SYZ TRANSFORMS FOR IMMERSED LAGRANGIAN MULTI-SECTIONS

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Abstract. In this paper, we study the geometry of the SYZ transform on a semi-flat Lagrangian torus fibration. Our starting point is an investigation on the relation between Lagrangian surgery of a pair of straight lines in a symplectic 2-torus and extension of holomorphic vector bundles over the mirror elliptic curve, via the SYZ transform for immersed Lagrangian multi-sections [5, 30]. This study leads us to a new notion of equivalence between objects in the immersed Fukaya category of a general compact symplectic manifold (M, ω), under which the immersed Floer cohomology is invariant; in particular, this provides an answer to a question of Akaho-Joyce [4, Question 13.15]. Furthermore, if M admits a Lagrangian torus fibration over an integral affine manifold, we prove that, under some additional assumptions, this new equivalence is mirror to isomorphism between holomorphic vector bundles over the dual torus fibration via the SYZ transform.

Contents

1. Introduction

Mirror symmetry was discovered by string theorists around 1990. It has been making a strong impact to mathematics after Candelas, de la Ossa, Green and Parkes [8] showed that mirror symmetry can be used to predict the number of rational curves in a generic quintic Calabi-Yau 3-fold. This mysterious phenomenon has been attracting the attention of many mathematicians since its was born.

The first mathematical approach towards understanding mirror symmetry was due to Kontsevich [27] in 1994. He suggested that mirror symmetry can be phrased

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as an equivalence between two triangulated categories, namely, the derived Fukaya category on one side and the derived category of coherent sheaves on the mirror side; this is known as the *homological mirror symmetry (HMS) conjecture*.

Two years later, Strominger, Yau and Zaslow proposed an entirely geometric approach to explain mirror symmetry, which is now known as the *SYZ conjecture* [32]. Roughly speaking, the SYZ conjecture states that mirror symmetry can be understood as a fiberwise duality between two special Lagrangian torus fibrations; moreover, symplectic (resp. complex) geometric data on one side could be transformed to complex (resp. symplectic) geometric data on the mirror side by a fiberwise Fourier–Mukai-type transform, which we call the *SYZ transform*.

The SYZ transform has been constructed and applied to understand mirror symmetry in the semi-flat case [5, 30, 29, 17] and the toric case [1, 3, 9, 11, 13, 10, 16, 15, 14, 19, 20, 21, 18]. But in all these works, mainly the SYZ transform for Lagrangian sections, which produces holomorphic line bundles over the mirror, was studied and used. Applications of the SYZ transform for Lagrangian multi-sections, which should produce higher rank holomorphic vector bundles over the mirror, is largely unexplored.

In this paper, we study the geometry of the SYZ transform, especially for immersed Lagrangian multi-sections, on a semi-flat Lagrangian torus fibration. The SYZ transform in the semi-flat setting will be reviewed in Section 2.

In view of the HMS conjecture, Thomas, Seidel and Fukaya, among others, have suggested that Lagrangian surgeries between two (graded) Lagrangian submanifolds should be mirror dual to extension between coherent sheaves over the mirror side. We refer this as the *surgery-extension correspondence*. In Section 3, we investigate this correspondence for the simplest example, namely, the 2-torus $X = T^2$. We are going to equip the Lagrangian submanifolds with $U(1)$-local systems, and we will see that this is going to play a key role in the proof of our correspondence theorem.

More precisely, we consider the Lagrangian straight lines

$$\begin{aligned} L_1 &:= L_{r_1,d_1}[c_1] := \{(e^{2\pi i r_1 x}, e^{2\pi i(d_1 x + c_1)}) \in T^2 : x \in \mathbb{R}\}, \\ L_2 &:= L_{r_2,d_2}[c_2] := \{(e^{2\pi i r_2 x}, e^{2\pi i(d_2 x + c_2)}) \in T^2 : x \in \mathbb{R}\} \end{aligned}$$

in T^2, which are equipped, respectively, with the $U(1)$-local systems

$$\mathcal{L}_{b_1} : d + 2\pi i \frac{b_1}{r_1} dx, \quad \mathcal{L}_{b_2} : d + 2\pi i \frac{b_2}{r_1} dx, \quad b_1, b_2 \in \mathbb{R}.$$

We write $L_{1,b_1} = (L_1, \mathcal{L}_{b_1}), L_{2,b_2} = (L_2, \mathcal{L}_{b_2})$ for the A-branes obtained in this way, and denote their SYZ transforms, which are holomorphic vector bundles over the mirror elliptic curve $\check{X}$, by $\check{L}_{1,b_1}, \check{L}_{2,b_2}$ respectively. We prove the following surgery-extension correspondence theorem in Section 3:

Theorem 1.1. *(=Theorem 3.4) Let r_1, d_1, r_2, d_2 be integers satisfying $r_1 d_2 > r_2 d_1$ and the gcd conditions $gcd(r_1, d_1) = gcd(r_2, d_2) = gcd(r_1 + r_2, d_1 + d_2) = 1$. Let $K \subset L_1 \cap L_2$ be a set of intersection points of L_1 and L_2 such that the (graded) Lagrangian surgery produces an immersed Lagrangian*

$$\mathbb{L}_K := L_2 \sharp_K L_1$$

with connected domain, which we then equip with the $U(1)$-local system

$$\mathcal{L}_b : d + 2\pi i \frac{b}{r_1 + r_2} dx, \quad b \in \mathbb{R}.$$

Then the SYZ mirror bundle $\check{\mathbb{L}}_{K,b}$ of the Lagrangian A-brane $(\mathbb{L}_K, \mathcal{L}_b)$ is an extension of $\check{L}_{1,b_1}$ by $\check{L}_{2,b_2}$, i.e. we have a short exact sequence:

$$0 \to \check{L}_{2,b_2} \to \check{\mathbb{L}}_{K,b} \to \check{L}_{1,b_1} \to 0$$

if and only if b satisfies the integrality condition

$$b_1 + b_2 - b - \frac{1}{2} \in \mathbb{Z}.$$

In particular, this theorem implies the intriguing phenomenon that the surgery-extension correspondence cannot hold unless we equip Lagrangian submanifolds with suitable *nontrivial* local systems (even in the case when we equip L_1, L_2 with trivial local systems).

In Floer-theoretic terms, the integrality condition in Theorem 1.1 can be regarded as a generalization of degree -1 marked points in Abouzaid's work. More precisely, in [2], Abouzaid considered immersed curves in Riemann surfaces with one marked point of prescribed degree -1, and proved that mapping cones in the Fukaya category can be geometrically realized as Lagrangian surgeries. One may think of the prescribed -1 degree for a marked point as the holonomy of a flat $U(1)$-connection concentrated at that point. Our integrality condition recovers Abouzaid's condition by taking $b_1 = b_2 = b = \frac{1}{2}$.

Remark 1.2. *Our theorem is a generalization of a recent result of K. Kobayashi* [26] *to* any *rank and degree that satisfy the gcd assumptions.*

Remark 1.3. *We believe that the above theorem is known to experts; see in particular* [33, Section 6].

In view of Atiyah's classification of indecomposable vector bundles over elliptic curves [6] and our surgery-extension correspondence theorem, we observe that, as long as the two sets of intersection points $K, K' \subset L_1 \cap L_2$ are chosen so that the surgeries $\mathbb{L}_K$ and $\mathbb{L}_{K'}$ satisfy the assumptions in Theorem 1.1, their SYZ mirror bundles $\check{\mathbb{L}}_{K,b}$ and $\check{\mathbb{L}}_{K',b}$ are isomorphic as holomorphic vector bundles, because both are indecomposable and share the same determinant line bundle $\det(\check{L}_{1,b_1}) \otimes \det(\check{L}_{2,b_2})$. For example, Figure 1 shows two immersed Lagrangian multi-sections $\mathbb{L}_1, \mathbb{L}_3$ in T^2 which share the same SYZ mirror bundles.

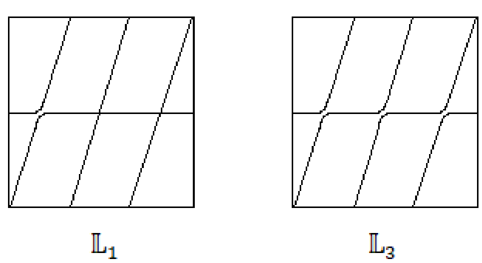

FIGURE 1. Two non-Hamiltonian equivalent immersed Lagrangian multi-sections in T^2.

A natural question is then:

Question 1.4. *What is the symplecto-geometric relation between $\mathbb{L}_K$ and $\mathbb{L}_{K'}$?*

First of all, the relation cannot be the ordinary Hamiltonian equivalence because $\mathbb{L}_K$ and $\mathbb{L}_{K'}$ may have different number of self-intersection points (like in the above example). A naïve guess is a weaker notion, called *local Hamiltonian equivalence* (Definition 4.6). However, Akaho and Joyce [4] have already pointed out that, in view of the *Lagrangian h-principle* [24, 28], local Hamiltonian equivalence is only a weak homotopical notion which cannot detect 'quantum' information, so it is too coarse for the immersed Floer cohomology to be invariant. On the other hand, since the SYZ mirror bundles of $\mathbb{L}_K$ and $\mathbb{L}_{K'}$ are isomorphic, the Floer cohomology of $\mathbb{L}_K$ and $\mathbb{L}_{K'}$ should also be isomorphic via homological mirror symmetry.

This leads us to make a digression from SYZ mirror symmetry to study the invariance property of immersed Floer cohomology in Section 4, in which we introduce a new equivalence relation on immersed Lagrangian submanifolds called *lifted Hamiltonian equivalence*.

Definition 1.5. *(=Definition 4.8) Let $\pi : \widetilde{M} \to M$ be a finite unramified covering of a symplectic manifold (M, ω). For two Lagrangian immersions $\mathbb{L}_1 = (L_1, \xi_1), \mathbb{L}_2 = (L_2, \xi_2)$ of M, we say $\mathbb{L}_1$ is $(\widetilde{M}, \pi)$-*lifted Hamiltonian isotopic *to $\mathbb{L}_2$ if there exists a diffeomorphism $\phi : L_1 \to L_2$ and Lagrangian immersions $\widetilde{\xi}_1 : L_1 \to \widetilde{M}$, $\widetilde{\xi}_2 : L_2 \to \widetilde{M}$ such that $\xi_1 = \pi \circ \widetilde{\xi}_1$, $\xi_2 = \pi \circ \widetilde{\xi}_2$ and $(L_1, \widetilde{\xi}_1)$ is globally Hamiltonian isotopic (see Definition 4.4 or* [4, Definition 13.14]*) to $(L_1, \widetilde{\xi}_2 \circ \phi)$ in $(\widetilde{M}, \pi^*\omega)$.*

We also make the following

Definition 1.6. *(=Definition 4.9) Let (M, ω) be a symplectic manifold. For two Lagrangian immersions $\mathbb{L}_1 = (L_1, \xi_1), \mathbb{L}_2 = (L_2, \xi_2)$ of M, we say $\mathbb{L}_1$ is* lifted Hamiltonian isotopic *to $\mathbb{L}_2$ if there exists an integer $l > 0$ and Lagrangian immersions $\mathbb{L}^{(1)} := \mathbb{L}_1, \mathbb{L}^{(2)}, \ldots, \mathbb{L}^{(l-1)}, \mathbb{L}^{(l)} := \mathbb{L}_2$ of M, such that $\mathbb{L}^{(j)}$ is $(\widetilde{M}_j, \pi_j)$-lifted Hamiltonian isotopic to $\mathbb{L}^{(j+1)}$, for some finite unramified covering $\pi_j : \widetilde{M}_j \to M$, $j = 1, \ldots, l-1$.*

This new notion of equivalence is weaker than the usual Hamiltonian equivalence but stronger than local Hamiltonian equivalence (as proved in Corollary 4.11). In Section 4, the following invariance property of immersed Floer cohomology under lifted Hamiltonian equivalences is proved:

Theorem 1.7. *(=Theorem 4.16) Let $\mathbb{L}_1, \mathbb{L}_2$ be Lagrangian immersions in (M, ω). The Floer cohomology $HF(\mathbb{L}_1, \mathbb{L}_2)$ is invariant under lifted Hamiltonian isotopy, i.e. if $\mathbb{L}_2$ is lifted Hamiltonian isotopic to $\mathbb{L}_2'$, then there is a quasi-isomorphism*

$$(CF(\mathbb{L}_1, \mathbb{L}_2), m_1) \simeq (CF(\mathbb{L}_1, \mathbb{L}_2'), m_1).$$

In particular, this gives an answer to a question of Akaho and Joyce [4, Question 13.15], asking for restricted classes of local Hamiltonian equivalences under which the immersed Lagrangian Floer cohomology is invariant.

In the final Section 5, we go back to SYZ mirror symmetry and Question 1.4; in fact, we would like to ask an even more general question:

Question 1.8. *Let $X \to B$ be a Lagrangian torus fibration and $\check{X} \to B$ be the dual torus fibration. What is the mirror analog of isomorphism between holomorphic vector bundles over $\check{X}$?*

We prove that, under certain conditions, the answer is, again, given by lifted Hamiltonian equivalence:

Theorem 1.9. *(=Theorem 5.3) Suppose that B is compact. Let $\mathbb{L}_1, \mathbb{L}_2$ be immersed Lagrangian multi-sections of $X \to B$ with the same connected domain L and unramified covering map $c_r : L \to B$. Assume that the group of deck transformations $Deck(L/B)$ acts transitively on fibers of $c_r : L \to B$. Then $\mathbb{L}_1$ is $(L \times_B X, \pi_X)$-lifted Hamiltonian isotopic to $\mathbb{L}_2$ if and only if their SYZ mirrors $\check{\mathbb{L}}_1$ and $\check{\mathbb{L}}_2$ are isomorphic as holomorphic vector bundles over $\check{X}$.*

Combining this with Theorem 3.4, we obtain an answer to Question 1.4:

Corollary 1.10. *(=Corollary 5.4) Let $L_1 = L_{r_1,d_1}[c_1]$ and $L_2 = L_{r_2,d_2}[c_2]$ be as in Theorem 1.1. If $K, K' \subset L_1 \cap L_2$ are sets of intersection points such that the Lagrangian surgeries $\mathbb{L}_K = L_2 \sharp_K L_1$ and $\mathbb{L}_{K'} = L_2 \sharp_{K'} L_1$ have connected domain and satisfy the gcd assumption $gcd(r_1 + r_2, d_1 + d_2) = 1$, then $\mathbb{L}_K$ and $\mathbb{L}_{K'}$ are $(S^1 \times_{S^1} T^2, \pi_{T^2})$-lifted Hamiltonian equivalent, and hence have isomorphic immersed Lagrangian Floer cohomologies.*

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2. Semi-flat mirror symmetry

In this section, we review the SYZ transform in the semi-flat setting, following [5] and [30] (see also [29], [12, Section 2] or [10, Section 2]).

2.1. **SYZ mirror construction.**

Let B be an n-dimensional integral affine manifold, that is, the transition functions of B belongs to $GL(n, \mathbb{Z}) \rtimes \mathbb{R}$, the group of $\mathbb{Z}$-affine linear map. Let $\Lambda \subset TB$ and $\Lambda^* \subset T^*B$ be the natural lattice bundles defined by the integral affine structure. More precisely, on a local affine chart $U \subset B$,

$$\Lambda(U) := \bigoplus_{j=1}^{n} \mathbb{Z} \cdot \frac{\partial}{\partial x^j}, \quad \Lambda^*(U) := \bigoplus_{j=1}^{n} \mathbb{Z} \cdot dx^j,$$

where (x^j) are affine coordinates of U. We set

$$X := T^*B/\Lambda^* \text{ and } \check{X} := TB/\Lambda.$$

Let $(y^j), (\check{y}_j)$ be fiber coordinates (which are dual to each other) of X and $\check{X}$ respectively. Then (x^j, y^j) and $(x^j, \check{y}_j)$ define a set of local coordinates on $T^*U/\Lambda^* \subset X$ and $TU/\Lambda \subset \check{X}$, respectively.

Equip X with the standard symplectic structure

$$\omega_\hbar := \hbar^{-1} \sum_j dy^j \wedge dx^j,$$

where $\hbar > 0$ is a small real parameter. There is a natural almost complex structure $\check{J}_\hbar$ on $\check{X}$ given by

$$\check{J}_\hbar\left(\frac{\partial}{\partial x^j}\right) = -\hbar^{-1}\frac{\partial}{\partial \check{y}_j} \text{ and } \check{J}_\hbar\left(\frac{\partial}{\partial \check{y}_j}\right) = \hbar\frac{\partial}{\partial x^j}.$$

It is easy to see that $\check{J}_\hbar$ is indeed integrable with local complex coordinates given by $z_j = \check{y}_j + ix^j$. Hence $(\check{X}, \check{J}_\hbar)$ defines a complex manifold.

Definition 2.1. *$(\check{X}, \check{J}_\hbar)$ is called the* SYZ mirror *of $(X, \omega_\hbar)$.*

The $\hbar$-parameter gives us a family of symplectic manifolds. As $\hbar \to 0$, the symplectic volume of $(X, \omega_\hbar)$ approaches infinity. This is the so called large volume limit of the family $\{(X, \omega_\hbar)\}_{\hbar>0}$. As this limiting process does not play any role in this paper, by absorbing $\hbar^{-1}$ into the (x^j)-coordinates, we simply assume that $\hbar = 1$ throughout this paper. Hence we just write ω for $\omega_\hbar$ and $\check{J}$ for $\check{J}_\hbar$ if there are no confusion.

2.2. **The SYZ transform of branes.**

In order for homological mirror symmetry to make sense, one needs to complexify the Fukaya category by unitary local systems [27]. Here, we just consider rank 1 local systems on Lagrangian submanifolds.

Definition 2.2. *A* Lagrangian immersion $\mathbb{L}$ *of (X, ω) is a pair (L, ξ), where L is an n-dimensional smooth manifold and $\xi : L \to X$ is an immersion with the following properties*

a) $\xi^*\omega = 0$.
b) *There is a discrete set of points $S \subset L$ such that $\xi : L\backslash S \to X$ is injective.*
c) *For all $p \in X$, the set $\xi^{-1}(p) \cap S \subset L$ is either empty or consists of two points.*

An A-brane *of X is a pair $(\mathbb{L}, \mathcal{L})$, where $\mathbb{L}$ is an immersed Lagrangian submanifold of X and $\mathcal{L}$ is a rank 1 unitary local system on L.*

We shall focus on the case that $\mathbb{L}$ is an immersed Lagrangian multi-section.

Definition 2.3. *An* immersed Lagrangian multi-section *of rank r is a triple $\mathbb{L} := (L, \xi, c_r)$, where $\xi : L \to X$ is a Lagrangian immersion and $c_r : L \to B$ is a r-fold unramified covering map such that $p \circ \xi = c_r$. We assume the image of L intersects transversally with each torus fiber.*

Remark 2.4. *We remark that L is not necessarily connected.*

We follow [5, 30] to define the SYZ transform of an immersed Lagrangian multi-section of a semi-flat Lagrangian torus fibration.

Let $\mathcal{P} \to X \times_B \check{X}$ be the Poincaré line bundle. The total space is defined as the quotient

$$\mathcal{P} := (T^*B \oplus TB) \times \mathbb{C}/\Lambda^* \oplus \Lambda.$$

The fiberwise action of $\Lambda^* \oplus \Lambda$ on $(T^*B \oplus TB) \times \mathbb{C}$ is given by

$$(\lambda, \check{\lambda}) \cdot (y, \check{y}, t) := (y + \lambda, \check{y} + \check{\lambda}, e^{i\pi((y,\check{\lambda})-(\lambda,\check{y}))} \cdot t).$$

Define a connection $\nabla_{\mathcal{P}}$ on $\mathcal{P}$ by

$$\nabla_{\mathcal{P}} := d + i\pi((y, d\check{y}) - (\check{y}, dy)).$$

The section $e^{i\pi(y,\check{y})}$ is invariant under the $\{0\} \oplus \Lambda$ action:

$$(0, \check{\lambda}) \cdot (y, \check{y}, t) = (y, \check{y} + \check{\lambda}, e^{i\pi(y, \check{\lambda}+\check{y})}).$$

Hence it descends to a section on $T^*B \times_B \check{X}$. With respective to this frame, the connection $\nabla_{\mathcal{P}}$ can be written as

$$\nabla_{\mathcal{P}} = d + 2\pi i(y, d\check{y}).$$

The remaining action of $\Lambda^* \oplus \{0\}$ then becomes

$$\begin{aligned}\lambda \cdot [(y, \check{y}, e^{i\pi(y,\check{y})})]_\Lambda &= [y + \lambda, \check{y}, e^{-i\pi(\lambda,\check{y})} \cdot e^{i\pi(y,\check{y})}]_\Lambda \\ &= e^{-2\pi i(\lambda,\check{y})}[y + \lambda, \check{y}, e^{i\pi(y+\lambda,\check{y})}]_\Lambda.\end{aligned}$$

Let $\mathbb{L} = (L, \xi, c_r)$ be an immersed Lagrangian multi-section of rank r and $\mathcal{L}$ be a $U(1)$-local system on L. Define

$$\check{\mathbb{L}} := (\pi_{\check{X}})_*((\xi \times id_{\check{X}})^*(\mathcal{P}) \otimes (\pi_L^*\mathcal{L})).$$

Note that the projection map $\pi_{\check{X}} : L \times_B \check{X} \to \check{X}$ is an unramified r-fold covering, $\check{\mathbb{L}}$ is a vector bundle of rank r. The connection on $\mathcal{P}$ induces a natural connection $\nabla_{\check{\mathbb{L}}}$ on $\check{\mathbb{L}}$. The following proposition is standard.

Proposition 2.5. *The connection $\nabla_{\check{\mathbb{L}}}$ satisfies $(\nabla^{0,2}_{\check{\mathbb{L}}})^2 = 0$ if and only if $\xi : L \to X$ is a Lagrangian immersion.*

Hence $\check{\mathbb{L}}$ carries a natural holomorphic structure.

Definition 2.6. $(\check{\mathbb{L}}, \nabla_{\check{\mathbb{L}}})$ *is called the* SYZ mirror bundle *of the A-brane* $(\mathbb{L}, \mathcal{L})$. *We simply write $\check{\mathbb{L}}$ for short.*

We give a local description of $\check{\mathbb{L}}$ and $\nabla_{\check{\mathbb{L}}}$ for the case $r = 1$.

Let's first suppose $\mathcal{L}$ is the trivial bundle with trivial connection. Let U be an affine chart of B. We can lift $L \cap T^*U/\Lambda^* \subset X$ to $\widetilde{L}_U \subset T^*U$. Let ξ_U be the defining equation of $\widetilde{L}_U$. The section $e^{i\pi(\xi_U,\check{y})}$ on $\widetilde{L}_U \times_B TU$ induces a section $\check{1}_U$ of $\check{L}$ on $\widetilde{L}_U \times_B \check{X}$ by taking its Λ-equivalence class. With respective to this local frame, the connection $\nabla_{\check{\mathbb{L}}}$ becomes

$$\nabla_{\check{\mathbb{L}}} = d + 2\pi i(\xi_U, d\check{y}).$$

We also compute the unitary and holomorphic transition functions of $(\check{\mathbb{L}}, \nabla_{\check{\mathbb{L}}})$. Let V be another affine chart of B such that $U \cap V \neq \phi$. Let ξ_V be the defining equation of the lift $\widetilde{L}_V \subset T^*V$. Since

$$[(\xi_U, dx_U)]_{\Lambda^*} = [(\xi_V, dx_V)]_{\Lambda^*},$$

there exists $\lambda_{UV} \in \Lambda^*|_{U\cap V}$ such that

$$(\xi_U, dx_U) = (\xi_V, dx_V) + (\lambda_{UV}, dx_V).$$

Then we have

$$\check{1}_V = e^{-2\pi i(\lambda_{UV},\check{y}_V)}\check{1}_U.$$

Therefore, $\tau_{UV}(x_V) = e^{2\pi i(\lambda_{UV},\check{y}_V)}$ are the unitary transition functions.

To compute the holomorphic one, let $f_U : U \to \mathbb{R}$ be a primitive of ξ_U. Then it is easy to check that

$$\check{e}_U = e^{-2\pi f_U}\check{1}_U$$

defines a local holomorphic frame of $\check{L}$. Since f_U, f_V are primitives of $(\xi_U, dx_U), (\xi_V, dx_V)$, respectively, we have

$$f_U(x_U) = f_V(x_V) + (\lambda_{UV}, x_V) + c_{UV},$$

for some $c_{UV} \in \mathbb{R}$. The holomorphic transition functions are then given by

$$g_{UV}(z_V) = e^{-2\pi c_{UV}} e^{2\pi i(\lambda_{UV}, z_V)},$$

where $z_V = \check{y}_V + ix_V$ is a holomorphic coordinate of TV/Λ.

Now, suppose $\mathcal{L} \to L$ is a $U(1)$-local system. Then $\mathcal{L}$ carries a natural flat connection $\nabla_{\mathcal{L}}$. Write

$$\nabla_{\mathcal{L}} = d + 2\pi i\beta, \quad \beta \in \Gamma(L, T^*L).$$

Since $\nabla_{\mathcal{L}}^2 = 0$, $d\beta = 0$. Let $b_U(x_U)$ be a primitive of β on U. Then a local holomorphic frame is given by

$$e^{-2\pi(f_U + ib_U)}\check{1}_U.$$

When $U \cap V \neq \phi$, we have

$$db_U(x_U) = \beta = db_V(x_V) \Rightarrow b_U(x_U) - b_V(x_V) = b_{UV} \in \mathbb{R}.$$

The holomorphic transition functions then become

$$g_{UV}(z_V) = e^{-2\pi(c_{UV} + ib_{UV})} e^{2\pi i(\lambda_{UV}, z_V)}.$$

The connection $\nabla_{\check{\mathbb{L}}}$ becomes

$$\nabla_{\check{\mathbb{L}}} = d + 2\pi i(\xi_U, d\check{y}) + 2\pi i\beta|_U.$$

For a general r, one can also write down the connection $\nabla_{\check{\mathbb{L}}}$ in terms of the data coming from the Lagrangian brane:

$$\nabla_{\check{\mathbb{L}}} = d + 2\pi i \sum_{j=1}^n \begin{pmatrix} \xi_{U_{1,j}}(x_1) & 0 & 0 & \dots & 0 \\ 0 & \xi_{U_{2,j}}(x_2) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \xi_{U_{r,j}}(x_r) \end{pmatrix} d\check{y}_U^j$$
$$+ 2\pi i \sum_{j=1}^n \begin{pmatrix} \beta_{U_{1,j}}(x_1) & 0 & 0 & \dots & 0 \\ 0 & \beta_{U_{2,j}}(x_2) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{U_{r,j}}(x_r) \end{pmatrix} d\check{x}_U^j,$$

where $c_r^{-1}(U) = \coprod_{k=1}^r U_k$ and $x_k \in U_k$, $k = 1, \dots, r$ are the preimage points of $x \in U \subset B$.

3. Surgery-extension correspondence for T^2

Let L_1 and L_2 be two graded immersed Lagrangian multi-sections and $\check{L}_1$ and $\check{L}_2$ be their mirror bundles. It is believed that performing Lagrangian surgeries at index 1 intersection points of L_1 and L_2 corresponds to forming a nontrivial extension of $\check{L}_1$ and $\check{L}_2$. More precisely, let $K := \{p_1, \cdots, p_k\} \subset CF(L_1, L_2)$ be a collection of index 1 intersection points of L_1 and L_2. We perform Lagrangian surgery at each point in K (see Figure 2) to obtain another graded immersed Lagrangian multi-section $\mathbb{L}_K := L_2 \sharp_K L_1$. Then the mirror bundle $\check{\mathbb{L}}_K$ of $\mathbb{L}_K$ should fit in an exact sequence:

$$0 \to \check{L}_2 \to \check{\mathbb{L}}_K \to \check{L}_1 \to 0.$$

In this section, we study this relation on the symplectic torus T^2 with standard symplectic structure and its mirror elliptic curve. We will see that the Lagrangian surgery and extension correspondence cannot be true in general if we do not include the brane structures, namely $U(1)$-local systems, to the Lagrangians.

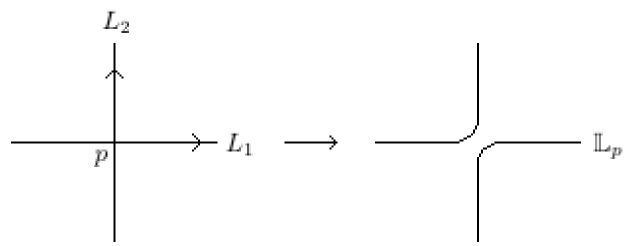


FIGURE 2. 1-dimensional Lagrangian surgery at an index 1 intersection point.

Let $X := T^2 = S^1 \times S^1$ be the product torus with standard symplectic structure given by

$$\omega := dy \wedge dx.$$

We begin with describing the SYZ transform of a general immersed Lagrangian multi-section in X.

Let $B = S^1$ and $p : X \to S^1$ be the projection onto the first factor. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\varphi(x + r) = \varphi(x) + d.$$

Then φ descends to an immersed Lagrangian multi-section $\mathbb{L}_\varphi$ of $p : X \to B$ which intersect the zero section $|d|$ times and each fiber r times. Since φ is smooth, the immersed Lagrangian multi-section $\mathbb{L}_\varphi$ intersects the fibers of $p : X \to B$ transversally. Clearly, every immersed Lagrangian multi-section with connected domain and intersects the fibers transversally arise in this manner.

Let $\{U, V\}$ be the following affine cover of the base $B = S^1$:

$$(0,1) \to V \subset S^1, \quad x \mapsto e^{2\pi i x},$$
$$(0,1) \to U \subset S^1, \quad x' \mapsto e^{2\pi i (x' + \epsilon)},$$

where $\epsilon \in (0,1)$ is fixed. Write $U \cap V = W_1 \amalg W_2$. The SYZ mirror bundle $\check{\mathbb{L}}_\varphi$ of $\mathbb{L}_\varphi$ is a rank r vector bundle with $U(r)$-connection

$$\nabla_{\check{\mathbb{L}}_\varphi} = d + 2\pi i \begin{pmatrix} \varphi(x) & 0 & 0 & \dots & 0 \\ 0 & \varphi(x+1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \varphi(x+r-1) \end{pmatrix} d\check{y}, \quad (x,\check{y}) \in TV/\Lambda.$$

With the complex structure $z = \check{y} + ix$, the degree of $\check{\mathbb{L}}_\varphi$ is given by

$$\begin{aligned} \frac{i}{2\pi}\int_0^1\int_0^1 \frac{d}{dx}\left(2\pi i \sum_{j=0}^{r-1}\varphi(x+j)\right) dx \wedge d\check{y} &= -\int_0^r \frac{d}{dx}\varphi(x)dx \\ &= \varphi(0) - \varphi(r) = -d. \end{aligned}$$

Let's write down the unitary and holomorphic transition functions of $\check{\mathbb{L}}_\varphi$ with from TV/Λ to TU/Λ. Let

$$c_r^{-1}(U) = \coprod_{j=1}^r U_j \text{ and } c_r^{-1}(V) = \coprod_{j=1}^r V_j.$$

By renaming, we can assume $U_1 \cap V_1, V_1 \cap U_2, U_2 \cap V_3, \dots, V_r \cap U_1$ are non-empty connected subsets of $L \times_B \check{X}$. Then we can assume

$$c_r^{-1}(W_1) = \coprod_{j=1}^r V_j \cap U_j \text{ and } c_r^{-1}(W_2) = \coprod_{j=1}^r U_j \cap V_{j+1},$$

where we put $V_{r+1} = V_1$. Then on TW_1/Λ, the unitary transition function is given by the identity matrix while on TW_2/Λ, it is given by

$$\tau_{UV}(x,\check{y}) = \left(\begin{array}{c|c} O & e^{2\pi i d\check{y}} \\ \hline I_{(r-1)\times(r-1)} & O \end{array}\right).$$

Choose a point $x_0 \in W_1 \subset V$. For each $j = 0, \dots, r-1$, let $f_j : \mathbb{R} \to \mathbb{R}$ be given by

$$f_j(x) := \int_{x_0}^{x+j} \varphi(u)du,$$

which is a primitive of $\varphi(x+j)$. A local holomorphic frame for $\check{\mathbb{L}}_\varphi$ on the chart TV/Λ is then given by

$$\{e^{-2\pi f_0(x)}\check{1}_0(x,\check{y}), \dots, e^{-2\pi f_{r-1}(x)}\check{1}_{r-1}(x,\check{y})\},$$

where $(x,\check{y}) \in TV/\Lambda$. On the U-chart, we have

$$f_j'(x') = \int_{x_0'+\epsilon}^{x'+\epsilon+j} \varphi(u)du.$$

On W_1, x' and x are related by $x'(x) = x - \epsilon$, so

$$f_j'(x') = \int_{x_0'+\epsilon}^{x'+\epsilon+j} \varphi(u)du = \int_{x_0}^{x+j} \varphi(u)du = f_j(x),$$

while on W_2, $x'(x) = x - \epsilon + 1$, so for $j = 0, \ldots, r-2$, we have

$$f'_j(x') = \int_{x_0}^{x+j+1} \varphi(u)du = f_{j+1}(x),$$

and for $j = r-1$, we have

$$f'_{r-1}(x') = \int_{x_0}^{x+r} \varphi(u)du = f_0(x) + \int_x^{x+r} \varphi(u)du.$$

Since

$$\frac{d}{dx}\int_x^{x+r} \varphi(u)du = \varphi(x+r) - \varphi(x) = d,$$

we have

$$\int_x^{x+r} \varphi(u)du = dx + \int_0^r \varphi(u)du.$$

The holomorphic transition functions are then give by the identity matrix on TW_1/Λ and

$$g_{UV}(z) = \left(\begin{array}{c|c} O & e^{-2\pi a}e^{2\pi i dz} \\ \hline I_{(r-1)\times(r-1)} & O \end{array}\right)$$

on TW_2/Λ, where c is given by

$$a = \int_0^r \varphi(u)du.$$

Suppose now we enrich $\mathbb{L}$ by a $U(1)$-local system $(\mathcal{L}, \nabla_{\mathcal{L}})$. Since the domain of $\mathbb{L}$ is a circle, the connection $\nabla_{\mathcal{L}}$ can always be written as

$$d + 2\pi i\frac{b}{r}dx,$$

for some $b \in \mathbb{R}$. Hence the transition functions of the SYZ mirror bundle of $(\mathbb{L}, \mathcal{L})$ are given by

$$g_{UV}(z) = \left(\begin{array}{c|c} O & e^{-2\pi(a+ib)}e^{2\pi i dz} \\ \hline I_{(r-1)\times(r-1)} & O \end{array}\right).$$

Example 3.1. *Let $r, d \in \mathbb{Z}$ with $gcd(r,d) = 1$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the straight line*

$$\varphi(x) = \frac{d}{r}x + \frac{c}{r}, \quad r > 0, c \in \mathbb{R}.$$

Then it descends to the Lagrangian multi-section

$$L_{r,d}[c] = \{(e^{2\pi i r x}, e^{2\pi i(dx+c)}) \in X : x \in \mathbb{R}\}.$$

One computes that

$$\int_0^r \varphi(x)dx = \frac{rd}{2} + c.$$

The transition function of the SYZ mirror bundle is given by

$$g_{UV}(z) = \left(\begin{array}{c|c} O & e^{-\pi dr}e^{-2\pi c}e^{2\pi i dz} \\ \hline I_{(r-1)\times(r-1)} & O \end{array}\right).$$

Let

$$L_1 := L_{r_1,d_1}[c_1] := \{(e^{2\pi i r_1 x}, e^{2\pi i(d_1 x + c_1)}) \in X : x \in \mathbb{R}\},$$
$$L_2 := L_{r_2,d_2}[c_2] := \{(e^{2\pi i r_2 x}, e^{2\pi i(d_2 x + c_2)}) \in X : x \in \mathbb{R}\}$$

be two distinct (embedded) Lagrangian multi-sections of X, where r_1, r_2, d_1, d_2 are integers such that $\gcd(r_1, d_1) = \gcd(r_2, d_2) = 1$ and $a_1, a_2 \in \mathbb{R}$. They intersect at $|r_1 d_2 - r_2 d_1|$ points. Let's assume $r_1 d_2 > r_2 d_1$. The base coordinate of the intersection points are given by equivalence classes of

$$x_{k,k'} := \frac{r_2(c_1 + k) - r_1(c_2 + k')}{r_1 d_2 - r_2 d_1}, \quad k, k' \in \mathbb{Z}.$$

We orientate L_1, L_2 such that both of them are pointing towards "right" in a fundamental domain of X. Since $r_1 d_2 > r_2 d_1$, using the degree convention of [2], all generators of $CF(L_1, L_2)$ is of index 1.

Let K be a subset of $L_1 \cap L_2$. Then we can perform Lagrangian surgery at each point in K to obtain a (graded) Lagrangian multi-section $\mathbb{L}_K := L_2 \sharp_K L_1$ (possibly with disconnected domain).

Remark 3.2. *For each surgery point, we have a parameter $\epsilon > 0$ which controls the size of the surgery. The surgery $\mathbb{L}_K$ we discuss here of course consists of the surgery parameters. However, these parameters do not play a role as we will see in the proof of our main theorem (Theorem 3.4).*

Now, we equip the domain of $L_1, L_2, \mathbb{L}_K$ by the $U(1)$-local system

$$\mathcal{L}_{b_1} : d + 2\pi i \frac{b_1}{r_1} dx,$$
$$\mathcal{L}_{b_2} : d + 2\pi i \frac{b_2}{r_2} dx,$$
$$\mathcal{L}_{b} : d + 2\pi i \frac{b}{r_1 + r_2} dx,$$

respectively. Here $b_1, b_2, b \in \mathbb{R}$. We denote the Lagrangian A-branes $(L_1, \mathcal{L}_{b_1}), (L_2, \mathcal{L}_{b_2})$, $(\mathbb{L}_K, \mathcal{L}_b)$ by $L_{1,b_1}, L_{2,b_2}, \mathbb{L}_{K,b}$, respectively. There is a simple obstruction for the SYZ mirror bundle $\check{\mathbb{L}}_{K,b}$ of $\mathbb{L}_{K,b}$ being an extension of $\check{L}_{1,b_1}$ and $\check{L}_{2,b_2}$. Note that if $\check{\mathbb{L}}_{K,b}$ is an extension of $\check{L}_{1,b_1}$ by $\check{L}_{2,b_2}$, then $\det(\check{\mathbb{L}}_{K,b}) \cong \det(\check{L}_{1,b_1}) \otimes \det(\check{L}_{2,b_2})$ as holomorphic line bundles.

Proposition 3.3. *Let $(L_{r_1,d_1}[c_1], \mathcal{L}_1)$ and $(L_{r_2,d_2}[c_2], \mathcal{L}_2)$ be Lagrangian A-branes with local system*

$$\mathcal{L}_{b_1} : d + 2\pi i \frac{b_1}{r_1} dx, \quad \mathcal{L}_{b_2} : d + 2\pi i \frac{b_2}{r_2} dx.$$

Let $\mathbb{L}'_1, \ldots, \mathbb{L}'_M$ be the components of an immersed Lagrangian multi-section $\mathbb{L}$ of rank $r_1 + r_2$ and degree $-d_1 - d_2$, equipped with $U(1)$-local systems

$$\mathcal{L}'_1 : d + 2\pi i \frac{b'_1}{r'_1} dx, \ldots, \mathcal{L}'_M : d + 2\pi i \frac{b'_M}{r'_M} dx.$$

Let $\varphi'_j : \mathbb{R} \to \mathbb{R}$ be the defining equation of $\mathbb{L}'_j$. Put

$$a'_j = \int_0^{r'_j} \varphi'_j(x) dx, \quad j = 1, \ldots, M,$$

Then $\det(\check{\mathbb{L}}_{b'}) \cong \det(\check{L}_{r_1,d_1}[c_1]_{b_1}) \otimes \det(\check{L}_{r_2,d_2}[c_2]_{b_2})$ *as holomorphic line bundles if and only if*

$$\sum_{j=1}^{M} a'_j - \frac{r_1 d_1}{2} - \frac{r_2 d_2}{2} - c_1 - c_2 \in \mathbb{Z} \textit{ and } b_1 + b_2 - \frac{M}{2} - \sum_{j=1}^{M} b'_j \in \mathbb{Z}.$$

Proof. Note that

$$\sum_{j=1}^{M} r'_j = r_1 + r_2, \quad \sum_{j=1}^{M} d'_j = d_1 + d_2.$$

The factor of automorphy [25] of $\det(\mathbb{L}) \otimes \det(\check{L}_1)^{-1} \otimes \det(\check{L}_2)^{-1}$ is generated by

$$A(1,u) = (-1)^M e^{-2\pi(\sum_{j=1}^{M} a'_j - \frac{r_1 d_1}{2} - \frac{r_2 d_2}{2} - c_1 - c_2)} e^{2\pi i(\sum_{j=1}^{M} b'_j + \frac{M}{2} - b_1 - b_2)},$$

where $u = e^{2\pi i z}$. Let $q = e^{2\pi i \tau}$, $\tau = i$. Then $A(1,u)$ is gauge equivalent to 1 if and only if

$$B(qu) = A(1,u)B(u) \tag{1}$$

for some holomorphic function $B : \mathbb{C}^\times \to \mathbb{C}^\times$. Let $\sum_{-\infty}^{\infty} B_k u^k$ be the Laurent series expansion of B. Then (1) holds if and only if

$$B_k(q^k - A(1,u)) = 0, \text{ for all } k.$$

Since $B(u)$ is nonzero, (1) holds if and only if $A(1,u) = q^N$ for some integer N and $B_k = 0$ for all $k \neq N$, which is equivalent to say M is even,

$$\sum_{j=1}^{M} a_j - \frac{r_1 d_1}{2} - \frac{r_2 d_2}{2} - c_1 - c_2 = N \in \mathbb{Z} \text{ and } b_1 + b_2 - \frac{M}{2} - \sum_{j=1}^{M} b'_j \in \mathbb{Z}.$$

□

We refer the conditions

$$\sum_{j=1}^{M} a_j - \frac{r_1 d_1}{2} - \frac{r_2 d_2}{2} - c_1 - c_2 \in \mathbb{Z} \text{ and } b_1 + b_2 - \frac{M}{2} - \sum_{j=1}^{M} b'_j \in \mathbb{Z}$$

as the first and second integrality condition for the triple $(L_{r_1,d_1}[c_1]_{b_1}, L_{r_2,d_2}[c_2]_{b_2}, \mathbb{L}_{b'})$, respectively.

Proposition 3.3 gives a necessary condition for the surgery-extension correspondence to hold. Next, we prove that under certain assumptions on the surgery $\mathbb{L}_K$, the second integrality condition is also sufficient.

Theorem 3.4. *Let* r_1, d_1, r_2, d_2 *be integers satisfying* $r_1 d_2 > r_2 d_1$ *and*

$$gcd(r_1, d_1) = gcd(r_2, d_2) = gcd(r_1 + r_2, d_1 + d_2) = 1.$$

Let $L_1 := L_{r_1,d_1}[c_1]$, $L_2 := L_{r_2,d_2}[c_2]$ *and* $K \subset L_1 \cap L_2$ *such that the (graded) Lagrangian surgery* $\mathbb{L}_K := L_2 \sharp_K L_1$ *has connected domain and equipped with the* $U(1)$*-local system*

$$\mathcal{L}_b : d + 2\pi i \frac{b}{r_1 + r_2} dx, \quad b \in \mathbb{R}.$$

Then the SYZ mirror bundle $\check{\mathbb{L}}_{K,b}$ *of the Lagrangian A-brane* $\mathbb{L}_{K,b}$ *is an extension of* $\check{L}_{1,b_1}$ *by* $\check{L}_{2,b_2}$ *if and only if* b *satisfies*

$$b_1 + b_2 - b - \frac{1}{2} \in \mathbb{Z}.$$

Proof. Since the domains of L_1, L_2 and $\mathbb{L}_K$ have connected domain and $\gcd(r_1, d_1) = \gcd(r_2, d_2) = \gcd(r_1 + r_2, d_1 + d_2) = 1$, $\check{L}_1, \check{L}_2, \check{\mathbb{L}}_K$ and $\check{L}_{1,b_1}, \check{L}_{2,b_2}, \check{\mathbb{L}}_{K,b}$ are indecomposable (stable in fact) bundles.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the defining equation of $\mathbb{L}_K$ such that $\varphi(0) = a_1/r_1$. We are going to prove (after the proof of this theorem) that

$$\int_0^{r_1+r_2} \varphi(x)dx = \frac{r_1 d_1}{2} + \frac{r_2 d_2}{2} + c_1 + c_2 + N, \text{ for some } N \in \mathbb{Z}.$$

This is the first integrality condition.

Now, we prove that $\check{\mathbb{L}}_{K,b}$ fits into the exact sequence. By the second integrality assumption and the integral formula (2) imply

$$\det(\check{\mathbb{L}}_{K,b}) \cong \det(\check{L}_{1,b_1}) \otimes \det(\check{L}_{2,b_2}).$$

Next, we need the following

Lemma 3.5. *(= Lemma 1.4 + Proposition 2.3 in* [7]*.) Let F, G be polystable vector bundles over an elliptic curve X with $rk(F) \geq rk(G)$ and $\mu(F) < \mu(G)$. Assume that no two among the indecomposable factors of F (resp. of G) are isomorphic. Define*

$$U := \{f \in Hom(F, G) : rk(Im(f)) = t, \deg(Im(F)) = h\},$$
$$t := \max_{f \in Hom(F,G)} rk(Im(f)),$$
$$h := \max_{f \in Hom(F,G), rk(Im(f))=t} \deg(Im(F)).$$

Then U is an open dense subset of $Hom(F, G)$. Moreover, if $rk(F) > rk(G)$, then each $f \in U$ is surjective.

Since $r_1 d_2 > r_2 d_1$, we have

$$\mu(\check{\mathbb{L}}_{K,b}) = -\frac{d_1 + d_2}{r_1 + r_2} < -d_1 \frac{1 + r_2/r_1}{r_1 + r_2} = -\frac{d_1}{r_1} = \mu(\check{L}_{1,b_1}).$$

Also, $\check{\mathbb{L}}_{K,b}$ and $\check{L}_{1,b_1}$ are stable by the gcd assumptions. Hence we can apply the lemma to find a surjective $f : \check{\mathbb{L}}_{K,b} \to \check{L}_{1,b_1}$. Let $\mathbb{K} := \ker(f)$. Then we have the exact sequence

$$0 \to \mathbb{K} \to \check{\mathbb{L}}_{K,b} \to \check{L}_{1,b_1} \to 0.$$

Since every vector bundle over an elliptic curve is the flat limit of a family of semi-stable bundles with same determinant, we can choose $\mathbb{K}(t)$ and $\check{L}_{1,b_1}(t)$ such that $\mathbb{K}(0) = \mathbb{K}$ and $\check{L}_{1,b_1}(0) = \check{L}_{1,b_1}$. By the classification result of Atiyah [6], any indecomposable vector bundle on an elliptic curve with gcd(rk,deg) = 1 is determined by its determinant line bundle. Hence we have $\check{L}_{1,b_1}(t) \cong \check{L}_{1,b_1}$ for all t by the openness of semi-stability (hence indecomposable). Since $\check{\mathbb{L}}_{K,b}$ is stable, we have $Hom(\check{L}_{1,b_1}, \mathbb{K}) = 0$ (see [31] Lemma 1.1). Then we can apply

Lemma 3.6. *(= Corollary 1.3 in* [7]*) Fix a flat family $\{X(t) : t \in T\}$ of smooth compact Riemann surfaces with T integral. Let H, Q be vector bundles on $X = X(0)$ such that $Hom(Q(0), H(0)) = 0$. Then any extension*

$$0 \to H(0) \to E \to Q(0) \to 0$$

is the limit of a flat family of extensions

$$0 \to H(t) \to E(t) \to Q(t) \to 0$$

with $H(t)$ and $Q(t)$ semi-stable and t in some open subset of T containing 0.

to obtain an exact sequence

$$0 \to \mathbb{K}(t) \to \check{\mathbb{L}}_{K,b}(t) \to \check{L}_{1,b_1}(t) \to 0 \tag{2}$$

with $\check{\mathbb{L}}_{K,b}(0) = \check{\mathbb{L}}_{K,b}$. Since $\check{\mathbb{L}}_{K,b}$ is stable, by openness of semistability, $\check{\mathbb{L}}_{K,b}(t)$ is semi-stable for all t near 0. But

$$\det(\check{\mathbb{L}}_{K,b}(t)) \cong \det(\mathbb{K}(t)) \otimes \det(\check{L}_{1,b_1}(t)) = \det(\mathbb{K}) \otimes \det(\check{L}_{1,b_1}) \cong \det(\check{\mathbb{L}}_{K,b}),$$

we have $\check{\mathbb{L}}_{K,b}(t) \cong \check{\mathbb{L}}_{K,b}$ for all t near 0. Note that we have

$$\det(\mathbb{K}(t)) = \det(\mathbb{K}) \cong \det(\check{\mathbb{L}}_{K,b}) \otimes \det(\check{L}_{1,b_1})^{-1} \cong \det(\check{L}_{2,b_2}),$$

and $\mathbb{K}(t)$ is semi-stable, we have $\mathbb{K}(t) \cong \check{L}_{2,b_2}$. Therefore, for $t \neq 0$, the exact sequence (2) reads

$$0 \to \check{L}_{2,b_2} \to \check{\mathbb{L}}_{K,b} \to \check{L}_{1,b_1} \to 0.$$

This completes the proof of the theorem. □

Next we prove the integral formula that we need in the proof of Theorem 3.4.

Lemma 3.7. *Let $\mathbb{L}_K$ be as in Theorem 3.4 Then*

$$\int_0^{r_1+r_2} \varphi(x)dx = \frac{r_1 d_1}{2} + \frac{r_2 d_2}{2} + c_1 + c_2 + N, \text{ for some } N \in \mathbb{Z},$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is the defining equation of $\mathbb{L}_K$.

Proof. By adding a sufficiently larger integer, we may assume $\varphi \geq 0$ on the interval $[0, r_1 + r_2]$.

Observe that

$$\int_0^{r_1+r_2} \varphi(x)dx,$$

nothing but the area bounded by φ and the x-axis, from 0 to $r_1 + r_2$. Since Lagrangian surgery is symmetric, the integral is the same as the area bounded by the piecewise linear function, obtained by replacing the non-linear portions by piecewise linear functions, with the x-axis, from 0 to $r_1 + r_2$.

We cut the area into several pieces as in Figure 3 .

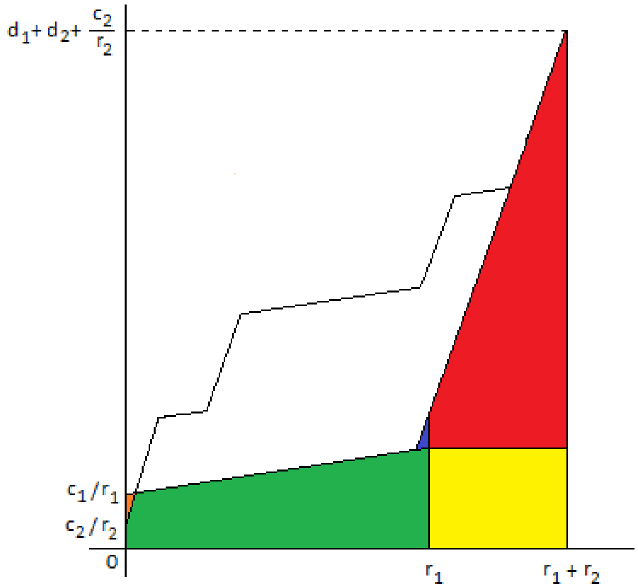


Figure 3

Denote the area of the red, blue, yellow, green, orange and white region by R, B, Y, G, O and W, respectively. Then we have

$$\int_0^{r_1+r_2} \varphi(x)dx = R + B + Y + G + W.$$

Note that we have the following

$$\begin{aligned} R &= \int_0^{r_1} \left(\frac{c_2}{r_2}x + \frac{c_2}{r_2} - \frac{c_1}{r_1} \right) dx = \frac{r_2 d_2}{2} + c_2 - \frac{c_1 r_2}{r_1}, \\ Y &= (d_1 + \frac{c_1}{r_1})r_2 = d_1 r_2 + \frac{c_1 r_2}{r_1}, \\ O + G &= \int_0^{r_1} \left(\frac{d_1}{r_1}x + \frac{c_1}{r_1} \right) dx = \frac{r_1 d_1}{2} + c_1, \\ B &= O. \end{aligned}$$

Hence

$$\int_0^{r_1+r_2} \varphi(x)dx = \frac{r_1 d_1}{2} + c_1 + \frac{r_2 d_2}{2} + c_2 + d_1 r_2 + W.$$

To see that W is an integer, take a look at Figure 4.

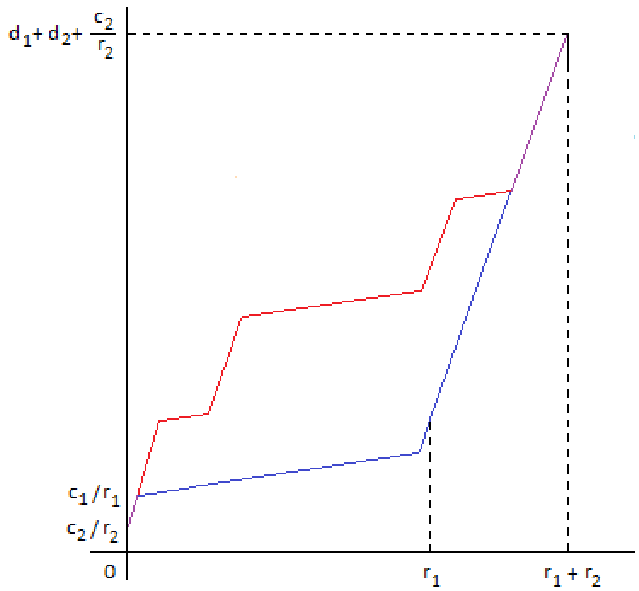


FIGURE 4

The red and blue lines can be viewed as two continuous maps $f_j : [0,1] \to X$, $j = 1, 2$ with the same image $L_1 \cup L_2$. They define the same homology class in $H_1(X;\mathbb{Z})$, namely, $(r_1 + r_2, d_1 + d_2) \in \mathbb{Z}^2 \cong H_1(X;\mathbb{Z})$. The white region serves as a 2-chain Δ such that

$$\partial\Delta = f_1 - f_2.$$

But f_1, f_2 also define the same image, Δ is indeed a 2-cycle, that is, $[\Delta] \in H_2(X;\mathbb{Z})$. Pulling back the symplectic form to $\mathbb{R}^2$, we then have $\omega = d(ydx)$. Let $\widetilde{f_1}, \widetilde{f_2} : [0,1] \to \mathbb{R}^2$ denote the lift of f_1, f_2 starting at c_2/r_2, respectively. Then

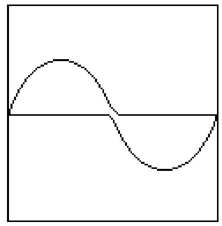
$$W = \int_{\widetilde{f_1} - \widetilde{f_2}} ydx = \int_{\Delta} \omega \in \mathbb{Z},$$

as $[\omega]$ is an integral class. This completes the proof of the lemma. □

The surgery-extension correspondence is not true for self-extension. For example, let L_0 be the zero section and L_0' be the Lagrangian

$$\{(e^{2\pi ix}, e^{2\pi i \sin(2\pi ix)}) : x \in \mathbb{R}\},$$

which is Hamiltonian equivalent to L_0. Hence both L_0 and L_0' have $\mathcal{O}_{\check{X}}$, the trivial line bundle, as the SYZ mirror bundle. Performing a Lagrangian surgery at the index 1 intersection point p to obtain an immersed Lagrangian multi-section $\mathbb{L}_p$ of rank 2. See Figure 5.



$\mathbb{L}_p$

FIGURE 5

For any choice of local system $d + 2\pi i \frac{b}{2} dx$ on the domain of $\mathbb{L}_p$, the transition function of $\check{\mathbb{L}}_{p,b}$ is given by

$$\begin{pmatrix} 0 & e^{2\pi i b} \\ 1 & 0 \end{pmatrix}$$

Since it is a constant matrix, it is gauge equivalent to its diagonalization:

$$\begin{pmatrix} e^{i\pi b} & 0 \\ 0 & -e^{i\pi b} \end{pmatrix}.$$

If $\mathbb{L}_{p,b}$ is a self-extension of $\mathcal{O}_{\check{X}}$, then we have $e^{2i\pi b} = -1$, which is equivalent to $b \in \frac{1}{2} + \mathbb{Z}$. In this case, it is easy to see that $\check{\mathbb{L}}_{p,b}$ is isomorphic to a non-trivial decomposable holomorphic vector bundle of rank 2. However, it is known that the only self-extensions of $\mathcal{O}_{\check{X}}$ are $\mathcal{O}_{\check{X}}^{\oplus 2}$ and the Atiyah bundle $\check{A}_2$, which is indecomposable. Hence $\check{\mathbb{L}}_{p,b}$ cannot be a self-extensions of $\mathcal{O}_{\check{X}}$ for any choice of b. So we expect $\mathbb{L}_{p,b}$ is not a mapping cone of $p : L_0 \to L_0'$. This does not violate the result of Abouzaid [2], because in there, he required two curves intersecting minimally within their isotopy class. In our example, L_0' does not intersect L_0 minimally within its isotopy class (the minimal intersection is empty).

Remark 3.8. *As we have mentioned in the introduction, the integrality condition*

$$b_1 + b_2 - b - \frac{1}{2} \in \mathbb{Z}$$

suggests that given two Lagrangian A-branes $(\mathbb{L}_1, \mathcal{L}_1), (\mathbb{L}_2, \mathcal{L}_2)$ *in a symplectic manifold* (M, ω)*, if* $\mathcal{L}_1$*,* $\mathcal{L}_2$ *have holonomy* $e^{2\pi i b_1}$*,* $e^{2\pi i b_2}$*, respectively, then the holonomy* $e^{2\pi i b}$ *of* $\mathcal{L}$ *should be chosen to satisfy*

$$e^{2\pi i b} e^{-2\pi i b_1} e^{-2\pi i b_2} = -1.$$

We believe that this relation can be understood Floer-theoretically.

4. INVARIANCE OF IMMERSED FLOER COHOMOLOGY

As we have seen in the introduction, the surgery-extension correspondence theorem (Theorem 3.4) gives us to a pair of non-Hamiltonian equivalent Lagrangian immersions that share the same SYZ mirror bundle. By homological mirror symmetry, we expect the two Lagrangian immersions should be equivalent in the immersed Fukaya category. Coincidentally, it was pointed out by Akaho and Joyce in their work [4] on immersed Floer theory that the immersed Floer cohomology should have invariance property under some equivalence that is weaker than global Hamiltonian equivalence. In this section, we will define a new notion called lifted Hamiltonian equivalence and prove that the immersed Floer cohomology is invariant under this new equivalence. So, let's forget about mirror symmetry for a moment and study symplectic geometry now.

Throughout this section, the notation (M, ω) will stand for a $2n$-dimensional compact symplectic manifold with symplectic form ω. The notion of Lagrangian immersions is defined as in Definition 2.2. We always assume the domain of a Lagrangian immersion is compact. Let's recall the definition of immersed Floer cohomology for a pair of Lagrangian immersions, which was introduced in by Akaho and Joyce in [4].

4.1. **Maslov index and the immersed Floer cohomology.** To have good Floer theory, Akaho and Joyce made the following assumption:

A) The intersection points of $\xi_1(L_1)$ and $\xi_2(L_2)$ are finite and do not coincide with their self-intersection points.

The Floer complex of two immersed Lagrangians $\mathbb{L}_1 = (L_1, \xi_1)$ and $\mathbb{L}_2 = (L_2, \xi_2)$ is defined by

$$CF(\mathbb{L}_1, \mathbb{L}_2) := \bigoplus_{p \in \xi_1(L_1) \cap \xi_2(L_2)} \Lambda_{nov} \cdot p,$$

where Λ_{nov} is the Novikov field given by

$$\Lambda_{nov} := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} : a_i \in k, \lim_{i \to \infty} \lambda_i = \infty \right\}$$

and k is a field ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_2$). Let $\{J_t\}_{t \in [0,1]}$ be a family of almost complex structure on M which are compatible with ω. Let $g_t(\cdot, \cdot) := \omega(J_t \cdot, \cdot)$ be the Riemannian metric associated to the pair (ω, J_t). The Floer differential m_1 is defined via counting J_t-holomorphic strips $u : \mathbb{R} \times [0,1] \to M$:

$$\frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} = 0$$

with finite energy

$$\int_{\mathbb{R} \times [0,1]} |du(s,t)|_{g_t} ds dt < +\infty$$

and boundary data

$$u(\mathbb{R} \times \{0\}) \subset \xi_1(L_1), \quad u(\mathbb{R} \times \{1\}) \subset \xi_2(L_2),$$

$$\lim_{s \to -\infty} u(s,t) = q, \quad \lim_{s \to +\infty} u(s,t) = p.$$

Since $\mathbb{L}_1, \mathbb{L}_2$ are immersed, they require that there are continuous liftings $u_1^- : \mathbb{R} \times \{0\} \to L_1$, $u_2^+ : \mathbb{R} \times \{1\} \to L_2$ such that $\xi_1 \circ u_1^- = u|_{\mathbb{R} \times \{0\}}$ and $\xi_2 \circ u_2^+ = u|_{\mathbb{R} \times \{1\}}$ (see Figure 6).

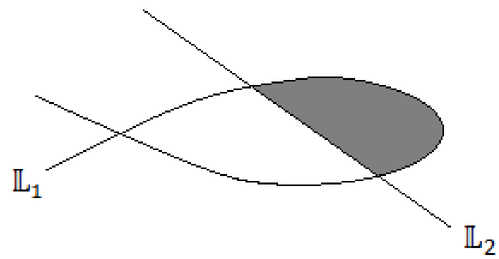


FIGURE 6. A J_t-holomorphic strip bounded by $\mathbb{L}_1$ and $\mathbb{L}_2$.

Let $\pi_2(M; L_1, L_2; p, q)$ be the space of all homotopy classes of strips $u : \mathbb{R} \times [0, 1] \to M$ that satisfy the above boundary conditions and lifting properties. Fix a homotopy class $\beta \in \pi_2(M; L_1, L_2; p, q)$. Let $\widetilde{\mathcal{M}}(p, q; \beta)$ be the moduli space of all J_t-holomorphic disks which represent the class β and satisfying the above boundary data. Quotienting the s-direction by translation, we obtain the moduli space

$$\mathcal{M}(p, q; \beta) := \widetilde{\mathcal{M}}(p, q; \beta)/\mathbb{R}.$$

In [22, 23], the authors proved that $\mathcal{M}(p, q; \beta)$ is a Kuranishi space and can be compactified. Moreover, if the Lagrangian immersions are relatively spin, then the compactified moduli space can be oriented. To describ the dimension of the moduli space, one need to introduce the Maslov index. Choose a symplectic trivialization $\Phi : u^*TM \cong (\mathbb{R} \times [0, 1]) \times T_pM$. Consider the Lagrangian paths

$$\gamma_1^- : s \mapsto \Phi(d\xi_1(T_{u_1^-(-s,0)}L_1)) \subset T_pM,$$
$$\gamma_2^+ : s \mapsto \Phi(d\xi_2(T_{u_2^+(s,1)}L_2)) \subset T_pM$$

in the Lagrangian Grassmanian $LGr(T_pM, \omega_p)$. Since p is not a self-intersection point for both $\mathbb{L}_1, \mathbb{L}_2$, it has unique preimage point $l_1 \in L_1$ and $l_2 \in L_2$. We identify (T_pM, ω_p) with $(\mathbb{C}^n, \omega_{std})$. There exists $A \in Sp(2n, \mathbb{R})$ such that

$$A(d\xi_1(T_{l_1}L_1)) = \mathbb{R}^n, \quad A(d\xi_2(T_{l_2}L_2)) = i\mathbb{R}^n.$$

The canonical short path $\lambda_p : [0, 1] \to LGr(T_pM, \omega_p)$ is defined to be

$$\lambda_p(t) := A^{-1}(e^{-\frac{\pi i}{2}t}(i\mathbb{R}^n)).$$

Then by concatenating the paths, $\gamma := \lambda_p^{-1} * \gamma_1^- * \Phi(\lambda_q) * \gamma_2^+$ defines a loop in $LGr(T_pM, \omega_p)$, based at $d\xi_1(T_{l_1}L_1)$. Recall that $\pi_1(LGr(T_pM, \omega_p)) \cong \mathbb{Z}$.

Definition 4.1. *The* Maslov index $\mu(u)$ *of the strip* $u : \mathbb{R} \times [0, 1] \to M$ *is defined to be the degree of the loop* $\gamma \subset LGr(T_pM, \omega_p)$.

It is a well-known fact that the Maslov index only depends on the homotopy class of the strip $u : \mathbb{R} \times [0, 1] \to M$. The (virtual) dimension of $\mathcal{M}(p, q; \beta)$ is given by $\mu(\beta) - 1$.

Let $\mathcal{M}^0(p, q; \beta)$ be the 0-dimensional component. The Floer differential $m_1 : CF(\mathbb{L}_1, \mathbb{L}_2) \to CF(\mathbb{L}_1, \mathbb{L}_2)$ is defined by

$$m_1(p) := \sum_{q \in \xi_1(L_1) \cap \xi_2(L_2)} \sum_{\beta : \mu(\beta) = 1} \sum_{u \in \mathcal{M}^0(p, q; \beta)} (-1)^{sign(u)} T^{\omega(u)} \cdot q,$$

where the sign $(-1)^{sing(u)}$ is determined by the orientation of the moduli space $\mathcal{M}^0(p,q;\beta)$ and

$$\omega(u) := \int_{\mathbb{R}\times[0,1]} u^*\omega$$

is the symplectic area of u. By Gromov compactness, the sum converges in Λ_{nov}.

Definition 4.2. *If $m_1^2 = 0$, then the* immersed Floer cohomology *is defined to be*

$$HF(\mathbb{L}_1, \mathbb{L}_2) := H(CF(\mathbb{L}_1, \mathbb{L}_2), m_1).$$

We assume all Lagrangian immersions we consider here are unobstructed, that is, the Floer differential m_1 satisfies $(m_1)^2 = 0$.

Remark 4.3. *Usually, the notion of unobstructed Lagrangian immersion involves a bounding cochain b on the domain of the immersion. In this paper, we will consider those Lagrangian immersions with $b = 0$.*

It is well known that the two limits

$$\lim_{s\to-\infty} u(s,t) = q, \quad \lim_{s\to+\infty} u(s,t) = p$$

converge uniformly in $t \in [0,1]$. Indeed, one has

$$dist(u(s,t),p) < Ce^{-\mu|s|}, \text{ for all } t \in [0,1],$$

where $C, \mu > 0$ are constants depend only on the energy $E(u)$ of u. By identifying $\mathbb{R} \times [0,1]$ with the closed unit disk Δ with punctures at ±1, the limits $\lim_{z\to-1} u(z), \lim_{z\to+1} u(z)$ exist and equal to q, p, respectively. Therefore, it makes sense to write $u(-1) = q$ and $u(1) = p$.

From now on, we replace the strip model by the disk model with finite energy and boundary data

$$u(\partial^-\Delta) \subset \xi_1(L_1), \quad u(\partial^+\Delta) \subset \xi_2(L_2),$$

$$u(-1) = q, \quad u(1) = p.$$

Here, we put $\partial^-\Delta = S^1 \cap \{z \in \mathbb{C} : Im(z) \leq 0\}$ and $\partial^+\Delta = S^1 \cap \{z \in \mathbb{C} : Im(z) \geq 0\}$.

4.2. **Three types of equivalences.** First, we recall the notion of global Hamiltonian equivalence introduced in [4].

Definition 4.4. *Let (L_1, ξ_2), (L_2, ξ_2) be two Lagrangian immersions in a symplectic manifold (M, ω). They are said to be* globally Hamiltonian equivalent *if there exists a diffeomorphism $\phi : L_1 \to L_2$ and a 1-parameter family of Hamiltonian diffeomorphism $\psi_t : M \to M$ such that $\psi_0 = id_M$ and $\psi_1 \circ \xi_1 = \xi_2 \circ \phi$.*

Remark 4.5. *Usually, Hamiltonian equivalence for a pair of embedded Lagrangian submanifolds (L_1, ξ_1), (L_2, ξ_2) is defined as follows: there exists a 1-parameter family of Hamiltonian diffeomorphism $\psi_t : M \to M$ such that $\psi_0 = id_M$ and $\psi_1(\xi_1(L_1)) = \xi_2(L_2)$. This definition is clearly equivalent to Definition 4.4. Indeed, by restricting ψ_1 to $\xi_1(L_1)$, which induces a diffeomorphism $\phi := \xi_2^{-1} \circ \psi_1 \circ \xi_1 : L_1 \to L_2$ such that*

$$\psi_1 \circ \xi_1 = \xi_2 \circ \phi.$$

However, in the immersed situation, these two notions can be different.

Akaho and Joyce proved that $HF(\mathbb{L}_1, \mathbb{L}_2)$ is indeed a global Hamiltonian invariant, that is, if $\mathbb{L}_2$ is globally Hamiltonian isotopic to $\mathbb{L}_2'$, then there is a quasi-isomorphism

$$(CF(\mathbb{L}_1, \mathbb{L}_2), m_1) \simeq (CF(\mathbb{L}_1, \mathbb{L}_2'), m_1').$$

In the immersed situation, there is another equivalence called local Hamiltonian equivalence. Let me recall its definition.

Definition 4.6. *Let (L_1, ξ_2), (L_2, ξ_2) be two Lagrangian immersions in a symplectic manifold (M, ω). They are said to be* locally Hamiltonian equivalent *if there exists a diffeomorphism $\phi : L_1 \to L_2$ and a smooth 1-parameter family $\Xi : [0,1] \times L_1 \to M$ such that $\Xi(0, -) = \xi_1$ and $\Xi(1, -) = \xi_2 \circ \phi$ and the 1-form*

$$\Xi^*(\omega)(\frac{d}{dt}, -)$$

on $\{t\} \times L_1$, is exact for all $t \in [0,1]$.

Note that local Hamiltonian equivalence can be implied by global Hamiltonian equivalence by pulling back the Hamiltonian function on M to the domain L via the immersion ξ but not the converse in general, as shown by the example below.

Example 4.7. *Consider the Lagrangian immersions $\mathbb{L}, \mathbb{L}'$ (Figure 7) in the standard symplectic 2-torus T^2.*

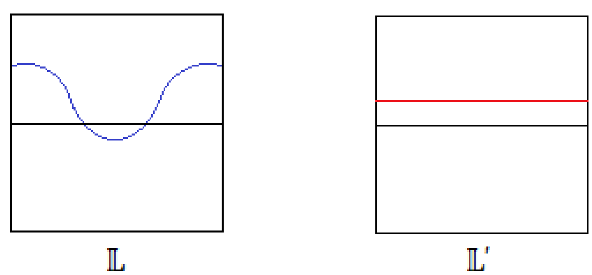


FIGURE 7

Clearly, $\mathbb{L}$ and $\mathbb{L}'$ are not Hamiltonian equivalent as they share different number of self-intersection points. Nevertheless, the blue curve can be Hamiltonian deformed into a horizontal Lagrangian section, namely, the red line. Hence $\mathbb{L}$ and $\mathbb{L}'$ are in fact locally Hamiltonian equivalent.

A natural question is that, does $HF(\mathbb{L}_1, \mathbb{L}_2)$ invariant under local Hamiltonian equivalence? It was pointed out by Akaho and Joyce that this not true for general local Hamiltonian isotopies (See Section 13 of [4]). The reason behind is the Lagrangian h-principle [24, 28], which states that two Lagrangian immersions (L, ξ_1), (L, ξ_2) are locally Hamiltonian equivalent if and only if there exists a smooth homotopy $\xi_t : L \to M$ from $(\xi_1, d\xi_1)$ to $(\xi_2, d\xi_2)$ and a bundle map $\widetilde{\xi_t} : TL \to \xi_t^* TM$ covering ξ_t which embeds TL as a Lagrangian subbundle in $\xi_t^* TM$. But $HF(\mathbb{L}_1, \mathbb{L}_2)$ consists of quantum data coming from holomorphic disks, which is invisible to classical algebraic topology, so one would not expect these quantum data can be preserved under general local Hamiltonian isotopies. Hence our goal is to find a new equivalence which is weaker than global Hamiltonian equivalence, but stronger than local Hamiltonian equivalence, such that $HF(\mathbb{L}_1, \mathbb{L}_2)$ is invariant under this equivalence. Let us introduce the following

Definition 4.8. *Let $\pi : \widetilde{M} \to M$ be a finite unramified covering of a symplectic manifold (M, ω). For two Lagrangian immersions $\mathbb{L}_1 = (L_1, \xi_1), \mathbb{L}_2 = (L_2, \xi_2)$ of M, we say $\mathbb{L}_1$ is* $(\widetilde{M}, \pi)$-lifted Hamiltonian isotopic *to $\mathbb{L}_2$ if there exists a diffeomorphism $\phi : L_1 \to L_2$ and Lagrangian immersions $\widetilde{\xi}_1 : L_1 \to \widetilde{M}$, $\widetilde{\xi}_2 : L_2 \to \widetilde{M}$ such that $\xi_1 = \pi \circ \widetilde{\xi}_1$, $\xi_2 = \pi \circ \widetilde{\xi}_2$ and $(L_1, \widetilde{\xi}_1)$ is globally Hamiltonian isotopic to $(L_1, \widetilde{\xi}_2 \circ \phi)$ in $(\widetilde{M}, \pi^*\omega)$.*

We remark that L_1, L_2 and $\widetilde{M}$ can all be disconnected. When $\mathbb{L}_1$ is $(\widetilde{M}, \pi)$-lifted Hamiltonian isotopic to $\mathbb{L}_2$, we may assume the immersions share the same domain, that is, $L_1 = L_2$. In this case we may take ϕ to be the identity map.

We also make the following

Definition 4.9. *Let (M, ω) be a symplectic manifold. For two Lagrangian immersions $\mathbb{L}_1 = (L_1, \xi_1), \mathbb{L}_2 = (L_2, \xi_2)$ of M, we say $\mathbb{L}_1$ is* lifted Hamiltonian isotopic *to $\mathbb{L}_2$ if there exists an integer $l > 0$ and Lagrangian immersions $\mathbb{L}^{(1)} := \mathbb{L}_1, \mathbb{L}^{(2)}, \dots, \mathbb{L}^{(l-1)}, \mathbb{L}^{(l)} := \mathbb{L}_2$ of M, such that $\mathbb{L}^{(j)}$ is $(\widetilde{M}_j, \pi_j)$-lifted Hamiltonian isotopic to $\mathbb{L}^{(j+1)}$, for some finite unramified covering $\pi_j : \widetilde{M}_j \to M$, $j = 1, \dots, l-1$.*

Clearly, $(\widetilde{M}, \pi)$-lifted Hamiltonian isotopy (resp. lifted Hamiltonian isotopy) defines an equivalence relation on the set of Lagrangian immersions. Hence it makes sense to say "$\mathbb{L}_1$ is $(\widetilde{M}, \pi)$-lifted Hamiltonian equivalent (resp. lifted Hamiltonian equivalent) to $\mathbb{L}_2$". Note that two Lagrangian immersions $\mathbb{L}_1, \mathbb{L}_2$ are globally Hamiltonian equivalent if and only if they are (M, id_M)-lifted Hamiltonian equivalent.

Proposition 4.10. *If $\mathbb{L}_1$ and $\mathbb{L}_2$ are Lagrangian immersions which are lifted Hamiltonian equivalent, then $\mathbb{L}_1$ and $\mathbb{L}_2$ are locally Hamiltonian equivalent.*

Proof. It suffices to prove that if $\mathbb{L}_1$ and $\mathbb{L}_2$ are $(\widetilde{M}, \pi)$-lifted Hamiltonian equivalent for some finite unramified covering $\pi : \widetilde{M} \to M$, then they are locally Hamiltonian equivalent. Let $\widetilde{\xi}_1, \widetilde{\xi}_2 : L \to \widetilde{M}$ be lifts of ξ_1, ξ_2, respectively. By assumption, there exists a family of Hamiltonian diffeomorphisms $\widetilde{\Xi}_t : \widetilde{M} \to \widetilde{M}$ such that $\widetilde{\Xi}_0 = id$ and $\widetilde{\Xi}_1 \circ \widetilde{\xi}_1 = \widetilde{\xi}_2$. Since $\pi : \widetilde{M} \to M$ is an unramified covering, $\Xi_t := \pi \circ \widetilde{\Xi}_t \circ \widetilde{\xi}_1 : L \to M$ defines a family of Lagrangian immersions such that

$$\Xi_0 = \pi \circ \widetilde{\Xi}_0 \circ \widetilde{\xi}_1 = \xi_1,$$
$$\Xi_1 = \pi \circ \widetilde{\Xi}_1 \circ \widetilde{\xi}_1 = \xi_2.$$

We shall prove that

$$\Xi^*(\omega)(\frac{d}{dt}, -)$$

is exact on $\{t\} \times L$ for all $t \in [0, 1]$. Let $h : [0, 1] \times \widetilde{M} \to \mathbb{R}$ be a Hamiltonian that generate $\widetilde{\Xi}_t$. We claim that

$$d_L(h_t \circ \widetilde{\Xi}_t \circ \widetilde{\xi}_1) = \Xi^*(\omega)(\frac{d}{dt}, -).$$

For any $v \in \Gamma(L, TL)$,

$$\begin{aligned}
\Xi^*(\omega)(\frac{d}{dt}, v) &= \omega(\Xi_* \frac{d}{dt}, \Xi_* v) \\
&= (\pi^*\omega)((\widetilde{\Xi} \circ \widetilde{\xi}_1)_* \frac{d}{dt}, (\widetilde{\Xi}_t \circ \widetilde{\xi}_1)_* v) \\
&= (\pi^*\omega)(X_{h_t}(\widetilde{\Xi}_t \circ \widetilde{\xi}_1), (\widetilde{\Xi}_t \circ \widetilde{\xi}_1)_* v) \\
&= d_{\widetilde{M}} h_t((\widetilde{\Xi}_t \circ \widetilde{\xi}_1)_* v) \\
&= d_L(h_t \circ \widetilde{\Xi}_t \circ \widetilde{\xi}_1)(v),
\end{aligned}$$

so we are done. □

As a summary, we have

Corollary 4.11. *Let $\mathbb{L}_1$, $\mathbb{L}_2$ Lagrangian immersions. Consider the following statements:*

a) $\mathbb{L}_1$ *and* $\mathbb{L}_2$ *are globally Hamiltonian equivalent.*
b) $\mathbb{L}_1$ *and* $\mathbb{L}_2$ *are lifted Hamiltonian equivalent.*
c) $\mathbb{L}_1$ *and* $\mathbb{L}_2$ *are locally Hamiltonian equivalent.*

We have a) $\Rightarrow$ b) $\Rightarrow$ c).

Remark 4.12. *When $\mathbb{L}_1, \mathbb{L}_2$ are embedded and locally Hamiltonian isotopic to each other, then they are Hamiltonian isotopic to each other if we can choose the isotopy Ξ_t to be embeddings for all $t \in [0,1]$. Hence a), b), c) in Corollary 4.11 are equivalent in such situation.*

4.3. **The invariance theorem.** Now, we study the invariance property of the immersed Floer cohomology under lifted Hamiltonian deformations.

Let $\mathbb{L}_1, \mathbb{L}_2$ be a pair of compact, unobstructed Lagrangian immersions of (M, ω). Let $\widetilde{\xi}_1 : L_1 \to \widetilde{M}_1$ and $\widetilde{\xi}_2 : L_2 \to \widetilde{M}_2$ be liftings of $\mathbb{L}_1, \mathbb{L}_2$ to some finite unramified covering $\pi_1 : \widetilde{M}_1 \to M$ and $\pi_2 : \widetilde{M}_2 \to M$ of M, respectively. Note that $(\widetilde{M}_j, \pi_j)$ may equal to (M, id_M), that is, the trivial covering of M.

Now, consider the following commutative diagram:

$$\begin{array}{ccccccc}
L_1 \times_M \widetilde{M}_2 & \xrightarrow{\widetilde{\xi}_1 \times id} & & \widetilde{M}_1 \times_M \widetilde{M}_2 & & \xleftarrow{id \times \widetilde{\xi}_2} & \widetilde{M}_1 \times_M L_2 \\
\Big\downarrow{\scriptstyle \pi_{L_1}} & & & \Big\downarrow{\scriptstyle \pi_M} & & & \Big\downarrow{\scriptstyle \pi_{L_2}} \\
L_1 & \xrightarrow{\widetilde{\xi}_1} & \widetilde{M}_1 & \longrightarrow M \longleftarrow & \widetilde{M}_2 & \xleftarrow{\widetilde{\xi}_2} & L_2
\end{array} .$$

Note that all the vertical maps are finite unramified covering maps (the domain of them can be disconnected in general, but they are still smooth manifolds).

Now, since (L_1, ξ_1) and (L_2, ξ_2) are Lagrangian immersions, by equipping $\widetilde{M}_1 \times_M \widetilde{M}_2$ with the pullback symplectic structure via π_M, it is easy to see that $(L_1 \times_M \widetilde{M}_2, \widetilde{\xi}_1 \times id)$ and $(\widetilde{M}_1 \times_M L_2, id \times \widetilde{\xi}_2)$ are Lagrangian immersions of $\widetilde{M}_1 \times_M \widetilde{M}_2$, whose images are given by $\widetilde{\xi}_1(L_1) \times_M \widetilde{M}_2$ and $\widetilde{M}_1 \times_M \widetilde{\xi}_2(L_2)$, respectively.

To simplify the notations, we let $\widetilde{M} = \widetilde{M}_1 \times_M \widetilde{M}_2$, $\widetilde{L}_1 = \widetilde{\xi}_1(L_1) \times_M \widetilde{M}_2$ and $\widetilde{L}_2 = \widetilde{M}_1 \times_M \widetilde{\xi}_2(L_2)$. Points in $\widetilde{M}$ are denoted by $\widetilde{p}$.

Lemma 4.13. *Under the assumptions A), we have*

a) $\pi_M : \widetilde{L}_1 \cap \widetilde{L}_2 \to \xi_1(L_1) \cap \xi_2(L_2)$ *is a 1-1 correspondence.*
b) *The map* $(\pi_M)_* : \pi_2(\widetilde{M}; \widetilde{L}_1, \widetilde{L}_2; \widetilde{p}, \widetilde{q}) \to \pi_2(M; \xi_1(L_1), \xi_2(L_2); p, q)$ *is bijective.*
c) $(\pi_M)_*$ *preseves the Maslov index, i.e.* $\mu(\widetilde{\beta}) = \mu((\pi_M)_* \widetilde{\beta})$.
d) *If* (M, ω, J) *is a Calabi-Yau manifold, then* π_M *preserves grading, i.e. for any* $\widetilde{p} \in \widetilde{L}_1 \cap \widetilde{L}_2$, $\deg(\pi_M(\widetilde{p})) = \deg(\widetilde{p})$.

Proof. a) Suppose $p \in \xi_1(L_1) \cap \xi_2(L_2)$. Then there exists $l_1 \in L_1$ and $l_2 \in L_2$ such that $\xi_1(l_1) = p = \xi_2(l_2)$. Since $(\pi_1 \circ \widetilde{\xi}_1)(l_1) = \xi_1(l_1) = p = (\pi_2 \circ \widetilde{\xi}_2)(l_2)$, we have

$$(\widetilde{\xi}_1(l_1); p; \widetilde{\xi}_2(l_2)) \in \widetilde{L}_1 \cap \widetilde{L}_2$$

and $\pi_M(\widetilde{\xi}_1(l_1); p; \widetilde{\xi}_2(l_2)) = p$. This proves surjectivity. For injectivity, note that any intersection point of $\widetilde{L}_1$ and $\widetilde{L}_2$ is of form $(\widetilde{\xi}_1(l_1); p; \widetilde{\xi}_2(l_2))$ for some $l_1 \in L_1, l_2 \in L_2$ and $\xi_1(l_1) = p = \xi(l_2)$. If

$$\pi_M(\widetilde{\xi}_1(l_1); p; \widetilde{\xi}_2(l_2)) = \pi_M(\widetilde{\xi}_1(l_1'); p'; \widetilde{\xi}_2(l_2')),$$

then $p = p'$ and so

$$\xi_1(l_1) = p = p' = \xi_1(l_1'),$$
$$\xi_2(l_2) = p = p' = \xi_2(l_2').$$

Since p is not a self-intersection point of $\xi_1(L_1)$ nor $\xi_2(L_2)$, we have $l_1 = l_1'$ and $l_2 = l_2'$. Hence $(\widetilde{\xi}_1(l_1); p; \widetilde{\xi}_2(l_2)) = (\widetilde{\xi}_1(l_1'); p'; \widetilde{\xi}_2(l_2'))$.

b) Let $\widetilde{u} : \Delta \to \widetilde{M}$ represent $\widetilde{\beta}$ with boundary data

$$\widetilde{u}(\partial^- \Delta) \subset \widetilde{L}_1, \quad \widetilde{u}(\partial^+ \Delta) \subset \widetilde{L}_2,$$
$$\widetilde{u}(-1) = \widetilde{q}, \quad \widetilde{u}(1) = \widetilde{p}.$$

Set $u := \pi_M \circ \widetilde{u}$. Then clearly, u has boundary data

$$u(\partial^- \Delta) \subset \xi_1(L_1), \quad u(\partial^+ \Delta) \subset \xi_2(L_2),$$
$$u(-1) = q, \quad u(1) = p.$$

To obtain the liftings on the boundary, we recall we already have the liftings $\widetilde{u}_1^- : \partial^- \Delta \to L_1 \times_M \widetilde{M}_2$ and $\widetilde{u}_2^+ : \partial^+ \Delta \to \widetilde{M}_1 \times_B L_2$ of $\widetilde{u}|_{\partial^- \Delta}$ and $\widetilde{u}|_{\partial^+ \Delta}$, respectively. By definition, they satisfy

$$(\widetilde{\xi}_1 \times id) \circ \widetilde{u}_1^- = \widetilde{u}|_{\partial^- \Delta},$$
$$(id \times \widetilde{\xi}_2) \circ \widetilde{u}_2^+ = \widetilde{u}|_{\partial^+ \Delta}.$$

We define

$$u_1^- := \pi_{L_1} \circ \widetilde{u}_1^- : \partial^- \Delta \to L_1,$$
$$u_2^+ := \pi_{L_2} \circ \widetilde{u}_2^+ : \partial^+ \Delta \to L_2.$$

Then $\xi_1 \circ u_1^- = \xi_1 \circ \pi_{L_1} \circ \widetilde{u}_1^- = \pi_M \circ (\widetilde{\xi}_1 \times id) \circ \widetilde{u}_1^- = \pi_M \circ \widetilde{u}|_{\partial^- \Delta} = u|_{\partial^- \Delta}$. Similarly, $\xi_2 \circ u_2^+ = u|_{\partial^+ \Delta}$. Hence $(\pi_M)_*$ is well-defined. Injectivity follows from the homotopy lifting property.

For surjectivity, let $u : \Delta \to M$ be a representative of β with boundary data

$$u(\partial^- \Delta) \subset \xi_1(L_1), \quad u(\partial^+ \Delta) \subset \xi_2(L_2),$$
$$u(-1) = q, \quad u(1) = p.$$

Let $\widetilde{u}:\Delta\to\widetilde{M}$ be the lift of u with $\widetilde{u}(1)=\widetilde{p}$. We claim that $\widetilde{u}(\partial^-\Delta)\subset\widetilde{L}_1$. Recall that we have a lift $u_1^-:\partial^-\Delta\to L_1$ of $u|_{\partial^-\Delta}$. Since $\pi_{L_1}:L_1\times_M\widetilde{M}_2\to L_1$ is an unramified covering of L_1 and $\xi_1^{-1}(p)$ consists of only one point, there is a lift $\widetilde{u_1^-}:\partial^-\Delta\to L_1\times_M\widetilde{M}_2$ of u_1^- such that $((\widetilde{\xi}_1\times id)\circ\widetilde{u_1^-})(1)=\widetilde{p}$. Note that

$$\pi_{\widetilde{M}_1}\circ(\widetilde{\xi}_1\times id)\circ\widetilde{u_1^-}=\widetilde{\xi}_1\circ\pi_{L_1}\circ\widetilde{u_1^-}=\widetilde{\xi}_1\circ u_1^-.$$

Hence

$$\begin{aligned}\pi_M\circ(\widetilde{\xi}_1\times id)\circ\widetilde{u_1^-}=&\pi_1\circ\pi_{\widetilde{M}_1}\circ(\widetilde{\xi}_1\times id)\circ\widetilde{u_1^-}\\ =&\xi_1\circ u_1^-\\ =&u|_{\partial^-\Delta}=\pi_M\circ\widetilde{u}|_{\partial^-\Delta}.\end{aligned}$$

By uniqueness, we have $\widetilde{u}|_{\partial^-\Delta}=(\widetilde{\xi}_1\times id)\circ\widetilde{u_1^-}$. In particular, we have $\widetilde{u}(\partial^-\Delta)\subset\widetilde{L}_1$. Similarly, we have $\widetilde{u}(\partial^+\Delta)\subset\widetilde{L}_2$. These two inclusions imply

$$\widetilde{u}(-1)\in\widetilde{u}(\partial^+\Delta\cap\partial^-\Delta)\subset\widetilde{L}_1\cap\widetilde{L}_2.$$

Because $u(-1)=q$, we must have $\widetilde{u}(-1)=\widetilde{q}$ by uniqueness.

c) Since, via the differential $d\pi_M:T\widetilde{M}\to\pi_M^*TM$, $T_{\widetilde{q}}\widetilde{M}$ can be indetified symplectically with T_qM, the Lagrangain Grassmanians $LGr(T_{\widetilde{q}}\widetilde{M},\widetilde{\omega}_{\widetilde{q}})$ and $LGr(T_qM,\omega_q)$ are naturally isomorphic via $d\pi_M$.Clearly, the Lagrangian paths

$$\begin{aligned}&s\mapsto d\widetilde{\xi}_1(T_{\widetilde{u_1^-}(s,0)}(L_1\times_M\widetilde{M}_2)),\\ &s\mapsto d\xi_1(T_{u_1^-(s,0)}L_1)\end{aligned}$$

can also be identified via $d\pi_M$. Similarly, $d\pi_M$ also identifies

$$\begin{aligned}&s\mapsto d\widetilde{\xi}_2(T_{\widetilde{u_2^+}(s,1)}(\widetilde{M}_1\times_M\widetilde{L}_2)),\\ &s\mapsto d\xi_1(T_{u_2^+(s,1)}L_2).\end{aligned}$$

Since π_M is an unramified covering map, it is a local symplectomorphism. It follows that the canonical short paths are also identified via $d\pi_M$. Hence the Maslov index are preserved under $(\pi_M)_*$.

d) Since π_M is an unramified covering map, $(\widetilde{M},\widetilde{\omega},\widetilde{J})$ is naturally a Calabi-Yau manifold with the pullback sturctures $\widetilde{\omega}=\pi_M^*\omega$, $\widetilde{J}=\pi_M^*J$ and so each intersection point between $\widetilde{L}_1$ and $\widetilde{L}_2$ can be graded. Let $\widetilde{p}\in\widetilde{L}_1\cap\widetilde{L}_2$. The grading of $\widetilde{p}$ only depends on the angles that $\widetilde{L}_1$ and $\widetilde{L}_2$ intersect (See [4], Section 12), which is a local property. Since π is an unramified $(\widetilde{J},J)$-holomorphic covering map, we see that the angles of intersection at $\widetilde{p}$ is the same as the angles of intersection at $\pi_M(\widetilde{p})$.

□

Next, we show that the immersed Floer cohomology $HF(\mathbb{L}_1,\mathbb{L}_2)$ can be computed by the immersed Floer cohomolgy of the liftings $(L_1\times_B\widetilde{M}_2,\widetilde{\xi}_1\times id),(\widetilde{M}_1\times_B L_2,id\times\widetilde{\xi}_2)$. Recall that we have chosen a family of ω-compatible almost complex structures $\{J_t\}_{t\in[0,1]}$. Let $\{\widetilde{J}_t\}_{t\in[0,1]}$ be the pullback almost complex structure of $\{J_t\}_{t\in[0,1]}$ via the unramified covering map $\pi_M:\widetilde{M}\to M$. Then we have the following

Lemma 4.14. *For each $p, q \in \xi_1(L_1) \cap \xi_2(L_2)$, $\beta \in \pi_2(M; L_1, L_2; p, q)$ and $u \in \mathcal{M}(p, q; \beta)$, there exists unique $\widetilde{p}, \widetilde{q} \in \widetilde{L}_1 \cap \widetilde{L}_2$, $\widetilde{\beta} \in \pi_2(\widetilde{M}; \widetilde{L}_1, \widetilde{L}_2; \widetilde{p}, \widetilde{q})$ and $\widetilde{u} \in \mathcal{M}(\widetilde{p}, \widetilde{q}; \widetilde{\beta})$ such that $\pi_M(\widetilde{p}) = p$, $\pi_M(\widetilde{q}) = q$ and $\pi_M \circ \widetilde{u} = u$. Moreover, the map π_M induces an isomorphism of Kuranishi spaces*

$$\mathcal{M}(p, q; \beta) \cong \mathcal{M}(\widetilde{p}, \widetilde{q}; \widetilde{\beta}).$$

Proof. The existence and uniqueness of $\widetilde{p}, \widetilde{q}$ follows from Lemma 4.13. Also, since π_M is an unramified $(\widetilde{J}_t, J_t)$-holomorphic covering map, the proof of Part b) of Lemma 4.13 have already proved the correspondence between $\widetilde{J}_t$-holomorphic disks in $\widetilde{M}$ and J_t-holomorphic disks in M with the given boundary data and lifting properties. □

Proposition 4.15. *Let $\mathbb{L}_1, \mathbb{L}_2$ be Lagrangian immersions of (M, ω). The projection map $\pi_M : \widetilde{M} \to M$ induces a canonical isomorphism between Floer complexes*

$$(CF(\widetilde{L}_1, \widetilde{L}_2), m_1) \cong (CF(\mathbb{L}_1, \mathbb{L}_2), m_1).$$

Proof. By Part b) of the Lemma 4.13, we have proved that the projection $\pi_M : \widetilde{M} \to M$ gives an identification between the Floer complexes:

$$\pi_M : CF(\widetilde{L}_1, \widetilde{L}_2) \to CF(\mathbb{L}_1, \mathbb{L}_2).$$

It suffices to prove that π_M is a chain map. By Lemma 4.13, for each $p \in \xi_1(L_1) \cap \xi_2(L_2)$, there exists unique $\widetilde{p} \in \widetilde{L}_1 \cap \widetilde{L}_2$ such that $\pi_M(\widetilde{p}) = p$. Furthermore, for each $q \in \xi_1(L_1) \cap \xi_2(L_2)$ and $u \in \mathcal{M}^0(p, q; \beta) \neq \phi$, there exists a unique $\widetilde{q} \in \widetilde{L}_1 \cap \widetilde{L}_2$ with $\pi_M(\widetilde{q}) = q$ and a unique $\widetilde{u} \in \mathcal{M}^0(\widetilde{p}, \widetilde{q}; \widetilde{\beta})$ such that $\pi_M \circ \widetilde{u} = u$. Hence we have

$$\begin{aligned} m_1(\pi_M(\widetilde{p})) &= \sum_{q \in \xi_1(L_1) \cap \xi_2(L_2)} \sum_{\beta : \mu(\beta) = 1} \sum_{u \in \mathcal{M}^0(p, q, \beta)} (-1)^{sign(u)} T^{\omega(u)} \cdot q \\ &= \sum_{\widetilde{q} \in \widetilde{L}_1 \cap \widetilde{L}_2} \sum_{\widetilde{\beta} : \mu(\widetilde{\beta}) = 1} \sum_{\widetilde{u} \in \mathcal{M}^0(\widetilde{q}, \widetilde{p}, \widetilde{\beta})} (-1)^{sign(\widetilde{u})} T^{\widetilde{\omega}(\widetilde{u})} \cdot \pi_M(\widetilde{q}). \end{aligned}$$

The last summation is exactly $\pi_M(m_1(\widetilde{p}))$. □

Theorem 4.16. *Let $\mathbb{L}_1, \mathbb{L}_2$ be Lagrangian immersions in (M, ω). The Floer cohomology $HF(\mathbb{L}_1, \mathbb{L}_2)$ is invariant under lifted Hamiltonian isotopy. That is, if $\mathbb{L}_2$ is lifted Hamiltonian isotopic to $\mathbb{L}_2'$, then there is a quasi-isomorphism*

$$(CF(\mathbb{L}_1, \mathbb{L}_2), m_1) \simeq (CF(\mathbb{L}_1, \mathbb{L}_2'), m_1).$$

Proof. It suffices to prove the theorem in the case when $\mathbb{L}_2$ is $(\widetilde{M}_2, \pi_2)$-lifted Hamiltonian equivalent to $\mathbb{L}_2'$ for some finite unramified covering $\pi_2 : \widetilde{M}_2 \to M$. Suppose $\mathbb{L}_2$ is $(\widetilde{M}_2, \pi_2)$-lifted Hamiltonian isotopic to $\mathbb{L}_2'$. By definition, there exist liftings $\widetilde{\xi}_2 : L_2 \to \widetilde{M}_2$ and $\widetilde{\xi}_2' : L_2 \to \widetilde{M}_2$ such that $(L_2, \widetilde{\xi}_2)$ is globally Hamiltonian equivalent to $(L_2, \widetilde{\xi}_2')$. In this case, $\widetilde{M} = M \times_M \widetilde{M}_2 \cong \widetilde{M}_2$, so we have a quasi-isomorphism

$$(CF(\widetilde{L}_1, \widetilde{L}_2), m_1) \simeq (CF(\widetilde{L}_1, \widetilde{L}_2'), m_1).$$

Together with the isomorphism obtained in Proposition 4.15, we have the quasi-isomorphism

$$(CF(\mathbb{L}_1, \mathbb{L}_2), m_1) \simeq (CF(\mathbb{L}_1, \mathbb{L}_2'), m_1).$$

This completes the proof. □

Theorem 4.16 shows that lifted Hamiltonian equivalent defines an equivalence on the immersed Fukaya category of (M, ω). This also provides an answer to Question 13.15 in [4].

When mirror symmetry is considered, one needs to complexify the Fukaya category by unitary local systems on the domain of the Lagrangian immersion. In this case, the differential m_1 on $CF((\mathbb{L}_1, \mathcal{L}_1), (\mathbb{L}_2, \mathcal{L}_2))$ should be coupled with the holonomy coming from the local systems $\mathcal{L}_1, \mathcal{L}_2$ on the boundary of the disks. The notion of lifted Hamiltonian isotopy can be generalized as follows

Definition 4.17. *Let $\mathbb{L}_1 = (L_1, \xi_1), \mathbb{L}_2 = (L_2, \xi_2)$ be two Lagrangian immersions of M and $\mathcal{L}_1, \mathcal{L}_2$ be local systems on L_1, L_2 respectively. Let $\pi : \widetilde{M} \to M$ be a finite unramified covering of M. We say $(\mathbb{L}_1, \mathcal{L}_1)$ is $(\widetilde{M}, \pi)$-*lifted Hamiltonian isotopic *to $(\mathbb{L}_2, \mathcal{L}_2)$ if*

(a) *There exists a diffeomorphism $\phi : L_1 \to L_2$ and liftings $\widetilde{\xi}_1 : L_1 \to \widetilde{M}$, $\widetilde{\xi}_2 : L_2 \to \widetilde{M}$ such that $(L_1, \widetilde{\xi}_1)$ is globally Hamiltonian isotopic to $(L_1, \widetilde{\xi}_2 \circ \phi)$ and*
(b) *$\phi^* \mathcal{L}_2 \cong \mathcal{L}_1$ as unitary bundles.*

With a slight modification, one can also prove the invariance of the Floer cohomology $HF((\mathbb{L}_1, \mathcal{L}_1), (\mathbb{L}_2, \mathcal{L}_2))$ under this generalized notion of lifted Hamiltonian isotopy for any pair of immersed Lagrangian branes $(\mathbb{L}_1, \mathcal{L}_1), (\mathbb{L}_2, \mathcal{L}_2)$. We omit the detailed proof.

5. Mirror of isomorphism between holomorphic vector bundles

Now, going back to the fibration $X \to B$. Let $\mathbb{L} = (L, \xi, c_r)$ be an immersed Lagrangian multi-section. In this section, we will prove, at least in the semi-flat and caustic free case, that certain lifted Hamiltonian equivalence is mirror to isomorphism between holomorphic vector bundles.

Recall that $c_r : L \to B$ is a finite unramified covering, the projection $\pi_X : L \times_B X \to X$ is also an finite unramified covering of X. A deck transformation $\tau_L \in \mathrm{Deck}(L/B)$ induces a deck transformation $\tau \in \mathrm{Deck}(L \times_B X/X)$ by

$$\tau : (l, x, y) \mapsto (\tau_L(l), x, y).$$

Hence we get an injective group homomorphism $\mathrm{Deck}(L/B) \to \mathrm{Deck}(L \times_B X/X)$. Let G be the image of this homomorphism. With respective to the pullback symplectic structure on $L \times_B X$, elements in G are symplectomorphisms. On the mirror side, we also have an finite unramified covering $\pi_{\check{X}} : L \times_B \check{X} \to \check{X}$. One can apply a similar construction to obtain an embedding $\mathrm{Deck}(L/B) \hookrightarrow \mathrm{Deck}(L \times_B \check{X}/\check{X})$. Denote the image by $\check{G}$. With respective to the pullback complex structure, $\check{G}$ is a subgroup of the group of biholomorphisms of $L \times_B \check{X}$. There is a natural bijection between G and $\check{G}$ given by $G \cong \mathrm{Deck}(L/B) \cong \check{G}$.

Lemma 5.1. *If L is connected and Deck(L/B) acts transitively on fibers of $c_r : L \to B$, then $G = Deck(L \times_B X/X)$ and $\check{G} = Deck(L \times_B \check{X}/\check{X})$.*

Proof. Since L is connected, $L \times_B X$ is also connected and so, $\mathrm{Deck}(L \times_B X/X)$ acts freely on the fiber of $\pi_X : L \times_B X \to X$. Hence G also acts freely on fibers of π_X. Fix $(x, y) \in X$. The fiber of π_X over (x, y) is in bijection to the fiber of c_r over $x \in B$. By assumption, $\mathrm{Deck}(L/B)$ acts transitively on the fiber of c_r. Hence G also acts transitively on the fiber of π_X. Therefore, $\mathrm{Deck}(L \times_B X/X)$ also acts transitively on fibers of π_X. Since both G and $\mathrm{Deck}(L \times_B X/X)$ acts transitively and freely on fibers, we must have $G = \mathrm{Deck}(L \times_B X/X)$. □

Remark 5.2. *The transitivity of the action of $Deck(L/B)$ on fibers of $c_r : L \to B$ is equivalent to the normality of $(c_r)_*(\pi_1(L))$ as a subgroup of $\pi_1(B)$.*

It is known in [17] that when B is compact, the Lagrangian sections L_1, L_2 are (globally) Hamiltonian equivalent if and only if their mirror line bundles $\check{L}_1, \check{L}_2$ are isomorphic as holomorphic vector bundles. In the higher rank situation, the following theorem shows, at least with a transitivity assumption, that $(L \times_B X, \pi_X)$-lifted Hamiltonian equivalence is the mirror analog of isomorphism of holomorphic vector bundles.

Theorem 5.3. *Suppose B is compact. Let $\mathbb{L}_1 = (L, \xi_1, c_r)$, $\mathbb{L}_2 = (L, \xi_2, c_r)$ be immersed Lagrangian multi-sections of $X \to B$ with same connected domain L and unramified covering map $c_r : L \to B$. Assume $Deck(L/B)$ acts transitively on fibers of $c_r : L \to B$. Then $\mathbb{L}_1$ is $(L \times_B X, \pi_X)$-lifted Hamiltonian isotopic to $\mathbb{L}_2$ if and only if $\check{\mathbb{L}}_1$ is isomorphic to $\check{\mathbb{L}}_2$ as holomorphic vector bundles.*

Proof. Given two immersed Lagrangian multi-sections $\mathbb{L}_1 = (L, \xi_1, c_r)$ and $\mathbb{L}_2 = (L, \xi_2, c_r)$ with the same connected domain L and finite unramified covering map $c_r : L \to B$, we can lift $\mathbb{L}_j$, $j = 1, 2$ to Lagrangian embeddings $\widetilde{\xi}_j : L \to \widetilde{L}_j \subset L \times_B X$, explicitly given by

$$\widetilde{\xi}_j : l \mapsto (l, x, \xi_j(l)).$$

Moreover, for any $\tau_L \in \mathrm{Deck}(L/B)$ and section $\widetilde{\xi} : L \to \widetilde{L} \subset L \times_B X$ of the fibration $\pi_L : L \times_B X \to L$, the composition

$$\tau \circ \widetilde{\xi} \circ \tau_L^{-1} : l \mapsto (l, x, \widetilde{\xi}(\tau_L^{-1}(l)))$$

defines a section of π_L.

Suppose $\mathbb{L}_1$ is $(L \times_B X, \pi_X)$-lifted Hamiltonian isotopic to $\mathbb{L}_2$. Then there exist liftings $\widetilde{\xi}_1', \widetilde{\xi}_2' : L \to L \times_B X$ such that $(L, \widetilde{\xi}_1')$ and $(L, \widetilde{\xi}_2')$ are globally Hamiltonian isotopic to each other. Since both $\widetilde{\xi}_1, \widetilde{\xi}_1'$ are liftings of ξ_1, by the transitivity assumption, there exists $\tau_1 \in \mathrm{Deck}(L \times_B X/X)$ such that

$$\widetilde{\xi}_1' = \tau_1 \circ \widetilde{\xi}_1.$$

Similarly, there exists τ_2 such that

$$\widetilde{\xi}_2' = \tau_2 \circ \widetilde{\xi}_2.$$

In particular, $(L, \widetilde{\xi}_1')$, $(L, \widetilde{\xi}_2')$ are embedded Lagrangian submanifolds and so, $\tau_1 \circ \widetilde{\xi}_1 \circ \tau_{L,1}^{-1}$ and $\tau_2 \circ \widetilde{\xi}_2 \circ \tau_{L,2}^{-1}$ are globally Hamiltonian equivalent Lagrangian sections of π_L. Let $\check{L}_1$, $\check{L}_2$ be the SYZ mirror line bundle of $(L, \widetilde{\xi}_1)$ and $(L, \widetilde{\xi}_2)$, respectively. Then the correspondence result of [17] gives $(\check{\tau}_1^{-1})^* \check{L}_1 \cong (\check{\tau}_2^{-1})^* \check{L}_2$ as holomorphic line bundles, where $\check{\tau}_j \in \check{G}$ corresponds to $\tau_j \in G$ under the natural isomorphism $G \cong \mathrm{Deck}(L/B) \cong \check{G}$. We have

$$\check{\mathbb{L}}_j(U) = ((\pi_{\check{X}})_* \check{L}_j)(U) = \check{L}_j(\pi_{\check{X}}^{-1}(U)), \quad j = 1, 2.$$

For $j = 1, 2$, we have

$$((\pi_{\check{X}})_* (\check{\tau}_j^{-1})^* \check{L}_j)(U) = \check{L}_j(\check{\tau}_j(\pi_{\check{X}}^{-1}(U))) = \check{L}_j((\pi_{\check{X}} \circ \check{\tau}_j^{-1})^{-1}(U)) = (\pi_{\check{X}})_* \check{L}_j(U),$$

so

$$\check{\mathbb{L}}_2 = (\pi_{\check{X}})_* \check{L}_2 \cong (\pi_{\check{X}})_* (\check{\tau}_2^{-1})^* \check{L}_2 \cong (\pi_{\check{X}})_* (\check{\tau}_1^{-1})^* \check{L}_1 = (\pi_{\check{X}})_* \check{L}_1 = \check{\mathbb{L}}_1.$$

Conversely, suppose $\check{\mathbb{L}}_1 \cong \check{\mathbb{L}}_2$. By the correspondence result of [17] again, it suffices to show that $\check{L}_1 \cong \check{\tau}^* \check{L}_2$ for some $\check{\tau} \in \check{G}$. Note that since $L \times_B X$ is connected, we have the following formula:

$$\pi_{\check{X}}^* (\pi_{\check{X}})_* \check{L}_j = \bigoplus_{\check{\tau} \in \check{G}} \check{\tau}^* \check{L}_j, \quad j = 1, 2.$$

Hence

$$\bigoplus_{\check{\tau} \in \check{G}} \check{\tau}^* \check{L}_1 \cong \bigoplus_{\check{\tau} \in \check{G}} \check{\tau}^* \check{L}_2.$$

In particular, $\check{L}_1$ defines a subbundle of $\oplus_{\check{\tau} \in \check{G}} \check{\tau}^* \check{L}_2$. Since $\check{L}_1$ is a subbundle, there exists $\check{\tau}_1 \in \check{G}$ such that the composition

$$\check{L}_1 \hookrightarrow \bigoplus_{\check{\tau} \in \check{G}} \check{\tau}^* \check{L}_2 \to \check{\tau}_1^* \check{L}_2$$

is not identically zero. Similarly, there exists $\check{\tau}_2 \in \check{G}$ such that the composition $\check{L}_2 \hookrightarrow \oplus_{\check{\tau} \in \check{G}} \check{\tau}^* \check{L}_1 \to \check{\tau}_2^* \check{L}_1$ is not identically zero. Then we obtain a chain of bundle maps

$$\check{L}_1 \to \check{\tau}_1^* \check{L}_2 \to \check{\tau}_1^* \check{\tau}_2^* \check{L}_1 \to \cdots \to (\check{\tau}_1^* \check{\tau}_2^*)^k \check{L}_1, \quad k \geq 1,$$

each of which is not identically zero. Take k to be the order of $\check{\tau}_2 \circ \check{\tau}_1$. Then we obtain a map $\check{L}_1 \to \check{L}_1$, which is again, not identically zero. Since B is compact, so is $\check{X}$. Hence $\check{L}_1 \to \check{L}_1$ corresponds to a nonzero holomorphic function which can only be a nonzero constant. Therefore, $\check{L}_1 \to \check{L}_1$ is an isomorphism and in particular, $\check{L}_1 \to \check{\tau}_1^* \check{L}_2$ is injective. Since $\check{L}_1$ and $\check{\tau}_1^* \check{L}_2$ are line bundles, $\check{L}_1 \to \check{\tau}_1^* \check{L}_2$ is an isomorphism. □

If we combine Theorem 5.3 with the surgery-extension correspondence theorem (Theorem 3.4), we obtain

Corollary 5.4. *Let $L_1 = L_{r_1,d_1}[c_1]$ and $L_2 = L_{r_2,d_2}[c_2]$. If $K, K' \subset L_1 \cap L_2$ are sets of intersection points such that the Lagrangian surgeries $\mathbb{L}_K = L_2 \sharp_K L_1$ and $\mathbb{L}_{K'} = L_2 \sharp_{K'} L_1$ have connected domain and satisfy the gcd assumption $gcd(r_1 + r_2, d_1 + d_2) = 1$, then $\mathbb{L}_K$ and $\mathbb{L}_{K'}$ are $(S^1 \times_{S^1} T^2, \pi_{T^2})$-lifted Hamiltonian equivalent, and hence have isomorphic immersed Lagrangian Floer cohomologies.*

We given an example to illustrate Corollary 5.4.

Example 5.5. *Let*

$$L_1 = L_{1,0}[1/2], \quad L_2 = L_{1,3}[0]$$

be Lagrangian straight lines in the standard symplectic tors T^2. Then L_1 intersects L_2 at three points, all of them are of index 1. We equip L_1 with the local system

$$d + 2\pi i \frac{1}{2} dx$$

and L_2 with the trivial one. Consider the Lagrangian immersions $\mathbb{L}_1, \mathbb{L}_3$, as shown in Figure 8.

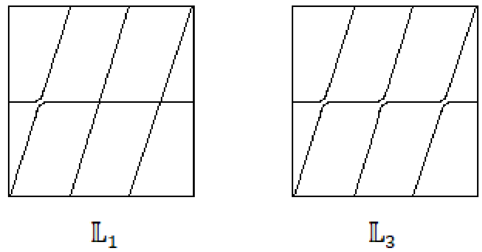


FIGURE 8. Two non-Hamiltonian equivalent but lifted Hamiltonian equivalent Lagrangian immersions in T^2.

Both $\mathbb{L}_1, \mathbb{L}_3$ have connected domain and satisfy the gcd assumption: $gcd(1+1, 0+3) = 1$ as in Theorem 3.4 If we equip their domain with the trivial local system, Theorem 3.4 can be applied to conclude both $\check{\mathbb{L}}_1$ and $\check{\mathbb{L}}_3$ fit into some exact sequences:

$$0 \to \check{L}_2 \to \check{\mathbb{L}}_1 \to \check{L}_{1,\frac{1}{2}} \to 0,$$
$$0 \to \check{L}_2 \to \check{\mathbb{L}}_3 \to \check{L}_{1,\frac{1}{2}} \to 0.$$

By Atiyah's classification on indecomposable bundles on elliptic curves, we know that $\check{\mathbb{L}}_1 \cong \check{\mathbb{L}}_3$. Hence by Theorem 5.3, $\mathbb{L}_1$ and $\mathbb{L}_3$ are $(S^1 \times_{S^1} T^2, \pi_{T^2})$-lifted Hamiltonian equivalent to each other.

We also compute the Floer cohomology of $\mathbb{L}_1$ and $\mathbb{L}_3$. Since $\mathbb{L}_3$ is embedded and bounds no holomorphic disk, we have

$$HF(\mathbb{L}_3, \mathbb{L}_3) \cong H(S^1; \Lambda_{nov}).$$

For $\mathbb{L}_1$, let $\xi_1 : S^1 \to T^2$ be the immersion map. The Floer complex is given by

$$H(S^1; \Lambda_{nov}) \oplus \Lambda_{nov}\{p^-, p^+, q^-, q^+\},$$

where $H(S^1; \Lambda_{nov})$ is the Λ_{nov}-valued cohomology of S^1 and p^-, p^+, q^-, q^+ are points on S^1 such that $\xi_1(p^-) = \xi_1(p^+)$ and $\xi_1(q^-) = \xi_1(q^+)$ are self-intersection points of $\mathbb{L}_1$ (see Corollary 11.4 of [4]). Points with a positive (resp. negative) sign are graded to have degree 0 (resp. 1). There is one holomorphic disk from p^+ to q^- and one from q^+ to p^- (See Figure 9).

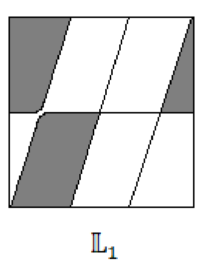


FIGURE 9. The holomorphic disk from p^+ (resp. q^-) to q^+ (resp. p^-).

Hence the Floer cohomology of $\mathbb{L}_1$ is given by $HF(\mathbb{L}_1, \mathbb{L}_1) \cong H(S^1; \Lambda_{nov})$, which is canonically isomorphic to $HF(\mathbb{L}_3, \mathbb{L}_3)$ as expected of Theorem 4.16. One can also use Hamiltonian perturbation to obtain the same result.

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