infty([0,1], \mathscr{J}_{\mathrm{ba}}(\overline{Y})) \tag{5.7.1}$$

of families of base-admissible Hamiltonians and base-admissible almost complex structures.

5.8. A *base-admissible perturbation data* (K, J) consists of

(1) a 1-form $K \in \Omega^1(\Sigma, H_{\mathrm{ba}}(\overline{Y}))$ on Σ with values in base-admissible Hamiltonians satisfying
$K|_{\partial\Sigma} = 0$,
outside of a compact set in the base, we have that $\pi_{N_{\mathbb{C}^\times},*}(K) = X_{H_{\mathrm{b}}} \otimes \gamma$ for γ sub-closed.

(2) a family $J \in C^\infty(\Sigma, \mathscr{J}_{\mathrm{ba}}(\overline{Y}))$ of base-admissible almost complex structures on $\overline{Y}$ parametrized by Σ.

It is *compatible* with a sequence $(\boldsymbol{H}, \boldsymbol{J}) = (H_j, J_j)_{j=0}^d$ of base-admissible Floer data if (4.10.1) holds for any $j \in \{0, \dots, d\}$ and any $t \in [0, 1]$.

Lemma 5.9. *Let $y \colon \Sigma \to \overline{Y}$ be a solution to Floer's equation* (4.11.1) *with respect to a base-admissible perturbation data. Set $p \coloneqq S \circ \overline{\pi}_{N_{\mathbb{C}^\times}} \circ y \colon \Sigma \to \mathbb{R}$. If p is not a constant function, then p does not have a maximum on Σ whose maximum value is outside of the compact set appearing in the definitions of $H_{\mathrm{ba}}(\overline{Y})$ and $\mathscr{J}_{\mathrm{ba}}(\overline{Y})$.*

Proof. It follows from the base-admissibility of K that the map $w \coloneqq \pi_{N_{\mathbb{C}^\times}} \circ y \colon \Sigma \to N_{\mathbb{C}^\times}$ satisfies the Floer's equation

$$(dw - X_{H_{\mathrm{b}}} \otimes \gamma)^{0,1} = 0 \tag{5.9.1}$$

on X for the Hamiltonian H_{b} outside of a compact set in $N_{\mathbb{C}^\times}$. We write the almost complex structures on $N_{\mathbb{C}^\times}$ and Σ as J and j respectively, and set $\beta \coloneqq -\partial^c S = -dS \circ J$ and $p \coloneqq S \circ w$. Applying dS to both sides of Floer's equation

$$(dw - X_{H_{\mathrm{b}}} \otimes \gamma) \circ j = J \circ (dw - X_{H_{\mathrm{b}}} \otimes \gamma) \tag{5.9.2}$$

and using $dS(X_{H_{\mathrm{b}}}) = 0$, one obtains

$$d^c p = -\beta \circ (dw - X_{H_{\mathrm{b}}} \otimes \gamma). \tag{5.9.3}$$

By applying d to both sides and using $\omega = d\beta = -dd^c S$, one obtains

$$-dd^c p = w^*\omega - d(\beta(X_{H_{\mathrm{b}}}) \cdot \gamma). \tag{5.9.4}$$

Since $\beta(X_{H_{\mathrm{b}}}) = -H_{\mathrm{b}}$ outside of a compact set in $N_{\mathbb{R}}$, one has

$$-dd^c p = w^*\omega - d(w^* H_{\mathrm{b}} \cdot \gamma) = w^*\omega - d(w^* H_{\mathrm{b}}) \wedge \gamma - w^* H_{\mathrm{b}} \cdot d\gamma \tag{5.9.5}$$

$$= \|dw - X_{H_{\mathrm{b}}} \otimes \gamma\|^2 - w^* H_{\mathrm{b}} \cdot d\gamma \geq 0 \tag{5.9.6}$$

because $H_{\mathrm{b}} \geq 0$ and $d\gamma \leq 0$. Now $-dd^c$ is an operator of the form (B.1.1), so the function p satisfies the strong maximum principle. If the function p attains a maximum at $\Sigma = \overline{\Sigma} \setminus \boldsymbol{\zeta}$, then Hopf's lemma implies that

$$dp(\nu) > 0 \tag{5.9.7}$$

for an outward normal vector ν of $\partial\Sigma$ at some point $x \in \partial\Sigma$. Let $\tau \in T_x(\partial\Sigma)$ be the tangent vector such that $\nu = j\tau$. Then one has

$$\begin{aligned} dp(\nu) &= dS \circ (dw \circ j)(\tau) = dS \circ (X_{H_{\mathrm{b}}} \otimes \gamma \circ j + J \circ (du - X_{H_{\mathrm{b}}} \otimes \gamma))(\tau) \\ &= -\beta \circ (dw - X_{H_{\mathrm{b}}} \otimes \gamma)(\tau), \end{aligned} \tag{5.9.8}$$

where we used $dS(X_{H_{\mathrm{b}}}) = 0$ and $\beta = -dS \circ J$. The first term vanishes by the Legendrian condition $\beta|_{\underline{L}} = 0$ at infinity, and the second term vanishes by $\gamma|_{\partial\Sigma} = 0$. This contradicts (5.9.7), and Lemma 5.9 is proved. $\square$

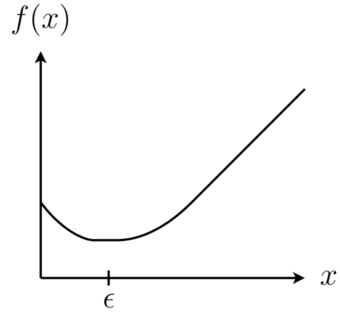

FIGURE 5.1. An admissible vertical Hamiltonian

5.10. An almost complex structure J on $\overline{Y}$ is *fibration-admissible* if

(1) J is base-admissible,
(2) the map $\overline{\pi}_{\mathbb{C}} : \overline{Y} \to \mathbb{C}$ is J-holomorphic on $\overline{\pi}_{\mathbb{C}}^{-1}(\{u \in \mathbb{C} \mid |u| > C_0\})$ for some C_0,
(3) the divisor $\overline{E}$ defined in (2.5.5) is J-holomorphic, and
(4) the almost complex structure $J|_{U_D}$ is the product of $J_{N_{\mathbb{C}^\times}}$ and the standard complex structure on $\mathbb{D}_\delta$ under the identification (2.5.13).

The set of fibration-admissible almost complex structures on $\overline{Y}$ will be denoted by $\mathscr{J}(\overline{Y})$.

The following stronger notion will be used later in Section 6:

5.11. An almost complex structure J on $\overline{Y}$ is *integrably fibration-admissible* if there exists an almost complex structure $J_{N_{\mathbb{C}^\times}}$ adapted to Z such that when one equips $N_{\mathbb{C}^\times} \times \mathbb{C}$ with the almost complex structure $(J_{N_{\mathbb{C}^\times}}, J_{\mathbb{C}})$, the structure map $p = (\overline{\pi}_{N_{\mathbb{C}^\times}}, \overline{\pi}_{\mathbb{C}}) \colon \overline{Y} \to N_{\mathbb{C}^\times} \times \mathbb{C}$ of the blow-up is pseudo-holomorphic on the union of

(1) $\overline{\pi}_{\mathbb{C}}^{-1}(\{u \in \mathbb{C} \mid |u| > C_0\})$ for some C_0,
(2) $\overline{\pi}_{N_{\mathbb{C}^\times}}^{-1}(\{\boldsymbol{w} \in N_{\mathbb{C}^\times} \mid S(\boldsymbol{w}) > C_1\})$ for some $C_1 > 0$, and
(3) $U_D \cup \pi^{-1}(U_Z)$.

The set of integrably fibration-admissible almost complex structures on $\overline{Y}$ will be denoted by $\mathscr{J}_{\mathrm{int}}(\overline{Y})$.

5.12. Fix $\mu_0, \mu_1 \in \mathbb{R}^{>0}$ such that $\mu_0 \ll \epsilon \ll \mu_1$. In the exact structure (2.6.10), if we set the boundary to be $\{\mu = \mu_0\}$ and $\{\mu = \mu_1\}$, then the Liouville coordinates become $c_-(\epsilon - \mu)$ and $c_+(\mu - \epsilon)$ for some constant c_- and c_+. A function $H_{\mathrm{v}} \colon \overline{Y} \to \mathbb{R}^{>0}$ is an *admissible vertical Hamiltonian* of slope $\lambda \in \mathbb{R}^{>0}$ if there exist a function $f \colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{>0}$ satisfying

(1) $f''(x) \geq 0$ for any $x \in \mathbb{R}^{\geq 0}$,
(2) $f(x) = \lambda c_-(\epsilon - x)$ when $x < \mu_0$,
(3) $f'(x) = 0$ in a neighborhood of $x = \epsilon$,
(4) $f(x) = \lambda c_+(x - \epsilon)$ when $x > \mu_1$, and
(5) $H_{\mathrm{v}} = f \circ \mu$ where $\mu \colon \overline{Y} \to \mathbb{R}^{\geq 0}$ is the moment map (2.6.7).

The set of admissible vertical Hamiltonians will be denoted by $\mathscr{H}_{\mathrm{v}}\left(\overline{Y}\right)$. The Hamiltonian vector field associated with H_{v} is given by

$$X_{H_{\mathrm{v}}} = f'(\mu) \cdot X_\mu, \tag{5.12.1}$$

where X_μ is the fundamental vector field for the $\mathbb{S}^1$-action (2.5.6). It follows that

$$\left(\pi_{N_{\mathbb{C}^\times}}\right)_* (X_{H_{\mathrm{v}}}) = 0. \tag{5.12.2}$$

5.13. A *fibration-admissible Hamiltonian* of slope $\lambda \in \mathbb{R}^{>0}$ is a function $H\colon \overline{Y} \to \mathbb{R}^{>0}$ satisfying the following conditions:

(1) One has $(\overline{\pi}_{N_{\mathbb{C}^\times}})_*(X_H) = X_{\lambda\rho}$ outside of a compact set in $N_{\mathbb{C}^\times}$.
(2) Whenever $|\mu| < \mu_0$ or $|\mu| > \mu_1$, one has $H = H_{\text{ba}} + H_{\text{v}}$, where H_{ba} is a base-admissible Hamiltonian of slope λ and H_{v} is an admissible vertical Hamiltonian of slope λ.
(3) X_H is tangent to $\overline{E}$.

The set of fibration-admissible Hamiltonians of slope λ will be denoted by $\mathscr{H}_\lambda\left(\overline{Y}\right)$.

Remark 5.14. To actually construct examples, we may assume that $H = H_{\text{ba}} + H_{\text{v}}$ holds everywhere. The extra flexibility of our definitions are included for Section 7.

5.15. A *fibration-admissible sequence of Hamiltonians* is a sequence $(H_m)_{m=1}^{\infty}$ of fibration-admissible Hamiltonians such that the slopes λ_m of H_m for $m \in \mathbb{Z}^{>0}$ satisfy $\lambda_m < \lambda_{m+1}$ and $\lambda_m + \lambda_{m'} \leq \lambda_{m+m'}$ for any $m, m' \in \mathbb{Z}^{>0}$.

5.16. A *fibration-admissible perturbation data* is a base-admissible perturbation data

$$(K, J) \in \Omega^1\left(\Sigma, H_{\text{ba}}\left(\overline{Y}\right)\right) \times C^\infty\left(\Sigma, \mathscr{J}\left(\overline{Y}\right)\right) \tag{5.16.1}$$

satisfying the following conditions:

(1) X_K is tangent to $\overline{E}$ and D.
(2) For $\mu \gg \epsilon$, one has $(\overline{\pi}_{\mathbb{C}})_*(X_K) = (\overline{\pi}_{\mathbb{C}})_* X_{H_{\text{v}}} \otimes \gamma_+$ for a subclosed one form γ_+ and a vertical Hamiltonian H_{v}.
(3) For points on $\overline{Y} \setminus \pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$ with $\mu \ll \epsilon$, one has

$$(\overline{\pi}_{\mathbb{C}})_*(X_K) = (\overline{\pi}_{\mathbb{C}})_* X_{H_{\text{v}}} \otimes \gamma_- \tag{5.16.2}$$

for a subclosed one form γ_- and a vertical Hamiltonian H_{v}.

5.17. Let $\overline{L}$ be the closure of a base-admissible Lagrangian section L in $\overline{Y}$. By base-admissibility, solutions to Floer's equation are now constrained to lie in a compact subspace in $\overline{Y}$ and so Gromov compactness applies as usual. Gromov compactness is typically stated for Lagrangians without boundary, but here it applies because we can extend $\overline{L}$ slightly in the negative real direction as well. Here we will assume that $d \leq 3$. Let $\mathcal{R}^d(\boldsymbol{x})$ be the moduli space of solutions to Floer's equation for perturbation data 4.14, 4.15, 4.16 and $\mathcal{R}_t^2(\boldsymbol{x})$ in the case 4.17. Because $\overline{L}$ is contractible, the relative homology classes of Floer curves in $\overline{Y}$ form a torsor over $\operatorname{Im}(\pi_2(\overline{Y})) \subset H_2(\overline{Y})$. The moduli spaces $\overline{\mathcal{R}}^d(\boldsymbol{x})$ or $\overline{\mathcal{R}}_t^2(\boldsymbol{x})$ embeds naturally into the Gromov compactifcation of maps into $\overline{Y}$ with relative homology class $A_{\boldsymbol{x}}$, where $A_{\boldsymbol{x}}$ can be characterized as the unique relative homology class such that $A_{\boldsymbol{x}} \cdot D = 0$. This class is unique since $\operatorname{Im}(\pi_2(\overline{Y}))$ is one dimensional generated by the class of the exceptional sphere $[E_{\boldsymbol{w}}]$. We will denote this moduli space by $\overline{\mathcal{R}}^d(\overline{Y}, \boldsymbol{x}, A_{\boldsymbol{x}})$ or $\overline{\mathcal{R}}_t^2(\overline{Y}, \boldsymbol{x}, A_{\boldsymbol{x}})$. The boundary $\partial\overline{\mathcal{R}}_t^2(\overline{Y}, \boldsymbol{x}, A_{\boldsymbol{x}})$, in addition to the usual trees of curves and spheres, includes maps with perturbation data corresponding to the limits $t \to \pm\infty$.

Definition 5.18. Hamiltonian chords for the Lagrangian $\overline{L}$ which are completely contained in D are called *divisor* chords.

5.19. We assume that the Hamiltonian flow preserves $U_{Z\times 0}$, so that all Hamiltonian chords are disjoint from $\pi^{-1}_{N_{\mathbb{C}^\times}}(U_Z)$.

Lemma 5.20. *Assume that our almost complex structure $J \in \mathcal{J}(\Sigma, D\cup\overline{E})$ and our Floer data are both fibration-admissible. Consider a sequence $(y_s)_{s=1}^{\infty}$ of maps $y_s : \Sigma \to Y$ in $\mathcal{R}^d(\boldsymbol{x})$ converging to $y_\infty \in \overline{\mathcal{R}}^d(\overline{Y}, \boldsymbol{x}, A_{\boldsymbol{x}})$. Assume this curve has components $y_{k,\infty}$. Then if a component $y_{k,\infty} : \Sigma_k \to \overline{Y}$ intersects D along a boundary then $y_{k,\infty}$ lies entirely in D.*

Proof. Observe that there are four essentially distinct ways that a limiting component $y_{k,\infty}$ could intersect the divisor D:

- In the interior of Σ,
- on the boundary $\partial\Sigma$,
- $y_{k,\infty}$ lies completely in D, or
- $y_{k,\infty}$ limits to a divisor Hamiltonian chord in D along some strip-like end ϵ.

It therefore remains to rule out intersections along a boundary, which we claim follows from the fact that $d^c(1/|u|)|_{\overline{L}} = 0$, where u is the base coordinate on $\mathbb{C}$ discussed above. Consider a subsequence $y_s\colon \Sigma \to Y$ with boundary on L which meets the exceptional divisor E at points $z_{k,s}$. We can define the intersection number

$$y_s \cdot E = \sum_{z_{k,s}} d_{k,s}, \tag{5.20.1}$$

where $d_{k,s} \geq 0$ are the local intersection numbers of y_s with E at $z_{k,s}$ on Σ. To define this number efficiently, observe that Gromov's trick [Gro85] (see e.g. [MS12, Section 8.1] for an exposition) allows us to view a solution y_s to Floer's equation as a pseudo-holomorphic section $\widetilde{y}_s\colon \Sigma \to \Sigma \times \overline{Y}$ for a specific almost complex structure on $\Sigma \times \overline{Y}$. When the perturbation data $(J, \mathbf{K})$ are admissible, both $\Sigma \times E$ and $\Sigma \times D$ are almost complex submanifolds of codimension 2. The local intersection number is then the intersection number of the section with $\Sigma\times E$. This number is constant in our sequence (y_s). Assume for contradiction that a sequence (y_s) has a convergent subsequence which limits to u_∞ that has a component $y_{k,\infty}$ intersecting D along some $\overline{L}$ at a point z_ℓ. Then the intersection points above limit to intersection points $z_{k,\infty}$ which are in the interior of y_∞.

Fix a small ball $\mathbb{D}_{\epsilon_i}(z_{k,\infty})$ about these points which avoids z_ℓ. We choose s large enough so that all of the points $z_{k,s}$ lie in $\mathbb{D}_{\epsilon_i}(z_{k,\infty})$. Then for large s, there must be a local maximum of $1/|u|$ near z_ℓ. This is impossible by the same calculation as in Lemma 5.9 if we note that $d^c(1/|u|)|_{\overline{L}} = 0$. □

5.21. There are two useful ways of grading Hamiltonian chords x_i. The first is used to grade Hamiltonian chords in Y. Namely, one considers an algebraic volume form Ω on Y and fixes "grading data" for these Lagrangians. One can choose this volume for so that the gradings agree with the standard gradings when the Lagrangian L are viewed as lying in $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ with respect the standard volume form. We denote this standard grading by $|x_i|$. We have that for generic $J \in \mathcal{J}(\Sigma, D \cup E)$ on Y,

$$\dim(\mathcal{R}^d(\boldsymbol{x})) = |x_0| - \sum_{i\neq 0} |x_i| + \dim \mathcal{R}^d, \tag{5.21.1}$$

where in the stable case $d \geq 2$, $\dim \mathcal{R}^d$ should be understood to be $d-2$ and when $d = 1$, it should be understood to be 0 in stable situations and -1 in the case of translation invariant perturbation data. In the case of perturbation data in 4.17, one has

$$\dim \mathcal{R}^2_t(\boldsymbol{x}) = |x_0| - \sum_{i\neq 0} |x_i| + 1. \tag{5.21.2}$$

The second allows us to grade all chords, including divisor chords. This is by grading chords x_i with respect to the standard volume form on $N_{\mathbb{C}^\times} \times \mathbb{C}$. Equivalently, one may view this as choosing an algebraic volume form Ω with a simple zero along E and restricting this to $\overline{Y} \setminus E$ and then grading the Lagrangians. It follows from the index computation in [She11, Lemma 3.22] that for a generic J, one has

$$\dim(\mathcal{R}^d(\boldsymbol{x})) = |x_0|_{\mathrm{rel}} - \sum_{i\neq 0} |x_i|_{\mathrm{rel}} + \dim(\mathcal{R}^d) - 2(A_{\boldsymbol{x}} \cdot E) \tag{5.21.3}$$

and similarly for $\mathcal{R}^2_t$.

Remark 5.22. Floer theoretic operations will respect this second grading after one inserts a formal parameter t of cohomological degree -2 and curves y are weighted by $t^{y\cdot E}$, where $y \cdot E$ is their intersection with the divisor E (see [She11, Lemma 3.22 and Definition 5.1]). We will be primarily interested in the first grading, but use this second grading to rule out certain breaking configurations and in certain arguments in Section 6.

5.23. Now we restrict our attention to the Lagrangian L_0, which is the example treated in this paper. Throughout the rest of this paper, we will equip L_0 with the trivial spin structure to view it as a Lagrangian brane. We may assume that all Hamiltonian chords are non-degenerate. For example, we may take a fixed H_{ba} and H_{v} of slope one and then take $H_{\mathrm{ba},m}$ and $H_{\mathrm{v},m}$ to be sufficiently small perturbations of mH_{ba} and mH_{v} supported away from Z.

Lemma 5.24. *The relative homotopy group $\pi_1(Y, L_0)$ is naturally isomorphic to N.*

Proof. The relative homotopy group $\pi_1(Y, L_0)$ is isomorphic to the fundamental group $\pi_1(Y)$ since L_0 is contractible. It is well-known that the blow-up $\overline{Y} \to N_{\mathbb{C}^\times} \times \mathbb{C}$ induces an isomorphism $\pi_1\left(\overline{Y}\right) \xrightarrow{\sim} \pi_1(N_{\mathbb{C}^\times} \times \mathbb{C}) \cong \pi_1(N_{\mathbb{C}^\times})$ of the fundamental group. The kernel of the map $\pi_1(Y) \to \pi_1\left(\overline{Y}\right)$ is the normal subgroup generated by the class of a loop of the form $|w| = \mathrm{pt}$, $|v_0| = \epsilon$ (cf. e.g. [Zai, Lemma 2.3(a)]). Such loops are contractible in Y due to the singular conic bundle structure. We therefore conclude that this map is an isomorphism. The fundamental group of $N_{\mathbb{C}^\times}$ is naturally isomorphic to N, and Lemma 5.24 is proved. □

Recall that we have assumed that near D, our Hamiltonians have the form

$$H_m = H_{\mathrm{ba},m} + H_{\mathrm{v},m} \tag{5.24.1}$$

for $H_{\mathrm{ba},m}$ and $H_{\mathrm{v},m}$ sufficiently generic so that all chords are non-degenerate. It follows that for every p in $\mathscr{X}(\underline{L}_0, \underline{L}_0, H_{\mathrm{ba},m})$, then there are divisor chords p_{d} in $\mathscr{X}(\bar{L}_0, \bar{L}_0, H_m)$.

Lemma 5.25. *One has $|p_{\mathrm{d}}|_{\mathrm{rel}} = 2\lfloor c_- \lambda_m \rfloor + 1 + |\overline{p}|_{N_{\mathbb{C}^\times}}$.*

Proof. This is obtained by observing that we have a product splitting for the Lagrangians, Hamiltonian flow, and the trivializations under the product decomposition of $N_{\mathbb{C}^\times} \times \mathbb{C}$. So it suffices to compute the contribution from the $\mathbb{C}$ factor. The short chord contributes 1 to the Maslov index and each rotation around the cylindrical end contributes $2\lfloor c_- \lambda_m \rfloor$(section (11e) of [Sei08]). □

5.26. For $\boldsymbol{n} \in \pi_1(Y, L)$, we set

$$\mathscr{X}(\underline{L}_0; H_{\mathrm{ba},m})_{\boldsymbol{n}} := \{p \in \mathscr{X}(\underline{L}_0; H_{\mathrm{ba},m}) \mid [p] = \boldsymbol{n}\}. \tag{5.26.1}$$

Let

$$\ell^\sharp \colon \pi_1(Y, L) \times \mathbb{Z}^{>0} \to \mathbb{Z} \tag{5.26.2}$$

be a function satisfying the following conditions:

(5.26.3) For any $\boldsymbol{n} \in \pi_1(Y, L)$, one has $\ell^\sharp(\boldsymbol{n}, m) < \inf_{p \in \mathscr{X}(\underline{L}_0; H_{\mathrm{ba},m})_{\boldsymbol{n}}} |p_{\mathrm{d}}|_{\mathrm{rel}} - 1$ for any $m \in \mathbb{Z}^{>0}$.

(5.26.4) $\ell^\sharp(\boldsymbol{n}, m) < \ell^\sharp(\boldsymbol{n}, m+1)$ for any $\boldsymbol{n} \in \pi_1(Y, L)$ and any $m \in \mathbb{Z}^{>0}$.

(5.26.5) For any $\boldsymbol{n} \in \pi_1(Y, L)$, one has $\lim_{m \to \infty} \ell^\sharp(\boldsymbol{n}, m) = \infty$.

We set

$$\mathrm{CF}^*(L_0; H_m) := \bigoplus_{x \in \mathscr{X}(L_0; H_m)} \mathfrak{o}_x, \tag{5.26.6}$$

where $\mathfrak{o}_x$ is a one-dimensional $\mathbb{C}$-normalized orientation space associated to x. We also set

$$\mathscr{X}(L_0; H_m)^{\leq \ell^\sharp} := \left\{x \in \mathscr{X}(L_0; H_m) \;\middle|\; |x|_{\mathrm{rel}} \leq \ell^\sharp([x], m)\right\} \tag{5.26.7}$$

and

$$\mathrm{CF}^*(L_0; H_m)^{\leq \ell^\sharp} := \bigoplus_{x \in \mathscr{X}(L_0; H_m)^{\leq \ell^\sharp}} \mathfrak{o}_x. \tag{5.26.8}$$

The vector space $\mathrm{CF}^*(L_0; H_m)^{> \ell^\sharp}$ is similarly defined.

5.27.

Definition 5.28. We say that $\{L_0, H_m, \ell^\sharp\}$ satisfies **Assumption A** if the following conditions are satisfied:

1. For every pair of chords $x_0 \in \mathscr{X}(\bar{L}_0, \bar{L}_0, H_{m'})$ and $x_1 \in \mathscr{X}(\bar{L}_0, \bar{L}_0, H_m)$ which lie in the same relative homotopy class, there is a topological strip y_{x_1, x_0} in $\overline{Y}$ between x_1 and x_0 such that y_{x_1, x_0} has intersection number zero with both components E and F of $\overline{E}$.
2. Given a pair of chords $x_0 \in \mathscr{X}(\bar{L}_0, \bar{L}_0, H_m)^{> \ell^\sharp}$ and $x_1 \in \mathscr{X}(\bar{L}_0, \bar{L}_0, H_m)^{\leq \ell^\sharp}$ such that $\operatorname{vdim} \mathcal{R}^1(x_0, x_1) = 0$, all moduli spaces $\mathcal{R}^1(x_0, x_1)$ are empty for generic J_t.

The following compactness lemma allows us to give $\mathrm{CF}^*(L_0; H_m)^{\leq \ell^\sharp}$ the structure of a complex and to define continuation maps which preserve the relative grading.

Lemma 5.29.
- *Fix a pair of chords $x_0 \in \mathscr{X}(L_0, L_0, H_{m'})^{\leq \ell^\sharp}$, $x_1 \in \mathscr{X}(L_0, L_0, H_m)^{\leq \ell^\sharp}$ such that $m \leq m'$ and such that*

$$|x_0|_{\mathrm{rel}} < \ell^\sharp(\boldsymbol{n}, m),$$

then for a generic almost complex structure J_t, the closure $\bar{\mathcal{R}}^1(x_0, x_1) \subset \overline{\mathcal{R}}^1(\overline{Y}, \boldsymbol{x}, A_{\boldsymbol{x}})$ consists of curves which are disjoint from D and with no strip-breaking along chords in $\mathscr{X}(L_0, L_0)^{> \ell^\sharp}$.

Proof. All Floer strips intersect both E and F with non-negative multiplicity and hence because the relative homology class of any strip can be obtained by connect summing a multiple of $[E_{\boldsymbol{w}}]$ onto y_{x_1, x_0}, must intersect both with multiplicity zero. Hence the all Floer strips must raise the relative grading in the unstable case and at least preserve the relative grading in the stable case. There can therefore be no strip breaking along such chords with higher relative grading. □

Definition 5.30. We say that $\{L_0, H_m, \ell^\sharp\}$ satisfies **Assumption B** if

(1) For every pair $(\boldsymbol{n}_1, \boldsymbol{n}_2)$, there exists a non-negative integer $s(\boldsymbol{n}_1, \boldsymbol{n}_2)$, such that for every pair of chords $x_1 \in \mathscr{X}(L_0, L_0, H_m)^{\leq \ell^\sharp}_{\boldsymbol{n}_1}$ and $x_2 \in \mathscr{X}(L_0, L_0, H_n)^{\leq \ell^\sharp}_{\boldsymbol{n}_2}$ and a third chord $x_0 \in \mathscr{X}(L_0, L_0, H_{m+n})^{\leq \ell^\sharp}_{\boldsymbol{n}_1+\boldsymbol{n}_2}$ such that $\mathrm{vdim}(\mathcal{R}(x_0, x_1, x_2)) = \{0, 1\}$ and

$$|x_0|_{\mathrm{rel}} \geq |x_1|_{\mathrm{rel}} + |x_2|_{\mathrm{rel}} + s(\boldsymbol{n}_1, \boldsymbol{n}_2) \tag{5.30.1}$$

the moduli space $\mathcal{R}(x_0, x_1, x_2)$ is empty for generic almost complex structures.

(2) For any chords $x_1 \in \mathscr{X}(L_0, L_0, H_m)^{\leq \ell^\sharp}_{\boldsymbol{n}_1}$ and $x_2 \in \mathscr{X}(L_0, L_0, H_n)^{\leq \ell^\sharp}_{\boldsymbol{n}_2}$,

$$\ell^\sharp(\boldsymbol{n}_1 + \boldsymbol{n}_2, m+n) \geq |x_1|_{\mathrm{rel}} + |x_2|_{\mathrm{rel}} + s(\boldsymbol{n}_1, \boldsymbol{n}_2) \tag{5.30.2}$$

(3) Let x_1, x_2 be as above and $x_0 \in \mathscr{X}(L_0, L_0, H_{m+n+r})^{\leq \ell^\sharp}$. We assume that whenever $\mathrm{vdim}(\mathcal{R}^2_t(x_0, x_1, x_2)) = \{0, 1\}$ and (5.30.1) holds, the moduli spaces $\mathcal{R}^2_t(x_0, x_1, x_2)$ are empty for generic complex structures. We also require that the same holds when the continuation maps are along the other positive strip like end.

Lemma 5.31.

- *Fix now a triple of chords* $x_0 \in \mathscr{X}(L_0, L_0, H_{m+n})^{\leq \ell^\sharp}$, $x_1 \in \mathscr{X}(L_0, L_0, H_m)^{\leq \ell^\sharp}$, $x_2 \in \mathscr{X}(L_0, L_0, H_n)^{\leq \ell^\sharp}$ *such that*

$$|x_1|_{\mathrm{rel}} + \inf_{p \in \mathscr{X}(\underline{L}_0; H_{\mathrm{ba},n})_{\boldsymbol{n}_2}} |p_{\mathrm{d}}|_{\mathrm{rel}} > |x_0|_{\mathrm{rel}}$$
$$|x_2|_{\mathrm{rel}} + \inf_{p \in \mathscr{X}(\underline{L}_0; H_{\mathrm{ba},m})_{\boldsymbol{n}_1}} |p_{\mathrm{d}}|_{\mathrm{rel}} > |x_0|_{\mathrm{rel}} \quad \textit{and}$$
$$\mathrm{vdim}(\mathcal{R}^2(x_0, x_1, x_2)) = \{0, 1\},$$

then for generic surface dependent almost complex structures, the closure $\bar{\mathcal{R}}^2(x_0, x_1, x_2)$ *consists of curves which are disjoint from* D *and with no breaking along chords in* $\mathscr{X}(L_0, L_0)^{>\ell^\sharp}$.

- *Let* $m_1 < m_2$ *and fix a triple of chords* $x_0 \in \mathscr{X}(L_0, L_0, H_{m_2+m_3})^{\leq \ell^\sharp}$, $x_1 \in \mathscr{X}(L_0, L_0, H_{m_1})^{\leq \ell^\sharp}$ *and* $x_2 \in \mathscr{X}(L_0, L_0, H_{m_3})^{\leq \ell^\sharp}$. *Assume that*

$$|x_1|_{\mathrm{rel}} + \inf_{p \in \mathscr{X}(\underline{L}_0; H_{\mathrm{ba},m_2})_{\boldsymbol{n}_2}} |p_{\mathrm{d}}|_{\mathrm{rel}} > |x_0|_{\mathrm{rel}}$$
$$|x_2|_{\mathrm{rel}} + \inf_{p \in \mathscr{X}(\underline{L}_0; H_{\mathrm{ba},m_1})_{\boldsymbol{n}_1}} |p_{\mathrm{d}}|_{\mathrm{rel}} > |x_0|_{\mathrm{rel}} \qquad |x_0|_{\mathrm{rel}} < \ell^\sharp(\boldsymbol{n}, m_1 + m_3) \ \textit{and}$$
$$\mathrm{vdim}(\mathcal{R}^2_t(x_0, x_1, x_2)) = \{0, 1\},$$

then for generic surface dependent almost complex structures, the closure $\bar{\mathcal{R}}^2_t(x_0, x_1, x_2)$ *consists of curves which are disjoint from* D *and which do not break along chords in* $\mathscr{X}(L_0, L_0)^{>\ell^\sharp}$.

Proof. A broken curve in this moduli space will have one component which is a Floer triangle Σ and possibly many components which are spheres and strips. The most difficult case to rule out is breaking along a divisor chord. The curve cannot break along a divisor chord at a position in our tree after or as outputs of the curve Σ for the same reason as in the preceding compactness lemma. Now we consider the case of breaking along divisor chords before or as inputs of Σ. The two bulleted conditions rule that out since the Floer triangle would necessarily have negative virtual dimension and hence not exist for generic almost complex structures. □

We define the product between two (orientation lines corresponding to) chords $x_1 \in \mathscr{X}(L_0, L_0, H_m)^{\leq \ell^\sharp}_{\boldsymbol{n}_1}$ and $x_2 \in \mathscr{X}(L_0, L_0, H_n)^{\leq \ell^\sharp}_{\boldsymbol{n}_2}$ as follows. Denote by m_1 and m_2 the smallest integers such that

$$\ell^\sharp(\boldsymbol{n}_1, m_1) > |x_1|_{\mathrm{rel}} + s(\boldsymbol{n}_1, \boldsymbol{n}_2)$$
$$\ell^\sharp(\boldsymbol{n}_2, m_2) > |x_2|_{\mathrm{rel}} + s(\boldsymbol{n}_1, \boldsymbol{n}_2)$$

If $m < m_1$ or $n < m_2$, we may define the Floer product by first applying the continuation maps into $\mathrm{CF}^*(L_0; H_{m_1})^{\leq \ell^\sharp}$ and $\mathrm{CF}^*(L_0; H_{m_2})^{\leq \ell^\sharp}$ and then applying the usual Floer product. This is compatible with continuation maps in both variables.

Definition 5.32. The triple $\{L_0, H_m, \ell^\sharp\}$ satisfies **Assumption C** if for a triple $\{\boldsymbol{n}_i\}$ there is an $s(\boldsymbol{n}_i) \geq 0$, such that for every triple of chords $x_i \in \mathscr{X}(\underline{L}_0; H_{m_i})^{\leq \ell^\sharp}_{\boldsymbol{n}_i}$ and any fourth chord $x_0 \in \mathscr{X}(L_0, L_0, H_{m_1+m_2+m_3+n})^{\leq \ell^\sharp}$ for $n \geq 0$ satisfying

(5.32.1) $$\mathrm{vdim}(\mathcal{R}^3(x_0, x_1, x_2, x_3)) = \{0, 1\}$$

and

(5.32.2) $$|x_0|_{\mathrm{rel}} \geq s(\boldsymbol{n}_i) + \sum_i |x_i|_{\mathrm{rel}}$$

the moduli spaces $\mathcal{R}^3(x_0, x_1, x_2, x_3)$ are empty.

A similar compactness result shows that under this assumption the multiplications defined give rise to an associative product.

5.33.

Definition 5.34. We say that a triple $\{L_0, H_m, \ell^\sharp\}$ is admissible if it satisfies Assumptions A-C.

Fix an admissible triple $\{L_0, H_m, \ell^\sharp\}$ as above. We define the cohomology $\mathrm{HF}_{\mathrm{ad}}(L_0; H_m)$ to be the homology with respect to $\mathfrak{m}_1$ on the complex (5.26.8). We let the *adapted wrapped Floer cohomology* $\mathrm{HW}^*_{\mathrm{ad}}(L_0; H_m)$ be the direct limit $\varinjlim_m \mathrm{HF}_{\mathrm{ad}}(L_0; H_m)$. The adapted wrapped Floer cohomology $\mathrm{HW}^*_{\mathrm{ad}}(L_0; H_m)$ has a structure of an associative ring coming from the triangle product $\mathfrak{m}_2$.

5.35. Let us assume that our data is chosen to satisfy (5.24.1) everywhere. For every $\overline{p}$ in $\mathscr{X}(\underline{L}_0; H_{\mathrm{ba},m})$, in addition to p_{d}, there are chords p_i where $|i| < c_{\pm}\lambda_m$.

Lemma 5.36. *We have* $|p_i|_{\mathrm{rel}} = 2i + |\overline{p}|_{N_{\mathbb{C}^\times}}$ *and* $|p_i| = |p_{\mathrm{d}}| = |\overline{p}|_{N_{\mathbb{C}^\times}}$

In fact, we can arrange that there is at most one generator $\overline{p}_{\boldsymbol{n},m} \in \mathscr{X}(\underline{L}_0, H_m)$ of grading 0 in each homotopy class $\boldsymbol{n} \in \pi_1(Y, L_0)$ (see the explicit constructions of Section 6 for further details on this point).

Lemma 5.37. *There exist admissible triples of the form* $\left(L_0, H_m, \ell^\sharp\right)$.

Proof. Let us assume that our data is chosen to satisfy (5.24.1) everywhere and to satisfy the condition of the sentence preceeding the statement of the lemma. Then, there are no differentials since everything lies in degree zero, so we only have to consider continuation solutions to define the wrapped Floer groups, Floer triangles to define the multiplication and one dimensional moduli spaces of curves in $\mathcal{R}^3$ to ensure that the multiplication is associative. A continuation solution y with input $p_{\boldsymbol{n}_1, m_1, i_1}$ does not intersect $\overline{E}$ and in particular does not intersect E. The output must be of the form $p_{\boldsymbol{n}_1, m_2, i_2}$, So we require that

$$\ell^\sharp(\boldsymbol{n}_1, m_1) < \ell^\sharp(\boldsymbol{n}_1, m_2).$$

Given a Floer triangle y with inputs $p_{\boldsymbol{n}_1,m_1,i_1}$, $p_{\boldsymbol{n}_2,m_2,i_2}$, the intersection of $y\cdot\overline{E}\geq y\cdot E$ is fixed and the output is necessarily of the form $p_{\boldsymbol{n}_1+\boldsymbol{n}_2,m_1+m_2+r,i_3}$. In view of (5.21.3), the degree of $p_{\boldsymbol{n}_1+\boldsymbol{n}_2,m_1+m_2+r,i_3}$ is at most

$$\sum_{i=1}^{2}|x_i|_{\text{rel}}+2y\cdot\overline{E}+1 \tag{5.37.1}$$

and condition (5.30.1) is satisfied. The condition (5.30.2) is also satisfied provided that for every $\boldsymbol{n}_1$, $\boldsymbol{n}_2$

$$\ell^\sharp(\boldsymbol{n}_1,m_1)+\ell^\sharp(\boldsymbol{n}_2,m_2)+2y\cdot\overline{E}+1\leq\ell^\sharp(\boldsymbol{n}_1+\boldsymbol{n}_2,m_1+m_2). \tag{5.37.2}$$

Conversely, these two conditions guarantee the admissibility of $\ell^\sharp$. To see this, we consider the curves in $\mathcal{R}^3$ and the equations (5.32.2). The equation (5.21.3) shows that for every $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$, we have that the grading for x_0 is at most

$$\sum_{i=1}^{3}|x_i|_{\text{rel}}+2y\cdot\overline{E} \tag{5.37.3}$$

where $y\cdot\overline{E}$ is again determined by $\boldsymbol{n}_i$. Setting $s(\boldsymbol{n}_i)=2y\cdot\overline{E}+1$ shows that (5.32.2) is satisfied. □

Remark 5.38. We will see that we can take the function $\ell^\sharp(\boldsymbol{n})=2m-2\ell_1(\boldsymbol{n})$, where ℓ_1 is the function defined in Section 6 provided $c_-\lambda_m>m$.

Definition 5.39. We say that two triples $\{L_0,H_m,\ell^\sharp_H\}$ and $\{L_0,G_m,\ell^\sharp_G\}$ are *equivalent* if

(1) for every m there are admissible homotopies: $H_{s,m}$ between H_m and $G_{m'}$ for some $m'>m$ with $\ell^\sharp_{m',G}>\ell^\sharp_{m,H}$, and $G_{s,m}$ between G_m and $H_{m''}$ for $m''>m$ with $\ell^\sharp_{m'',H}>\ell^\sharp_{m,H}$, and
(2) for any m and m', and $x\in\mathscr{X}(L_0;H_m)$ and $y\in\mathscr{X}(L_0;G_{m'})$ which lie in the same relative homotopy class, there is a topological strip Σ in $\overline{Y}$ between x and y satisfies $\Sigma\cdot E=0$ and $\Sigma\cdot F=0$.

Lemma 5.40. *Given an equivalence between two fibered admissible families, the induced maps* $\mathrm{HW}^*_{\mathrm{ad}}(L_0;H_m)\to\mathrm{HW}^*_{\mathrm{ad}}(L_0;G_m)$ *is an equivalence.*

Proof. The continuation maps induced by $G_{s,m}$ and $H_{s,m}$ preserve the relative gradings by (5.21.3) and hence the equivalence follows by a standard Floer theoretic argument. □

Remark 5.41. Fix finitely many base-admissible Lagrangian sections L_i. By restricting our attention to suitable Hamiltonians of the form $H=H_{\text{ba}}+H_{\text{v}}$, it is possible to formulate a generalization of the functions $\ell^\sharp$ for every pair L_i and L_j, to construct an A_∞ category with objects L_i. As we have emphasized, in this paper we will only discuss the lagrangian L_0 and hence do not pursue this generalization here.

6. Wrapped Floer cohomology ring of L_0

6.1. Approximations of a quadratic Hamiltonian. In this section, we compute the adapted wrapped Floer cohomology ring of the admissible Lagrangian submanifold L_0. The first step in our computation is to construct a particular sequence $(H_k)_{k=1}^\infty$ of fibration-admissible Hamiltonians. Our calculation will be modeled on the following simplified situation: The positive real part $N_{\mathbb{R}^{>0}}\times\mathbb{R}^{>0}$ is an open Lagrangian submanifold of $N_{\mathbb{C}^\times}\times\mathbb{C}$, which can be separated from the center $Z\times 0$ of the blow-up by an open set.

By choosing a function supported in this open set as the cut-off function χ appearing in the symplectic form (2.5.15) on the blow-up, this Lagrangian submanifold naturally lifts diffeomorphically to the Lagrangian submanifold L_0 of Y. Note that $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ has a natural quadratic Hamiltonian

$$H_{\text{quad}}(\boldsymbol{w}, u) := \frac{1}{2}|\boldsymbol{r}|^2 + \frac{1}{2}(\log|u|)^2. \tag{6.1.1}$$

The image of $N_{\mathbb{R}^{>0}} \times \mathbb{R}^{>0}$ under the time 1 Hamiltonian flow with respect to this quadratic Hamiltonian is given by $\theta_1 = r_1$, $\theta_2 = r_2$, and $\arg u = \log|u|$, so that the set of Hamiltonian chords is naturally in bijection with $N \times \mathbb{Z}$. This bijection sends a chord to its class in the relative homotopy group $\pi_1(N_{\mathbb{C}^\times} \times \mathbb{C}^\times, N_{\mathbb{R}^{>0}} \times \mathbb{R}^{>0})$, which can naturally be identified with $N \times \mathbb{Z}$. It is well known in this case that one can identify the wrapped Floer cohomology $\mathrm{HW}^*_{N_{\mathbb{C}^\times} \times \mathbb{C}^\times}(L_0)$ with the group ring $\mathbb{C}[\pi_1(N_{\mathbb{C}^\times} \times \mathbb{C}^\times)]$.

We now regard L_0 as a Lagrangian submanifold of Y and construct a sequence $(H_k)_{k=1}^\infty$ of Hamiltonians, such that H_k behaves like H_{quad} in a compact set and has a slope λ_k elsewhere. To be more precise, for each $k \in \mathbb{Z}^{>0}$, we consider a function of the form $H_k = H_{\text{ba},k} + H_{\text{v},k}$. We require that $H_{\text{v},k} = f_k \circ \mu$ and that there exist sequences $\{b_k\}_{k=1}^\infty$, $\{c_k\}_{k=1}^\infty$ of positive real numbers

- $\lim_{k\to\infty} b_k = \infty$ and $\lim_{k\to\infty} c_k = 0$,
- $f_k\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{>0}$ is a convex function,
- $f_k(x) = \lambda_k x$ for $x > b_k$,
- $f_k(x) = \lambda_k(\epsilon - x)$ for $x < c_k$,
- if $k < k'$, then any Hamiltonian chord of $f_k \circ \mu$ is also a Hamiltonian chord of $f_{k'} \circ \mu$.

To construct the function $H_{\text{ba},k}$, we may embed the neighborhoods U_{Π_c} and U_{Π_i} in neighborhoods W_{Π_i} and W_{Π_c}, where

$$W_{\Pi_c} := \{\boldsymbol{r} \in N_{\mathbb{R}} \mid |\boldsymbol{r}| < R_1\} \tag{6.1.2}$$

for $R_0 < R_1 \ll R$ and

$$W_{\Pi_i} = \left\{\boldsymbol{r} \in N_{\mathbb{R}} \;\middle|\; \left|\boldsymbol{r}_{\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i}\right| < 2\varepsilon_n \text{ and } \boldsymbol{r}_{(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^\perp} \geq a_i + \varepsilon_n\right\} \tag{6.1.3}$$

Let $g_{\text{deg}}\colon Y \to \mathbb{R}$ be a smooth convex function satisfying

$$g_{\text{deg}} = \begin{cases} 0 & \boldsymbol{r} \in U_c, \\ \frac{1}{2}\rho^2 + \psi_i(r_{\alpha_i - \beta_i}) & \boldsymbol{r} \in W_i \setminus (W_i \cap W_c),\ i = 1, \dots, \ell, \\ \frac{S(\boldsymbol{r})}{R^2} & \boldsymbol{r} \in N_{\mathbb{R}} \setminus \left(W_c \cup \bigcup_{i=1}^{\ell} W_i\right) \end{cases} \tag{6.1.4}$$

where $\psi_i\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is a smooth convex function satisfying

- $\psi_i^{-1}(0) = [0, \varepsilon_n]$ and
- $\psi_i(r) = \frac{c_i r^2}{R^2}$ if $r \geq 2\varepsilon_n$

We let $\overline{H}_{\text{ba},k}$ be a function of the form:

$$\overline{H}_{\text{ba},k} = \begin{cases} g_{\text{deg}} & \rho < \lambda_k - \varepsilon_k, \\ \lambda_k \rho + \text{constant} & \rho > \lambda_k + \varepsilon_k \end{cases} \tag{6.1.5}$$

for some $0 < \varepsilon_k << \lambda_k$ and in each tubular neighborhood U_i, $\overline{H}_{\text{ba},k} = h_i(\rho)$. Finally, we let $H_{\text{ba},k}$ be a small perturbation of this function, where the perturbation is supported outside of $\overline{\pi}^{-1}(U_Z)$ (in W_i, it suffices to take a small perturbation of the function ψ). We assume that the Hamiltonian flow preserves $U_{Z\times 0}$, so that the Hamiltonian chords for L_0

live entirely in the region where the fibration $\overline{\pi}_{N_{\mathbb{C}^\times}} : \overline{Y} \to N_{\mathbb{C}^\times}$ is trivial. By choosing suitable slopes λ_k and perturbations in the definition of $H_{\mathrm{ba},k}$, we can assume

(6.1.6) all Hamiltonian chords are in the region where $H_{\mathrm{ba},k}$ is quadratic,

(6.1.7) for any $k \in \mathbb{Z}^{>0}$ and any $\boldsymbol{n} \in N$ satisfying $|\boldsymbol{n}| \le \lambda_k$, there are exactly $2\lambda_k+1$ Hamiltonian chords $\{p^k_{\boldsymbol{n},i}\}_{i=-k}^{k}$ of H_k in the homotopy class $\boldsymbol{n}$, and

(6.1.8) all $p^k_{\boldsymbol{n},i}$ are non-degenerate.

We assume $\lambda_k > k$ and define $\ell^\sharp(\boldsymbol{n}) = 2k - 2\ell_1(\boldsymbol{n})$. These functions are admissible and we may consider the adapted Floer groups $\mathrm{HF}_{\mathrm{ad}}(L_0; H_k)$. For each Hamiltonian chord $p_{\boldsymbol{n},i}$, there is an associated orientation space $\mathfrak{o}_{p_{\boldsymbol{n},i}}$. However, as noted by Pascaleff [Pas14, Section 4.6], in our situation, the kernel and cokernel of the local orientation operator $D_{p_{\boldsymbol{n},i}}$ are trivial. It follows that $\mathfrak{o}_{p_{\boldsymbol{n},i}}$ is canonically trivialized and has a canonical generator. We denote this generator by $p_{\boldsymbol{n},i}$. In constructing homotopies between H_k and $H_{k'}$, we can and will assume that our family of Hamiltonians satisfies $\partial_s H_{s,t} \le 0$ everywhere.

Proposition 6.2. *If $k < k'$, then the continuation map*

$$\mathrm{HF}_{\mathrm{ad}}(L_0; H_k) \to \mathrm{HF}_{\mathrm{ad}}(L_0; H_{k'}) \tag{6.2.1}$$

sends $p^k_{\boldsymbol{n},i}$ to $p^{k'}_{\boldsymbol{n},i}$.

Proof. Note that the continuation map preserves homotopy classes. Since there is a unique Hamiltonian chord on $N_{\mathbb{C}^\times}$ in each homotopy class, we have that the limits of the curve are the same once projected to $N_{\mathbb{C}^\times}$. The projection of such a solution $\pi_{N_{\mathbb{C}^\times}}(y)$ is therefore a topological cylinder such that $\pi_{N_{\mathbb{C}^\times}}(y) \cdot Z = 0$. It follows that such solutions y have intersection number zero with $\overline{E}$ and in particular E. This means that these solutions lie in $(N_{\mathbb{C}^\times} \setminus Z) \times \mathbb{C}^\times$.

However, in this space, the orbits of our Hamiltonian all lie in different homotopy classes (or alternatively we can reach the same conclusion by comparison of $|-|_{\mathrm{rel}}$). Our family of Hamiltonians satisfies $\partial_s H_{s,t} \le 0$ everywhere. Therefore, we have that the $E^{\mathrm{geom}}(y) \le E^{\mathrm{top}}(y)$. The quantity on the right-hand side is zero since the solution has the same asymptotes at the positive and negative ends. We therefore conclude that the solution is constant. □

Corollary 6.3. *One has*

$$\mathrm{HW}_{\mathrm{ad}}(L_0) := \varinjlim_k \mathrm{HF}_{\mathrm{ad}}(L_0; H_k) = \mathrm{span}\{p_{\boldsymbol{n},i}\}_{(\boldsymbol{n},i)\in N\times\mathbb{Z}}, \tag{6.3.1}$$

where $p_{\boldsymbol{n},i} = \lim_{k\to\infty} p^k_{\boldsymbol{n},i}$.

6.4. We may also consider the family of functions $\{2H_k\}$. We equip these with the function $\ell^\sharp(\boldsymbol{n}, k) = 4k - 2\ell_1(\boldsymbol{n})$. For any positive integer k and any $\boldsymbol{n} \in N$ satisfying $|\boldsymbol{n}| \le 2\lambda_k$, there are exactly $4\lambda_k + 1$ Hamiltonian chords of $2H_k$ in the homotopy class $\boldsymbol{n}$. All of them are non-degenerate, and will be denoted as $q^k_{\boldsymbol{n},i}$ for $-2\lambda_k \le i \le 2\lambda_k$.

6.5. Calculation of the product. Let ϕ_{H_k} denote the time-one flow of X_{H_k}. For this section, we adopt the notation that $L_1 = \phi_{H_k}(L_0)$ and $L_2 = \phi_{2H_k}(L_0)$. Time-one chords in $\mathscr{X}_{H_k}(L_0)$ (respectively $2H_k$) correspond to intersection points of $\phi_{H_k}(L_0)$ and $\phi_{2H_k}(L_0)$. The first step in our proof is to relate our problem of calculating the product in wrapped Floer cohomology to the ordinary product in Lagrangian Floer cohomology for these Lagrangians.

For the purposes of our calculation we will define the Floer triangle product with respect to perturbation data of the form

$$X_K = X_{H_k} \otimes \gamma,$$

where γ is a *closed* form on the pair of pants such that $\epsilon_0^*\gamma = 2dt$, $\epsilon_1^*\gamma = dt$ and $\epsilon_2^*\gamma = dt$. This implies that a Floer triangle can then be recast as a pseudo-holomorphic curve for a *domain-dependent* almost complex structure on Y. Given $y : \Sigma \to (Y, J, H)$ we consider $\widetilde{y} = \phi^\tau_{H_k} \circ y$, where $\tau : \Sigma \to [0, 2]$ is the primitive of the closed one form γ used to define the pair of pants product. In particular, the product structure then agrees with the usual product operation

$$\mathrm{Hom}(L_1, L_0) \otimes \mathrm{Hom}(L_2, L_1) \to \mathrm{Hom}(L_2, L_0)$$

in Lagrangian Floer theory for the domain dependent almost complex structure $J_\tau = (\phi^\tau)^{-1}_*(J)$. We remark that, after perhaps modifying our original J at infinity, we may assume that J_τ is fibration-admissible. As a consequence of this, for this section we will consider only pseudoholomorphic maps and all perturbation data X_K will be taken to be trivial.

Proposition 6.6. *Let J' be another fibration-admissible almost complex structure. The continuation maps*

$$\mathrm{HF}_{\mathrm{ad}}(L_i, L_0; J') \to \mathrm{HF}_{\mathrm{ad}}(L_i, L_0; J_\tau) \tag{6.6.1}$$

are the identity.

Proof. The proof follows that of Proposition 6.2 quite closely. Namely, observe that such continuation solutions have intersection number zero with E. By comparing the relative indices $|-|_{\mathrm{rel}}$, we see that the continuation maps must be constant. □

Lemma 6.7. *Any Floer triangle y with inputs $p_{\boldsymbol{n},i}$, $p_{\boldsymbol{n}',i'}$ has a fixed intersection number $y \cdot \overline{E}$ with $\overline{E}$.*

Proof. The projected triangle $\pi_{N_{\mathbb{C}^\times}}(y)$ is a topological triangle with fixed intersection number $\pi_{N_{\mathbb{C}^\times}}(y) \cdot Z$ with Z. This is equal to $y \cdot \overline{E}$ since $\overline{E} = E \cup F$ is the total transform of $Z \times \mathbb{C}$. □

We set $j \coloneqq y \cdot E$, which is a non-negative integer satisfying $j \leq j_{\max} \coloneqq y \cdot \overline{E}$. Note that $j_{\max}$ depends only on $\boldsymbol{n}$ and $\boldsymbol{n}'$. The relative gradings imply that one has

$$p_{\boldsymbol{n},i} \cdot p_{\boldsymbol{n}',i'} = \sum_{j=0}^{j_{\max}} N_j(p_{\boldsymbol{n},i}, p_{\boldsymbol{n}',i'}) q_{\boldsymbol{n}+\boldsymbol{n}',i+i'+j} \tag{6.7.1}$$

for some integers $N_j(p_{\boldsymbol{n},i}, p_{\boldsymbol{n}',i'})$.

The rest of this section is devoted to the computation of $N_j(p_{\boldsymbol{n},i}, p_{\boldsymbol{n}',i'})$. To simplify notation, we set $p_{\boldsymbol{n},i} = \mathbf{p}_1$, $p_{\boldsymbol{n}',i'} = \mathbf{p}_2$, $\vec{\mathbf{p}} = (\mathbf{p}_1, \mathbf{p}_2)$, and $\mathbf{q} = q_{\boldsymbol{n}+\boldsymbol{n}',i+i'+j}$. Fix $J \in \mathscr{J}_{\mathrm{int}}(D \cup \overline{E})$ and $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ such that p is pseudoholomorphic. We now introduce some relative moduli spaces which will be needed in our argument.

Definition 6.8. Let $\mathcal{R}^{2,j}$ be the moduli space of closed disks with 3 ordered punctures $(\zeta_0, \zeta_1, \zeta_2)$ on the boundary and j marked points $(z_1, \cdots, z_j)$ in the interior. We write $\Sigma_j = (\Sigma, (z_1, \ldots, z_j))$, so that the forgetting map $\mathcal{R}^{2,j} \to \mathcal{R}^2$ sends $\Sigma_j \to \Sigma$.

Definition 6.9. Let $\mathcal{R}^{2,j}(Y, \mathbf{q}, \vec{\mathbf{p}})$ be the moduli space of pseudoholomorphic maps $y \colon \Sigma_j \to Y$ from closed disks with 3 boundary punctures and j interior marked points.

Definition 6.10. Let $\mathcal{R}^{2,j}(Y, E, \mathbf{q}, \vec{\mathbf{p}})$ be the subspace of $\mathcal{R}^{2,j}(Y, \mathbf{q}, \vec{\mathbf{p}})$ consisting of $y\colon \Sigma_j \to Y$ such that

- $\{z_i\}_{i=1}^{j} = y^{-1}(E)$, and
- y intersects E transversely at z_i.

Signs can be associated to these moduli spaces exactly like their non-relative counterparts. Namely, given $\mathbf{p}_1$ and $\mathbf{p}_2$ and $\mathbf{q}$, there exists an isomorphism

$$\mathfrak{o}_{\mathbf{q}} \cong \det(D_y) \otimes \mathfrak{o}_{\mathbf{p}_1} \otimes \mathfrak{o}_{\mathbf{p}_2}, \tag{6.10.1}$$

where D_y is the extended linearized operator at y. Whenever all curves in $\mathcal{R}^{2,j}(Y, \mathbf{q}, \vec{\mathbf{p}})$ are regular and the natural evaluation map

$$\mathrm{ev}\colon \mathcal{R}^{2,j}(Y, \mathbf{q}, \vec{\mathbf{p}}) \to Y^j \tag{6.10.2}$$

is transverse to E^j this enables us to assign a map between orientation spaces for each isolated map $y \in \mathcal{R}^{2,j}(Y, E, \mathbf{q}, \vec{\mathbf{p}})$. This is because in this situation, the determinant of the linearized operator can be expressed as $\det(D_y) \cong \otimes_i \det(TY_{y(z_i)}) \otimes \det(TE_{y(z_i)})^\vee$ which is canonically oriented. Notice that there is a surjection from $\mathcal{R}^{2,j}(Y, E, \mathbf{q}, \vec{\mathbf{p}})$ to $\mathcal{R}^{2}(Y, \mathbf{q}, \vec{\mathbf{p}})$, which is $j! : 1$ because of the choice of an ordering of the marked points. This map is sign-preserving if we give $\mathcal{R}^{2,j}$ its natural orientation.

Definition 6.11. Let $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{q}, \vec{\mathbf{p}})$ be the moduli space of pseudoholomorphic maps

$$\begin{cases} y\colon \Sigma_j \to N_{\mathbb{C}^\times} \times \mathbb{C}, \\ y(\partial_i \Sigma) \subset L_i, \\ \lim\limits_{s \to \pm\infty} y(\epsilon_i(s, -)) = \mathbf{q}, \mathbf{p}_1, \mathbf{p}_2. \end{cases} \tag{6.11.1}$$

The evaluation maps are denoted by

$$\mathrm{ev}_{N_{\mathbb{C}^\times} \times \mathbb{C}}\colon \mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{q}, \vec{\mathbf{p}}) \to (N_{\mathbb{C}^\times} \times \mathbb{C})^j. \tag{6.11.2}$$

Definition 6.12. Let $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{q}, \vec{\mathbf{p}})$ be the moduli space of pseudoholomorphic maps $y\colon \Sigma_j \to N_{\mathbb{C}^\times} \times \mathbb{C}$ such that

- $\{z_i\}_{i=1}^{j} = y^{-1}(Z \times 0)$ and
- at each point z_i above, y intersects the divisors $\{u = 0\}$ and Z transversely.

For suitably generic $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$, the linearized operator is surjective at all curves in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{q}, \vec{\mathbf{p}})$ and the evaluation map (6.11.2) is transverse to $(Z \times 0)^j$. For such $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ and when $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{q}, \vec{\mathbf{p}})$ is isolated, the moduli space can be given a relative orientation in a similar fashion to $\mathcal{R}^{2,j}(Y, E, \mathbf{q}, \vec{\mathbf{p}})$. We now show that split almost complex structures of the form $J_{N_{\mathbb{C}^\times} \times \mathbb{C}} = (J_B, J_{\mathbb{C}})$ on $N_{\mathbb{C}^\times} \times \mathbb{C}$ are sufficiently generic to achieve these two conditions. The Lagrangians L_i are also split Lagrangians, i.e., can be written as $\underline{L}_i \times \pi_{\mathbb{C}}(L_i)$. Thus, for this class of almost complex structures, we view our holomorphic curve $p \circ y$ as pairs (y_1, y_2) where

$$\begin{cases} y_1\colon \Sigma \to N_{\mathbb{C}^\times}, \\ y_1(\partial_k \Sigma) \subset \underline{L}_k, \\ \lim\limits_{s \to \pm\infty} y_1(\epsilon_k(s, -)) = \bar{q}_{\boldsymbol{n}+\boldsymbol{n}'}, \overline{p}_{\boldsymbol{n}}, \overline{p}_{\boldsymbol{n}'}, \end{cases} \tag{6.12.1}$$

and

$$(6.12.2)\qquad \begin{cases} y_2 \colon \Sigma \to \mathbb{C}, \\ y_2(\partial_k \Sigma) \subset \pi_{\mathbb{C}}(L_k), \\ \lim_{s \to \pm\infty} y_2(\epsilon_k(s, -)) = q_{i+i'+j}, p_i, p'_i. \end{cases}$$

We have

$$(6.12.3)\qquad y_1 \cdot Z = \overline{E} \cdot y = j_{\max}.$$

Moreover, for generic J, we can assume that these points are distinct points. We fix an ordering of these points $z_1, \cdots, z_{j_{\max}}$ on Σ. Now fix an integer j and a subcollection of $[1, \cdots, j_{\max}]$ of size j. Let $z_{i_1}, \cdots, z_{i_j}$ be the corresponding collection of points on Σ. Note that this data determines a domain $\Sigma_j \in \mathcal{R}^{2,j}$. The condition that the map y lies in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{q}, \vec{\mathbf{p}})$ is then equivalent to y_2 lying in

$$(6.12.4)\qquad \begin{cases} y_2 \colon \Sigma \to \mathbb{C}, \\ y_2(\partial_k \Sigma) \subset \pi_{\mathbb{C}}(L_k), \\ \lim_{s \to \pm\infty} y_2(\epsilon_k(s, -)) = q_{i+i'+j}, p_i, p'_i, \\ y_2^{-1}(0) = z_{i_1}, \cdots, z_{i_j}. \end{cases}$$

Lemma 6.13. *If $J_{N_{\mathbb{C}^\times} \times \mathbb{C}} = (J_B, J_{\mathbb{C}})$ is the product of a generic almost complex structure J_B adapted to Z and the standard almost complex structure $J_{\mathbb{C}}$, then all curves in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{q}, \vec{\mathbf{p}})$ are transverse and the evaluation map* (6.11.2) *is transverse to $(Z \times 0)^j$.*

Proof. We have split Lagrangian boundary conditions and no curve is constant when projected to either of the product factors, so it is clear that we can achieve surjectivity of the operator D_y using split J. The transversality of the evaluation of y to $\{u = 0\}$ and Z is equivalent to the transversality of y_2 with Z, which as mentioned above can be achieved by a small perturbation of J_B and the transversality of the evaluation map to $(\{u = 0\})^j$ at curves of the form (6.12.4). It is sufficient to prove that the tangent space to y_1 of the form (6.12.4) is trivial. An element of this tangent space is precisely an element of $\ker(D_{y_1})$ which vanishes at $z_{i_1}, \cdots, z_{i_j}$. Key for us is that $y_1^*(T\mathbb{C}_u)$ is a line bundle and our element is a section satisfying a $\bar{\partial}$-equation with a prescribed number of zeroes. We may therefore apply [Sei08, Proposition 11.5], which controls the number of zeroes on such solutions on line bundles, to show that this element must be trivial. □

Fix $J \in \mathscr{J}_{\text{int}}(D \cup \overline{E})$ and $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ be a split almost complex structure such that p is pseudoholomorphic. Given a pseudoholomorphic triangle y in $\mathcal{R}^2(Y, E, \mathbf{q}, \vec{\mathbf{p}})$ with boundary on L_i, we can compose it with the structure morphism $p \colon \overline{Y} \to N_{\mathbb{C}^\times} \times \mathbb{C}$ to obtain a curve $p \circ y$. Moreover given a curve $p \circ y$, we can reconstruct y as the proper transform of $p \circ y$. The condition that $y \in R^{2,j}(Y, E, \mathbf{q}, \vec{\mathbf{p}})$ is equivalent to $p \circ y \in \mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{q}, \vec{\mathbf{p}})$. In particular, whenever curves in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{q}, \vec{\mathbf{p}})$ are isolated, so are curves in $R^{2,j}(Y, E, \mathbf{q}, \vec{\mathbf{p}})$. In fact, this persists at the level of linearized operators.

Lemma 6.14. *Let $J \in \mathscr{J}_{\text{int}}(D \cup \overline{E})$ and $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ be a split almost complex structure satisfying the conditions of Lemma 6.13 such that p is pseudoholomorphic. Let further y be the proper transform of a curve in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{q}, \vec{\mathbf{p}})$. Then the linearized operator D_y is surjective.*

Proof. Since all of our moduli spaces are isolated, it suffices to check that any element $u \in \ker(D_y)$ such that the linearized evaluation map is tangent to E is zero. Given such a u, we may push this down to an element $p \circ u \in \ker(D_{p\circ y})$ such that the linearized evaluation maps at z_i are tangent to $Z \times 0$. By our hypothesis, such an element must be zero. For every point $z \neq z_i$ the map $y^*(TY)_z \to (p \circ y)^*(T(N_{\mathbb{C}^\times} \times \mathbb{C})_{p}(z)$ induced by the projection from y to $p \circ y$ is injective and hence it, it follows that $u = 0$. □

The key enumerative calculation is the following:

Lemma 6.15. *The numbers $N_j(p_{\boldsymbol{n},i}, p_{\boldsymbol{n}',i'})$ appearing in* (6.7.1) *are* $\binom{j_{\max}}{j}$.

Proof. We first determine the signed count of elements in the moduli space $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{q}, \vec{\mathbf{p}})$ up to dividing by $j!$ to remove for the redundancy from the ordering of the marked points. Consider curves y_2 as above. After picking j points $z_{i_1} \cdots, z_{i_j}$, it suffices to prove that the signed count of pairs (y_1, y_2) is 1. Let μ be the standard moment map on the disc and consider a vertically admissible Hamiltonian $v(\mu)$ which is of slope $\lambda_{\mathrm{v}} > 1$ near $\mu = 0$. Let α be the orbit whose homotopy class in $\mathbb{C}^\times$ is primitive. Let S be $\mathbb{P}^1 \setminus \{0\}$ with a distinguished marked point at $z = \infty$ and a negative cylindrical end near $z = 0$ given by

$$(s,t) \to e^{s+\sqrt{-1}t}. \tag{6.15.1}$$

Fix a subclosed one-form β which restricts to dt on the cylindrical end and which restricts to zero in a neighborhood of $z = 0$. To be explicit, we consider a non-negative, monotone non-increasing cutoff function $\rho(s)$ such that

$$\rho(s) = \begin{cases} 0 & s \gg 0, \\ 1 & s \ll 1, \end{cases} \tag{6.15.2}$$

and let $\beta = \rho(s)dt$. The remainder of our proof is divided into two steps:

Step 1. Consider solutions y_I to the equation

$$\begin{cases} y_I : S \to \mathbb{C} \\ (dy_I - X_{v(\mu)} \otimes \beta)^{0,1} = 0 \\ y_I(0) = 0 \\ \lim_{s \to -\infty} u(\epsilon(s,-)) = \alpha \end{cases} \tag{6.15.3}$$

There is a well-known isomorphism

$$\mathrm{SH}^0(\mathbb{C}^\times) \cong H_0(C_{1-*}(\mathcal{L}S^1)) \cong \mathbb{C}[\alpha, \alpha^{-1}]. \tag{6.15.4}$$

We wish to prove that the signed count of curves y_I as above is $+1$. Let S_2 be the disc with a fixed boundary marked point ζ and with one interior puncture. Equip S_2 with a positive strip-like end near the puncture. Let $L = S^1$ denote the zero section of $T^*S^1 \cong \mathbb{C}^\times$ and $x \in L$. A consequence of the isomorphism (6.15.4) is that for suitable choices of Floer data, the signed count of curves with domain S_2 with Hamiltonian input α and with the marked point ζ passing through x is $+1$. Now view $\mathbb{C}$ as a compactification of $\mathbb{C}^\times$ and glue S to S_2 along α to obtain a disc with boundary on L and with a marked point passing through $z = 0$. If we then deform the perturbation data to zero, the result then follows from the obvious fact that there is a unique disc in $\mathbb{C}$ with a marked point at $z = 0$ and a boundary marked point at any point on the Lagrangian L.

Step 2. With this established, let S_3 be a Riemann surface with j positive cylindrical ends, two positive strip like ends and one negative strip like end. There is also an isomorphism

$$\mathrm{HW}^0(T_q) \cong \mathbb{C}[p_1, p_1^{-1}] \tag{6.15.5}$$

After choosing suitable perturbation data it follows from (6.15.4), (6.15.5), together with the TQFT structure that the signed count of curves S_3 with chord inputs p_i, $p_{i'}$, Hamiltonian orbit inputs $\boldsymbol{\alpha}$ and output $q_{i+i'+j}$ must be $+1$. At each Hamiltonian input, we may then glue on the above curves with domain S to obtain maps satisfying all of the requirements of (6.12.4) except that the marked points z'_{i_j} may be different and there will be non-trivial perturbation data X_K (these depend on the gluing parameters along the ends).

We may deform away the Floer data and deform the marked points to z_{i_j} as above. The signed count of these curves does not change because there is no possible breaking and it follows that the signed count of elements y_1 satisfying (6.12.4) is $+1$. The signed count of curves y_2 is also $+1$. Since we have split Lagrangian boundary conditions and almost complex structures, the count of configurations of curves of the form (y_1, y_2) contribute $+1$. One readily checks that the bijection between $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{q}, \vec{\mathbf{p}})$ and $\mathcal{R}^{2,j}(Y, E, \mathbf{q}, \mathbf{p})$ preserves orientations, so the signed count of curves (y_1, y_2) contributes $+1$ to the Floer product as well. □

Theorem 6.16. *Given two generators, $p_{\boldsymbol{n},i}$ and $p_{\boldsymbol{n}',i'}$, let $m = \overline{E} \cdot y$ we have that*

$$p_{\boldsymbol{n},i} \cdot p_{\boldsymbol{n}',i'} = \sum_{j=0}^{m} \binom{m}{j} q_{\boldsymbol{n}+\boldsymbol{n}',i+i'+j} \tag{6.16.1}$$

Proof. This is a consequence of Lemma 6.15. □

We next observe the following fact concerning continuation maps. The above theorem allows us to calculate the continuation map:

$$\mathfrak{c} : \mathrm{HF}_{\mathrm{ad}}(L_0; H) \to \mathrm{HF}_{\mathrm{ad}}(L_0; 2H) \tag{6.16.2}$$

Lemma 6.17. *For an arbitrary element $p_{\boldsymbol{n},i}$, one has $\mathfrak{c}(p_{\boldsymbol{n},i}) = q_{\boldsymbol{n},i}$.*

Proof. The arguments concerning relative index in Proposition 6.2 and Proposition 6.6 together with the fact that all of our Floer products are in fact defined over $\mathbb{Z}$ imply that the continuation map must send $p_{\boldsymbol{n},i} \to \pm q_{\boldsymbol{n},i}$. A standard Floer theoretic argument shows that the continuation map sends $\mathfrak{c}(p_{\boldsymbol{n},i}) = p_{0,0} \cdot p_{\boldsymbol{n},i}$. The result follows immediately from the fact that the Floer triangle y with inputs $p_{0,0}$, $p_{\boldsymbol{n},i}$ and outputs $q_{\boldsymbol{n},i}$ must have intersection number zero since we already knew that $\mathfrak{c}(p_{\boldsymbol{n},i}) = \pm q_{\boldsymbol{n},i}$. □

Lemma 6.18. *The intersection number is given by*

$$\overline{E} \cdot y = \ell_2(\boldsymbol{n}, \boldsymbol{n}'). \tag{6.18.1}$$

It follows that one has an isomorphism

$$\mathrm{HW}_{\mathrm{ad}}(L_0) \xrightarrow{\sim} H^0(\mathcal{O}_{\check{Y}}), \qquad p_{\boldsymbol{n},i} \mapsto p^i \chi_{-\boldsymbol{n},\ell_1(\boldsymbol{n})} \tag{6.18.2}$$

of algebras, and Theorem 1.3 is proved.

7. Comparison of Floer theories

7.1. The Liouville domain. For this section, we define

$$\hat{\pi}_B \colon Y \to \hat{B} \coloneqq N_{\mathbb{R}} \times \mathbb{R}, \quad x \mapsto (\log|w_1|, \log|w_2|, \log|\mu|), \tag{7.1.1}$$

and write its restriction to $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ as

$$\hat{\pi}^0_B \colon N_{\mathbb{C}^\times} \times \mathbb{C}^\times \to \hat{B}. \tag{7.1.2}$$

Set $r_3 \coloneqq \log|\mu|$ and fix $R \gg 0$. Consider as well the natural projection $\pi_{N_{\mathbb{R}}} : \hat{B} \to N_{\mathbb{R}}$. For some sufficiently small μ_0 and sufficiently large μ_1, consider the region in $\hat{B}$ defined by $\epsilon - \mu_0 \le |\mu| \le \epsilon + \mu_1$ and $H_{\mathrm{b}} \le R$. We choose μ_1 sufficiently large so that $\chi(u, \boldsymbol{w}) = 0$ for $\mu \ge \epsilon + \mu_1/2$. This forms a cornered region in $\hat{B}$, and we may round corners to obtain a region $D_R \in \hat{B}$ such that the boundary ∂D_R is smooth and we may construct Liouville domains $T_R \coloneqq (\hat{\pi}^0_B)^{-1}(D_R)$ and $Y_R \coloneqq \hat{\pi}_B^{-1}(D_R)$. Outside of U_Z, these manifolds are canonically identified. We view $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ as $T^*B/\mathbb{Z}^3$ which we identify using the standard Euclidean inner product with $B \times B/\mathbb{Z}^3$. For any point $x \in \partial D_R$, let $\vec{n}_x$ denote the normal vector to ∂D_R in the diagonal metric on B with coefficients

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{r_3} \end{pmatrix}.$$

Elementary calculus shows that the Reeb flow on ∂T_R fibers over ∂D_R and points in the direction of $\vec{n}_x$ for every $x \in \partial D_R$. We will make a few technical assumptions concerning our smoothing:

- (D1) For every leg Π_i and every point $x \in \partial D_R \cap \pi_{N_{\mathbb{R}}}^{-1}(U_{\Pi_i})$, the normal vector to ∂D_R at x, $\vec{n}_x$ lies parallel to Π_i.
- (D2) In a neighborhood of the preimage of the vertices under $\pi_{N_{\mathbb{R}}}$, ∂D_R agrees with $(\mu = \epsilon - \mu_0) \cup (\mu = \epsilon + \mu_1)$.
- (D2') Along Π_i, when $\epsilon \le \mu \le \epsilon + \mu_1/2$, ∂D_R agrees with $H_{\mathrm{b}} = R$.
- (D3) Every Reeb orbit of ∂D_R of homotopy class α projects to the chamber of B corresponding to α.

We next deform the Liouville form θ_ϵ to a Liouville form θ'_ϵ on Y_R so that $\overline{E}$ is preserved by both the Liouville flow and the Reeb flow along the boundary ∂Y_R. This will be necessary since for some arguments it will be necessary to choose the almost complex structure to be both adapted with respect to the Liouville domain and to preserve the divisor $\overline{E}$.

Lemma 7.2. *There is a Liouville form $\theta'_{N_{\mathbb{C}^\times}}$ on $N_{\mathbb{C}^\times}$ which agrees with $\theta_{N_{\mathbb{C}^\times}}$ outside of U_Z and such that the Liouville vector field preserves Z.*

Proof. Let $\mathbb{D}_\delta$ denote a disc of radius δ with coordinates x, y. Using the symplectic tubular neighborhood theorem we may find a possibly smaller tubular neighborhood which is of the form $Z \times \mathbb{D}_\delta$ on which the symplectic form is of the form

$$\omega_{N_{\mathbb{C}^\times}} = dx \wedge dy + \omega_{N_{\mathbb{C}^\times}}|_Z. \tag{7.2.1}$$

Moreover, outside of a compact set, we have that the primitive is given by

$$\theta_{N_{\mathbb{C}^\times}} = (x - x_0)dy + \theta_{N_{\mathbb{C}^\times}}|_Z \tag{7.2.2}$$

for some $x_0 \in \mathbb{R}$. Therefore, after possibly adding a compactly supported closed form to $\theta_{N_{\mathbb{C}^\times}}|Z$, we have $\theta_{N_{\mathbb{C}^\times}} - (x - x_0)dy - \theta_{N_{\mathbb{C}^\times}}|_Z = dh$, where h vanishes along the preimages of the legs of Z. Let ρ be the radial coordinate on $\mathbb{D}_\delta$ and consider a cutoff function $\chi(\rho)$ such that $\chi(\rho) = 1$ for $\rho < \delta/4$ and $\chi(\rho) = 0$ for $\rho > \delta/2$. Then define

$$\theta'_{N_{\mathbb{C}^\times}} \coloneqq (x - x_0)dy + x_0 d(\chi y) + \theta_{N_{\mathbb{C}^\times}}|_Z + d(\chi h). \tag{7.2.3}$$

This patches together with $\theta_{N_{\mathbb{C}^\times}}$ and satisfies all of the properties required. □

Remark 7.3. If we assume that the non-compact legs Π_i pass through the origin, then such a form can be chosen to agree with $\theta_{N_{\mathbb{C}^\times}}$ outside of small neighborhoods of the vertices of Z.

We let $\theta'_{N_{\mathbb{C}^\times}\times\mathbb{C}}$ be the induced primitive one form of $\omega_{N_{\mathbb{C}^\times}\times\mathbb{C}}$ on $N_{\mathbb{C}^\times} \times \mathbb{C}$. Finally, set

$$\theta'_\epsilon \coloneqq \theta_{\mathrm{vc}} + p^*\theta'_{N_{\mathbb{C}^\times}\times\mathbb{C}}. \tag{7.3.1}$$

Lemma 7.4. *The Liouville flow on Y_R and the Reeb flow on ∂Y_R both preserve $\overline{E}$.*

Proof. Consider equation (2.6.1). We may expand this out further to see that at points along points lying in E,

$$\omega_\epsilon = p^*\omega_{N_{\mathbb{C}^\times}} + \frac{\sqrt{-1}}{4}|v|^2 dh \wedge d\bar{h} + \frac{\sqrt{-1}\epsilon}{2\pi}\partial\bar{\partial}\left(\log\left(1 + |v|^2\right)\right) \tag{7.4.1}$$

Similarly, for the primitive θ_ϵ, we have that along points lying in E,

$$\theta'_\epsilon = p^*\theta'_{N_{\mathbb{C}^\times}} - \epsilon d^c\left(\log\left(1 + |v|^2\right)\right). \tag{7.4.2}$$

The second term of (7.4.1) vanishes on vector fields tangent to Z. Because the Liouville vector field $V_{\theta'}$ on $N_{\mathbb{C}^\times}$ is tangent to Z, we have that

$$p^*\omega_{N_{\mathbb{C}^\times}} + \frac{\sqrt{-1}}{4}|v|^2 dh \wedge d\bar{h}(V_{\theta'}, -) = p^*\theta'_{N_{\mathbb{C}^\times}} \tag{7.4.3}$$

as required. Similarly, for the Reeb flow, observe that in a neighborhood of the vertices it is tangent to the fibers of the map $Y_R \to N_{\mathbb{C}^\times}$ and preserves the level sets $\mu = \epsilon + \mu_0 \cup \mu = \epsilon + \mu_1$ by Condition (D2). Along the legs, Condition (D1) ensures that the projection of the Reeb vector field is tangent to Z in view of (7.4.1).

On the other component of $\overline{E} \cap \partial Y_R$, we have that the function $\chi(u, \boldsymbol{w})$ vanishes except when ∂D_R agrees with $H_{\mathrm{b}} = R$ by Condition (D2'). There we have that the $\boldsymbol{w}$ components of $d^c\overline{\chi}\left(|u|^2 + \left|t^{-\nu(\boldsymbol{\alpha})}\boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})}\boldsymbol{w}^{\boldsymbol{\beta}}\right|^2\right)$ vanish (see Proposition 5.5 for more details) and the $\boldsymbol{w}$ components of (5.5.3) vanish on vectors tangent to $h_t(\boldsymbol{w})$ so the Liouville field is as described. Moreover, by Condition (D2'), the Reeb vector field has no θ_3 component and points in the direction $(\partial\theta_{\boldsymbol{\alpha}-\boldsymbol{\beta}}, 0)$, and is hence unaffected by the additional term (5.5.3). □

We also note that

Lemma 7.5. *The Reeb flow on ∂Y_R preserves the neighborhood $\pi^{-1}_{N_{\mathbb{C}^\times}}(U_Z)$.*

Proof. It follows immediately from Conditions (D1) and (D2) *mutatis mutandis* compared to Lemma 7.4. □

The Liouville coordinate in the fiber is given by $C|\mu - \epsilon|$ for some $C > 0$. We also have the Hamiltonians H_{ba}. Let $V_{\theta'_\epsilon}$ be the Liouville vector field which is the ω_ϵ dual of θ'_ϵ.

Lemma 7.6. *There is a constant B such that when $H_{\mathrm{b}} \geq R$ and $\mu \leq \epsilon - \mu_0$ or $\mu \leq \epsilon + \mu_1$, we have estimates*

$$d\rho(V_{\theta'_\epsilon}) \leq B\rho \tag{7.6.1}$$

and

$$d\,|\mu-\epsilon|\,(V_{\theta'_\epsilon}) \leq |\mu-\epsilon|\,. \tag{7.6.2}$$

Proof. The key to this estimate is again the local T^2 action in the tubular neighborhoods. Both (7.6.1) and (7.6.2) are immediate outside of the tubular neighborhood. Let us first consider (7.6.1), when, inside of the tubular neighborhood, we have

$$d\rho(V_{\theta'_\epsilon}) = \omega(V_{\theta'_\epsilon}, \partial^\perp_{\theta_{\boldsymbol{\alpha}-\boldsymbol{\beta}}}) = \theta'_\epsilon(\partial^\perp_{\theta_{\boldsymbol{\alpha}-\boldsymbol{\beta}}}). \tag{7.6.3}$$

We have

$$p^*\theta'_{N_{\mathbb{C}^\times}\times\mathbb{C}}(\partial_{\theta_{\boldsymbol{\alpha}-\boldsymbol{\beta}}})^\perp = c_i^\perp r_{\boldsymbol{\alpha}-\boldsymbol{\beta}}, \tag{7.6.4}$$

and one easily calculates that

$$\overline{\chi}'\left(|u|^2 + \left|t^{-\nu(\boldsymbol{\alpha})}\boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})}\boldsymbol{w}^{\boldsymbol{\beta}}\right|^2\right) d^c\overline{\chi}\left(|u|^2 + \left|t^{-\nu(\boldsymbol{\alpha})}\boldsymbol{w}^{\boldsymbol{\alpha}} + t^{-\nu(\boldsymbol{\beta})}\boldsymbol{w}^{\boldsymbol{\beta}}\right|^2\right)\left(\partial_{\theta_{\boldsymbol{\alpha}-\boldsymbol{\beta}}}\right) \tag{7.6.5}$$

is bounded which completes the first estimate since $d\rho$ differs from $r^\perp_{\boldsymbol{\alpha}-\boldsymbol{\beta}}$ by a bounded function. The second estimate (7.6.2) is simpler since $\theta'_\epsilon(\partial_\theta) = \mu-\epsilon$ and hence $d\,|\mu-\epsilon|\,(V_{\theta'_\epsilon}) = |\mu-\epsilon|\,.$ □

Let r_Y denote the Liouville coordinate on the completion of Y_R. It is not difficult to see that the Liouville flow is not complete as we approach the divisor D. However, given μ_0, we can assume that we have an embedding of the region of the symplectic completion

$$Y_R \cup \partial Y_R \times [1, c_{\mu_0}) \to Y. \tag{7.6.6}$$

Lemma 7.6 implies the following direct analogue of [McL09, Lemma 5.7], which is crucial in McLean's proof of [McL09, Theorem 5.5].

Corollary 7.7. *When $H_{\mathrm{b}} \geq R$ and $\mu \leq \epsilon - \mu_0$ or $\mu \leq \epsilon + \mu_1$, one has*

$$\rho \leq e^B r_Y \tag{7.7.1}$$

and

$$C|\mu-\epsilon| \leq r_Y. \tag{7.7.2}$$

Proof. This follows from Lemma 7.6 because r_Y is defined to be the integration of the Liouville vector field for time $\log(r_Y)$. □

As a consequence of Corollary 7.7, we have that we may take $c_{\mu_0} \to \infty$ as $\mu_0 \to 0$. For the moment take μ_0 sufficiently small so that $c_{\mu_0} \geq 2$.

Definition 7.8. Let $\hat{Y}_R$ denote the Liouville completion of Y_R and let $\hat{\theta'}_\epsilon$ denote the standard extension of the Liouville form to the completion. We say that an almost complex structure on $\hat{Y}_R$ is Liouville admissible in a neighborhood of a level set $r_Y = c$ if for for some $\delta > 0$, we have that whenever $c-\delta < r_Y < c+\delta$, we have $\hat{\theta'}_\epsilon \circ J = dr_Y$.

It is these almost complex structures which are used in [AS10]. We will make use of almost complex structure which combine Definition 7.8 and 5.10.

Definition 7.9. We let $J(Y_R, \overline{E})$ denote the set of admissible almost complex structures on $\overline{Y}$ which are Liouville admissible in a neighborhood of the hypersurface where $r_Y = 2$.

Lemma 7.10. *In the above situation, the space $J(Y_R, \overline{E})$ is non-empty.*

Proof. By Lemma 7.4, the Liouville vector fields and Reeb fields are tangent to the two components of $\overline{E}$. It suffices to check that the almost complex structure on the contact distributions can be chosen so as to preserve both components of $\overline{E}$. This is a routine calculation left to the reader. □

7.11. Floer theory for the Liouville domain. The Lagrangian L_0 satisfies $\theta'_\epsilon|L = 0$ everywhere on Y and is in particular Legendrian at infinity for the Liouville structure. In this subsection, we recall in more detail the Liouville admissible families of Hamiltonians which are relevant Abouzaid and Seidel's theory for Liouville domains. We then recall a variant of their construction which uses Hamiltonians which are constant outside of a compact region and review the equivalence of the two approaches. This is all completely standard material in the closed string version of wrapped Floer cohomology, symplectic cohomology, and it is easily adapted to the Lagrangian setting. Choose a generic family of Liouville admissible Hamiltonians H^λ. We first compute the wrapped Floer cohomology

$$\mathrm{HW}^*(L_0) := \varinjlim_{\lambda} \mathrm{HF}^*(L_0; H^\lambda). \tag{7.11.1}$$

For any Liouville admissible Hamiltonian H^λ, notice that the flow of H^λ and preserve the neighborhood $\pi^{-1}_{N_{\mathbb{C}^\times}}(U_Z)$. Therefore all Hamiltonian chords of $\mathscr{X}(L_0, H^\lambda)$ lie in $Y \setminus \pi^{-1}_{N_{\mathbb{C}^\times}}(U_Z)$. We may label generators by $\theta^\lambda_{\boldsymbol{n},i}$, which corresponds to the unique chord of period less than or equal to λ whose homotopy class in $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ is given by $(\boldsymbol{n}, i)$ (though an additional argument is needed to know that the continuation maps send $\theta^\lambda_{\boldsymbol{n},i} \to \theta^{\lambda'}_{\boldsymbol{n},i}$ see Lemma 7.12). As $\lambda \to \infty$, all homotopy classes are eventually realized.

Lemma 7.12. *If the Hamiltonian chord corresponding to $\theta^\lambda_{\boldsymbol{n},i}$ has period λ_0, then for any $\lambda' \gg \lambda \gg \lambda_0$, the continuation map $\mathrm{HF}^*(L_0; H^\lambda) \to \mathrm{HF}^*(L_0; H^{\lambda'})$ sends $\theta^\lambda_{\boldsymbol{n},i} \to \theta^{\lambda'}_{\boldsymbol{n},i}$.*

Proof. Because $d = 0$ for grading reasons, we may use any Hamiltonians of slope λ and λ' to compute this continuation map. We again approximate by quadratic Hamiltonians, e.g. by a Hamiltonian which is

- r_Y^2 for $r_Y < a_\lambda$
- $\lambda' r_Y$ for $r_Y \geq a_\lambda + d_\lambda$.

Then the continuation maps are the identity on the Reeb chords which arise in the quadratic part. □

Corollary 7.13. $\mathrm{HW}^*(L_0)$ *is freely generated by the images $\theta_{\boldsymbol{n},i}$ of $\theta^\lambda_{\boldsymbol{n},i}$ for sufficiently high λ.*

We now turn to the Hamiltonians which are constant outside of a compact set following [McL09]. We assume

- $H^\lambda(x) = h^\lambda(r_Y)$ for $r_Y \geq 1$,
- $H^\lambda(x) = \lambda r_Y$ for $r_Y \geq 2$
- $(h^\lambda)' \geq 0$ for $r \geq 1$, and
- H^λ is C^2 small on Y_R.

Choose a generic almost complex structure which is Liouville admissible in a neighborhood of $r_Y = 2$. For any λ, define $\mu(\lambda) > 0$ to be some constant smaller than the distance between λ and the action spectrum. Define $A(\lambda) := 3\lambda/\mu(\lambda)$. By choosing $\mu(\lambda)$ small enough, we may assume that $A(\lambda) \geq 4$. We let H^λ_c be a function satisfying the following conditions. For $r_Y \geq 1$, H^λ_c depends only on r_Y and

- for $r_Y \leq A - 1$, $H^\lambda_c = H^\lambda$,
- for $r_Y > A$ we assume that $H^\lambda_c = \lambda(A - 1)$
- for $r_Y \geq A - 1$, $(H^\lambda_c)'' \leq 0$.

Recall that the Floer differential increases the action. We therefore have a subcomplex $\mathrm{CF}^*(L_0; H_c^\lambda)_{(0,\infty)}$ of generators whose action is greater than zero. Define

$$\mathrm{CF}^*(L_0; H_c^\lambda)_p := \mathrm{CF}^*(L_0; H_c^\lambda)/\,\mathrm{CF}^*(L_0; H_c^\lambda)_{(0,\infty)}. \tag{7.13.1}$$

We may choose H^λ so that all time one chords x satisfy

$$A_{H^\lambda}(x) \leq 0$$

An elementary calculation shows that the action of any time one chord when $r_Y \geq 2$ all satisfy $A_{H_c^\lambda}(x) \geq \lambda$. It therefore follows that:

Lemma 7.14. *We have an isomorphism of cochain-complexes*

$$\mathrm{CF}^*(L_0; H_c^\lambda)_p \cong \mathrm{CF}^*(L_0; H^\lambda). \tag{7.14.1}$$

In the limit, we obtain an isomorphism

$$\mathrm{HW}^*(L_0) \cong \varinjlim_{\lambda} \mathrm{HF}^*(L_0; H_c^\lambda)_p \tag{7.14.2}$$

of vector spaces.

Proof. The above discussion gives the identification on generators, so it only remains to give the identification of differentials. Since all generators of both complexes lie in the region where $r_Y \leq 2$, this follows from the "no-escape lemma" in [AS10, Lemma 7.2]. This lemma also applies to continuation solution to the result passes to the limit as claimed. □

7.15. The main comparison theorem. This section compares the ring structure on the two flavours of wrapped Floer cohomology. For this section, we choose all of our almost complex structures to lie in $J(Y_R, \overline{E})$. As we will need to vary μ_0 in this section, for clarity we use the notation Y_{R,μ_0} to indicate the dependence on μ_0. Notice that for $\mu_0' < \mu_0''$ the manifolds $Y_{R,\mu_0'}$ are Liouville isomorphic in a way which preserves L_0. As a result, there are continuation isomorphisms

$$\mathrm{HW}^*_{Y_{R,\mu_0'}}(L_0) \to \mathrm{HW}^*_{Y_{R,\mu_0''}}(L_0) \tag{7.15.1}$$

between the wrapped Floer cohomologies. In this case, the continuation maps are given by interpolating between μ_0' and μ_0'' so that for every s in the continuation family, the Hamiltonian preserves the divisor E. It follows that continuation map solutions have intersection number zero with the divisor E and thus the isomorphism respects the free homotopy class of the chord in $Y \setminus \pi^{-1}_{N_{\mathbb{C}^\times}}(U_Z)$. We finally are in a position to state and prove our main comparison theorem.

Theorem 7.16. *There is an isomorphism*

$$\mathrm{HW}^*(L_0) \cong \mathrm{HW}_{\mathrm{ad}}(L_0) \tag{7.16.1}$$

of $\mathbb{C}$-algebras.

Proof. This is a modification of the proof of [McL09, Theorem 5.5]. We may consider a family of fibered admissible Hamiltonians H_{λ_k} which has slope λ_k in both directions, which vanish on Y_R, μ_0 and such that for any time-one chord x, $A_{H_{\lambda_k}}(x) \leq 0$. Next, by shrinking μ_0 to $\mu_{0,k}$, we may ensure that $c_{\mu_{0,k}}$ above is arbitrarily large. Thus, for any slope λ_k' and for sufficiently small $\mu_{0,k}$, the Hamiltonian $H_c^{\lambda_k'}$ makes sense as a Hamiltonian on Y. Given $H_c^{\lambda_k'}$, we will construct a fibered admissible Hamiltonian $H_{h,\lambda_k'}$ which for $r < A$ agrees with H_c^λ and for which all chords in $r_Y \geq 2$ have very positive action. To do this, we consider a shifted version of a fibered admissible Hamiltonian H_A which vanishes when

$$(C\,|\mu-\epsilon| \leq A+1) \cup (\rho \leq B \cdot A+1) \tag{7.16.2}$$

We also require that whenever $\rho - B \cdot A \geq 1$ or $C\,|\mu-\epsilon| - A \geq 1$, H_A takes the form:

$$v(C\,|\mu-\epsilon| - A) + h(\rho - B\cdot A) \tag{7.16.3}$$

where h and v are suitable base-admissible and admissible vertical Hamiltonians. Finally, we require that $v(C|\mu-\epsilon|-A)$ and $h(\rho - B\cdot A)$ are linear of slope $\sqrt(\lambda_k')$ whenever $\rho - B\cdot A \geq 2$ or $C|\mu-\epsilon| - A \geq 2$. It follows as in McLean Lemma 5.6 that the chords of the function H_A have action have action $A_{H_A}(x) \geq -\sqrt(\lambda_k')(A + B\cdot A)$. In fact, the argument here is even easier because the chords live over the part of the manifold where the symplectic fibration is trivial. Now the function $H_A + H_c^{\lambda_k'}$ is admissible. If follows that for any chord which lies in the region where $r_Y \geq 2$

$$A_H(x) \geq \lambda_k'(A-1) - \sqrt(\lambda_k')(A+B\cdot A) > 0. \tag{7.16.4}$$

Furthermore, it is a consequence of Corollary 7.7 that for any λ_k, we may choose λ_k', $\mu_{0,k}$ and $H_{h,\lambda_k'}$ such that $H_{h,\lambda_k'} > H_{\lambda_k}$ and the slopes in both direction of $H_{h,\lambda_k'}$ are bigger than H_{λ_k}. Next, we can construct fibered admissible Hamiltonians $H^{\lambda_k''}$ which have slope λ_k'' in both directions, which vanish on $Y_{R,\mu_{0,k}}$ and such the the action of all chords is ≤ 0. We assume that $\lambda_k'' \gg \lambda_k'$. We therefore have, after choosing suitable functions $\ell^\sharp$, continuation maps

$$\mathrm{HF}^*_{\mathrm{ad}}(L_0; H_{\lambda_k}) \to \mathrm{HF}^*_{\mathrm{ad}}(L_0; H_{h,\lambda_k'})_p \to \mathrm{HF}^*_{\mathrm{ad}}(L_0; H_{\lambda_k''}). \tag{7.16.5}$$

In the limit, the composition of these two maps becomes an isomorphism and, as a result, the second map is an isomorphism as well. Since these maps are ring maps, this concludes the proof. □

Appendix A. A priori energy bounds

A.1. The *geometric energy* and the *topological energy* of solutions to Floer's equation (4.11.1) are respectively defined by

$$E^{\mathrm{top}}(u) := \int_\Sigma u^*\omega - d(u^*K) \tag{A.1.1}$$

and

$$E^{\mathrm{geom}}(u) := \frac{1}{2}\int_\Sigma \|du - X_K\|\,. \tag{A.1.2}$$

Using $\omega(X,X) = 0$ and $dH = \iota_X\omega$, a standard calculation in local holomorphic coordinates $z = s + \sqrt{-1}t$ on Σ yields

$$E^{\mathrm{geom}}(u) = E^{\mathrm{top}}(u) + \int_\Sigma \Omega_K \tag{A.1.3}$$

where the curvature Ω_K of a perturbation datum K is the exterior derivative of K in the Σ direction.

Definition A.2. We say that a perturbation datum K is *monotonic* if $\Omega_K \leq 0$.

In practice, the most typical monotonic perturbation data are of the form $H \otimes \gamma$ for $H : X \to \mathbb{R}^{\geq 0}$ and γ a sub-closed one form on Σ. Whenever the perturbation datum is chosen to be monotonic, we have that

$$E^{\text{geom}}(u) \leq E^{\text{top}}(u) \tag{A.2.1}$$

Note that the topological energy is determined by the action

$$A_{H_i}(x) = \int_0^1 \big(- x^*\theta + H_i(x(t))dt\big) + h(x(1)) - h(x(0)) \tag{A.2.2}$$

as

$$E^{\text{top}}(u) = A_{H_0}(x_0) - \sum_{k=1}^{d} A_{H_k}(x_k), \tag{A.2.3}$$

where $h \colon L \to \mathbb{R}$ is defined by $\theta|_L = dh$. This gives an a priori bound for the geometric energy of a solution to Floer's equation (4.11.1).

Appendix B. Maximum principle

B.1. Let $u \colon \Omega \to \mathbb{R}^{\geq 0}$ be a smooth function on a connected open set $\Omega \subset \mathbb{R}^n$ satisfying $Lu \geq 0$ for an elliptic second order differential operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i \tag{B.1.1}$$

such that

- the matrix-valued function $(a_{ij}(x))_{i,j=1}^n$ is locally uniformly positive-definite, and
- the functions $b_i(x)$ are locally bounded for $i = 1, \dots, n$.

The *strong maximum principle* states that if u attains a maximum in Ω, then it must be constant. *Hopf's lemma* states that if

- u extends smoothly to the boundary, and
- there exists $x_0 \in \partial\Omega$ that $u(x_0) > u(x)$ for any $x \in \Omega$,

then one has $du(\nu) > 0$, where ν is the unit outward normal vector to $\partial\Omega$ at x_0. Proofs of the strong maximum principle and Hopf's lemma can be found, e.g., in [Eva10, Section 6.4.2].

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