GEOMETRY OF THE MAURER-CARTAN EQUATION NEAR DEGENERATE CALABI-YAU VARIETIES

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Abstract. In this paper, we construct a differential graded Batalin-Vilkovisky (dgBV) algebra $PV^{\cdot,\cdot}(X)$ associated to a possibly degenerate Calabi-Yau variety $X$ equipped with local deformation data. This gives a singular version of the (extended) Kodaira-Spencer dgLa. We work in an abstract framework and use a local-to-global Čech-de Rham–type gluing construction. Applying standard techniques in BV algebras [33, 30, 47], and under both the Hodge-to-de Rham degeneracy assumption and a local assumption that guarantees freeness of the Hodge bundle, we prove an unobstructedness theorem, which can be regarded as a singular version of the famous Bogomolov-Tian-Todorov theorem [2, 48, 49], in the spirit of the work of Katzarkov-Kontsevich-Pantev [33, 30]. In particular, we recover the existence of smoothing for both log smooth Calabi-Yau varieties (studied by Kawamata-Namikawa [32]) and maximally degenerate Calabi-Yau varieties (studied by Gross-Siebert [22]). We also demonstrate how our construction can be applied to produce a log Frobenius manifold structure on a formal neighborhood of the extended moduli space using Barannikov’s technique [1].

1. Introduction

1.1. Background. Deformation theory of Calabi-Yau manifolds $X$ plays an important role in algebraic geometry, mirror symmetry and mathematical physics. There are two major approaches to deformation theory, namely, the Čech approach [44] and the Kodaira-Spencer approach [41]. The advantage of the Kodaira-Spencer differential graded Lie algebra (abbrev. dgLa) $\Omega^{0,\cdot}(X, T^0_X)$ is that it can be upgraded to a differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra $\Omega^{0,\cdot}(X, \bigwedge T^1)$ which is equipped with a BV operator $\Delta$ constructed from the holomorphic volume form $\omega$ satisfying the famous Bogomolov-Tian-Todorov Lemma [2, 48, 49]

$$(-1)^{|v|}[v, w] := \Delta(v \wedge w) - \Delta(v) \wedge w - (-1)^{|v|}v \wedge \Delta(w).$$

This naturally leads to unobstructedness of deformations, or equivalently, local smoothness of the moduli space of compact Calabi-Yau manifolds. Indeed the geometry of the holomorphic volume form not only proves the unobstructedness result but also gives a local structure of Frobenius manifold for the moduli space by the work of Barannikov [1].

By degenerating a family of Calabi-Yau manifolds $\{X_q\}$ to limit points in the compactified moduli space ($q \to \infty$), one would obtain a degenerate Calabi-Yau variety which is usually equipped with a log structure introduced by Fontaine-Illusie and K. Kato [28]. Smoothing of degenerate log Calabi-Yau varieties via log deformations is thus fundamental in understanding the global moduli space of Calabi-Yau manifolds. Two celebrated results in this direction are, the unobstructedness of log smooth Calabi-Yau varieties studied by Kawamata-Namikawa [32] and that of maximally degenerate Calabi-Yau varieties studied by Gross-Siebert [20, 21, 22], where two entirely different methods, namely, the $T^1$-lifting technique in Kawamata-Namikawa [32] and the scattering diagram technique in Gross-Siebert [22], are used respectively.

After pioneering works of Quillen, Deligne and Drinfeld, it is now a universally accepted philosophy that any deformation problem should be governed by the Maurer-Cartan equation in a dgLa (or...
$L_\infty$-algebra). Lots of works have been done in this direction; see e.g. [12][13][14][24][39][38][40]. Using this framework, Bogomolov-Tian-Todorov-type theorems have been formulated and proven in various settings, e.g. in [33][30][31][27][25][26]. In many cases, the existence of an underlying dgLa and the Maurer-Cartan equation provides a tool to solve a geometric deformation problem using algebraic techniques.

On the other hand, our earlier work [4] and [5], motivated by Fukaya's asymptotic approach [16], showed how asymptotic expansions of Maurer-Cartan solutions for the Kodaira-Spencer dgLa $\Omega^{0,*}(X, \Lambda^* T^1,0_X)$, where $X$ is a torus bundle over a base $B$, give rise to scattering diagrams and tropical disk counts as the torus fibers shrink. Scattering diagrams are the combinatorial structures used by Kontsevich-Soibelman [34] and Gross-Siebert [22] to solve the important reconstruction problem in mirror symmetry. Therefore, it is natural to ask whether there exists a dgLa (or better, a dgBV) which governs the smoothing of maximally degenerate Calabi-Yau varieties. The difficulty arises from the fact that there is non-trivial topology change for such a degeneration, which is dictated by (the residue of) the Gauss-Manin connection. Taking the trivial product $\Omega^{0,*}(X, \Lambda^* T^{1,0})[[q]]$ cannot record the change in topology and hence does not lead to smoothing, in contrast to deformations of smooth manifolds where the topology is unchanged.

In this paper, we provide a construction of the desired dgBV $PV^{*,*}(X)$ associated to degenerate Calabi-Yau varieties $X$, which plays the same role as $\Omega^{0,*}(X, \Lambda^* T^{1,0})[[q]]$ for smooth $X$. We prove that the classical part of the Maurer-Cartan equation (see Definition 5.10) indeed governs the geometric smoothing of $X$. We develop an abstract framework construct the dgBV algebra from local deformation (or thickening) data via a local to global Čech-(simplicial) de Rham–type gluing construction. Such a simplicial construction of $PV^{*,*}(X)$ allows us to directly link the smoothing of degenerate Calabi-Yau varieties with the Hodge theoretic viewpoint as developed in [1][3][30][37]. It also paves the way towards studying Calabi-Yau varieties via BV algebra and developing the BCOV higher genus $B$-model [6]. Moreover, various known techniques for dgBV algebras can then be applied. As applications, we prove unobstructedness or Bogomolov-Tian-Todorov–type theorems using algebraic techniques.

1.2. Main results.
1.2.1. Construction of the singular Kodaira-Spencer complex. To begin with, we let $Q$ be a monoid and $\mathbb{C}[Q]$ be the universal coefficient ring with the monomial ideal $\mathfrak{m} = \langle Q \setminus \{0\} \rangle$. We consider a complex analytic space $(X, O_X)$ and a covering $V = \{V_\alpha\}$ by Stein open subsets, together with local deformation (or thickening) data on each $V_\alpha$ as in [2] which consist of a $k$th-order coherent sheaf of BV algebras $(kG^*_\alpha, \Lambda, k\Delta_\alpha)$ over $kR := \mathbb{C}[Q]/\mathfrak{m}^{k+1}$ (Definition 2.13) acting on a $k$th-order coherent sheaf of de Rham modules $(k\mathcal{K}^*_\alpha, \Lambda, k\partial_\alpha)$ (Definition 2.17). We further fix another covering $U = \{U_i\}_{i \in \mathbb{N}}$ by Stein open subsets together with higher order local patching data, which is an isomorphism $k\psi^\alpha\beta_i : kG^*_\alpha|_{U_i} \rightarrow kG^*_\beta|_{U_i}$ (Definition 2.15). At this stage, we do not require these patching data $k\psi^\alpha\beta_i$'s to be compatible on the nose, but rather the differences between them are being captured by a collection of local sections from $kG^*_\alpha$ (see Definition 2.15).

The ordinary Čech approach to deformation theory is done by solving for compatible gluings $k\gamma^\alpha\beta : kG^*_\alpha \rightarrow kG^*_\beta$ and understanding the obstruction in doing so. Instead of gluing these sheaves directly, we first take the dg resolution given as a dgBV algebra $kPV^{*,*}_\alpha$ (which can be viewed as a dg Calabi-Yau variety) for each $kG^*_\alpha$ using the Thom-Whitney construction [51][9], and then
solve for compatible gluings \( k g_{\alpha \beta} \cdot k PV^* \alpha \rightarrow k PV^* \beta \) which satisfy the cocycle condition in Lemma 3.18 and are also compatible for different orders \( k \). This is possible since the local sheaves of dgBV algebras \( k PV^* \alpha \)'s are more topological than the sheaves \( k G^* \alpha \)'s themselves. This gives our first main theorem which is a construction of the global dgBV algebras \( k PV^* \alpha (X)[[t]] \) and \( PV^* \alpha (X)[[t]] = \lim_{\leftarrow k} k PV^* \alpha (X)[[t]] \):

**Theorem 1.1** (=Theorem 3.18 + Proposition 3.24 + Theorem 3.34). There exists a differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra of polyvector fields \( (PV^* \alpha (X), \bar{\partial}, \Delta, \wedge) \) over \( \mathbb{C}[[Q]] \) satisfying all the dgBV algebra identities except that the equation \( (\bar{\partial} + t \Delta)^2 = 0 \) is only satisfied in \( 0 PV^* \alpha (X) := PV^* \alpha (X) \otimes_{\mathbb{C}[Q]} (\mathbb{C}[[Q]] / \langle Q \setminus \{0\} \rangle) \), i.e. when it is restricted to the 0-th order.

In geometric situations such as those considered by Kawamata-Namikawa [32] and Gross-Siebert [21], the above theorem produces a singular version of the Kodaira-Spencer complex governing the smoothing.

1.2.2. Unobstructedness from Maurer-Cartan equation. Consider the extended Maurer-Cartan equation

\[
(\bar{\partial} + t \Delta + [\varphi, \cdot])^2 = 0
\]

for \( \varphi \in PV^0(X)[[t]] \) for all order, which can be thought of solving \( \bar{\partial}^2 = 0 \) (or more precisely the volume form equation \( d(e^{2t/3} \omega) \)) when compared to classical Kodaira-Spencer theory. After setting these up, the unobstructedness theorem can be proven using standard techniques in BV algebras as in [33, 30, 47] under the Hodge-to-de Rham degeneracy Assumption 5.4 that \( H^*(0 PV^*(X)[[t]], \bar{\partial} + t \Delta) \) is a free \( \mathbb{C}[[t]] \) and a local Assumption 4.15 that guarantees the freeness of the Hodge bundle (Lemma 4.16).

**Theorem 1.2** (Theorem 5.5). The extended Maurer-Cartan equation 1.2 can be solved order by order for \( \varphi \in PV^* \alpha (X) \otimes_{\mathbb{C}[[t]]} \), under Assumptions 5.4 and 4.15. In particular, the geometric deformations (i.e. smoothing) of \( X \) over \( \text{Spf}(\mathbb{C}[[Q]]) \) (see Definition 5.12) by adapting the framework set up by Katzarkov-Kontsevich-Pantev [33, 30].

This can be regarded as an abstract, singular version of the famous Bogomolov-Tian-Todorov (BTT) theorem [2, 48, 49] for smoothing degenerate Calabi-Yau varieties, formulated in the spirit of the framework set up by Barannikov [1] together with the extra data of the mixed Hodge structure on \( \mathbb{H}^*(X, \Omega^*) \).

1.2.3. Construction of semi-infinite log variation of Hodge structure. Another benefit of having the polyvector field dgBV algebra \( PV^* \alpha (X) \) associated to \( X \) is that we can have a direct construction of the semi-infinite log variation of Hodge structures (see 6.1 for its definition) by adapting the techniques developed by Barannikov [1] together with the extra data of the mixed Hodge structure on \( \mathbb{H}^*(X, \Omega^*) \).

From the log structure on \( X \), one can construct the residue action \( N_\nu \) of the Gauss-Manin connection acting on the cohomology \( \mathbb{H}^*(X, \Omega^*) \) for each constant vector field on \( \text{Spec}(\mathbb{C}[Q]) \) given by \( \nu \in (Q^\text{gp})^\vee \otimes_{\mathbb{Z}} \mathbb{R} \). We require that there is a weight filtration of the form

\[
\{ 0 \} \subset W_{-1} \subset \cdots \subset W_{-r} \subset \cdots \subset W_d = \mathbb{H}^*(X, \Omega^*)
\]

indexed by weights \( r \in \frac{1}{2} \mathbb{Z} \) such that it is opposite to the Hodge filtration \( \mathcal{F}^* (\mathbb{H}^*(X, \Omega^*)) \) (introduced in Definition 2.10) in the sense of Assumption 6.12. We further require that there is a trace map

\[1\] Note that here \( \bar{\partial} \) is not the Dolbeault operator on \( X \), while it plays the role of \( \bar{\partial} \) in the case when \( X \) is smooth and therefore we write it as \( \bar{\partial} \).

\[2\] Here \( t \) is the descendant parameter as in [1].
tr : \( \mathbb{H}^*(X, \Omega^*) \rightarrow \mathbb{C} \) (playing the role of integration in the smooth case) such that the associated pairing \( 0 \langle p(\alpha, \beta) \rangle := \text{tr}(\alpha \wedge \beta) \) is non-degenerate as in Assumption 6.17.

Our second main theorem shows that the above information can be incorporated into our dgBV algebra \( PV^{*,*}(X) \) to obtain a log Frobenius manifold structure in a formal neighborhood of the extended complex moduli space near \( X \). To do so, we suitably enlarge our coefficient ring \( \mathbb{C}[Q] \) to encode all the extended moduli parameters (which by abuse of notations will be denoted as \( \mathbb{C}[Q] \) again).

With a versal solution \( \varphi \) to the Maurer-Cartan equation \( 1.2 \), we therefore define the semi-infinite Hodge bundle \( \mathcal{H}_+ \) over \( \text{Spf} (\mathbb{C}[Q][[t]]) \) to be the \( \mathbb{C}[Q][[t]] \) module \( \mathcal{H}_+ := \lim \mathbb{H}^*(kPV^{*,*}(X)[[t]], \partial + t \Delta + [\varphi, \cdot]) \). Over the central fiber \( \text{Spf}(\mathbb{C}) \hookrightarrow \text{Spf}(\mathbb{C}[[Q]]) \), the above weighted filtration can be used to define an opposite filtration which is a \( \mathbb{C}[t^{-1}] \) submodule of the form \( 0\mathcal{H}_- := \bigoplus_{r \in \mathbb{Z}} \mathcal{W}_{r} C[t^{-1}]t^{-r+d-2} \subset \mathbb{H}^*(X, \Omega^*)[[t^\frac{1}{2}]]t^{-\frac{1}{2}} \), to the Hodge bundle over \( \text{Spf}(\mathbb{C}) \). Making use of the flatness of the Gauss-Manin connection, it is possible to extend the opposite filtration \( 0\mathcal{H}_- \) to the whole moduli \( \text{Spf}(\mathbb{C}[[Q]]) \) as a \( \mathbb{C}[Q][t^{-1}] \) submodule which is preserved by it. Furthermore, the pairing \( 0\langle \cdot, \cdot \rangle \) can also be extended to a flat pairing on \( \langle \cdot, \cdot \rangle : \mathcal{H}_+ \times \mathcal{H}_+ \rightarrow \mathbb{C}[Q][[t]] \) with respect to the Gauss-Manin connection.

**Theorem 1.3** (=Theorem 6.28). The triple \( (\mathcal{H}_+, \nabla, \langle \cdot, \cdot \rangle) \) forms a semi-infinite log variation of Hodge structures in the sense of Definition 6.2, and \( \mathcal{H}_- \) is an opposite filtration to the Hodge bundle \( \mathcal{H}_+ \) in the sense of Definition 6.5. Furthermore, there exists a versal solution to the Maurer-Cartan equation \( 1.2 \) such that \( e^{\varphi/\hbar} \) gives a miniversal section of the Hodge bundle in the sense of Definition 6.26. As a consequence, there is a structure of log Frobenius manifold on the formal neighborhood \( \text{Spf}(\mathbb{C}[[Q]]) \) of \( X \) in the extended moduli space constructed from these data.

1.2.4. Geometric applications. In \( \S 7 - \S 8 \), we demonstrate how to apply our abstract framework to the geometric situations studied by Kawamata-Namikawa \([22] \) (see \([7] \) for description of \( X \)) and Gross-Siebert \([21, 22] \) (see \([5] \) for description of \( X \)) by extracting the local deformation (or thickening) data from their works. In both cases, there is a covering \( \mathcal{V}_\alpha \) on \( X \) together with a local thickening \( V_\alpha \) (which is toric in both cases) of each \( V_\alpha \) over \( \text{Spec} (\mathbb{C}[Q]) \) as

\[
\begin{array}{ccc}
\mathcal{V}_\alpha & \xrightarrow{\pi} & \mathcal{V}_\alpha \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{C}) & \xleftarrow{\psi} & \text{Spec}(\mathbb{C}[Q])
\end{array}
\]

which serves a local model for smoothing of \( X \). By writing \( Z \subset X \) be the possible codimension 2 singular locus of log-structure of \( X \) with \( j : X \setminus Z \rightarrow X \). We take \( ^{\bar{6}}G_\alpha^* \) to be the push-forward of the sheaf of relative log polyvector fields on \( X \setminus Z \) to \( X \), and \( ^{\bar{6}}K_\alpha^* \) to be the push-forward of the sheaf of total log holomorphic de Rham complex on \( X \setminus Z \) to \( X \). The higher order patching data \( ^{k}\psi_{\alpha,i} \) come from uniqueness of the local model near a point in \( X \). These data fit into our framework, and the Hodge-to-de Rham degeneracy is proven in \([21, 3.26] \) in the Gross-Siebert case and in \([22, 4.1.1] \) in the Kawamata-Namikawa case. Therefore, we recover the following smoothing result in both cases as a corollary of Theorem 1.2.

**Corollary 1.4.** In both the cases studied by Kawamata-Namikawa and Gross-Siebert, the complex analytic space \( (X, \mathcal{O}_X) \) is smoothable, i.e. there exist \( k^\text{th} \)-order thickening \( ^{k}X, ^{k}\mathcal{O} \) over \( ^{k}S^\dag \) locally modeled on \( ^{k}V_\alpha \)'s, and is compatible for each \( k \)’s.

**Remark 1.5.** After finishing this paper, we were informed by Mark Gross and Helge Ruddat that Simon Felten, Matej Filip and Helge Ruddat have a closely related work \([10] \) where they smooth

\[\text{Here } \mathbb{C}[Q][[t]] \text{ refers to a suitable completion as in } [6].\]
maximally degenerate Calabi-Yaus and Fanos via log geometry and the algebraic BTT techniques by Iacono-Manetti. We thank Mark Gross and Helge Ruddat for communicating this to us.

Notation Summary

Notations 1.6. We fix a rank \( s \) lattice \( K \) together with a strictly convex \( s \)-dimensional rational polyhedral cone \( Q_R \subset K \subset K_R := K \otimes \mathbb{Z} \). We let \( Q := Q_R \cap K \) and call it the universal monoid. We consider the ring \( R := \mathbb{C}[Q] \) and write a monomial element as \( q^m \in R \) for \( m \in Q \), and consider the maximal ideal given by \( m := \mathbb{C}[Q \setminus \{0\}] \).

We let \( kR := R/mk^{k+1} \) be the Artinian ring, and \( \hat{R} := \varprojlim_k kR \) be the completion of \( R \). We further equip \( R \), \( kR \) and \( \hat{R} \) with the natural monoid homomorphism \( Q \to R \), \( m \mapsto q^m \), giving them the structure of a log ring (see [22, Definition 2.11]); the corresponding log spaces will be denoted as \( S^\dagger \), \( kS^\dagger \) and \( \hat{S}^\dagger \) respectively.

Furthermore, we let \( \Omega^*_S := R \otimes C \wedge^*(K_C) \), \( k\Omega^*_S := kR \otimes C \wedge^*(K_C) \) and \( \hat{\Omega}^*_S := \hat{R} \otimes C \wedge^*(K_C) \) (here \( K_C = K \otimes C \)) be the spaces of log de Rham differentials on \( S^\dagger \), \( kS^\dagger \) and \( \hat{S}^\dagger \) respectively, where we write \( 1 \otimes m = d\log q^m \) for \( m \in K \), and they are equipped with the de Rham differential \( \partial \) satisfying \( \partial(q^m) = q^m d\log q^m \). Similarly, we denote the spaces of log derivations by \( \Theta_S^\dagger := R \otimes C K_C^\dagger \), \( \Theta_S^\dagger \) and \( \hat{\Theta}^\dagger \) respectively, which are equipped with a natural Lie bracket \([\cdot, \cdot]\). We will write \( \partial_n \) for the element \( 1 \otimes n \) with the action \( \partial_n(q^m) = (m,n)q^m \), where \((m,n)\) is the natural pairing between \( K_C \) and \( K_C^\dagger \).

For a \( \mathbb{Z}^2 \)-graded vector space \( V^{*,*} = \bigoplus_{p,q} V^{p,q} \), we often write \( V^k = \bigoplus_{p+q=k} V^{p,q} \), and \( V^* = \bigoplus_k V^k \) if we only care about the total degree. We also often simply write \( V \) if we do not need to consider the grading.

Throughout this paper, we are dealing with two Čech covers \( V = (V_\alpha)_{\alpha} \) and \( U = (U_i)_{i \in \mathbb{Z}^+} \) at the same time and also \( k \)-th order thickenings, so we will use the following (rather unusual) notations: The top left hand corner in a notation \( ^k\bullet \) refers to the order of \( \bullet \). The bottom left hand corner in a notation \( \bullet \) will stand for something constructed on the Koszul filtration on \( K_C \)'s (as in Definitions 2.9 and 2.17), where \( \bullet \) can be \( r, r_1 : r_2 \) or \( | \) (meaning relative forms). The bottom right hand corner is reserved for the Čech indices. We write \( ^{\bullet}_{\alpha_0 \cdots \alpha_{k}} \) for the Čech indices of \( V \) and \( ^{\bullet}_{i_0 \cdots i_{k}} \) for the Čech indices of \( U \), and if they appear at the same time, we write \( ^{\bullet}_{\alpha_0 \cdots \alpha_{k}, i_0 \cdots i_{k}} \).

2. The abstract setup

2.1. BV algebras and modules.

Definition 2.1. A graded Batalin-Vilkovisky (abbrev. BV) algebra is a unital \( \mathbb{Z} \)-graded \( \mathbb{C} \)-algebra \((V^*, \wedge)\) together with a degree 1 operator \( \Delta \) such that \( \Delta(1) = 0 \), \( \Delta^2 = 0 \) and the operator \( \delta_v : V^* \to V^{*+|v|+1} \) defined by \( \delta_v(w) := \Delta(v \wedge w) - \Delta(v) \wedge w - (-1)^{|v|} v \wedge \Delta(w) \) is a derivation of degree \( |v|+1 \) for any homogeneous element \( v \in V^* \) (here \(|v|\) denotes the degree of \( v \)).

Definition 2.2. A differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra is a graded BV algebra \((V^*, \wedge, \Delta)\) together with a degree 1 operator \( \partial \) satisfying

\[
\tilde{\partial}(\alpha \wedge \beta) = (\tilde{\partial}\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (\tilde{\partial}\beta), \quad \tilde{\partial}^2 = \tilde{\partial} \Delta + \Delta \tilde{\partial} = 0.
\]

Definition 2.3. A differential graded Lie algebra (abbrev. dgLa) is a triple \((L^*, d, [\cdot, \cdot])\), where \( L = \bigoplus_{i \in \mathbb{Z}} L^i \), \([\cdot, \cdot] : L^* \otimes L \to L^* \) is a graded skew-symmetric pairing satisfying the Jacobi identity \([a, [b, c]] + (-1)^{|a||b|+|a||c|}[b, [c, a]] + (-1)^{|a||c|+|b||c|}[c, [a, b]] = 0\) for homogeneous elements \( a, b, c \in L^* \),
and $d : L^* \to L^{*+1}$ is a degree 1 differential satisfying $d^2 = 0$ and the Leibniz rule $d[a,b] = [da,b] + (-1)^{|a|}[a,db]$ for homogeneous elements $a, b \in L^*$.

Given a BV algebra $(V^*, \wedge, \Delta)$, the map $[,] : V \otimes V \to V$ defined by $[v,w] = (-1)^{|v||w|}\delta_v(w)$ is called the associated Lie bracket. Using this bracket, the triple $(V^*[-1], \Delta, [\cdot,\cdot])$ forms a dgLa.

**Notations 2.4.** Given a nilpotent graded Lie algebra $L^*$, we define a product $\odot$ by the Baker-Campbell-Hausdorff formula: $v \odot w := v + w + \frac{1}{2}[v,w] + \cdots$ for $v, w \in V^*$. The pair $(L^*, \odot)$ is called the exponential group of $L^*$ and is denoted by $\exp(L^*)$.

**Lemma 2.5** (See e.g. [33]). For a dgLa $(L^*, d, [\cdot,\cdot])$, we consider the endomorphism $ad_\partial := [\partial,\cdot]$ for an element $\partial \in L^0$ such that $ad_\partial$ is nilpotent. Then we have the formula

$$e^{ad_\partial}(d + [\xi, -])e^{-ad_\partial} = d + [e^{ad_\partial}(\xi),\cdot] - [\frac{e^{ad_\partial} - 1}{ad_\partial}(d\partial),\cdot]$$

for $\xi \in L^*$. For a nilpotent element $\partial \in L^0$, we define the gauge action

$$\exp(\partial) \ast \xi := e^{ad_\partial}(\xi) - \frac{e^{ad_\partial} - 1}{ad_\partial}(d\partial)$$

for $\xi \in L^*$. Then we have $\exp(\partial_1) \ast (\exp(\partial_2) \ast \xi) = \exp(\partial_1 \odot \partial_2) \ast \xi$, where $\odot$ is the Baker-Campbell-Hausdorff product as in Notation 2.4.

**Definition 2.6** (see e.g. [33]). A BV module $(M^*, \partial)$ over a BV algebra $(V^*, \wedge, \Delta)$ is a complex of $\mathbb{C}$-vector spaces equipped with a degree 1 differential $\partial$ and a graded action by $(V^*, \wedge)$, which will be denoted as $\nu_v = v \wedge : M^* \to M^{*+|v|}$ (for a homogeneous element $v \in V^*$) and called the interior multiplication or contraction by $v$, such that if we let $(-1)^{|v|}\mathcal{L}_v := [\partial, v \wedge] := \partial \circ (v \wedge) - (-1)^{|v|}(v \wedge) \circ \partial$, where $[\cdot,\cdot]$ is the graded commutator for operators, then $[\mathcal{L}_{v_1}, v_2 \wedge] = [v_1, v_2] \wedge$.

Given a BV module $(M^*, \partial)$ over $(V^*, \wedge, \Delta)$, we have $[\partial, \mathcal{L}_v] = 0$, $\mathcal{L}_{[v_1,v_2]} = \{\mathcal{L}_{v_1}, \mathcal{L}_{v_2}\}$ and $\mathcal{L}_{v_1 \wedge v_2} = (-1)^{|v_2|}\mathcal{L}_{v_1} \circ (v_2 \wedge) + (v_1 \wedge) \circ \mathcal{L}_{v_2}$.

**Definition 2.7.** A BV module $(M^*, \partial)$ over $(V^*, \wedge, \Delta)$ is called a de Rham module if there is a unital differential graded algebra (abbrev. dga) structure $(M^*, \partial, \Delta)$ such that for $v \in V^{-1}$, $\nu_v$ acts as a derivation, i.e. $v \wedge (w_1 \wedge w_2) = (v \wedge w_1) \wedge w_2 + (-1)^{|w_1|}w_1 \wedge (v \wedge w_2)$. If in addition there is a finite decreasing filtration of BV submodules $\{0\} = \cdots \subseteq \cdots \subseteq \mathbb{N}^{-1} \mathbb{M}^* \subseteq \cdots \subseteq \mathbb{M}^* = M^*$, then we call it a filtered de Rham module.

**Lemma 2.8.** Given a de Rham module $(M^*, \partial)$ over $(V^*, \wedge, \Delta)$, it is easy to check that for $v \in V^{-1}$, $\nu_v$ acts as a derivation, i.e. $\mathcal{L}_v(w_1 \wedge w_2) = (\mathcal{L}_v w_1) \wedge w_2 + w_1 \wedge (\mathcal{L}_v w_2)$.

**Lemma 2.9.** Given a BV algebra $(V^*, \wedge, \Delta)$ acting on a BV module $(M^*, \partial)$ both with bounded degree, together with an element $v \in V^{-1}$ such that the operator $\nu_v$ is nilpotent and element $\omega$ such that $\partial \omega = 0$ satisfying $\Delta(\alpha \wedge \omega) = (\partial \alpha \wedge \omega)$, we have the following identities

$$\exp(\Delta, v \wedge)(1) = \exp\left(\sum_{k=0}^{\infty} \frac{\delta^k_v}{(k+1)!}(\Delta v)\right), \quad \exp([\partial, v \wedge]) \omega = \exp\left(\sum_{k=0}^{\infty} \frac{\delta^k_v}{(k+1)!}(\Delta v)\right) \wedge \omega$$

where $\delta_v$ is the operator defined in Definition 2.7.

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4 For polyvector fields on a Calabi-Yau manifold, we have $[\cdot,\cdot] = [-\cdot,\cdot]_{\text{SN}}$, where $[-\cdot,\cdot]_{\text{SN}}$ is the Schouten-Nijenhuis bracket; see e.g. [23, §6.4].

5 This is motivated by the de Rham complex equipped with the Koszul filtration associated to a family of varieties; see e.g. [12, Chapter 10.4].
Proof. To prove the first identity, notice that \([\Delta, v \wedge] = \delta_v + (\Delta v) \wedge\) and
\[
\exp \left( \sum_{k=0}^{\infty} \frac{\delta_v^k}{(k+1)!} (\Delta v) \right) = 1 + \sum_{m \geq 1} \sum_{0 \leq k_1 < \cdots < k_m, \ s_1, \cdots, s_m > 0} \frac{\left( \frac{\delta_v^{k_1}}{(k_1+1)!} (\Delta v) \right)^{s_1} \cdots \left( \frac{\delta_v^{k_m}}{(k_m+1)!} (\Delta v) \right)^{s_m}}{(s_1)! \cdots (s_m)!},
\]
so it suffices to establish the equality
\[
\left( \frac{\delta_v + (\Delta v) \wedge)}{L!} \right) (1) = \sum_{0 \leq k_1 < \cdots < k_m, \ s_1, \cdots, s_m > 0; \ (k_1+1)s_1 + \cdots + (k_m+1)s_m = L} \frac{\left( \frac{\delta_v^{k_1}}{(k_1+1)!} (\Delta v) \right)^{s_1} \cdots \left( \frac{\delta_v^{k_m}}{(k_m+1)!} (\Delta v) \right)^{s_m}}{(s_1)! \cdots (s_m)!},
\]
which can be proven by induction on \(L\). Essentially the same proof gives the second identity. \(\square\)

2.2. The 0th-order data. Let \((X, \mathcal{O}_X)\) be a \(d\)-dimensional compact complex analytic space.

Definition 2.9. A 0th-order datum over \(X\) consists of:

- a coherent sheaf of graded BV algebras \(0^G^*, [], \wedge, 0^\Delta\) over \(X\) (with \(-d \leq * \leq 0\)), called the 0th-order complex of polyvector fields, such that \(0^G^0 = \mathcal{O}_X\) and the natural Lie algebra morphism \(0^G^-1 \rightarrow \text{Der}(\mathcal{O}_X), v \mapsto [v, \cdot]\) is injective,

- a coherent sheaf of dga's \((0^K^*, [], 0^\partial)\) over \(X\) (with \(0 \leq * \leq d + s\)) endowed with a dg module structure over the dga \(0^\Omega^*_S\), called the 0th-order de Rham complex, and equipped with the natural filtration \(0^K^*\) defined by \(0^K^* := 0^\Omega^*_S \wedge 0^K^*\) (here \(\wedge\) denotes the dga action),

- a de Rham module structure on \(0^K^*\) over \(0^G^*\) such that \([\varphi, \alpha \wedge] = 0\) for any \(\varphi \in 0^G^*\) and \(\alpha \in 0^\Omega^*_S\), and

- an element \(0^\omega \in \Gamma(X, 0^K^*/0^K^d)\) with \(0^\partial(0^\omega) = 0\), called the 0th-order volume element, such that

1. the map \(0^\omega : (0^G^*[-d], 0^\Delta) \rightarrow (0^K^*/0^K^d, 0^\partial)\) is an isomorphism, and
2. the map \(0^\sigma^{-1} : 0^\Omega^*_S \otimes_C (0^K^*/0^K^d[-r]) \rightarrow 0^K^*/0^K^d,\) given by taking wedge product by \(0^\Omega^*_S\) (here \([-r]\) is the upshift of the complex by degree \(r\)), is also an isomorphism.

Note that \((0^K^*, [], 0^\partial)\) is a filtered de Rham module over \(0^G^*\) using the filtration \(0^K^*\), and the map \(0^\sigma^{-1}\) is an isomorphism of BV modules. We write \((0^K^*, 0^\partial) := (0^K^*/0^K^d, 0^\partial)\) and \(0^\sigma := (0^\sigma^{-1})^{-1}\).

We consider the hypercohomology \(H^\ast(0^K^*, 0^\partial)\) of the complex of sheaves \((0^K^*, 0^\partial)\).

Definition 2.10. For each \(r \in \frac{1}{2}\mathbb{Z}\), let \(\mathcal{F}^{2r}\mathbb{H}^l\) be the image of the linear map \(H^l(0^K^*/0^K^d, 0^\partial) \rightarrow H^l(0^K^*, 0^\partial)\), where \(p\) is the smallest integer such that \(2p \geq 2r + l - d\). Then
\[
0 \subset \mathcal{F}^{2d} \subset \mathcal{F}^{2d - \frac{1}{2}} \subset \cdots \subset \mathcal{F}^{2r} \subset \cdots \subset \mathcal{F}^{20} = H^\ast(0^K^*, 0^\partial)
\]
is called the Hodge filtration.

We usually write \(\mathcal{F}^{2r}\) instead of \(\mathcal{F}^{2r}\mathbb{H}^l\) when there is no confusion.

We have the following exact sequence of sheaves from Definition 2.9.

\[
0 \rightarrow 0^\Omega^1_S \otimes 0^K^[-1] \cong 0^K^*/0^K^d \rightarrow 0^K^*/0^K^d \rightarrow 0^K^*/0^K^d \rightarrow 0^K^*/0^K^d \rightarrow 0
\]
\[\text{We follow Barannikov} \quad \text{for the convention on the index} \ r \ \text{of the Hodge filtration, which differs from the usual one by a shift.}\]
Definition 2.11. Suppose we take the long exact sequence associated to the hypercohomology of \([2.1]\), we obtain the 0\(^{th}\)-order Gauss-Manin (abbrev. GM) connection:

\[(2.2) \quad 0\nabla : \mathbb{H}^*\left(\Omega^0_{\mathcal{S}^1}, \mathcal{O}\right) \to 0\Omega^1_{\mathcal{S}^1} \otimes \mathbb{H}^*\left(\Omega^0_{\mathcal{S}^1}, \mathcal{O}\right).\]

Note that the 0\(^{th}\)-order GM connection is indeed the residue of the usual GM connection.

Proposition 2.12. Griffith’s transversality holds for \(0\nabla\), i.e. \(0\nabla(\mathcal{F}^{\geq r}) \subset 0\Omega^1_{\mathcal{S}^1} \otimes \mathcal{F}^{\geq r-1}\).

The proof of this is the same as [12] Corollary 10.31. With \([0\omega] \in \mathcal{F}^{\geq d} \mathbb{H}^0\) and Griffith’s transversality, we then define the Kodaira-Spencer map as \(0\nabla([0\omega]) : 0\Omega^1_{\mathcal{S}^1} \to \mathcal{F}^{\geq d-1} \mathbb{H}^0\).

2.3. The higher order data. We fix an open cover \(\mathcal{V}\) of \(X\) which consists of Stein open subsets \(V_\alpha \subset X\).

Definition 2.13. A local thickening datum of the complex of polyvector fields (with respect to \(\mathcal{V}\)) consists of, for each \(k \in \mathbb{Z}_{\geq 0}\) and \(V_\alpha \in \mathcal{V}\),

- a coherent sheaf of BV algebras \((kG^*_\alpha, [\cdot, \cdot], \wedge, k\Delta_\alpha)\) over \(V_\alpha\) such that \(kG^*_\alpha\) is also a sheaf of algebras over \(kR\) so that \([\cdot, \cdot], \wedge\) are \(kR\)-bilinear and \(k\Delta_\alpha\) is \(kR\)-linear, and
- a surjective morphism of sheaves of BV algebras \(k^{+1} \beta_\alpha : k^{+1}G^*_\alpha \to kG^*_\alpha\) inducing a sheaf isomorphism upon tensoring with \(kR\)

satisfying the following conditions

\[(1) \quad (0G^*_\alpha, [\cdot, \cdot], \wedge, 0\Delta_\alpha) = (0G^*_\alpha, [\cdot, \cdot], \wedge, 0\Delta)|_{V_\alpha},\]

\[(2) \quad kG^*_\alpha\text{ is flat over }kR, \text{ i.e. the stalk } (kG^*_\alpha)_x \text{ is flat over }kR \text{ for any } x \in V_\alpha, \text{ and}\]

\[(3) \quad \text{the natural Lie algebra morphism } kG^*_{\alpha} \rightarrow \text{Der}(kG^0_{\alpha}) \text{ is injective}.\]

We write \(k^{+l} \beta_\alpha := l+1 \beta_\alpha \circ \cdots \circ k \beta_\alpha \circ k^{+1} \beta_\alpha : kG^*_\alpha \rightarrow kG^*_\alpha\) for every \(k > l\), and \(k \beta_\alpha \equiv id\). We also introduce the following notation: Given two elements \(a \in kG^*_\alpha, b \in kG^*_\alpha\) and \(l \leq \min\{k_1, k_2\}\), we say that \(a \equiv b \mod \mathfrak{m}^{l+1}\) if and only if \(k^{+l} \beta_\alpha(a) = k^{+l} \beta_\alpha(b)\).

Notations 2.14. We also fix, once and for all, a cover \(\mathcal{U}\) of \(X\) which consists of a countable collection of Stein open subsets \(\mathcal{U} = \{U_i\}_{i \in \mathbb{Z}^+}\) forming a basis of topology. We refer readers to [8] Chapter IX Theorem 2.13 for the existence of such a cover. Note that an arbitrary finite intersection of Stein open subsets remains Stein.

Definition 2.15. A patching datum of the complex of polyvector fields (with respect to \(\mathcal{U}, \mathcal{V}\)) consists of, for each \(k \in \mathbb{Z}_{\geq 0}\) and triple \((U_i; V_\alpha, V_\beta)\) with \(U_i \subset V_{\alpha \beta} := V_\alpha \cap V_\beta\), a sheaf isomorphism \(k\psi_{\alpha \beta, i} : kG^*_{\alpha}|_{U_i} \rightarrow kG^*_{\beta}|_{U_i}\) over \(kR\) preserving the structures \([\cdot, \cdot], \wedge\) and fitting into the diagram

\[
\begin{array}{ccc}
&\ast &\ast \\
&\ast &\ast \\
0G^*|_{U_i} & k\psi_{\alpha \beta, i} & kG^*|_{U_i} \\
k\beta_{\alpha} & | & k\beta_{\beta} \\
& k\psi_{\alpha \beta, i} & \\
\end{array}
\]

and an element \(k\omega_{\alpha \beta, i} \in kG^0(\mathcal{U}_i)\) with \(k\omega_{\alpha \beta, i} = 0 \mod \mathfrak{m}\) such that

\[(2.3) \quad k\psi_{\beta \alpha, i} \circ k\Delta_\beta \circ k\psi_{\alpha \beta, i} - k\Delta_\alpha = [k\omega_{\alpha \beta, i}, \cdot] \quad \text{satisfying the following conditions:}\]

\[(1) \quad k\psi_{\beta \alpha, i} = k\psi_{\alpha \beta, i}^{-1}, k\psi_{\alpha \beta, i} \equiv id;\]
(2) for $k > 1$ and $U_i \subset V_{\alpha \beta}$, there exists $k^l b_{\alpha \beta, i} \in \mathcal{G}_{\alpha}^{-1}(U_i)$ with $k^l b_{\alpha \beta, i} = 0 \pmod{m}$ such that

$$l_{\psi_{\alpha \beta, i}} \circ k^{l_\beta} \circ k^{l_{\alpha \beta, i}} = \exp\left(\left[k^l b_{\alpha \beta, i}, \cdot \right] \right) \circ k^{l_\alpha};$$

(3) for $k \in \mathbb{Z}_{\geq 0}$ and $U_i, U_j \subset V_{\alpha \beta}$, there exists $k p_{\alpha \beta, ij} \in \mathcal{G}_{\alpha}^{-1}(U_i \cap U_j)$ with $k p_{\alpha \beta, ij} = 0 \pmod{m}$ such that

$$\left(k \psi_{\alpha \beta, i}|_{U_i \cap U_j}\right) \circ \left(k \psi_{\alpha \beta, i}|_{U_i \cap U_j}\right) = \exp\left(\left[k p_{\alpha \beta, ij}, \cdot \right] \right);$$

(4) for $k \in \mathbb{Z}_{\geq 0}$ and $U_i \subset V_{\alpha \beta} := V_\alpha \cap V_\beta \cap V_\gamma$, there exists $k o_{\alpha \beta \gamma, i} \in \mathcal{G}_{\alpha}^{-1}(U_i)$ with $k o_{\alpha \beta \gamma, i} = 0 \pmod{m}$ such that

$$\left(k \psi_{\alpha \beta, i}|_{U_i}\right) \circ \left(k \psi_{\alpha \beta, i}|_{U_i}\right) \circ \left(k \psi_{\alpha \beta, i}|_{U_i}\right) = \exp\left(\left[k o_{\alpha \beta \gamma, i}, \cdot \right] \right).$$

Lemma 2.16. The elements $k^l b_{\alpha \beta, i}$'s, $k p_{\alpha \beta, ij}$'s and $k o_{\alpha \beta \gamma, i}$'s are uniquely determined by the patching isomorphisms $k \psi_{\alpha \beta, i}$'s.

Proof. We just prove the statement for the elements $k p_{\alpha \beta, ij}$'s as the other cases are similar. Suppose we have another set of elements $k p_{\alpha \beta, ij}$'s satisfying $(2,5)$, then we have $\exp\left(\left[k p_{\alpha \beta, ij} - k p_{\alpha \beta, ij}, \cdot \right] \right) \equiv \text{id}$ as actions on $\mathcal{G}_{\alpha}^0(U_{ij})$ where $U_{ij} = U_i \cap U_j$. The result then follows from an order-by-order argument using the assumptions that $k \supseteq k p_{\alpha \beta, ij} - k p_{\alpha \beta, ij} = 0$ and that the map $k \mathcal{G}_{\alpha}^{-1} \to \text{Der}(k \mathcal{G}_{\alpha}^0)$ is injective.

Definition 2.17. A local thickening datum of the de Rham complex (with respect to $\mathcal{V}$) consists of, for each $k \in \mathbb{Z}_{\geq 0}$ and $V_\alpha \in \mathcal{V}$,

- a coherent sheaf of dgas $(k^s K_{\alpha}^*, \wedge, k^s \partial_\alpha)$ with a dg module structure over $k^s \Omega_{S^r}$ equipped with the natural filtration $k^s K_{\alpha}^*$ defined by $k^s K_{\alpha}^* := k^s \Omega_{S^r} \wedge k^s K_{\alpha}^*$,
- a de Rham module structure on $k^s K_{\alpha}^*$ over $k \mathcal{G}_{\alpha}^r$ such that $[\varphi, \alpha \wedge] = 0$ for any $\varphi \in k \mathcal{G}_{\alpha}^r$ and $\alpha \in k \Omega_{S^r}$,
- a surjective morphism $k^{+1} b_{\alpha} : k^{+1} K_{\alpha}^* \to k^s K_{\alpha}^*$ inducing an isomorphism upon tensoring with $k^r R$ which is compatible with both $k^{+1} b_{\alpha} : k^{+1} \mathcal{G}_{\alpha}^r \to k \mathcal{G}_{\alpha}^r$ and $k^{+1} \Omega_{S^r} \to k \Omega_{S^r}$ under the contraction and dg module respectivelyootnote{Here we abuse notations and use $k^{+1} b_{\alpha}$ for both $k^{+1} K_{\alpha}^*$ and $k^{+1} \mathcal{G}_{\alpha}^r$.} and
- an element $k^s \omega_\alpha \in \Gamma(V_\alpha, k^{d}/k^{d} \mathcal{K}_{\alpha}^*)$ satisfying $k^s \partial_\alpha (k^s \omega_\alpha) = 0$ called the local $k^s$-order volume element

such that

1. $k^s K_{\alpha}^*$ is flat over $k^r R$ for $0 \leq r \leq s$;
2. $k^{+1} b_{\alpha} : k^{+1} \Omega_{S^r} \to k^s \omega_\alpha$;
3. $(0 K_{\alpha}^*, 0 K_{\alpha}^*, \wedge, 0 \partial_\alpha) \equiv (0 K_{\alpha}^*, 0 K_{\alpha}^*, \wedge, 0 \partial_\alpha)|_{V_\alpha}$ and $0 \omega_\alpha = 0 \omega|_{V_\alpha}$;
4. the map $k^s \omega_\alpha : (k \mathcal{G}_{\alpha}^r[-d], k \Delta_{\alpha}) \to (k K_{\alpha}^*/k K_{\alpha}^*, k \partial_\alpha)$ is an isomorphism, and
5. the map $k^{r} \sigma_{\alpha}^{-1} : k \Omega_{S^r} \otimes_{k^r R} (k^s K_{\alpha}^*/k^s K_{\alpha}^*[-r]) \to k^s K_{\alpha}^*/k^s K_{\alpha}^*$, given by taking wedge product by $k^s \Omega_{S^r}$, is also an isomorphism.

Note that $k^s K_{\alpha}^*$ is a filtered de Rham module over $k \mathcal{G}_{\alpha}^r$ using the filtration $k^s K_{\alpha}^*$.

We write $k^s K_{\alpha}^* := 0 K_{\alpha}^*/k^s K_{\alpha}^*$ and $k^{r} \sigma_{\alpha} = (k^{r} \sigma_{\alpha}^{-1})^{-1}$. 
We also write \( k,l b_\alpha := l+1,l b_\alpha \circ \cdots \circ k,k-1 b_\alpha \) for every \( k > l \) and \( k,k b_\alpha \equiv \text{id} \), and introduce the following notation: Given two elements \( a \in k,k K^*_\alpha \), \( b \in k,k K^*_{\hat{\alpha}} \) and \( l \leq \min\{k_1,k_2\} \), we say that \( a = b \) (mod \( m^{l+1} \)) if and only if \( k,l b_\alpha(a) = k,l b_\alpha(b) \).

From Definition 2.17 we have the following diagram of BV modules

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K \otimes_{\mathbb{Z}}^{k+1} K^*_{\alpha}[1] & \longrightarrow & K^*/2^*_{\alpha} & \longrightarrow & K^*/k^*_{\alpha} & \longrightarrow & 0 \\
\downarrow \text{id} \otimes^{k+1} K^*_{\alpha} & & \downarrow \text{id} \otimes^{k+1} K^*_{\alpha} & & \downarrow \text{id} \otimes^{k+1} K^*_{\alpha} & & \downarrow \text{id} \otimes^{k+1} K^*_{\alpha} & & \downarrow \text{id} \otimes^{k+1} K^*_{\alpha} \\
0 & \longrightarrow & K \otimes_{\mathbb{Z}}^{k} K^*_{\alpha}[1] & \longrightarrow & K^*/2^*_{\alpha} & \longrightarrow & K^*/k^*_{\alpha} & \longrightarrow & 0 \\
\end{array}
\]

\[(2.7)\]

**Definition 2.18.** A patching datum of the de Rham complex \((\text{with respect to } \mathcal{U}, \mathcal{V})\) consists of, for each \( k \in \mathbb{Z}_{\geq 0} \) and triple \((U_i; V_\alpha, V_\beta)\) with \( U_i \subset V_\alpha \beta \), a sheaf isomorphism \( k \hat{\psi}_{\alpha, i} \) of dg modules over \( k \Omega^*_S \) such that it fits into the diagram

\[
\begin{array}{ccc}
k^*_{\alpha, i} & \longrightarrow & k^*_{\beta, i} \\
\downarrow k,0_{\alpha} & & \downarrow k,0_{\beta} \\
0^* \big| U_i & \longrightarrow & 0^* \big| U_i,
\end{array}
\]

and satisfying the following conditions:

1. \( k \hat{\psi}_{\alpha, i} \) is an isomorphism of de Rham modules meaning that the diagram

\[
\begin{array}{ccc}
k^*_{\alpha, i} \big| U_i & \longrightarrow & k^*_{\beta, i} \big| U_i \\
\downarrow k,\psi_{\alpha, i} & & \downarrow k,\psi_{\beta, i} \\
k^*_{\alpha, i} \big| U_i & \longrightarrow & k^*_{\beta, i} \big| U_i,
\end{array}
\]

(2.8)

is commutative;

2. \( k \hat{\psi}_{\alpha, i} = k \hat{\psi}_{\alpha, i}^{-1}, k \hat{\psi}_{\alpha, i} \equiv \text{id}; \)

3. we have

\[
k^*_{\psi_{\beta, i}}(k^*_{\gamma, i}) = \exp(k^*_{w_{\alpha, i}})(k^*_{\omega_{\alpha, i}}), \tag{2.9}
\]

where the elements \( k^*_{w_{\alpha, i}} \)’s are as in Definition 2.15;

4. for \( k > l \) and \( U_i \subset V_\alpha \beta \), we have

\[
l^* \hat{\psi}_{\alpha, i} \circ k^*_{\beta, i} \circ k^*_{\alpha, i} = \exp \left( L_{k,l b_{\alpha, i}} \right) \circ k^*_{\alpha, i}, \tag{2.10}
\]

where the elements \( k^*_{l b_{\alpha, i}} \)’s are as in (2.4);

5. for \( k \in \mathbb{Z}_{\geq 0} \) and \( U_i, U_j \subset V_\alpha \beta \), we have

\[
k^*_{\psi_{\alpha, i}}(U_i \cap U_j) \circ k^*_{\psi_{\beta, i}}(U_i \cap U_j) = \exp \left( L_{k,p_{\alpha, \beta, i}} \right), \tag{2.11}
\]

where the elements \( k^*_{p_{\alpha, \beta, i}} \)’s are as in (2.5), and

6. for \( k \in \mathbb{Z}_{\geq 0} \) and \( U_i \subset V_\alpha \gamma \), we have

\[
k^*_{\psi_{\alpha, i}}(U_i) \circ k^*_{\psi_{\beta, i}}(U_i) \circ k^*_{\psi_{\gamma, i}}(U_i) = \exp \left( L_{k,\varphi_{\alpha, \beta, i}} \right), \tag{2.12}
\]

where the elements \( k^*_{\varphi_{\alpha, \beta, i}} \)’s are as in (2.6).
Since both \( k\omega_\alpha \) and \( k\omega_\beta \) are nowhere-vanishing, \( k\hat{\psi}_{\alpha\beta,i} \) actually determines \( k\psi_{\alpha\beta,i} \). Also observe that for every \( k \in \mathbb{Z}_{\geq 0} \) and any \( U_i \subseteq V_{\alpha\beta} \), we have a commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
0 \rightarrow K \otimes \mathbb{Z} \left( \|K\|^n \left[ -1 \right] \right) |_U \rightarrow \left( \frac{kK^n}{2} \right) |_U \rightarrow \frac{kK^n}{2} |_U \rightarrow 0 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
0 \rightarrow K \otimes \mathbb{Z} \left( \|K\|^n \left[ -1 \right] \right) |_U \rightarrow \left( \frac{kK^n}{2} \right) |_U \rightarrow \frac{kK^n}{2} |_U \rightarrow 0.
\end{array}
\end{array}
\]

**Remark 2.19.** We can deduce (2.4) from (2.9) as follows: From (2.9), we have \( (k\hat{\psi}_{\alpha,i} \circ (\frac{k\omega_{\beta,U_i}}{k\omega_{\beta,U_i}}) \circ k\psi_{\alpha,i}))(\gamma) = (\gamma \wedge \exp(k\psi_{\alpha,i})) \circ (\frac{k\omega_{\beta,U_i}}{k\omega_{\beta,U_i}}) \circ k\psi_{\alpha,i} \), so \( (k\hat{\psi}_{\alpha,i} \circ k\Delta_\beta \circ k\psi_{\alpha,i})(\gamma) \wedge \exp(k\psi_{\alpha,i}) = k\Delta_\alpha (\gamma \wedge \exp(k\psi_{\alpha,i})) = (k\Delta_\alpha (\gamma) + [k\psi_{\alpha,i}, \gamma]) \wedge \exp(k\psi_{\alpha,i}) \) for any \( \gamma \in k\mathcal{G}_U(U_i) \), which gives \( k\Delta_\alpha (\gamma) + [k\psi_{\alpha,i}, \gamma] = (k\hat{\psi}_{\alpha,i} \circ k\Delta_\beta \circ k\psi_{\alpha,i})(\gamma) \).

### 3. Abstract construction of the Čech-Thom-Whitney complex

#### 3.1. The simplicial set \( \mathcal{A}^*(\Delta_n) \)

In this subsection, we recall some notions and facts about the simplicial sets \( \mathcal{A}^*_k(\Delta_n) \) of polynomial differential forms with coefficient \( k = \mathbb{Q}, \mathbb{R}, \mathbb{C} \); we will simply write \( \mathcal{A}^*(\Delta_n) \) when \( k = \mathbb{C} \), which will be the case for all other parts of this paper.

**Notations 3.1.** We let \( \text{Mon} \) (resp. \( \text{sMon} \)) be the category of finite ordinals \([n] = \{0, 1, \ldots, n\}\) in which morphisms are non-decreasing maps (resp. strictly increasing maps). We denote by \( d_{i,n} : [n-1] \rightarrow [n] \) the unique strictly increasing map which skips the \( i \)-th element, and by \( e_{i,n} : [n+1] \rightarrow [n] \) the unique non-decreasing map sending both \( i \) and \( i+1 \) to the same element \( i \).

Note that every morphism in \( \text{Mon} \) can be decomposed as a composition of the maps \( d_{i,n} \)'s and \( e_{i,n} \)'s, and any morphism in \( \text{sMon} \) can be decomposed as a composition of the maps \( d_{i,n} \)'s.

**Definition 3.2 (50).** Let \( C \) be a category. A (semi-)simplicial object in \( C \) is a contravariant functor \( A(\bullet) : \text{Mon} \rightarrow C \) (resp. \( A(\bullet) : \text{sMon} \rightarrow C \)), and a (semi-)cosimplicial object in \( C \) is a covariant functor \( A(\bullet) : \text{Mon} \rightarrow C \) (resp. \( A(\bullet) : \text{sMon} \rightarrow C \)).

**Definition 3.3 (17).** Let \( k \) be a field which is either \( \mathbb{Q}, \mathbb{R} \) or \( \mathbb{C} \). Consider the dga

\[ A_k^*(\Delta_n) := \mathbb{K}[x_0, \ldots, x_n, dx_0, \ldots, dx_n] \]  

with \( \deg(x_i) = 0, \deg(dx_i) = 1 \), and equipped with the degree 1 differential \( d \) defined by \( d(x_i) = dx_i \) and the Leibniz rule. Given \( a : [n] \rightarrow [m] \) in \( \text{Mon} \), we let \( a^* \) := \( A_k(a) : A_k^*(\Delta_m) \rightarrow A_k^*(\Delta_n) \) be the unique dga morphism satisfying \( a^*(x_j) = \sum_{i \in [n]: a(i) = j} x_i \) and \( a^*(x_j) = 0 \) if \( j \neq a(i) \) for any \( i \in [n] \). From this we obtain a simplicial object in the category of dga’s, which we denote as \( A_k^*(\Delta_n) \).

**Notations 3.4.** We denote by \( \Delta_n \) the boundary of \( \Delta_n \), and let

(3.1) \[ A_k^*(\Delta_n) := \{ (\alpha_0, \ldots, \alpha_n) \mid \alpha_i \in A_k^*(\Delta_{n-1}), d^*_{i,n-1}(\alpha_j) = d^*_{i-1,n-1}(\alpha_i) \text{ for } 0 \leq i < j \leq n \} \]

be the space of polynomial differential forms on \( \Delta_n \). There is a natural restriction map defined by \( \beta |_{\Delta_n} : (d_{i,n}(\beta), \ldots, d_{n,n}(\beta)) \) for \( \beta \in A_k^*(\Delta_n) \).

The following extension lemma will be frequently used in subsequent constructions:

\[^8\text{In the case } k = \mathbb{R}, \text{ this can be thought of as the space of polynomial differential forms on } \mathbb{R}^{n+1} \text{ restricted to the } n\text{-simplex } \Delta_n.\]
Lemma 3.5 (Lemma 9.4 in [17]). For any \( \bar{\alpha} = (\alpha_0, \ldots, \alpha_n) \in A^*_k(\Delta_n) \), there exists \( \beta \in A^*_k(\Delta_n) \) such that \( \beta|_{\Delta_n} = \bar{\alpha} \).

Notations 3.6. We let \( \blacksquare_n := \Delta_1 \times \Delta_n \), where \( \Delta_1 := \{(t_0, t_1) \mid 0 \leq t_1 \leq 1, t_0 + t_1 = 1\} \), and
\[
A^*_k(\blacksquare_n) := A^*_k(\Delta_1) \otimes_k A^*_k(\Delta_n) = \{ \sum_{i=0}^n x_i - 1, \sum_{i=0}^n dx_i, t_0 + t_1 - 1, dt_0 + dt_1 \}.
\]
Besides the restriction maps \( d^*_j : A^*_k(\blacksquare_n) \rightarrow A^*_k(\blacksquare_{n-1}) \) induced from that on \( \Delta_n \), we also have the maps \( r^*_j : A^*_k(\blacksquare_n) \rightarrow A^*_k(\Delta_n) \) defined by putting \( t_j = 1 \) (and \( t_{1-j} = 0 \)).

Notations 3.7. We denote by \( \square_n \) the boundary of \( \blacksquare_n \), and let
\[
A^*_k(\square_n) := \left\{ (\alpha_0, \ldots, \alpha_n, \beta_0, \beta_1) \mid \alpha_i \in A^*_k(\blacksquare_{n-1}), \beta_i \in A^*(\Delta_1), \right. \left. r^*_i(\alpha_j) = d^*_{i-1, n-1}(\alpha_i) \text{ for } 0 \leq i < j \leq n \right\}
\]
be the space of polynomial differential forms on \( \square_n \). There is a natural restriction map defined by \( \gamma|_{\square_n} := (d^*_0(\gamma), \ldots, d^*_n(\gamma), r^*_0(\gamma), r^*_1(\gamma)) \) for \( \gamma \in A^*_k(\blacksquare_n) \).

Lemma 3.8. For any \( (\alpha_0, \ldots, \alpha_n, \beta_0, \beta_1) \in A^*_k(\square_n) \), there exists \( \gamma \in A^*_k(\blacksquare_n) \) such that \( \gamma|_{\square_n} = (\alpha_0, \ldots, \alpha_n, \beta_0, \beta_1) \).

This variation of Lemma 3.5 can be proven by the same technique as in [17] Lemma 9.4.

3.2. Local Thom-Whitney complexes. Consider a sheaf of BV algebras \( (\mathcal{G}, \wedge, \Delta) \) on a topological space \( V \) together with an acyclic cover \( \mathcal{U} = \{U_i\}_{i \in \mathbb{Z}_+} \) of \( V \) such that \( H^0(U_{i_0 \cdots i_l}, \mathcal{G}) = 0 \) for all \( j \) and all finite intersections \( U_{i_0 \cdots i_j} := U_{i_0} \cap \cdots \cap U_{i_j} \). In particular, this allows us to compute the sheaf cohomology \( H^\bullet(V, \mathcal{G}) \) and the hypercohomology \( H^\bullet(V, \mathcal{G}) \) using the Čech complex \( \check{C}^\bullet(\mathcal{U}, \mathcal{G}) \) and the total complex of \( \check{C}^\bullet(\mathcal{U}, \mathcal{G}) \) respectively.

Let \( I = \{(i_0, \ldots, i_l) \mid i_j \in \mathbb{Z}_+, i_0 < i_1 < \cdots < i_{l}\} \) be the index set. Let \( \Delta_I \) be the standard \( l \)-simplex in \( \mathbb{R}^{l+1} \) and \( A^q(\Delta_I) \) be the space of \( \mathbb{C} \)-valued polynomial differential \( q \)-forms on \( \Delta_I \). Also let \( d^*_{j,l} : \Delta_{l-1} \rightarrow \Delta_I \) be the inclusion of the \( j \)-th facet in \( \Delta_I \) and let \( d^*_j \) be the pullback map. See Definition 3.3 and Notation 3.1 for details.

Definition 3.9 (see e.g. [51] 9 [11]). The Thom-whitney complex is defined as \( TW^{*,*}(\mathcal{G}) := \bigoplus_{p,q} TW^{p,q}(\mathcal{G}) \) where
\[
TW^{p,q}(\mathcal{G}) := \left\{ (\varphi_{i_0 \cdots i_l})_{(i_0, \ldots, i_l)} \in I \mid \varphi_{i_0 \cdots i_l} \in A^q(\Delta_I) \otimes \mathcal{G}^p(U_{i_0 \cdots i_l}) \right\},
\]
\( TW^{p,q}(\mathcal{G}) \) is equipped with the structures \( \wedge, \Delta_I \) defined componentwise by
\[
(\alpha_I \otimes v_I) \wedge (\beta_I \otimes w_I) := (-1)^{|v_I||\beta_I|}(\alpha_I \wedge \beta_I) \otimes (v_I \wedge w_I),
\]
\( \Delta_I(\alpha_I \otimes v_I) := (d\alpha_I) \otimes v_I, \Delta_I(\alpha_I \otimes v_I) := (-1)^{|v_I|}(\alpha_I \otimes (\Delta_I v_I)) \)
for \( \alpha_I, \beta_I \in A^*(\Delta_I) \) and \( v_I, w_I \in \mathcal{G}^*(U_I) \), where \( I = (i_0, \ldots, i_l) \in I \) and \( l = |I| - 1 \).

Remark 3.10. We use the notation \( \tilde{\partial} \) since it plays the role of the Dolbeault operator in the classical deformation theory of smooth Calabi-Yau manifolds.

\( (TW^{*,*}(\mathcal{G}), \tilde{\partial}, \Delta, \wedge) \) forms a dgBV algebra in the sense of Definition 2.2. From Definitions 2.2 and 3.9, the Lie bracket on the Thom-Whitney complex is determined componentwise by the formula
\[
[\alpha_I \otimes v_I, \beta_I \otimes w_I] := (-1)^{|v_I|+1}|\beta_I|(\alpha_I \wedge \beta_I) \otimes [v_I, w_I],
\]
for \( \alpha_I, \beta_I \in A^*(\Delta_I) \) and \( v_I, w_I \in \mathcal{G}^*(U_I) \) where \( l = |I| - 1 \).  

\footnote{Readers may assume that \( \mathcal{G}^* \) is a bounded complex for the purpose of this paper.}
We consider the integration map $I : TW^{p,q}(\mathcal{G}) \rightarrow \check{C}^{q}(\mathcal{U}, \mathcal{G}^p)$ defined by

$$I(\alpha_{i_0...i_l}) := \left( \int_{\Delta} \otimes \text{id} \right)(\alpha_{i_0...i_l})$$

for each component $\alpha_{i_0...i_l} \in \mathcal{A}^q(\Delta_i) \otimes \mathcal{G}^p(U_{i_0...i_l})$ of $(\alpha_{i_0...i_l})_{(i_0...i_l) \in I} \in TW^{p,q}(\mathcal{G})$. Notice that $I$ is a chain morphism from $(TW^{p,q}(\mathcal{G}), \bar{\partial})$ to $(\check{C}^{q}(\mathcal{U}, \mathcal{G}^p), \bar{\partial})$, where $\bar{\partial}$ is the Čech differential. Taking the total complexes gives a chain morphism from $(TW^{*,*}(\mathcal{G}), \bar{\partial} \pm \Delta)$ to $\check{C}^{*}(\mathcal{U}, \mathcal{G}^*)$, which is equipped with the total Čech differential $\Delta$.

**Lemma 3.11**. The maps $I : TW^{p,*}(\mathcal{G}) \rightarrow \check{C}^{*}(\mathcal{U}, \mathcal{G}^p)$ and $I : TW^{*,*}(\mathcal{G}) \rightarrow \check{C}^{*}(\mathcal{U}, \mathcal{G}^*)$ are quasi-isomorphisms.

**Remark 3.12**. Comparing to the standard construction of the Thom-Whitey complex in e.g. [11], where one considers $(\varphi_{i_0...i_l})_{(i_0...i_l) \in I} \in \prod_{l \geq 0} \left( \mathcal{A}^l(\Delta_i) \otimes \bigotimes_{i_0<...<i_l} \mathcal{G}^p(U_{i_0...i_l}) \right)$, we are taking a bigger complex in Definition 3.9 for the purpose of later constructions. However, the original proof of Lemma 3.11 works in exactly the same way for this bigger complex, and hence $TW^{p,*}(\mathcal{G})$ also serves as a resolution of $\mathcal{G}^p$.

**Definition 3.13**. Given the $0$th-order complex of polyvector fields $(0\mathcal{G}^*, \Lambda, 0\mathcal{A})$ over $X$ (Definition 2.9), we use the cover $\mathcal{U}$ in Notation 2.14 to define the $0$th-order Thom-Whitey complex $(TW^{0,*}(\mathcal{G}), \bar{\partial}, 0\mathcal{A}, \Lambda)$. To simplify notations, we write $0TW^{*,*}$ to stand for $TW^{0,*}(0\mathcal{G})$.

Given a finite intersection of open subsets $V_{\alpha_0...\alpha_\ell} := V_{\alpha_0} \cap \cdots \cap V_{\alpha_\ell}$ of the cover $\mathcal{V}$, and local thickenings of the complex of polyvector fields $(k\mathcal{G}_{\alpha_0}^*, \Lambda, k\mathcal{A}_{\alpha_0})$ over $V_{\alpha_\ell}$ for each $k \in \mathbb{Z}_{>0}$ (Definition 2.13), we use the cover $\mathcal{U}_{\alpha_0...\alpha_\ell} := \{ U \in \mathcal{U} | U \subset V_{\alpha_0...\alpha_\ell} \}$ to define the local Thom-Whitey complex $(TW^{*,*}(k\mathcal{G}_{\alpha_0}|_{V_{\alpha_0...\alpha_\ell}}), \bar{\partial}, k\mathcal{A}_{\alpha_0}, \Lambda)$ over $V_{\alpha_0...\alpha_\ell}$. To simplify notations, we write $kTW^{*,*}$ to stand for $TW^{*,*}(k\mathcal{G}_{\alpha_0}|_{V_{\alpha_0...\alpha_\ell}})$.

The covers $\mathcal{U}$ and $\mathcal{U}_{\alpha_0...\alpha_\ell}$ satisfy the acyclic assumption at the beginning of this section because $0\mathcal{G}^*$ and $k\mathcal{G}_{\alpha_0}^*$ are coherent sheaves and all the open sets in these covers are Stein.

**Theorem 3.14** (Cartan’s Theorem B [3]; see e.g. Chapter IX Corollary 4.11 in [8]). For a coherent sheaf $\mathcal{F}$ over a Stein space $U$, we have $H^0(U, \mathcal{F}) = 0$.

### 3.3. The gluing morphisms

3.3.1. Existence of a set of compatible gluing morphisms. The aim of this subsection is to construct, for each $k \in \mathbb{Z}_{\geq 0}$ and any pair $V_{\alpha}, V_{\beta} \in \mathcal{V}$, an isomorphism

$$k g_{\alpha \beta} : kTW^{*,*}_{\alpha \alpha} \rightarrow kTW^{*,*}_{\beta \alpha \beta},$$

as a collection of maps $(k g_{\alpha \beta, I})_{I \in \mathcal{I}}$ so that for each $\varphi = (\varphi_I)_{I \in \mathcal{I}} \in kTW^{*,*}_{\alpha \alpha}$ with $\varphi_I \in \mathcal{A}^*(\Delta_i) \otimes k\mathcal{G}_{\alpha | U_{\alpha}}$ we have $(k g_{\alpha \beta}(\varphi))_I = k g_{\alpha \beta, I}(\varphi_I)$, which preserves the algebraic structures $[\cdot, \cdot], \Lambda$ and satisfies the following condition:

**Condition 3.15.** (1) for $U_I \subset V_{\alpha} \cap V_{\beta}$, we have

$$k g_{\alpha \beta, i} = \exp([k a_{\alpha \beta, i}, \cdot]) \circ k \psi_{\alpha \beta, i}$$

for some element $k a_{\alpha \beta, i} \in k \mathcal{G}_{\alpha | U_{\alpha}}^{-1}(U_I)$ with $k a_{\alpha \beta, i} = 0 (mod \mathfrak{m})$;

(2) for $U_{i_0}, \ldots, U_{i_l} \subset V_{\alpha} \cap V_{\beta}$, we have

$$k g_{\alpha \beta, i_0 \cdots i_l} = \exp([k \vartheta_{\alpha \beta, i_0 \cdots i_l}, \cdot]) \circ \left( k g_{\alpha \beta, i_0} | U_{i_0 \cdots i_l} \right),$$

for some element $k \vartheta_{\alpha \beta, i_0 \cdots i_l} \in \mathcal{A}^0(\Delta_i) \otimes k \mathcal{G}_{\beta | U_{i_0 \cdots i_l}}^{-1}(U_{i_0 \cdots i_l})$ with $k \vartheta_{\alpha \beta, i_0 \cdots i_l} = 0 (mod \mathfrak{m})$; and
(3) the elements \( k \partial_{\alpha, i_0 \cdots i_l} \)’s satisfy the relation\(^{10}\)

\[
\begin{align*}
\mathbf{d}^*(I^j) & = \begin{cases} 
 k \partial_{\alpha, i_0 \cdots i_l} & 	ext{for } j > 0, \\
 k \partial_{\alpha, i_0 \cdots i_l} \circ k \phi_{\alpha, i_0 i_1} & 	ext{for } j = 0,
\end{cases}
\end{align*}
\]

where \( k \phi_{\alpha, i_0 i_1} \in k G_{\beta}^{-1}(U_{i_0 i_1}) \) is the unique element such that

\[
\exp((k \phi_{\alpha, i_0 i_1}, \cdot)) k g_{\alpha, i_0} = k g_{\alpha, i_1}.
\]

**Lemma 3.16.** Suppose that the morphisms \( k g_{\alpha, \beta} \)’s, each of which is a collection of maps \( (k g_{\alpha, \beta})_I \in \mathcal{I} \), all satisfy Condition\(^{13}\) For any \( \varphi = (\varphi_I)_{I \in \mathcal{I}} \in k TW_{\alpha, \alpha, \beta}^* \), we have \( (k g_{\alpha, \beta})(\varphi_I)_{I \in \mathcal{I}} \in k TW_{\beta, \alpha, \beta}^* \).

**Proof.** Suppose that we have \( (\varphi_I)_{I \in \mathcal{I}} \in k TW_{\alpha, \alpha, \beta}^* \) such that \( \varphi_{i_0 \cdots i_l} \in A^0(\mathbf{1}_l) \otimes k G_{\alpha}^*(U_{i_0 \cdots i_l}) \) and \( \varphi_{i_0 \cdots i_l} = \mathbf{d}^*_j(\varphi_{i_0 \cdots i_l}) \). Letting \( (k g_{\alpha, \beta})(\varphi)_{i_0 \cdots i_l} := (\exp(k \partial_{\alpha, i_0 \cdots i_l}, \cdot)) k g_{\alpha, \beta}(\varphi)_{i_0 \cdots i_l} \), we have

\[
\begin{align*}
(k g_{\alpha, \beta})(\varphi)_{i_0 \cdots i_l} &= (\exp(k \partial_{\alpha, i_0 \cdots i_l}, \cdot)) k g_{\alpha, \beta}(\varphi)_{i_0 \cdots i_l} \\
&= (\exp(k \partial_{\alpha, i_0 \cdots i_l}, \cdot)) k g_{\alpha, \beta}(\varphi)_{i_0 \cdots i_l} \\
&= \mathbf{d}^*_j(\varphi)_{i_0 \cdots i_l},
\end{align*}
\]

and

\[
\begin{align*}
(k g_{\alpha, \beta})(\varphi)_{i_0 \cdots i_l} &= (\exp(k \partial_{\alpha, i_0 \cdots i_l}, \cdot)) k g_{\alpha, \beta}(\varphi)_{i_0 \cdots i_l} \\
&= (\exp(k \partial_{\alpha, i_0 \cdots i_l}, \cdot)) k g_{\alpha, \beta}(\varphi)_{i_0 \cdots i_l} \\
&= \mathbf{d}^*_j(\varphi)_{i_0 \cdots i_l},
\end{align*}
\]

which are the required conditions for \( (k g_{\alpha, \beta})(\varphi)_I \in k TW_{\beta, \alpha, \beta}^* \).

Given a multi-index \((\alpha_0 \cdots \alpha_\ell)\), we have, for each \( j = 0, \ldots, \ell \), a natural restriction map

\[
\tau_{\alpha_j} : k TW_{\alpha; \alpha_0 \cdots \alpha_j \cdots \alpha_\ell}^* \to k TW_{\alpha_0 \cdots \alpha_\ell}^*
\]

defined componentwise by

\[
\tau_{\alpha_j}(\varphi_I) = \begin{cases} 
 \varphi_I & \text{if } U_I \subset V_{\alpha_0 \cdots \alpha_\ell} \text{ for all } I \in \mathcal{I}, \\
 0 & \text{otherwise,}
\end{cases}
\]

for \( (\varphi_I)_{I \in \mathcal{I}} \in k TW_{\alpha; \alpha_0 \cdots \alpha_j \cdots \alpha_\ell}^* \). The map \( \tau_{\alpha_j} \) is a morphism of dgBV algebras.

Now for a triple \( V_\alpha, V_\beta, V_\gamma \in \mathcal{V} \), we define the restriction of \( k g_{\alpha, \beta} \) to \( k TW_{\alpha; \alpha, \beta}^* \) as the unique map \( k g_{\alpha, \beta} : k TW_{\alpha; \alpha, \beta}^* \to k TW_{\beta; \alpha, \beta}^* \) that fits into the diagram

\[
\begin{array}{ccc}
\begin{array}{c} 
\tau_{\alpha_j}
\end{array} & \begin{array}{c} k TW_{\alpha; \alpha_0 \cdots \alpha_j \cdots \alpha_\ell}^* \to k TW_{\alpha_0 \cdots \alpha_\ell}^*
\end{array} & \begin{array}{c} k g_{\alpha, \beta}
\end{array} \\
\downarrow & \downarrow & \downarrow \\
\begin{array}{c} k TW_{\beta; \alpha_0 \cdots \alpha_j \cdots \alpha_\ell}^* \to k TW_{\beta_0 \cdots \alpha_\ell}^*
\end{array} & \begin{array}{c} k g_{\alpha, \beta}
\end{array} \\
\end{array}
\]

**Definition 3.17.** The morphisms \( \{ k g_{\alpha, \beta} \} \) satisfying Condition\(^{13}\) are said to form a set of compatible gluing morphisms if in addition the following conditions are satisfied:

\(^{10}\)Here \( \mathbf{d}^*_j \) is induced by the corresponding map \( \mathbf{d}^*_j : A^*(\mathbf{1}_l) \to A^*(\mathbf{1}_{l-1}) \) on the simplicial set \( A^*(\mathbf{1}_l) \) introduced in Definition\(^{3}\) and \( \circ \) is the Baker-Campbell-Hausdorff product in Notation\(^{2}\).
Theorem 3.18. There exists a set of compatible gluing morphisms \( \{ k g_{\alpha \beta} \} \).

We will construct the gluing morphisms \( k g_{\alpha \beta} \)’s inductively. To do so, we need a couple of lemmas.

Lemma 3.19. Fixing \( U_{i_0}, \ldots, U_{i_l} \in \mathcal{U} \) and \(-d \leq j \leq 0\), we consider the index set \( I_{i_0 \ldots i_l} := \{ \alpha \mid U_{i_\alpha} \subset V_{\alpha} \text{ for all } 0 \leq r \leq l \} \) and the following Čech complex \( \check{C}^*(I_{i_0 \ldots i_l}, 0 \mathcal{G}^j) \) of vector spaces

\[
0 \to \prod_{\alpha \in I_{i_0 \ldots i_l}} \mathcal{G}^j(U_{i_0 \ldots i_l} \cap V_{\alpha}) \to \prod_{\alpha, \beta \in I_{i_0 \ldots i_l}} \mathcal{G}^j(U_{i_0 \ldots i_l} \cap V_{\alpha \beta}) \to \prod_{\alpha, \beta, \gamma \in I_{i_0 \ldots i_l}} \mathcal{G}^j(U_{i_0 \ldots i_l} \cap V_{\alpha \beta \gamma}) \cdots,
\]

where each arrow is the Čech differential associated to the index set \( I_{i_0 \ldots i_l} \). Then we have

\[
H^0(\check{C}(I_{i_0 \ldots i_l}, 0 \mathcal{G}^j)) = 0 \text{ and } H^0(\check{C}(I_{i_0 \ldots i_l}, 0 \mathcal{G}^j)) = 0 \mathcal{G}^j(U_{i_0 \ldots i_l});
\]

the same holds for \( 0 \mathcal{G}^j \otimes \mathcal{V} \) for any vector space \( \mathcal{V} \).

Proof. We consider the topological space \( pt \) consisting of a single point and an indexed cover \((\mathcal{V}_\alpha)_{\alpha \in I_{i_0 \ldots i_l}}\) such that \( \mathcal{V}_\alpha = pt \) for each \( \alpha \). Then we take a constant sheaf \( F \) over \( pt \) with \( F(pt) = 0 \mathcal{G}^j(U_{i_0 \ldots i_l}) \). Since \( 0 \mathcal{G}^j(U_{i_0 \ldots i_l} \cap V_{\alpha_0 \ldots \alpha_\ell}) = 0 \mathcal{G}^j(U_{i_0 \ldots i_l}) = F(V_{\alpha_0 \ldots \alpha_\ell}) \) for any \( \alpha_0, \ldots, \alpha_\ell \in I_{i_0 \ldots i_l} \), we have a natural isomorphism \( \check{C}^*(I_{i_0 \ldots i_l}, 0 \mathcal{G}^j) \cong \check{C}^*(I_{i_0 \ldots i_l}, F) \). The result then follows by considering the Čech cohomology of \( pt \).

Lemma 3.20 (Lifting Lemma). Let \( \mathcal{U} \) be a surjective morphism of sheaves over \( V := V_{\alpha_0 \ldots \alpha_\ell} \). For a Stein open subset \( U := U_{i_0 \ldots i_l} \subset V \) let \( \mathfrak{v} : A^q(\mathcal{U}) \otimes \mathcal{H}(U) \to \partial \mathcal{V} \in A^q(\mathcal{U}) \otimes \mathcal{F}(U) \) such that \( \mathfrak{v}(\mathfrak{u}) = \mathfrak{w} \). Then there exists \( \mathfrak{v} \in \mathcal{F}(\mathcal{U}) \) such that \( \mathfrak{v}|_{\mathcal{U}} = \partial \mathcal{V} \) and \( \mathfrak{v}(\mathfrak{u}) = \mathfrak{w} \). The same holds if \( A^q(\mathcal{U}) \otimes \mathcal{H}(U) \) is replaced by \( A^q(\mathcal{U}) \) and \( A^q(\mathcal{U}) \) respectively.

Proof. By Lemma 3.5, there is a lifting \( \tilde{\mathfrak{v}} \in A^q(\mathcal{U}) \otimes \mathcal{F}(U) \) such that \( \tilde{\mathfrak{v}}|_{\mathcal{U}} = \partial \mathcal{V} \). Let \( u := \mathfrak{v} - \partial \tilde{\mathfrak{v}} \in A^q(\mathcal{U}) \otimes \mathcal{H}(U) \), where \( A^q(\mathcal{U}) \) is the space of differential 0-forms whose restriction to \( \mathcal{U} \) is 0. Since \( U \) is Stein, the map \( \mathcal{U} : \mathcal{F}(U) \to \mathcal{H}(U) \) is surjective. So we have a lifting \( \tilde{u} \) of \( u \) to \( A^q(\mathcal{U}) \otimes \mathcal{F}(U) \). Now the element \( \mathfrak{v} := \tilde{\mathfrak{v}} + \tilde{\mathfrak{u}} \) satisfies the desired properties. The same proof applies to the case involving \( U \) and \( U \).

Lemma 3.21 (Key Lemma). Suppose we are given a set of gluing morphisms \( \{ k g_{\alpha \beta} \} \) for some \( k \geq 0 \) satisfying Condition 3.15 and the cocycle condition (3.10). Then there exists a set of \( \{ k+1 g_{\alpha \beta} \} \) satisfying Condition 3.15, the compatibility condition (3.9) as well as the cocycle condition (3.10).

Proof. We will prove by induction on \( l \) where \( l = |I| - 1 \) for a multi-index \( I = (i_0, \ldots, i_l) \in \mathcal{I} \).

For \( l = 0 \), we fix \( i = i_0 \). From (3.5) in Condition 3.15, we have \( k g_{\alpha \beta, i} = \exp([k a_{\alpha \beta, i}, \cdot]) \circ k \psi_{\alpha \beta, i} \) for some \( k a_{\alpha \beta, i} \in k \mathcal{G}^{-1}_{\alpha}(U_i) \). Also, from (2.4) in Definition 2.15, there exist elements \( k+1 k b_{\alpha \beta, i} \in k \mathcal{G}^{-1}_{\alpha}(U_i) \) such that

\[
k+1 k b_{\alpha \beta, i} \circ k+1 k \psi_{\alpha \beta, i} = k \psi_{\alpha \beta, i} \circ \exp([k+1 k b_{\alpha \beta, i}, \cdot]) \circ k+1 k g_{\alpha} = \exp \left( [k \psi_{\alpha \beta, i}, k+1 k b_{\alpha \beta, i}] \right) \circ k \psi_{\alpha \beta, i} \circ k+1 k g_{\alpha}.
\]
where we use the fact that \( k\psi_{\alpha\beta,i} \) is an isomorphism preserving the Lie bracket \([\cdot, \cdot]\). Therefore we have \( k\gamma_{\alpha\beta,i} \circ k^{+1}\beta_{\alpha} = \exp([k\alpha_{\alpha\beta,i}, \cdot]) \circ \exp([-([k\psi_{\alpha\beta,i}]/k^{+1}k\beta_{\alpha\beta,i}, \cdot)]) \circ k^{+1}\beta_{\alpha} \circ k^{+1}\psi_{\alpha\beta,i} \). By taking a lifting \( U_{\alpha\beta,i} \) of the term \( k\alpha_{\alpha\beta,i} \otimes (k\psi_{\alpha\beta,i}(-k^{+1}k\beta_{\alpha\beta,i})) \) from \( kG^{+1}_{\beta} \times kG^{+1}_{\alpha} \) to \( k^{+1}G^{+1}_{\beta}(U_{i}) \) in the above equation (using the surjectivity of the map \( k^{+1}k\beta_{\alpha} : k^{+1}G^{+1}_{\alpha} \to k^{+1}G^{+1}_{\alpha} \)), we define a lifting

\[
k^{+1}g_{\alpha\beta,i} := \exp([\gamma_{\alpha\beta,i}, \cdot]) \circ k^{+1}\psi_{\alpha\beta,i} : k^{+1}G^{+1}_{\alpha}|_{U_{i}} \to k^{+1}G^{+1}_{\beta}|_{U_{i}}
\]
of \( k\gamma_{\alpha\beta,i} \).

As endomorphisms of \( k^{+1}G^{+1}_{\alpha} \), we have \( k^{+1}\gamma_{\alpha\beta,i} \circ k^{+1}g_{\beta\gamma,i} \circ k^{+1}\beta_{\alpha} = \exp([k^{+1}\alpha_{\alpha\beta,i}, \cdot]) \) for some \( k^{+1}\gamma_{\alpha\beta,i} \in k^{+1}G^{+1}_{\alpha}(U_{i}) \). From the fact that \( k^{+1}\gamma_{\alpha\beta,i} \circ k^{+1}g_{\beta\gamma,i} \circ k^{+1}\beta_{\alpha} = \text{id} \pmod{m^{+1}} \), we have \( \exp([k^{+1}\alpha_{\alpha\beta,i}, \cdot])(v) = v \pmod{m^{+1}} \) for all \( v \in kG^{+1}_{\alpha}(U_{i}) \). Therefore \( k^{+1}\gamma_{\alpha\beta,i} = 0 \pmod{m^{+1}} \) as the map \( kG^{+1}_{\alpha} \to \text{Der}(kG^{+1}_{\alpha}) \) is injective (see Definition \ref{definition:cohomology}). Since every stalk \( (k^{+1}G^{+1}_{\alpha})x \) is a free \( k^{+1}R \)-module and \( k^{+1}0_{\alpha} \) induces a sheaf isomorphism upon tensoring with the residue field \( 0R \cong \mathbb{C} \) (over \( k^{+1}R \)), we have a sheaf isomorphism

\[
(3.11) \quad k^{+1}G^{+1}_{\alpha} \cong \mathbf{0} \mathcal{G}^{+1}_{\alpha} \oplus \bigoplus_{j=1}^{k^{+1}} (m^{j}/m^{j+1}) \otimes \mathbb{C}^{0}G^{+1}_{\alpha} .
\]

Hence we have \( k^{+1}\gamma_{\alpha\beta,i} \in (m^{+1}/m^{+2}) \otimes \mathbb{C}^{0}G^{+1}_{\alpha}(U_{i}) \).

Now we consider the Čech complex \( C^{*}(I_{U_{i}}, 0G^{+1}) \otimes \mathbb{C} \) as in Lemma \ref{lemma:cohomology}. The collection \( (k^{+1}\gamma_{\alpha\beta,i})_{(\alpha\beta, \gamma) \in I_{U_{i}}} \) is a 2-cocycle in \( C^{2}(I_{U_{i}}, 0G^{+1}) \otimes \mathbb{C} \) as \ref{lemma:cohomology}. By Lemma \ref{lemma:cohomology} for the case \( l = 0 \), there exists \( (k^{+1}\alpha_{\alpha\beta,i})_{(\alpha, \beta) \in I_{U_{i}}} \in C^{1}(I_{U_{i}}, 0G^{+1}) \otimes \mathbb{C} \) whose image under the Čech differential is precisely \( (k^{+1}\gamma_{\alpha\beta,i})_{(\alpha\beta, \gamma) \in I_{U_{i}}} \). By the identification \ref{equation:cohomology}, we can regard \( (k^{+1}\alpha_{\alpha\beta,i}) \) as an element in \( k^{+1}G^{+1}_{\beta}(U_{i}) \) such that \( k^{+1}\gamma_{\alpha\beta,i} = 0 \pmod{m^{+1}} \). Therefore letting \( k^{+1}\gamma_{\alpha\beta,i} := \exp([k^{+1}\alpha_{\alpha\beta,i}, \cdot]) \circ k^{+1}g_{\alpha\beta,i} \), we have the cocycle condition \( k^{+1}\gamma_{\alpha\beta,i} \circ k^{+1}g_{\beta\gamma,i} \circ k^{+1}g_{\alpha\beta,i} = \text{id} \).

For the induction step, we assume that maps \( k^{+1}\gamma_{\alpha\beta,i} \circ \cdots \) satisfying all the required conditions have been constructed for each multi-index \( (i_{0}, \ldots, i_{j}) \) with \( j \leq l - 1 \). We shall construct \( k^{+1}\gamma_{\alpha\beta,i} \circ \cdots \) for any multi-index \( (i_{0}, \ldots, i_{l}) \).

In view of Condition \ref{condition:cohomology}, what we need are elements \( k^{+1}\gamma_{\alpha\beta,i} \circ \cdots \in \mathcal{A}^{0}((\mathbf{A}_{l})) \otimes k^{+1}G^{+1}_{\beta}(U_{i_{0}} \cdots i_{l}) \) satisfying \ref{equation:cohomology}, and the cocycle condition \ref{equation:cohomology}, the latter of which can be written explicitly as

\[
\exp([k^{+1}\gamma_{\alpha\beta,i} \circ \cdots]) \circ k^{+1}\gamma_{\alpha\beta,i} \circ \exp([k^{+1}\gamma_{\beta\gamma,i} \circ \cdots]) \circ k^{+1}\gamma_{\beta\gamma,i} \circ \exp([k^{+1}\gamma_{\alpha\beta,i} \circ \cdots]) \circ k^{+1}\gamma_{\alpha\beta,i} = \exp([k^{+1}\gamma_{\alpha\beta,i} \circ \cdots]) \circ \exp([k^{+1}\gamma_{\alpha\beta,i} \circ \cdots]) \circ \exp([k^{+1}\gamma_{\alpha\beta,i} \circ \cdots]) = \text{id} .
\]

Using the \( k^{+1}\gamma_{\alpha\beta,i} \circ \cdots \) that were defined previously, we let

\[
\partial(k^{+1}\gamma_{\alpha\beta,i} \circ \cdots) := (k^{+1}\gamma_{\alpha\beta,i} \circ \cdots) \circ \partial k^{+1}\gamma_{\alpha\beta,i} \circ \cdots \circ \partial k^{+1}\gamma_{\alpha\beta,i} \circ \cdots,
\]

where \( k^{+1}\gamma_{\alpha\beta,i} \circ \cdots \) is defined in Condition \ref{condition:cohomology}. For \( 0 \leq r_{1} < r_{2} \leq l \), we have

\[
(3.12) \quad d^{*}_{r_{1}, l - 1}(k^{+1}\gamma_{\alpha\beta,i} \circ \cdots) = \begin{cases} 
(k^{+1}\gamma_{\alpha\beta,i} \circ \cdots) \circ \partial(m^{r_{1} - 1}k^{+1}\gamma_{\alpha\beta,i} \circ \cdots) & \text{if } r_{1} \neq 0, \\
(k^{+1}\gamma_{\alpha\beta,i} \circ \cdots) \circ \partial k^{+1}\phi_{\alpha\beta,i} & \text{if } r_{1} = 0, r_{2} \neq 1, \\
(k^{+1}\gamma_{\alpha\beta,i} \circ \cdots) \circ \partial k^{+1}\phi_{\alpha\beta,i} & \text{if } r_{1} = 0, r_{2} = 1,
\end{cases}
\]
where the last case follows from the identity \( k+1 \varphi_{\alpha\beta,i_1}\circ k+1 \varphi_{\alpha\beta,i_0} = k+1 \varphi_{\alpha\beta,i_0} \), which in turn follows from the definition of \( k+1 \varphi_{\alpha\beta,i_0} \) in Condition 3.15. Therefore we have \( \partial(k+1 \varphi_{\alpha\beta,i_0}) \in A^0(\partial \iota) \) \( \otimes k+1 G_\beta^{-1}(U_{i_0\cdots i_l}) \). By Lemma 3.20, we obtain \( k+1 \varphi_{\alpha\beta,i_0} \in A^0(\iota) \) \( \otimes k+1 G_\beta^{-1}(U_{i_0\cdots i_l}) \) satisfying

\[
(\partial(k+1 \varphi_{\alpha\beta,i_0})|_{\iota}) = \partial(k+1 \varphi_{\alpha\beta,i_0}) \quad \text{mod } m^{k+1}.
\]

Therefore, we have an obstruction term \( k+1 O_{\alpha\beta\gamma\iota_0\cdots i_l} \in A^0(\iota) \) \( \otimes k+1 G_\alpha^{-1}(U_{i_0\cdots i_l}) \) given by

\[
k+1 O_{\alpha\beta\gamma\iota_0\cdots i_l} = (k+1 \varphi_{\alpha\beta\gamma\iota_0\cdots i_l}) \otimes (k+1 \varphi_{\alpha\beta\gamma\iota_0\cdots i_l}) \quad \text{mod } m^{k+1}
\]

which satisfies \( k+1 O_{\alpha\beta\gamma\iota_0\cdots i_l} = 0 \) \( \text{mod } m^{k+1} \). Direct computation gives \( |(d^*_k(k+1 O_{\alpha\beta\gamma\iota_0\cdots i_l}), \cdot)| = \text{id} \) for all \( r = 0, \ldots, l \). Using injectivity of \( kG_\alpha \rightarrow \text{Der}(kG_\alpha) \) we deduce \( (k+1 O_{\alpha\beta\gamma\iota_0\cdots i_l})|_{\iota} = 0 \).

Via (3.11) again, we may regard the term \( k+1 O_{\alpha\beta\gamma\iota_0\cdots i_l} \) as lying in \( A^0(\iota) \otimes B_1(U_{i_0\cdots i_l}) \) \( \otimes (m^{k+1}/m^{k+2}) \). By a similar argument as in the \( l = 0 \) case, we obtain an element \( (k+1 c_{\alpha\beta\iota_0\cdots i_l}) \alpha_\beta \) whose image under the Čech differential is precisely \( (k+1 O_{\alpha\beta\gamma\iota_0\cdots i_l}) \alpha_\beta \), and such \( (k+1 c_{\alpha\beta\iota_0\cdots i_l}) \alpha_\beta \) is 0. Therefore setting \( k+1 \varphi_{\alpha\beta\gamma\iota_0\cdots i_l} = (k+1 c_{\alpha\beta\iota_0\cdots i_l}) \alpha_\beta \) solves the required cocycle condition (3.10). Also we have \( k+1 \varphi_{\alpha\beta\gamma\iota_0\cdots i_l} \in (k+1 \varphi_{\alpha\beta\gamma\iota_0\cdots i_l}) \alpha_\beta, \] \( \text{mod } m^{k+1} \) since \( k+1 c_{\alpha\beta\iota_0\cdots i_l} = 0 \) \( \text{mod } m^{k+1} \) by our construction, and \( (k+1 \varphi_{\alpha\beta\iota_0\cdots i_l}) \alpha_\beta \) which is the required compatibility condition (3.7). This completes the proof of the lemma.

**Proof of Theorem 3.18** We prove by induction on the order \( k \). For the initial case \( k = 0 \), as \( G^* \) is globally defined on \( X \) with \( G^*_\alpha = G^*_\alpha|_V \) (see Definition 2.13), we can (and have to) set \( g_{\alpha\beta,i} = 0 \) \( \varphi_{\alpha\beta,i} = 0 \), \( \varphi_{\alpha\beta,i} = 0 \), and \( \varphi_{\alpha\beta,i} = 0 \). The induction step is proven in Lemma 3.21.

### 3.3.2. Homotopy between two sets of gluing morphisms.

The set of compatible gluing morphisms \( \{ k g_{\alpha\beta}(0) \} \) and \( \{ k g_{\alpha\beta}(1) \} \), we need, for each \( k \in \mathbb{Z}_\geq 0 \) and any pair \( V_\alpha, V_\beta \subseteq V \), an isomorphism

\[
k h_{\alpha\beta} : k TW^*_{\alpha\alpha\beta}(\iota) \rightarrow k TW^*_{\beta\alpha\beta}(\iota),
\]

as a collection of maps \( (k h_{\alpha\beta})_{1 \in I} \) so that for each \( \varphi = (\varphi_1)_{1 \in I} \in k TW^*_{\alpha\alpha\beta}(\iota) \) with \( \varphi_1 \in A^*(\iota) \otimes \otimes A^*(\iota) \otimes G^*_\alpha(U_1) \) we have \( (k h_{\alpha\beta}(\varphi))_1 = (k h_{\alpha\beta}(\varphi_1)) \), which preserves the algebraic structures \( [\cdot, \cdot], \wedge \) obtained via tensoring with the dga \( A^*(\iota) \) and fits into the following commutative diagram

\[
\begin{array}{ccc}
k TW^*_{\alpha\alpha\beta}(\iota) & \xrightarrow{\tau_0} & k TW^*_{\beta\alpha\beta}(\iota) \\
\downarrow k g_{\alpha\beta}(0) & & \downarrow k g_{\alpha\beta}(1) \\
k TW^*_{\beta\alpha\beta}(\iota) & \xrightarrow{\tau_1} & k TW^*_{\beta\alpha\beta}(\iota)
\end{array}
\]

here \( k TW^*_{\alpha\alpha\beta}(\iota) \) is the Thom-Whitney complex constructed from the sheaf \( A^*(\iota) \otimes G^*_\alpha \), where the degree \( * \) in \( k TW^*_{\alpha\alpha\beta}(\iota) \) refers to the total degree on \( A^*(\iota) \otimes A^*(\iota) \), and \( \tau_1 : A^*(\iota) \rightarrow A^*(\iota) \) is induced by the evaluation \( A^*(\iota) \rightarrow \mathbb{C} \). 1. for \( j = 1 \) as in Notation 3.7. The isomorphisms \( k h_{\alpha\beta} \)'s are said to constitute a homotopy from \( \{ k g_{\alpha\beta}(0) \} \) to \( \{ k g_{\alpha\beta}(1) \} \) if they further satisfy the following condition (cf. Condition 3.15): **Condition 3.22.**

1. for \( U_{i_0, \ldots, i_l} \subset V_\alpha \cap V_\beta \), we have

\[
k h_{\alpha\beta,i_0\cdots i_l} = \exp ([k z_{\alpha\beta,i_0\cdots i_l}, \cdot]) \circ (k g_{\alpha\beta,i_0}(0)|_{U_{i_0\cdots i_l}}).
\]
for some element $k\gamma_{\alpha\beta,i_0\cdots i_1} \in \mathcal{A}_0(\mathbf{1}_1) \otimes \mathcal{A}_0(\mathbf{1}_1) \otimes kG^{-1}_\beta(U_{i_0\cdots i_1})$ with $k\gamma_{\alpha\beta,i_0\cdots i_1} = 0 \pmod{m}$;
(2) the elements $k\gamma_{\alpha\beta,i_0\cdots i_1}$ satisfy the relation (cf. (3.7));

(3.16) \[ d^*_j(k\gamma_{\alpha\beta,i_0\cdots i_1}) = \begin{cases} k\gamma_{\alpha\beta,i_0\cdots i_1} & \text{for } j > 0, \\ k\gamma_{\alpha\beta,i_0\cdots i_1} \circ k\phi_{\alpha\beta,i_0\cdots i_1}(0) & \text{for } j = 0, \end{cases} \]

where $k\phi_{\alpha\beta,i_0i_1}(0) \in kG^{-1}(U_{i_0i_1})$ is the unique element such that $\exp([k\phi_{\alpha\beta,i_0i_1}(0), \cdot]) \circ k\gamma_{\alpha\beta,i_0}(0) = k\gamma_{\alpha\beta,i_0}(0)$, and the relation

(3.17) \[ r^*_j(k\gamma_{\alpha\beta,i_0\cdots i_1}) = \begin{cases} k\gamma_{\alpha\beta,i_0\cdots i_1}(0) & \text{for } j = 0, \\ k\gamma_{\alpha\beta,i_0\cdots i_1}(1) \circ k\gamma_{\alpha\beta,i_0}(1) & \text{for } j = 1, \end{cases} \]

where $k\gamma_{\alpha\beta,i_0\cdots i_1}(j) \in \mathcal{A}_0(\mathbf{1}_1) \otimes kG^{-1}(U_{i_0\cdots i_1})$ is the element associated to $k\gamma_{\alpha\beta,i_0\cdots i_1}(j)$ as in (3.6) for $j = 0, 1$, and $k\gamma_{\alpha\beta,i_0} \in kG^{-1}(U_{i_0})$ is the unique element such that $\exp([k\gamma_{\alpha\beta,i_0}, \cdot]) \circ k\gamma_{\alpha\beta,i_0}(0) = k\gamma_{\alpha\beta,i_0}(1)$.

**Definition 3.23.** A homotopy $\{kh_{\alpha\beta}\}$ from $\{k\gamma_{\alpha\beta}(0)\}$ to $\{k\gamma_{\alpha\beta}(1)\}$ is said to be compatible if in addition the following conditions are satisfied:

(1) $0h_{\alpha\beta} = id$ for all $\alpha, \beta$;
(2) (compatibility between different orders) for each $k \in \mathbb{Z}_{\geq 0}$ and any pair $V_\alpha, V_\beta \in \mathcal{V}$,

(3.18) \[ kh_{\alpha\beta} \circ k^{+1}h_{\alpha\beta} = k^{+1}h_{\alpha\beta} \circ kh_{\alpha\beta}; \]
(3) (cocycle condition) for each $k \in \mathbb{Z}_{\geq 0}$ and any triple $V_\alpha, V_\beta, V_\gamma \in \mathcal{V}$,

(3.19) \[ kh_{\alpha\beta} \circ kh_{\beta\gamma} \circ kh_{\gamma\alpha} = id \]

when $kh_{\alpha\beta}, kh_{\beta\gamma}, kh_{\gamma\alpha}$ are restricted to $kTW_{\alpha\beta\gamma}(\mathbf{1}_1), kTW_{\beta\alpha\gamma}(\mathbf{1}_1), kTW_{\gamma\alpha\beta}(\mathbf{1}_1)$ resp.

The same induction argument as in Theorem 3.18 proves the following:

**Proposition 3.24.** Given any two sets of compatible gluing morphisms $\{k\gamma_{\alpha\beta}(0)\}$ and $\{k\gamma_{\alpha\beta}(1)\}$, there exists a compatible homotopy $\{kh_{\alpha\beta}\}$ from $\{k\gamma_{\alpha\beta}(0)\}$ to $\{k\gamma_{\alpha\beta}(1)\}$.

### 3.4. The Čech-Thom-Whitney complex

The goal of this subsection is to construct a Čech-Thom-Whitney complex $k\check{C}^*(TW, g)$ for each $k \in \mathbb{Z}_{\geq 0}$ from a given set $g = \{k\gamma_{\alpha\beta}\}$ of compatible gluing morphisms.

**Definition 3.25.** For $\ell \in \mathbb{Z}_{\geq 0}$, we let $kTW_{\alpha_0\cdots\alpha_\ell}(g) \subset \bigoplus_{i=0}^{\ell} kTW_{\alpha_i\cdots\alpha_\ell}(g)$ be the set of elements $(\varphi_0, \cdots, \varphi_\ell)$ such that $\varphi_j = k\gamma_{\alpha_j}(\varphi_i)$. Then the $k$th-order Čech-Thom-Whitney complex over $X$ $k\check{C}^*(TW, g)$ is defined by setting $k\check{C}^\ell(TW_{p,q}, g) := \prod_{\alpha_0, \cdots, \alpha_\ell} kTW_{\alpha_0\cdots\alpha_\ell}(g)$ and $k\check{C}^\ell(TW, g) := \bigoplus_{p,q} k\check{C}^\ell(TW_{p,q}, g)$ for each $k \in \mathbb{Z}_{\geq 0}$.

This is equipped with the Čech differential $k\delta_\ell := \sum_{j=0}^{\ell+1} (-1)^j \tau_{j,\ell+1} : k\check{C}^\ell(TW, g) \to k\check{C}^{\ell+1}(TW, g)$, where $\tau_{j,\ell} : k\check{C}^{\ell-1}(TW, g) \to k\check{C}^\ell(TW, g)$ is the natural restriction map defined compositionwise by the map $\tau_{j,\ell} : kTW_{\alpha_0\cdots\alpha_{\ell-1}}(g) \to kTW_{\alpha_0\cdots\alpha_{\ell}}(g)$ which in turn comes from (3.8).

We define the $k$th-order complex of polyvector fields over $X$ by $kPV^*(g) := \text{Ker}(k\delta_0)$ and denote the natural inclusion $kPV^*(g) \to k\check{C}^0(TW, g)$ by $k\delta_{-1}$, so we have the following sequence of maps

(3.20) $0 \to kPV_{p,q}(g) \to k\check{C}^0(TW_{p,q}, g) \to k\check{C}^1(TW_{p,q}, g) \to \cdots \to k\check{C}^\ell(TW_{p,q}, g) \to \cdots$. 
For \( \ell \in \mathbb{Z}_{\geq 0} \) and \( k \geq l \), there is a natural map \( k,l : kC^\ell(TW_{p,q}, g) \to lC^\ell(TW_{p,q}, g) \) defined componentwise by the map \( k,l : kC^\ell(TW_{p,q}, g) \to lC^\ell(TW_{p,q}, g) \) obtained from \( k,l : kG^* \to lG^* \) (see Definition 2.13). Similarly, we have the natural maps \( k,l : kPV_{p,q}(g) \to lPV_{p,q}(g) \).

**Definition 3.26.** The Čech-Thom-Whitney complex \( \check{C}^\ell(TW, g) = \bigoplus_{p,q} C^\ell(TW_{p,q}, g) \) is defined by taking the inverse limit \( \check{C}^\ell(TW_{p,q}, g) := \lim_{k} kC^\ell(TW_{p,q}, g) \) along the maps \( k,l : kC^\ell(TW_{p,q}, g) \to lC^\ell(TW_{p,q}, g) \).

The complex of polyvector fields \( PV_{p,q}(g) = \bigoplus_{p,q} PV_{p,q}(g) \) is defined by taking the inverse limit \( PV_{p,q}(g) := \lim_{k} kPV_{p,q}(g) \) along the maps \( k,l : kPV_{p,q}(g) \to lPV_{p,q}(g) \).

The maps \( k,l \)'s commute with the Čech differentials \( k\delta_{\ell} \)'s and \( l\delta_{\ell} \)'s, so we have the following sequence of maps

\[
(3.21) \quad 0 \to PV_{p,q}(g) \to \check{C}^0(TW_{p,q}, g) \to \check{C}^1(TW_{p,q}, g) \to \cdots \to \check{C}^\ell(TW_{p,q}, g) \to \cdots
\]

**Lemma 3.27.** Given \( k+1, w \in \check{C}^{k+1}(TW, g) \) with \( k+1\delta_{k+1}(k+1, w) = 0 \) and \( k,v \in \check{C}^\ell(TW, g) \) satisfying \( k\delta_{\ell}(k,v) = k+1, w \pmod{m^{k+1}} \), there exists a lifting \( k+1, v \in \check{C}^{k+1}(TW, g) \) such that \( k+1\delta_{\ell}(k+1, v) = k+1, w \). As a consequence, both (3.20) and (3.21) are exact sequences.

**Proof.** We only need to prove the first statement of the lemma because the second statement follows by induction on \( k \) (note that the initial case for this induction is \( k = -1 \) where we take the trivial sequence whose terms are all zero).

Without loss of generality, we can assume that \( k+1, w \in \check{C}^{k+1}(TW_{p,q}, g) \) and \( k,v \in \check{C}^\ell(TW_{p,q}, g) \) for some fixed \( p \) and \( q \). We need to construct \( k+1, v_{\alpha_0 \cdots \alpha_\ell} \in \check{C}^\ell(TW_{p,q}, g) \) for every multi-index \( (\alpha_0, \ldots, \alpha_\ell) \) which, by Definition 3.25, can be written as

\[
k+1, v_{\alpha_0 \cdots \alpha_\ell} = (k+1, v_{\alpha_0}, \ldots, k+1, v_{\alpha_\ell})
\]

satisfying \( k+1, v_{\alpha_j; \alpha_0 \cdots \alpha_{j-1}} = k+1, g_{\alpha_j}(k+1, v_{\alpha_0 \cdots \alpha_{j-1}}) \), and each component \( k+1, v_{\alpha_j; \alpha_0 \cdots \alpha_{j-1}} \) is of the form \( k+1, v_{\alpha_j; \alpha_0 \cdots \alpha_{j-1}} = (k+1, v_{\alpha_j; \alpha_0 \cdots \alpha_{j-1}}) \), where \( U_{ij} \subset V_{\alpha_0 \cdots \alpha_{j-1}} \) and \( k+1, v_{\alpha_j; \alpha_0 \cdots \alpha_{j-1}} \in A^q(\alpha_j) \otimes k+1, G^p_{\alpha_j}(U_{ij}) \), such that \( d_{ij}^* (k+1, v_{\alpha_j; \alpha_0 \cdots \alpha_{j-1}}) = k+1, v_{\alpha_j; \alpha_0 \cdots \alpha_{j-1}} \otimes U_{ij} \) (see Definition 3.9).

We will use induction on \( l \) to prove the existence of such an element.

The initial case is \( l = q \). We fix \( U_{ij} \) and consider all the multi-indices \( (\alpha_0, \ldots, \alpha_\ell) \) such that \( U_{ij} \subset V_{\alpha_0 \cdots \alpha_{j-1}} \) for \( r = 0, \ldots, q \). Using the fact that \( k+1, G^p_{\alpha_0} \) is free over \( k+1, R \), we can take a lifting \( k+1, w_{\alpha_0 \cdots \alpha_{j-1}} \in A^q(\alpha_j) \otimes k+1, G^p_{\alpha_0}(U_{ij}) \) of \( k+1, v_{\alpha_0 \cdots \alpha_{j-1}} \). Then we let \( k+1, v_{\alpha_j; \alpha_0 \cdots \alpha_{j-1}} := k+1, g_{\alpha_j; \alpha_0 \cdots \alpha_{j-1}} \) for \( j = 1, \ldots, \ell \) and set

\[
k+1, v_{\alpha_0 \cdots \alpha_{j-1}} := (k+1, v_{\alpha_0 \cdots \alpha_{j-1}}, \ldots, k+1, v_{\alpha_{j-1}}).
\]

Now the element

\[
k+1, w_{\alpha_0 \cdots \alpha_{j-1}; \alpha_{j+1} \cdots \alpha_{j+1}} := \frac{k+1, v_{\alpha_0 \cdots \alpha_{j-1}; \alpha_{j+1} \cdots \alpha_{j+1}}}{-k+1, v_{\alpha_0 \cdots \alpha_{j-1}; \alpha_{j+1} \cdots \alpha_{j+1}}}
\]

\[
+ \cdots + (-1)^j k+1, v_{\alpha_0 \cdots \alpha_{j+1} \cdots \alpha_{j+1}} + \cdots + (-1)^{\ell+1} k+1, v_{\alpha_0 \cdots \alpha_{j+1} \cdots \alpha_{j+1}}
\]

satisfies the condition that \( k+1, w_{\alpha_0 \cdots \alpha_{j-1}; \alpha_{j+1} \cdots \alpha_{j+1}} = 0 \pmod{m^{k+1}} \).

Under the identification (3.11), we can treat \( k+1, w_{\alpha_0 \cdots \alpha_{j-1}; \alpha_{j+1} \cdots \alpha_{j+1}} \) as \( \ell \)-cocycle in the Čech complex \( C^{\ell+1}(U_{ij} \cdots, 0,G^p) \otimes \).

So the collection \( (k+1, w_{\alpha_0 \cdots \alpha_{j-1}; \alpha_{j+1} \cdots \alpha_{j+1}})_{\alpha_0 \cdots \alpha_{j-1}} \) is an \( (\ell+1) \)-cocycle in the Čech complex \( C^{\ell+1}(U_{ij} \cdots, 0,G^p) \otimes \).
\(A^q(\mathbf{\Delta}_q) \otimes (\mathbf{m}^{k+1}/\mathbf{m}^{k+2})\). By Lemma 3.19 there exists \((k+1)c_{\alpha_0 \cdots \alpha_l \cdots \alpha_k} \in \tilde{\mathcal{C}}^l(I_{i_l \cdots i_q}^0 \mathcal{G}^p) \otimes A^q(\mathbf{\Delta}_q) \otimes (\mathbf{m}^{k+1}/\mathbf{m}^{k+2})\) whose image under the Čech differential is precisely \((k+1)w_{\alpha_0 \cdots \alpha_{l+1} i_l \cdots i_q} \alpha_0 \cdots \alpha_{l+1}\). Therefore if we let
\[
\kappa + 1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q} := \kappa + 1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q} - (k+1)c_{\alpha_0 \cdots \alpha_l \cdots \alpha_k},
\]
then its image under the Čech differential is \((k+1)w_{\alpha_0 \cdots \alpha_{l+1} i_l \cdots i_q} \alpha_0 \cdots \alpha_{l+1}\) as desired.

Next we suppose that we are given \(k+1)w_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q} \) for some \(l \geq q\). Then we need to construct \(k+1)w_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q} \) for any \(U_{i_l \cdots i_q}^0 \) and \(V_{\alpha_0 \cdots \alpha_l}:\) such that \(U_{i_r} \subset V_{\alpha_0 \cdots \alpha_l}\) for \(r = 1, \ldots, l+1\). We fixed \(U_{i_l \cdots i_q}^0\) and consider one such \(V_{\alpha_0 \cdots \alpha_l}\). Letting
\[
\partial(k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}) := \left(k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}, \ldots, k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}\right)
\]
gives an element in \(A^q(\Delta_{l+1}) \otimes k+1 \mathcal{G}^p_{\alpha_0}(U_{i_l \cdots i_q})\). Using Lemma 3.20 we can construct an element
\[
k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q} \in A^q(\mathbf{\Delta}_{l+1}) \otimes k+1 \mathcal{G}^p_{\alpha_0}(U_{i_l \cdots i_q})
\]
such that
\[
k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q} = k v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q} \mod \mathbf{m}^{k+1},
\]

\[
k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}(\Delta_{l+1}) = \partial(k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}).
\]

We then let \(k+1)\delta_{k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}} = k+1 g_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q} \) for \(j = 1, \ldots, \ell\) and set
\[
k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q} := \left(k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}, \ldots, k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}\right),
\]

Now the elements \(k+1)\delta_{k+1 v_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}}\) and \(k+1)w_{\alpha_0 \cdots \alpha_l : i_0 \cdots i_q}\) agree modulo \(\mathbf{m}^{k+1}\) and on the boundary \(\Delta_{l+1}\) of the simplex \(\mathbf{\Delta}_{l+1}\), so the rest of the proof of this induction step would be the same as the initial case \(l = q\).

**Corollary 3.28.** For all \(k, \ell \in \mathbb{Z}_{\geq 0}\), the map \(k+1)\mathcal{C}^l(TW^{p,q}, g) \rightarrow k+1)\mathcal{C}^l(TW^{p,q}, g)\) and hence the induced map \(\mathcal{C}^l(TW^{p,q}, g) \rightarrow k+1)\mathcal{C}^l(TW^{p,q}, g)\) are surjective; in particular, both \(k+1)PV^{p,q}(g) \rightarrow kPV^{p,q}(g)\) and \(k+1)PV^{p,q}(g) \rightarrow kPV^{p,q}(g)\) are surjective.

**Proof.** It suffices to show that for any \(k)\mathbf{v} \in k)\mathcal{C}^l(TW, g)\), there exists \(k+1)\mathbf{v} \in k+1)\mathcal{C}^l(TW, g)\) such that \(k+1)\mathcal{C}^l(TW, g) = k)\mathbf{v}\). If \(k)\mathcal{C}^l(TW, g) = 0\), then applying Lemma 3.27 with \(k+1)\mathbf{v} = 0\) we get the desired \(k+1)\mathbf{v}\). For a general \(k)\mathbf{v}\), we let \(k)\mathbf{w} = k)\mathbf{v}\). Since \(k)\delta_{k)\mathbf{w}} = 0\), we can find a lifting \(k+1)\mathbf{w}\) such that \(k+1)\mathcal{C}^l(TW, g) = k+1)\mathbf{w}\). Applying Lemma 3.27 again, we obtain \(k+1)\mathbf{v}\) satisfying \(k+1)\mathcal{C}^l(TW, g) = k)\mathbf{v}\).

**Definition 3.29.** Let \(g(0) = \{k)g_{\alpha_0} \}\) and \(g(1) = \{k)g_{\alpha_0} \}\) be two sets of compatible gluing morphisms, and \(h = \{k)h_{\alpha_0} \}\) be a compatible homotopy from \(g(0)\) to \(g(1)\). For \(l \geq 0\), we let \(k)TW_{\alpha_0 \cdots \alpha_l}(h) \subset \bigoplus_{l=0}^\ell k)TW_{\alpha_0 \cdots \alpha_l}(\mathbf{\Delta}_1)\) be the set of elements \((\varphi_0, \ldots, \varphi_\ell)\) such that \(\varphi_l = k)h_{\alpha_0 \cdots \alpha_l}(\varphi_0)\). Then, for each \(k \in \mathbb{Z}_{\geq 0}\), the \(k\)th-order homotopy Čech-Thom-Whitney complex is defined by setting \(k)\mathcal{C}^l(TW^{p,q}, h) := \prod_{\alpha_0 \cdots \alpha_l} k)TW_{\alpha_0 \cdots \alpha_l}(h)\) and \(k)\mathcal{C}^l(TW, h) = \bigoplus_{p,q} k)\mathcal{C}^l(TW^{p,q}, h)\). We have the natural restriction map \(\gamma_{k,l} : k)\mathcal{C}^l(TW, h) \rightarrow k)\mathcal{C}^l(TW, h)\) as in Definition 3.25.

Let \(k)\gamma_{k,l} := \sum_{j=0}^{l+1} (-1)^j \gamma_{j,k,l+1} : k)\mathcal{C}^l(TW, h) \rightarrow k)\mathcal{C}^l(TW, h)\) be the Čech differential acting on \(k)\mathcal{C}^l(TW, h)\). Then the \(k\)th order homotopy polyvector field on \(X\) is defined as \(k)PV^{*,*}(h) := \text{Ker}(k)\gamma_{k,0})\). So we have the following sequences

\[
0 \rightarrow k)PV^{p,q}(h) \rightarrow k)\mathcal{C}^{0}(TW^{p,q}, h) \rightarrow \cdots \rightarrow k)\mathcal{C}^{l}(TW^{p,q}, h) \rightarrow \cdots \rightarrow k)\mathcal{C}^{\ell}(TW^{p,q}, h) \rightarrow \cdots,
\]

\[
0 \rightarrow PV^{p,q}(h) \rightarrow \mathcal{C}^{0}(TW^{p,q}, h) \rightarrow \cdots \rightarrow \mathcal{C}^{l}(TW^{p,q}, h) \rightarrow \cdots \rightarrow \mathcal{C}^{\ell}(TW^{p,q}, h) \rightarrow \cdots,
\]
where (3.23) is obtained from (3.22) by taking the inverse limit. We also write \( \tilde{\mathcal{C}}^l(TW, h) := \bigoplus_{p,q} \tilde{\mathcal{C}}^l(TW^{pq}, h) \).

We further let \( k\mathbf{r}_j^* : k\tilde{\mathcal{C}}^l(TW^{pq}, h) \rightarrow k\tilde{\mathcal{C}}^l(TW^{pq}, g(j)) \) and \( k\mathbf{r}_j^* : k\mathcal{P}^{pq}(h) \rightarrow k\mathcal{P}^{pq}(g(j)) \) be the maps induced by \( \mathbf{r}_j^* : \mathcal{A}^*(\mathbf{A}) \rightarrow \mathcal{A}^*(\mathbf{A}) \) for \( j = 0, 1 \), and let \( \mathbf{r}_j^* := \lim_k k\mathbf{r}_j^* \). Then we have the following diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & PV^{*,*}(g(0)) & \longrightarrow & \mathcal{C}^0(TW, g(0)) & \longrightarrow & \cdots \\
\downarrow^{r_0^*} & & \downarrow^{r_0^*} & & \downarrow^{r_0^*} & & \\
0 & \longrightarrow & PV^{*,*}(h) & \longrightarrow & \mathcal{C}^0(TW, h) & \longrightarrow & \cdots \\
\downarrow^{r_1^*} & & \downarrow^{r_1^*} & & \downarrow^{r_1^*} & & \\
0 & \longrightarrow & PV^{*,*}(g(1)) & \longrightarrow & \mathcal{C}^0(TW, g(1)) & \longrightarrow & \cdots \\
\end{array}
\]

Similar proofs as those of Lemma 3.27 and Corollary 3.28 yield the following lemma:

**Lemma 3.30.** Given \( k\mathbf{w} \in k^{k+1}\tilde{\mathcal{C}}^{l+1}(TW, h) \) with \( k^{k+1}\delta_{l+1}(k\mathbf{w}) = 0 \), \( k^{k+1}\mathbf{a}_j \in k^{k+1}\tilde{\mathcal{C}}^l(TW, g(j)) \) satisfying \( k^{k+1}\delta_l(k\mathbf{a}_j) = k^{k+1}\mathbf{r}_j^*(k\mathbf{w}) \) and \( k\mathbf{v} \in k\tilde{\mathcal{C}}^l(TW, h) \) such that \( k\delta_l(k\mathbf{v}) = k\mathbf{w} \) (mod \( \mathbf{m}^{k+1} \)) and \( k\mathbf{r}_j^*(k\mathbf{v}) = k^{k+1}\mathbf{a}_j \) (mod \( \mathbf{m}^{k+1} \)), there exists \( k^{k+1}\mathbf{v} \in k^{k+1}\tilde{\mathcal{C}}^l(TW, h) \) such that \( k^{k+1}\mathbf{v} = k\mathbf{u} \) and \( k^{k+1}\delta_l(k^{k+1}\mathbf{v}) = k\mathbf{w} \). As a consequence, both (3.22) and (3.23) are exact sequences.

Furthermore, the maps \( \omega_{k\mathbf{w}} : k\tilde{\mathcal{C}}^l(TW^{pq}, h) \rightarrow k\tilde{\mathcal{C}}^l(TW^{pq}, h) \) and \( \omega_{k\mathbf{u}} : PV^{pq}(h) \rightarrow PV^{pq}(h) \), as well as \( k\mathbf{r}_j^* : kPV^{*,*}(h) \rightarrow kPV^{*,*}(g(j)) \) and \( r_j^* : PV^{*,*}(h) \rightarrow PV^{*,*}(g(j)) \) are all surjective.

### 3.5. The dgBV algebra structure

The complex \( PV^{*,*}(g) \) (as well as \( PV^{*,*}(h) \)) constructed in [3.4] is only a graded vector space. In this subsection, we equip it with two differential operators \( \partial \) and \( \Delta \), turning it into a dgBV algebra.

We fix a set of compatible gluing morphisms \( g = \{ k\mathbf{g}_{\alpha\beta} \} \) consisting of isomorphisms \( k\mathbf{g}_{\alpha\beta} : k\mathcal{W}^{*,*}_{\alpha\beta} \rightarrow k\mathcal{W}^{*,*}_{\beta\alpha} \) for each \( k \in \mathbb{Z}_{\geq 0} \) and pair \( \mathcal{V}_\alpha, \mathcal{V}_\beta \subset \mathcal{V} \). Both \( k\mathcal{W}^{*,*}_{\alpha\beta} \) and \( k\mathcal{W}^{*,*}_{\beta\alpha} \) are dgBV algebras with differentials and BV operators given by \( k\bar{\partial}_\alpha, k\bar{\partial}_\beta \) and \( k\bar{\Delta}_\alpha, k\bar{\Delta}_\beta \) respectively.

**Lemma 3.31.** For each \( k \in \mathbb{Z}_{\geq 0} \) and pair \( \mathcal{V}_\alpha, \mathcal{V}_\beta \subset \mathcal{V} \), there exists \( k\mathbf{u}_{\alpha\beta} \) such that \( k\mathbf{u}_{\alpha\beta} = 0 \) (mod \( \mathbf{m} \)), \( k^{k+1}\mathbf{u}_{\alpha\beta} = k\mathbf{u}_{\alpha\beta} \) (mod \( \mathbf{m}^{k+1} \)) and \( k\mathbf{g}_{\alpha\beta} \circ k\bar{\partial}_\beta \circ k\mathbf{g}_{\alpha\beta} = k\bar{\Delta}_\alpha, k\bar{\Delta}_\beta \). Furthermore, if we let \( k\mathbf{u}_{\alpha\beta} := \{ k\mathbf{u}_{\alpha\beta}: k\mathbf{g}_{\alpha\beta}(k\mathbf{u}_{\alpha\beta}) \} \), then \( k\mathbf{u}_{\alpha\beta} \) is a Čech 1-cocycle in \( \tilde{\mathcal{C}}^1(TW^{*,*}, g) \).

**Proof.** By Lemma 2.5 we have

\[
\exp(-k\bar{\partial}_{\alpha,\beta, 0\cdots 0\cdots i_1}) \circ d \circ \exp\left(k\bar{\partial}_{\alpha, \beta, 0\cdots 0\cdots i_1}\right) = d - \left[ \exp\left(-k\bar{\partial}_{\alpha, \beta, 0\cdots 0\cdots i_1}\right) - 1 \right] \left( d k\mathbf{g}_{\alpha\beta, 0\cdots 0\cdots i_1} \right),
\]

where \( d \) is the de Rham differential acting on \( \mathcal{A}^*(\mathbf{A}) \) (recall that \( k\bar{\partial}_\beta \) is induced by the de Rham differential on \( \mathcal{A}^*(\mathbf{A}) \)). Then using (3.6) in Condition 3.15 (i.e. \( k\mathbf{g}_{\alpha, 0\cdots 0\cdots i_1} = \exp\left(-k\bar{\partial}_{\alpha, 0\cdots 0\cdots i_1}\right) \circ \left( k\mathbf{g}_{\alpha, 0\cdots 0\cdots i_1} \right) \)), we obtain

\[
k\mathbf{g}_{\alpha, 0\cdots 0\cdots i_1} \circ d \circ k\mathbf{g}_{\alpha, 0\cdots 0\cdots i_1} = d - \left[ k\mathbf{g}_{\alpha, 0\cdots 0\cdots i_1} \left( \exp\left(-k\bar{\partial}_{\alpha, 0\cdots 0\cdots i_1}\right) - 1 \right) \left( d k\mathbf{g}_{\alpha, 0\cdots 0\cdots i_1} \right) \right].
\]
Now we put
\[ k w_{\alpha;\beta,i_0 \ldots i_l} := -k g_{\beta,i_0} \left( \exp \left( -\left[ k \partial_{\alpha;\beta,i_0 \ldots i_l} \right] \right) - 1 \right) \frac{1}{\left[ k \partial_{\alpha;\beta,i_0 \ldots i_l} \right]^2} \].

Then \( k+1 w_{\alpha;\beta,i_0 \ldots i_l} \equiv k w_{\alpha;\beta,i_0 \ldots i_l} \mod m^{k+1} \). To check that it is well-defined as an element in \( k T W^1_{\alpha;\beta} \), we compute using Lemma 2.5 to get \( d + [d^* d (k w_{\alpha;\beta,i_0 \ldots i_l})] = d + [k w_{\alpha;\beta,i_0 \ldots i_l}] \). By injectivity of \( k G^\alpha_\alpha \rightarrow \text{Der}(k G^\alpha_\alpha) \), this implies that \( d^* d (k w_{\alpha;\beta,i_0 \ldots i_l}) = k w_{\alpha;\beta,i_0 \ldots i_l} \). To see that it is a Čech cocycle, we deduce from its definition that \( k g_{\beta} (k w_{\alpha;\beta}) \cdot k = k \Delta_\beta - k g_{\beta} \circ k \partial_{\alpha} \circ k g_{\beta} \). Thus, by direct computation, we have \( k w_{\alpha;\beta} - k w_{\alpha;\gamma} + k g_{\beta} (k w_{\beta;\gamma}) \cdot \cdot \cdot = 0 \) and can conclude that \( k w_{\alpha;\beta} - k w_{\alpha;\gamma} + k g_{\beta} (k w_{\beta;\gamma}) = 0 \).

We have a similar result concerning the difference between the BV operators \( k \Delta_\alpha \) and \( k \Delta_\beta \):

**Lemma 3.32.** For each \( k \in \mathbb{Z}_{\geq 0} \) and pair \( V_\alpha, V_\beta \subset V \), there exists \( k f_{\alpha;\beta} \in k T W^0_{\alpha;\beta} \) such that \( k f_{\alpha;\beta} = 0 \) (mod \( m \)), \( k+1 f_{\alpha;\beta} = k f_{\alpha;\beta} \mod m^{k+1} \), and \( k g_{\beta} \circ k \Delta_\beta \circ k g_{\beta} - k \Delta_\alpha = [k f_{\alpha;\beta}] \). Furthermore, if we let \( k f_{\alpha;\beta} := (k f_{\alpha;\beta}, k g_{\beta}(k f_{\alpha;\beta})) \), then \( k f_{\alpha;\beta} \) is a Čech 1-cocycle in \( k C^1(T W^0_{\alpha;\beta}, \beta) \).

**Proof.** To simplify notations, we introduce the power series
\[ T(x) := \frac{e^{-x} - 1}{x} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^k}{(k+1)!} \]

Similar to the previous proof, we have
\[
\begin{align*}
\exp(-[k a_{\alpha;\beta,i_0} \cdot]) \circ \exp(-[k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) \circ k \Delta_\beta \circ \exp([k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) \circ \exp([k a_{\alpha;\beta,i_0} \cdot])
= k \Delta_\beta - \left( \exp(-[k a_{\alpha;\beta,i_0} \cdot]) \circ T([k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) \circ k \Delta_\beta ([k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) \right) \\
- \left( T([k a_{\alpha;\beta,i_0} \cdot]) \circ k \Delta_\beta ([k a_{\alpha;\beta,i_0} \cdot]) \right)
\end{align*}
\]
using Lemma 2.5. By (3.5) in Condition 3.15, we can write \( k g_{\beta,i} = \exp([k a_{\beta,i} \cdot]) \circ k \psi_{\beta,i} \), and from (2.3) in Definition 2.15 we have \( k \psi_{\beta,i} \circ k \Delta_\beta \circ k \psi_{\beta,i} - k \Delta_\alpha = [k w_{\alpha;\beta,i} \cdot] \), so
\[
\begin{align*}
[k g_{\beta,i_0 \ldots i_l} \circ k \Delta_\beta \circ k g_{\beta,i_0 \ldots i_l} & - k \Delta_\alpha = - \left( [k g_{\beta,i_0} \circ T([k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) \circ k \Delta_\beta ([k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) \right) \right] \\
& - \left( T([k a_{\alpha;\beta,i_0} \cdot]) \circ k \Delta_\beta ([k a_{\alpha;\beta,i_0} \cdot]) \right) + [k w_{\alpha;\beta,i} \cdot] \right)
\end{align*}
\]
Now we put
\[
\begin{align*}
k f_{\alpha;\beta,i_0 \ldots i_l} := - (k g_{\beta,i_0} \circ T([k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) \circ k \Delta_\beta ([k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) - k \psi_{\beta,i_0} \circ T([k a_{\alpha;\beta,i_0} \cdot]) \circ k \Delta_\beta ([k a_{\alpha;\beta,i_0} \cdot]) & + k w_{\alpha;\beta,i_0} \\
= - (k g_{\beta,i_0} \circ T([k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) \circ k \Delta_\beta ([k \partial_{\alpha;\beta,i_0 \ldots i_l} \cdot]) + 1 c \otimes k f_{\alpha;\beta,i_0} U_{i_0 \ldots i_l} \right)
\end{align*}
\]
where \( 1 c \) denotes the constant function with value 1 on \( c \). We need to check the following conditions for the elements \( k f_{\alpha;\beta,i_0 \ldots i_l} \)’s:

1. \( k+1 f_{\alpha;\beta,i_0 \ldots i_l} \equiv k f_{\alpha;\beta,i_0 \ldots i_l} \mod m^{k+1} \);
2. \( k f_{\alpha;\beta} := (k f_{\alpha;\beta,i_0 \ldots i_l}) \) \( (i_0, \ldots, i_l) \in I \subset k T W^0_{\alpha;\beta} \) (see Definition 3.9);
(3) letting \( k\beta_{\alpha} := (k\beta_{\alpha}\beta, kg_{\alpha\beta}(k\beta_{\alpha})), \) we have

\[
(3.26) \quad kg_{\alpha\beta,i_{0}}\cdots-i_{1}(k\beta_{\alpha\beta,i_{0}}\cdots-i_{1}) = (kg_{\alpha\beta,i_{0}} \circ T([kg_{\beta_{\alpha\beta,i_{0}}\cdots-i_{1}}]) \circ k\Delta_{\alpha}(kg_{\beta_{\alpha\beta,i_{0}}\cdots-i_{1}}) + 1_{\alpha} \otimes kg_{\beta_{\alpha\beta,i_{0}}},
\]

where \( kg_{\beta_{\alpha\beta,i_{0}}} = (kg_{\beta_{\alpha\beta,i_{0}}} \circ T([kg_{\beta_{\alpha\beta,i_{0}}}, \cdot]) \circ k\Delta_{\alpha}(kg_{\beta_{\alpha\beta,i_{0}}}) - kg_{\beta_{\alpha\beta,i_{0}}}, \) and

(4) that \( k\beta_{\alpha} \alpha_{\beta} \alpha_{\beta} \) is a Čech 1-cocycle in \( k\hat{\mathcal{C}}(TW^{0,0}, g). \)

The properties (1)-(4) are proven by applying the comparison \( (2.9) \) of the volume forms in Definition 2.18 (which can be regarded as a more refined piece of information than the comparison of BV operators in \( (2.3) \)) and Lemma 2.8 in the same manner, together with some rather tedious (at least notationally) calculations. For simplicity, we shall only present the proof of (1) here.

To prove (1), first notice that the term \( (kg_{\beta_{\alpha\beta,i_{0}}} \circ T([kg_{\beta_{\alpha\beta,i_{0}}\cdots-i_{1}}]) \circ k\Delta_{\alpha}(kg_{\beta_{\alpha\beta,i_{0}}\cdots-i_{1}}) \) already satisfies the equality, so we only need to consider the case for \( l = 0. \) In the rest of this proof, we shall work (mod \( m^{k+1} \)), meaning that all equalities hold (mod \( m^{k+1} \)). First of all, the equation \( k^{+1}k\beta \circ kg_{\alpha\beta,i_{0}} = kg_{\alpha\beta,i_{0}} \circ k^{+1}k\beta (mod m^{k+1}) \) can be rewritten as

\[
\exp(-[a_{\beta,i_{0}}, \cdot]) \circ \exp([k^{+1}a_{\beta,i_{0}}, \cdot]) \circ k^{+1}k\beta = kg_{\beta_{\alpha\beta,i_{0}}} \circ k^{+1}k\beta \circ \exp([k^{+1}b_{\beta_{\alpha\beta,i_{0}}, \cdot}] \circ k^{+1}k\beta
\]

using \( (3.5) \) and \( (2.4) \), so we have

\[
-k^{+1}a_{\beta,i_{0}} = (-k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}) \otimes (-k\beta_{\alpha\beta,i_{0}})
\]

by the injectivity of \( k\mathcal{G}_{\beta}^{-1} \rightrightarrows \text{Der}(k\mathcal{G}_{\beta}^{\ast}). \)

Applying Lemma 2.5 to the dgLa \( (k\mathcal{G}_{\beta}, [-], k\Delta_{\beta}) \), we get \( \exp(-k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}) \ast \exp(-k\beta_{\alpha\beta,i_{0}}) \ast 0 = \exp(-k^{+1}a_{\beta,i_{0}} \ast 0) \), which can be expanded as

\[
0 = -\left( \exp(-[k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}, \cdot]) \circ T([k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}, \cdot]) \circ k\Delta_{\beta} \right)(k^{+1}a_{\beta,i_{0}})
\]

Applying \( k^{+1}k\beta_{\alpha\beta,i_{0}} \) to both sides (note that \( \gamma \in k\mathcal{G}_{\beta}^{\ast} \)), so when we write \( k^{+1}\beta_{\beta_{\alpha\beta,i_{0}}}(\gamma) \), we mean \( k^{+1}\beta_{\beta_{\alpha\beta,i_{0}}}(\tilde{\gamma}) \) where \( \tilde{\gamma} \in k^{+1}\mathcal{G}_{\beta}^{\ast} \) is an arbitrary lifting of \( \gamma \); as we are working (mod \( m^{k+1} \)), \( k^{+1}\beta_{\beta_{\alpha\beta,i_{0}}}(\tilde{\gamma}) \) is independent of the choice of \( \tilde{\gamma} \), we obtain

\[
0 = -\left( k^{+1}\beta_{\alpha\beta,i_{0}} \circ T([k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}, \cdot]) \circ k\Delta_{\beta}(k^{+1}a_{\beta,i_{0}}) + k^{+1}k\beta_{\alpha\beta,i_{0}} \circ T([k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}, \cdot]) \circ k\Delta_{\beta}(k^{+1}a_{\beta,i_{0}}) \right)
\]

\[
= k\beta_{\alpha\beta,i_{0}} - k\beta_{\alpha\beta,i_{0}} - k^{+1}k\beta_{\alpha\beta,i_{0}} + k^{+1}k\beta_{\alpha\beta,i_{0}} = -k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}(\tilde{\gamma}) k\beta_{\beta_{\alpha\beta,i_{0}}}(\tilde{\gamma}) \circ k\Delta_{\beta}(k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}).
\]

From \( (2.9) \), we learn that \( k^{+1}k\beta_{\alpha\beta,i_{0}}(k\beta_{\alpha\beta,i_{0}}) = -k\beta_{\alpha\beta,i_{0}}. \) Hence it remains to show that

\[
k\beta_{\alpha\beta,i_{0}} + k^{+1}k\beta_{\alpha\beta,i_{0}}(k^{+1}k\beta_{\alpha\beta,i_{0}}) = T(-[k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}, \cdot}] \circ k\Delta_{\beta})(k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}),
\]

which follows from the relation

\[
\exp(k\beta_{\alpha\beta,i_{0}} + k^{+1}k\beta_{\alpha\beta,i_{0}}) = \exp(k^{+1}k\beta_{\alpha\beta,i_{0}}) = \exp([k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}, \cdot]) \circ k\Delta_{\beta}(k^{+1}k\beta_{\beta_{\alpha\beta,i_{0}}}),
\]

coming from Definition 2.18 and using Lemma 2.8.

The same results hold with exactly the same proofs for the homotopy Čech-Thom-Whitney complex with gluing morphisms \( kh_{\alpha\beta} : kTW_{\alpha\beta}(\mathbf{1}) \rightarrow kTW_{\beta\alpha}(\mathbf{1}), \) where \( kTW_{\alpha\beta}(\mathbf{1}) \) and \( kTW_{\beta\alpha}(\mathbf{1}) \) are equipped with the differentials \( kD_{\alpha} := d_{\mathbf{1}} \otimes 1 + 1 \otimes k\delta_{\alpha} \) and \( kD_{\beta} := d_{\mathbf{1}} \otimes 1 + 1 \otimes k\delta_{\beta} \) and BV operators \( k\Delta_{\alpha} \) and \( k\Delta_{\beta} \) respectively. Such results are summarized in the following Lemma.
Lemma 3.33. There exist $kW_{\alpha\beta} \in kTW_{-1,1}(\mathbf{1}_1)$ and $kF_{\alpha\beta} \in kTW_{0,0}(\mathbf{1}_1)$ such that $kW_{\alpha\beta} = 0 \pmod{m}$ and $kF_{\alpha\beta} = 0 \pmod{m}$, and

$$k\beta \circ kD_{\beta} \circ k\alpha - kD_{\alpha} = [kW_{\alpha\beta}] \cdot, \quad k\beta \circ k\Delta_{\beta} \circ k\alpha - k\Delta_{\alpha} = [kF_{\alpha\beta}] \cdot.$$ 

Furthermore, if we let

$$kW_{\alpha\beta} := (kW_{\alpha\beta}, kF_{\alpha\beta}), \quad kF_{\alpha\beta} := (kF_{\alpha\beta}, kF_{\alpha\beta}),$$

then $(kW_{\alpha\beta})_{\alpha\beta}$ and $(kF_{\alpha\beta})_{\alpha\beta}$ are Čech 1-cocycles in the complex $k\mathcal{C}(TW, h)$.

We conclude this subsection by the following theorem:

Theorem 3.34. There exist elements $\vartheta = (\vartheta_{\alpha})_{\alpha} = \lim_{k}(k\vartheta_{\alpha})_{\alpha} \in \mathcal{C}(TW, g)$ and $\bar{\vartheta} = (\bar{\vartheta}_{\alpha})_{\alpha} = \lim_{k}(k\vartheta_{\alpha})_{\alpha} \in \mathcal{C}(TW^0, g)$ such that

$$g_{\beta \alpha} \circ (\bar{\vartheta}_{\beta} + [\vartheta_{\beta}]) \circ g_{\alpha\beta} = \bar{\vartheta}_{\alpha} + [\vartheta_{\alpha}].$$

Also, $(\vartheta_{\alpha} + [\vartheta_{\alpha}])$ and $(\bar{\vartheta}_{\alpha} + [\vartheta_{\alpha}])$ glue to give operators $\bar{\vartheta}$ and $\Delta$ on $PV^*_*(g)$ such that

1. $\bar{\vartheta}$ is a derivation of $[\cdot, \cdot]$ and $\wedge$ in the sense that

$$\bar{\vartheta}[u, v] = [\bar{\vartheta}u, v] + (-1)^{|u|+1}[u, \bar{\vartheta}v], \quad \bar{\vartheta}(u \wedge v) = (\bar{\vartheta}u) \wedge v + (-1)^{|u|}u \wedge (\bar{\vartheta}v),$$

where $|u|$ and $|v|$ denote respectively the total degrees (i.e. $|u| = p + q$ if $u \in PV^{p,q}(g)$) of the homogeneous elements $u, v \in PV^*(g)$;

2. the BV operator $\Delta$ satisfies the BV equation and is a derivation for the bracket $[\cdot, \cdot]$, i.e.

$$\Delta[u, v] = [\Delta u, v] + (-1)^{|u|+1}[u, \Delta v], \quad \Delta(u \wedge v) = (\Delta u) \wedge v + (-1)^{|u|}u \wedge (\Delta v) + (-1)^{|u|}[u, v],$$

for homogeneous elements $u, v \in PV^*(g)$; and

3. we have

$$\bar{\vartheta}^2 = 0 = \Delta^2 = 0 = \bar{\vartheta} \Delta + \Delta \bar{\vartheta} \pmod{m},$$

so $(PV^*_*, \wedge, \bar{\vartheta}, \Delta)$ (mod $m$) forms a dgBV algebra.

Moreover, if $(\bar{\vartheta}', \bar{\vartheta}')$ is another pair of such elements defining operators $\bar{\vartheta}'$ and $\Delta'$, then we have

$$\bar{\vartheta}' - \bar{\vartheta} = [\nu_1, \cdot], \quad \Delta' - \Delta = [\nu_2, \cdot],$$

for some $\nu_1 \in PV^{*-1}(g)$ and $\nu_2 \in PV^{0,0}(g)$.

Proof. In view of Lemmas 3.31 and 3.32, we have a Čech 1-cocycle $\varpi = (\varpi_{\alpha\beta})_{\alpha\beta} = \lim_{k}(k\varpi_{\alpha\beta})_{\alpha\beta}$ and $f = (f_{\alpha\beta})_{\alpha\beta} = \lim_{k}(k\varpi_{\alpha\beta})_{\alpha\beta}$. Using the exactness of the Čech-Thom-Whitney complex in Lemma 3.27, we obtain $\varrho \in \mathcal{C}(TW^{-1,1}, g)$ and $f \in \mathcal{C}(TW^0, g)$ such that the images of $-\varrho$ and $-f$ under the Čech differential $\delta_0$ are $\varpi$ and $f$ respectively, and also $\varrho = 0 \pmod{m}$ and $f = 0 \pmod{m}$. Therefore we obtain the identities

$$g_{\beta \alpha} \circ (\bar{\varrho}_{\beta} + [\varrho_{\beta}]) \circ g_{\alpha\beta} = \bar{\varrho}_{\alpha} + [\varrho_{\alpha}], \quad g_{\beta \alpha} \circ (\vartheta_{\beta} + [\varpi_{\beta}]) \circ g_{\alpha\beta} = \vartheta_{\alpha} + [\varpi_{\alpha}].$$

Also notice that if we have another choice of $\vartheta'$ and $\bar{\vartheta}'$ such that the images of $-\varrho'$ and $-\bar{\vartheta}'$ under the Čech differential $\delta_0$ are $\varpi$ and $f$ respectively, then we must have $\bar{\vartheta}' - \bar{\vartheta} = \delta_{-1}(\nu_1)$ and $\bar{\vartheta}' - \bar{\vartheta}' = \delta_{-1}(\nu_2)$ for some elements $\nu_1 \in PV^{-1,1}(g)$ and $\nu_2 \in PV^{0,0}(g)$.

It remains to show that the operators $\bar{\vartheta}$ and $\Delta$ defined by $\bar{\vartheta}$ and $\Delta$ satisfying the desired properties. First note that we have an injection $\delta_{-1} : PV^{p,q}(g) \hookrightarrow \mathcal{C}(TW^{p,q}, g) = \prod_{\alpha} TW^{p,q}_\alpha$, where we write $TW^{p,q}_\alpha := \lim_{k}(kTW^{p,q}_\alpha)$. Also the product $\wedge$ and the Lie bracket $[\cdot, \cdot]$ on $PV^*_*(g)$ are induced by those on each $TW^{p,q}_\alpha$. Since $\bar{\vartheta}$ and $\Delta$ are defined by gluing the operators $(\bar{\vartheta}_{\alpha} + [\varrho_{\alpha}])_{\alpha}$ and

There exist elements $\mathcal{D} = (\mathcal{D}_\alpha)_\alpha = \lim_k (k\mathcal{D}_\alpha)_\alpha \in \mathcal{O}(TW^{-1}, h)$ and $\tilde{\mathcal{D}} = (\tilde{\mathcal{D}}_\alpha)_\alpha = \lim_k (k\tilde{\mathcal{D}}_\alpha)_\alpha \in \mathcal{O}(TW^0, h)$ such that
\[ h_{\beta\alpha} \circ (D_\beta + [\mathcal{D}_\beta, \cdot]) \circ h_{\alpha\beta} = D_\alpha + [\mathcal{D}_\alpha, \cdot], \quad h_{\beta\alpha} \circ (\Delta_\beta + [\tilde{\mathcal{D}}_\beta, \cdot]) \circ h_{\alpha\beta} = \Delta_\alpha + [\tilde{\mathcal{D}}_\alpha, \cdot]. \]
Furthermore, $(D_\alpha + [\mathcal{D}_\alpha, \cdot])_\alpha$ and $(\Delta_\alpha + [\tilde{\mathcal{D}}_\alpha, \cdot])_\alpha$ glue to give operators $D$ and $\Delta$ on $PV^{*,*}(h)$ so that $(PV^{*,*}(h), \wedge, D, \Delta)$ satisfies (1)–(3) of Theorem 3.34 (with $D$ playing the role of $\tilde{\partial}$).

4. Abstract construction of the de Rham differential complex

4.1. The de Rham complex. Given a set of compatible gluing morphisms $g = \{g_{\alpha\beta}\}$, the goal of this subsection is to glue the local filtered de Rham modules $k\mathcal{K}_\alpha$ over $V_\alpha$ to form a global differential graded algebra over $X$. Similar to §3.2, we consider a sheaf of filtered de Rham modules $(\mathcal{K}^*, \mathcal{K}^\bullet, \wedge, \partial)$ over a sheaf of BV algebras $(\mathcal{G}^*, \wedge, \Delta)$ on $V$ and a countable acyclic cover $U = \{U_i\}_{i \in \mathbb{Z}_+}$ of $V$ which satisfies the condition that $H^{>0}(U_{i_0\cdots i_r}, \mathcal{K}^3) = 0$ for all $j, r$ and all finite intersections $U_{i_0\cdots i_l} := U_{i_0} \cap \cdots \cap U_{i_l}$.

Definition 4.1. We let
\[ TW^{p,q}(\mathcal{K}) := \{(\eta_{i_0\cdots i_l})_{(i_0, \ldots, i_l)} \in \mathcal{T} \mid \eta_{i_0\cdots i_l} \in \mathcal{A}^q(\mathcal{A}_l) \otimes_{\mathcal{C}} \mathcal{K}^p(U_{i_0\cdots i_l}), \quad d_{j,l}(\eta_{i_0\cdots i_l}) = \eta_{i_0\cdots i_l} |_{U_{i_0\cdots i_l}}, \quad (\mathcal{D}_\alpha)_\alpha \text{ defined componentwise by} \]
\[ (\alpha_l \otimes u_l) \wedge (\beta_l \otimes w_l) := (-1)^{|u_l||\beta_l|} (\alpha_l \wedge \beta_l) \otimes (u_l \wedge w_l), \quad \partial(\alpha_l \otimes u_l) := (-1)^{|\alpha_l|} \alpha_l \otimes (\partial u_l), \]
for $\alpha_l, \beta_l \in \mathcal{A}^*(\mathcal{A}_l)$ and $u_l, w_l \in \mathcal{K}^*(U_l)$ where $l = |I| - 1$. Furthermore, there is an action
\[ \iota_\varphi = \varphi : TW^*(\mathcal{K}) \to TW^{*+|\varphi|}(\mathcal{K}) \text{ defined componentwise by} \]
\[ (\alpha_l \otimes v_l) \partial_l(\beta_l \otimes u_l) := (-1)^{|\beta_l||v_l|} (\alpha_l \wedge \beta_l) \otimes (v_l \wedge u_l), \]
for $\alpha_l, \beta_l \in \mathcal{A}^*(\mathcal{A}_l)$, $v_l \in \mathcal{G}^*(U_l)$ and $u_l \in \mathcal{K}^*(U_l)$, where $|\varphi| = p + q$ for $\varphi \in TW^{p,q}(\mathcal{K})$ and $l = |I| - 1$.

Direct computation shows that $(TW^*(\mathcal{K}), \wedge, \partial)$ is a filtered de Rham complex over the BV algebra $(TW^*(\mathcal{G}), \wedge, \Delta)$ with the identity
\[ L_{\alpha} \otimes v_l(\beta_l \otimes u_l) = (-1)^{|v_l||\beta_l|} (\alpha_l \wedge \beta_l) \otimes (L_{\gamma} v_l) \text{ for } \alpha_l, \beta_l \in \mathcal{A}^*(\mathcal{A}_l), v_l \in \mathcal{G}^*(U_l) \text{ and } u_l \in \mathcal{K}^*(U_l) \text{ where } l = |I| - 1. \]

Proposition 4.2. There is an exact sequence
\[ 0 \to TW^{p,q}(r+1, \mathcal{K}) \to TW^{p,q}(\mathcal{K}) \to TW^{p,q}(\mathcal{K} / r+1, \mathcal{K}) \to 0 \]
induced naturally by the exact sequence $0 \to r+1\mathcal{K} \to r\mathcal{K} \to \mathcal{K} / r+1\mathcal{K} \to 0$.

Proof. The only nontrivial part is the surjectivity of the map $p : TW^{p,q}(\mathcal{K}) \to TW^{p,q}(\mathcal{K} / r+1, \mathcal{K})$, which is induced from surjective maps $p : r\mathcal{K}^p(U_{i_0\cdots i_l}) \to (r\mathcal{K}^p / r+1\mathcal{K}^p)(U_{i_0\cdots i_l})$. We fix $\eta = (\eta_{i_0\cdots i_l})_{(i_0, \ldots, i_l)} \in TW^{p,q}(\mathcal{K} / r+1, \mathcal{K})$ with $\eta_{i_0\cdots i_l} \in \mathcal{A}^q(\mathcal{A}_l) \otimes r\mathcal{K}^p(U_{i_0\cdots i_l})$, and show by induction on $l$ that there exists a lifting $\tilde{\eta}_{i_0\cdots i_l} \in \mathcal{A}^q(\mathcal{A}_l') \otimes r\mathcal{K}^p(U_{i_0\cdots i_l'})$ for any $l' \leq l$ satisfying
Given a set of compatible gluing morphisms \( g = \{g_{\alpha \beta}\} \) as in §3.4, we can extend them to gluing morphisms acting on \( \mathcal{A}_{\alpha \beta}^{*,*,*} := TW^{*,*}(\mathcal{K}_\alpha |_{V_{0\ldots\alpha}}) \).

**Definition 4.4.** For each pair \( V_\alpha, V_\beta \subset V \), the morphism \( \hat{g}_{\alpha \beta} = (\hat{g}_{\alpha \beta,i_{0\ldots\beta}})_{(i_0, \ldots, i_\ell) \in I} : \mathcal{A}_{\alpha \beta}^{*,*,*} \to \mathcal{A}_{\alpha \beta}^{*,*,*} \) is defined componentwise by

\[
\hat{g}_{\alpha \beta,i_{0\ldots\beta}}(\eta_{0\ldots\beta}) := (\exp(L_{\eta_{0\ldots\beta},i_{0\ldots\beta}}) \circ \hat{g}_{\alpha \beta,i_{0\ldots\beta}}(\eta_{0\ldots\beta}))
\]

for any multi-index \( (i_0, \ldots, i_\ell) \in I \) such that \( U_{i_\ell} \subset V_{\alpha \beta} \) for every \( 0 \leq j \leq \ell \).

From Definition 2.18, we see that the differences between the morphisms \( \hat{\psi}_{\alpha \beta} \)'s are captured by taking Lie derivatives of the same elements \( \hat{p}_{\alpha \beta,k} \)'s, \( \hat{p}_{\alpha \beta,j} \)'s and \( \hat{p}_{\alpha \beta} \)'s as for the morphisms \( \hat{\psi}_{\alpha \beta} \)'s. So the morphisms \( \hat{p}_{\alpha \beta} : \mathcal{A}_{\alpha \beta}^{*,*,*} \to \mathcal{A}_{\alpha \beta}^{*,*,*} \) are well-defined and satisfying \( \hat{g}_{\alpha \beta} = \hat{g}_{\alpha \beta} \pmod{m^{k+1}}, \hat{g}_{\alpha \beta} \circ \hat{g}_{\alpha \beta} \circ \hat{g}_{\alpha \beta} = \text{id} \). As a result, we can define the Čech-Thom-Whitney complex \( \tilde{C}^*(\mathcal{A}, \hat{g}) \) as in §3.4.

**Definition 4.5.** For \( \ell \geq 0 \), we let \( \mathcal{A}_{\alpha \beta}^{*,*,*}(\hat{g}) \subset \bigoplus_{\ell=0}^\infty \mathcal{A}_{\alpha \beta}^{*,*,*} \) be the set of elements \( (\eta_0, \ldots, \eta_\ell) \) such that \( \eta_j = \hat{g}_{\alpha \beta,j}(\eta_j) \). Then we set \( \tilde{C}^\ell(\mathcal{A}_{\alpha \beta}^{*,*,*}(\hat{g})) := \prod_{\alpha \beta} \mathcal{A}_{\alpha \beta}^{*,*,*}(\hat{g}) \) for each \( k \in \mathbb{Z}_{\geq 0} \) and \( \tilde{C}^\ell(\mathcal{A}, \hat{g}) := \bigoplus \tilde{C}^\ell(\mathcal{A}_{\alpha \beta}^{*,*,*}(\hat{g})) \) which is equipped with the natural restriction maps \( r_{\alpha \beta} : k \tilde{C}^{\ell-1}(\mathcal{A}, \hat{g}) \to k \tilde{C}^\ell(\mathcal{A}, \hat{g}) \).

We let \( k \delta := \sum_{j=0}^{\ell+1} (-1)^j r_{j+1} : k \tilde{C}(\mathcal{A}, \hat{g}) \to k \tilde{C}^{\ell+1}(\mathcal{A}, \hat{g}) \) be the Čech differential. The \( k \delta \)th order total de Rham complex over \( X \) is then defined to be \( k \tilde{C}^*(\mathcal{A}, \hat{g}) := \text{Ker}(k \delta_0) \). Denoting the natural inclusion \( k \tilde{C}^*(\mathcal{A}, \hat{g}) \to k \tilde{C}^0(\mathcal{A}, \hat{g}) \) by \( k \delta \), we obtain the following sequence of maps

\[
0 \to k \tilde{C}^{p,q}(\mathcal{A}, \hat{g}) \to k \tilde{C}^{0}(\mathcal{A}, \hat{g}) \to k \tilde{C}^{1}(\mathcal{A}, \hat{g}) \to \cdots \to k \tilde{C}^{\ell}(\mathcal{A}, \hat{g}) \to \cdots
\]

where the second sequence is obtained by taking inverse limits \( \lim_{\leftarrow k} \). The first sequence.

Furthermore, we let \( k \tilde{\partial} \) and \( k \partial \) be the operators on \( k \tilde{C}^*(\mathcal{A}, \hat{g}) \) obtained by gluing of the operators \( (k \tilde{\partial}_\alpha + L_{\delta_\alpha}) \)'s (where \( (k \delta_\alpha) \in k \tilde{C}^1(TW^{-1,1}, g) \) is the element obtained from Theorem 3.34) and \( k \partial_\alpha \)'s on \( k \mathcal{A}_{\alpha \beta} \), and \( \tilde{\partial} := \lim_{\leftarrow k} k \tilde{\partial} \) and \( \partial := \lim_{\leftarrow k} k \partial \) be the corresponding inverse limits.

**Proposition 4.6.** Let \( k \tilde{C}^*(\mathcal{A}, \hat{g}) \) be the filtration inherited from that of \( \mathcal{K} \) for \( \bullet = 0, \ldots, s \). Then \( (k \mathcal{A}_{\alpha \beta}^{*,*}(\hat{g}), \wedge, \partial) \) is a filtered de Rham module over the BV algebra \( \text{PV}^{*,*}(\hat{g}, \wedge, \Delta) \), and we have \( \partial^2 = 0 = \partial \tilde{\partial} + \tilde{\partial} \partial \pmod{m} \) as well as the relations

\[
\tilde{\partial}(\eta \wedge \mu) = \partial(\eta \wedge \mu + (-1)^{|\eta|} \eta \wedge (\tilde{\partial} \mu), \quad \tilde{\partial}(\partial \eta) = (\partial \eta) \wedge \eta + (-1)^{|\eta|} \partial \eta)
\]

for \( \phi \in \text{PV}^{*,*}(g) \) and \( \eta, \mu \in \mathcal{A}_{\alpha \beta}^{*,*}(\hat{g}) \). Furthermore, the filtration \( \mathcal{A}_{\alpha \beta}^{*,*}(\hat{g}) \) satisfies the relation

\[
\mathcal{A}_{\alpha \beta}^{*,*}(\hat{g}) / r_{\alpha \beta+1} \mathcal{A}_{\alpha \beta}^{*,*}(\hat{g}) = \mathcal{A}_{\alpha \beta}^{*,*}(\hat{g}) / r_{\alpha \beta+1} \mathcal{A}_{\alpha \beta}^{*,*}(\hat{g})
\]
Proof. Since $\partial$ and $\partial$ are constructed from the operators $(k^\partial + L_{k\theta})$’s and $k^\partial$’s on $k\mathcal{A}_{\alpha;\alpha}$, we only have to check the relations for each $(k\mathcal{A}_{\alpha;\alpha}, \wedge, k^\partial)$, which is a filtered de Rham module over the BV algebra $(k\mathcal{T}W_{\alpha;\alpha}, \wedge, k^\Delta)$. To see the last relation, note that there is an exact sequence of Čech-Thom-Whitney complexes $0 \rightarrow k^\mathcal{C}^*(r_{r+1}A^{p,q}, \hat{g}) \rightarrow k^\mathcal{C}^*(r_{r+1}A^{p,q}, \hat{g})$ $\rightarrow k^\mathcal{C}^*(r_{r+1}A^{p,q}, \hat{g}) \rightarrow 0$ using Proposition 4.2 Taking the kernel and inverse limits then gives the exact sequence $0 \rightarrow r_{r+1}A^{p,q}(\hat{g}) \rightarrow r_{r+1}A^{p,q}(\hat{g}) \rightarrow r_{r+1}A^{p,q}(\hat{g}) \rightarrow 0$.

The result follows. 

Notations 4.7. We will simplify notations by writing $k\mathcal{A}_{\alpha}^{*,*} = \mathcal{A}(\hat{g})$ and $PV^{*,*} = PV^{*,*}(g)$ if the gluing morphisms $\hat{g} = \{k\hat{g}_{\alpha;\beta}\}$ and $g = \{k\hat{g}_{\alpha;\beta}\}$ are clear from the context. We will further denote the relative de Rham complex (over $\text{spf}(\hat{R})$) as $A^{*,*} := 0_{1}A^{*,*} = 0_{1}A^{*,*}$.

Proposition 4.8. Using the element $(k\phi_{\alpha})_{\alpha} \in k\mathcal{C}^0(TW^{0,0}, \hat{g})$ obtained from Theorem 3.32, we obtain the element $\left(\exp(k\phi_{\alpha})\right)_{\alpha} \in k\mathcal{C}^0_{\alpha;\alpha}(A^{0,0}, \hat{g})$ whose components glue to form a global element $k\omega \in \mathcal{A}_{0;0}$, i.e. we have $(k\omega_{\alpha;\beta})_\alpha = \exp(k\phi_{\alpha})_\alpha$. Furthermore, we have $k\omega = k\omega$ (mod $m^{k+1}$). In view of this, we define the relative volume form to be $\omega := \lim \frac{k\omega}{k}$. 

Proof. Similar to the proof of Lemma 3.32, we use the power series $T(x)$ in (3.24) to simplify notations. We fix $V_{\alpha;\beta}$ and $U_{0..i}$ such that $U_{i} \subset V_{\alpha;\beta}$ for all $0 \leq j \leq l$. We need to show that 

$$(\exp(L_{k\theta_{\alpha;\beta}, i_{0}..i_{l}}))(k\omega_{\alpha}) = \exp(k\phi_{\alpha})_\alpha$$

We begin with the case $l = 0$. Making use of the identities 

$$(\exp([\theta, \cdot])(u \wedge v)) = (\exp([\theta, \cdot])(u)) \wedge (\exp([\theta, \cdot])(v))$$

for $\theta \in k\mathcal{G}_{\beta}^{-1}(U_{0..i})$, $u, v \in k\mathcal{G}_{\beta}^{0}(U_{0..i})$ and $w \in 0_{1}k\mathcal{G}_{\beta}^{0}(U_{0..i})$, we have

$$(k\omega_{\alpha}) = (\exp(L_{k\theta_{\alpha;\beta}, i_{0}..i_{l}}))(k\omega_{\alpha}) = \exp(\exp(k\phi_{\alpha}))(k\omega_{\alpha}) = \exp(\exp(k\phi_{\alpha}))(k\omega_{\alpha}) = \exp(\exp(k\phi_{\alpha}))(k\omega_{\alpha}) = \exp(\exp(k\phi_{\alpha}))(k\omega_{\alpha})$$

where $k\phi_{\beta;\alpha,i} = (k\phi_{\beta;\alpha,i})(k\phi_{\alpha;\alpha,i}) = k\phi_{\beta;\alpha,i}((k\phi_{\alpha;\alpha,i}) \circ k\Delta_{\beta})(k\phi_{\alpha;\alpha,i})$ is the component of the term $k\phi_{\beta;\alpha,i}$ obtained in Lemma 3.32.

The general case $l \geq 0$ is similar, as we have

$$(k\omega_{\alpha}) = (\exp(L_{k\theta_{\alpha;\beta}, i_{0}..i_{l}}))(k\omega_{\alpha}) = \exp(\exp(k\phi_{\alpha}))(k\omega_{\alpha}) = \exp(\exp(k\phi_{\alpha}))(k\omega_{\alpha}) = \exp(\exp(k\phi_{\alpha}))(k\omega_{\alpha})$$

□
Lemma 4.9. The elements $k\eta = \delta + \beta + k\Delta, \beta = k\Delta + k\Delta_0 + k\Delta_1$ glue to give global elements $k\eta \in \kappa \in PV^{-1.2}$ and $\kappa \in PV^{0.1}$ respectively.

Proof. For the element $k\eta$, we have

\[
\delta + \beta + k\Delta_0 + k\Delta_1 \in \kappa \in PV^{0.1}
\]

and since $k\eta \in \kappa$, we deduce that $k\eta \in \kappa$. Hence $k\eta$ is the injectivity of $k\Delta \mapsto k\partial_0$.

For the element $k\eta$, we have $k\eta = k\partial_0 + k\partial_1$, which implies that $k\eta = k\partial_0 + k\partial_1$ from the constructions in Theorem 3.34. Making use of the relations $k\partial_0 \circ k\eta = k\partial_0 + k\partial_0$, and $k\partial_0 = k\partial_0$, we deduce that $k\eta$ from Lemmas 3.31 and 3.32 we have

\[
k\eta = k\partial_0 + k\partial_1
\]

Hence it remains to show that $k\eta \in \kappa$, $k\eta$, and $k\eta$ are in $k\partial_0$.

We fix a multi-index $(i_0, \ldots, i_l) \in \mathcal{I}$, and recall from the proofs of Lemmas 3.31 and 3.32 the formulas:

\[
k\eta = k\partial_0 + k\partial_1
\]

where $T$ is the formal series introduced in (3.24). Now we consider the dgLa $(A^\ast(\eta) \otimes k\partial_0(U_{i_0, \ldots, i_l}, k\Delta) + k\partial_1, \cdot_1))$. We apply Lemma 2.5 and notice that

\[
\Delta = \exp(k\partial_0 + k\partial_1 + k\partial_1) \ast 0 = (T(-[k\partial_0, \partial_1]) \circ (k\Delta + k\partial_0))((k\partial_0, \partial_1).
\]

Since $\Delta$ is gauge equivalent to the above dgLa, we have the equation $(k\Delta + k\partial_0) \ast 0 = \frac{1}{2}[\Delta, \partial_0] = 0$ whose component in $A^1(\eta) \otimes k\partial_0(U_{i_0, \ldots, i_l})$ can be extracted as

\[
k\eta = k\partial_0 + k\partial_1
\]

Therefore, the $(i_0, \ldots, i_l)$-component of the term $k\Delta + k\partial_0$ is given by

\[
k\eta = k\partial_0 + k\partial_1
\]

Definition 4.10. We let $l = \lim_{k} k! \in PV^{-1, 2}$ and $\eta = \lim_{k} k! \in PV^{0, 1}$. The operator $d = \partial + \partial + L$, which acts on $\mathfrak{A}^\ast$, preserves the filtration, is called the total de Rham differential. We also denote the pull back of $d$ to $PV^{\ast, \ast}$ under the isomorphism $\omega : PV^{\ast, \ast} \rightarrow \mathfrak{A}^{d + \ast, \ast}$ by $d$.

Proposition 4.11. The pair $(\mathfrak{A}^\ast, d)$ forms a filtered complex, i.e. $d^2 = 0$ and $d$ preserves the filtration. We also have $d = \partial + \Delta + (1 + \eta) \omega$ on $PV^{\ast}$. 
Proof. From the discussion right before Proposition 4.2 we compute $d^2 = (\bar{\partial} + \partial + 1)^2 = (\bar{\partial} + \partial)^2 - L_1$. If we compute $(\bar{\partial} + \partial)^2$ locally on $A_{\alpha a}^*$, we obtain $(\bar{\partial}_\alpha + L_{\partial a} + \partial_\alpha)^2 = L_{\partial \alpha} (\omega_\alpha) + L_0^2 = L_{\partial_\alpha \omega_\alpha + \frac{1}{2} [\omega_\alpha, \omega_\alpha]} = L_{\partial_\alpha}$. So we get $(\bar{\partial} + \partial)^2 = L_1$ and hence $d^2 = 0$. As for $d$, we compute locally on $TW_{\alpha a}^*$. Taking $\gamma \in TW_{\alpha a}^*$, we have

$$d(\gamma \exp(f_\alpha) \omega) = (\bar{\partial}_\alpha + L_{\partial a} + \partial_\alpha + f_\alpha) \omega + [\partial_\alpha, \gamma] + \Delta_\alpha (\gamma) + [f_\alpha, \gamma] + \eta_\alpha \gamma \omega,$$

which gives the identity $d = \bar{\partial} + \Delta + (1 + \eta)\omega$. □

4.2. The Gauss-Manin connection. Using the natural isomorphisms $k^1 \sigma_\alpha : (k^1 K_{\alpha a}^*, k \partial_\alpha) \cong K \otimes \mathbb{Z} (0:1 K_{\alpha a}^*[-1], k \partial_\alpha)$ from Definition 2.17, we obtain isomorphisms

$$k^1 \sigma_\alpha : k^1 A_{\alpha a}^* \to K \otimes \mathbb{Z} 0:1 A_{\alpha a}^*[-1],$$

which can be patched together to give an isomorphism of complexes $k^1 \sigma : k^1 A^* = k^1 A^* \otimes k^1 A^* \to K \otimes \mathbb{Z} k^1 A^*[-1]$ which is equipped with the differential $k^1 d$. This produces an exact sequence of complexes:

$$(4.3) \quad 0 \to K \otimes \mathbb{Z} k^1 A^*[-1] \to k^1 A^* \otimes k^1 A^* \to k^1 A^* \to 0,$$

which we use to define the Gauss-Manin connection (cf. Definition 2.11).

Definition 4.12. Taking the long exact sequence associated to (4.3), we get the map

$$k^1 \nabla : H^* (k^1 A, d) \to K \otimes \mathbb{Z} H^* (k^1 A, d),$$

which is called the $k^1$-th order Gauss-Manin (abbrev. GM) connection over $k R$. Taking inverse limit over $k$ gives the formal Gauss-Manin connection over $\hat{R}$:

$$\nabla : \hat{H}^* (\hat{k} A, \hat{d}) \to \hat{K} \otimes \mathbb{Z} \hat{H}^* (\hat{k} A, \hat{d}).$$

The modules $H^* (\hat{k} A, \hat{d})$ and $\hat{H}^* (\hat{k} A, \hat{d})$ over $k R$ and $\hat{R}$ are respectively called the $k^1$-th order Hodge bundle and the formal Hodge bundle.

Remark 4.13. By its construction, the complex $(0:1 K^*, 0 \partial)$ serves as a resolution of the complex $(0:1 K^*, 0 \partial)$, and the cohomology $H^* (0:1 K^*, 0 \partial)$ computes the hypercohomology $\hat{H}^* (0:1 K, 0 \partial)$. So the 0-th order Gauss-Manin connection 0$\nabla$ agrees with the ones introduced in Definition 2.11.

Proposition 4.14. The Gauss-Manin connection $\nabla$ defined in Definition 4.12 is a flat connection, i.e. the map $\nabla^2 : \hat{H}^*(\hat{k} A, \hat{d}) \to \Lambda^2 (K_{\mathbb{Z}}) \otimes \mathbb{Z} \hat{H}^*(\hat{k} A, \hat{d})$ is a zero map.

Proof. It suffices to show the $k^1$-th order Gauss-Manin connection $k^1 \nabla$ is flat for every $k$. Consider the short exact sequence (4.3), and take a cohomology class $[\eta] \in H^* (\hat{k} A, \hat{d})$ represented by an element $\eta \in k^1 A^*$. Then we take a lifting $\bar{\eta} \in k^0 A^*$ so that $k^1 \nabla ([\eta])$ is represented by the element $k^1 d(\bar{\eta}) \in k^0 A^* \otimes k^2 A^*$. We write $k^1 \nabla (\eta) = \sum \alpha_i \otimes [\xi_i]$ for $\alpha_i \in k^1 \Omega_{S_i}^1$ and $[\xi_i] \in H^* (k^1 A, \hat{d})$. Once again we take a representative $\xi_i \in k^1 A^*$ for $[\xi_i]$ and by our construction we have an element $e \in k^1 A^*$ such that $\sum \alpha_i \otimes \xi_i = k^1 d(\bar{\eta}) + e$. Therefore if we consider the exact sequence of complexes

$$0 \to k^1 A^* / k^2 A^* \to k^1 A^* / k^3 A^* \to k^1 A^* / k^2 A^* \to 0,$$
we have \( k \mathbf{d} (\sum \alpha_i \otimes \xi_i) = k \mathbf{d} (e) \in k \mathcal{A}^*/k\mathcal{A}^* \). Note that \((k\nabla)^2([\eta])\) is represented by the cohomology class of the element \( k \mathbf{d} (\sum \alpha_i \otimes \xi_i) \in k \mathcal{A}^*/k\mathcal{A}^* \cong k \Omega^2_{\mathcal{S}^1} \otimes (k R) \mathcal{A}^*[-2] \) using the isomorphism induced by \( \frac{\partial}{\partial \sigma} \)'s from Definition 2.17. Hence we have \( [k \mathbf{d} (\sum \alpha_i \otimes \xi_i)] = [k \mathbf{d} (e)] = 0 \) in \( k \Omega^2_{\mathcal{S}^1} \otimes (k R) \mathcal{A}^*[-2] \).

### 4.3. Freeness of the Hodge bundle from a local criterion.

To prove the desired unobstructedness result, we need freeness of the Hodge bundle; in geometric situations, this has been established in various cases [29, 46, 32, 21]. In this subsection, we generalize the techniques in [21, 32, 46] to prove the freeness of the \( k \)-th order Hodge bundle \( H^*(k \mathcal{A}, k \mathbf{d}) \) over \( k R \) in our abstract setting (Lemma 4.16) under a local criterion (Assumption 4.15).

#### 4.3.1. A local criterion.

Recall from Notations 1.6 that we have a strictly convex polyhedral cone \( \mathcal{Q} \subset \mathcal{K} \), the coefficient ring \( R = \mathbb{C}[\mathcal{Q}] \), and the log space \( \mathcal{S}^1 \) (or the formal log space \( \mathcal{S}^1 \)) parametrizing the moduli space near the degenerate Calabi-Yau variety \( \mathcal{X} \). In our abstract setting, the existence of such restrictions can be parametrized morphisms for the \( k \mathcal{A}^* \) parametrizing the moduli space near the degenerate Calabi-Yau variety \( \mathcal{X} \). In this subsection, we generalize the techniques in [21, 32, 46] to prove the freeness of the \( k \)-th order Hodge bundle \( H^*(k \mathcal{A}, k \mathbf{d}) \) over \( k R \) in our abstract setting (Lemma 4.16) under a local criterion (Assumption 4.15).

Geometrically, taking base change with the map \( \mathcal{A} \to \mathcal{S}^1 \) should be viewed as restricting the family to the 1-dimensional family determined by \( \mathcal{N} \). In our abstract setting, we consider the tensor product \( k \mathcal{G}^*_{\mathcal{N}, \alpha} := k \mathcal{G}^*_{\alpha} \otimes_{k R} (\mathbb{C}[q]/(q^{k+1})) \). Then tensoring the maps \( k \psi_{n,\alpha,i} \)'s with \( \mathbb{C}[q]/(q^{k+1}) \) give patching morphisms for the \( k \mathcal{G}^*_{\mathcal{N}, \alpha} \)'s which will be denoted as \( k \psi_{n,\alpha,\beta,i} \). Similarly we use \( k \mathcal{b}_{n,\alpha,i} \)'s, \( k \mathcal{b}_{n,\alpha,\beta,j} \)'s, \( k \mathcal{b}_{n,\alpha,\beta,i,j} \)'s and \( k \mathcal{b}_{n,\alpha,\beta,j,i} \)'s to denote the tensor products of the corresponding terms appearing in Definition 2.15 with \( \mathbb{C}[q]/(q^{k+1}) \). Note that all the relations in Definition 2.15 still hold after taking the tensor products.

In view of the isomorphism \( \mathcal{J}^R : k \mathcal{P}^* \to k \mathcal{A}^* \) in Definition 4.10 and the fact that the complex \((k \mathcal{P}^*, k \mathcal{d})\) is free over \( k R \) (meaning that the differential is \( k R \)-linear), we see that \((k \mathcal{A}^*, k \mathcal{d})\) is also free over \( k R \). Then taking tensor product with \( \mathbb{C}[q]/(q^{k+1}) \) (for a fixed \( \mathcal{N} \)), we obtain the relative de Rham complex \((\mathcal{A}^* \otimes_{k R} (\mathbb{C}[q]/(q^{k+1})), \mathcal{d}) \) over \( \mathbb{C}[q]/(q^{k+1}) \).

Now the filtered de Rham module \( k \mathcal{K}^*_{\mathcal{N}, \alpha} \) plays the role of the sheaf of holomorphic de Rham complex on the thickening of \( \mathcal{V}_\alpha \). We need to consider restrictions of these holomorphic differential forms to the 1-dimensional family Spec(\( \mathbb{C}[q]/(q^{k+1}) \)), but naively taking tensor product with \( \mathbb{C}[q]/(q^{k+1}) \) does not give the desired answer. In our abstract setting, the existence of such restrictions can be formulated as the following assumption (which is motivated by the proof of [21] Theorem 4.1):

**Assumption 4.15.** For each \( \mathcal{N} \in \text{int}(\mathcal{Q}^\vee) \cap \mathcal{K}^\vee \), \( k \in \mathbb{Z}_{\geq 0} \) and \( \mathcal{V}_\alpha \in \mathcal{V} \), we assume there exists a coherent sheaf of dga’s

\[
(k \mathcal{K}^*_{\mathcal{N}, \alpha}, k \mathcal{d}_{\mathcal{N}, \alpha})
\]

equipped with a dg module structure over \( k \mathcal{Q}^\vee \), the natural filtration \( k \mathcal{K}^*_{\mathcal{N}, \alpha} \to k \mathcal{K}^*_{\mathcal{N}, \alpha} \to k \mathcal{K}^*_{\mathcal{N}, \alpha} \) where \( k \mathcal{K}^*_{\mathcal{N}, \alpha} = d \log(\mathcal{Q}) \wedge k \mathcal{K}^*_{\mathcal{N}, \alpha} \), and a de Rham module structure over \((k \mathcal{G}^*_{\mathcal{N}, \alpha}, \mathcal{d}_{\mathcal{N}, \alpha})\) satisfying all the conditions in Definitions 2.17 and 2.18 (in particular, we have surjective morphisms \( k \mathcal{b}_{\mathcal{N}, \alpha,i} : k \mathcal{K}^*_{\mathcal{N}, \alpha} \to k \mathcal{K}^*_{\mathcal{N}, \alpha} \) for \( k \geq 1 \), a volume element \( k \omega_{\mathcal{N}, \alpha} \in k \mathcal{K}^*_{\mathcal{N}, \alpha} / k \mathcal{K}^*_{\mathcal{N}, \alpha} \), an isomorphism \( k \sigma_{\mathcal{N}, \alpha} : (k \mathcal{K}^*_{\mathcal{N}, \alpha} / k \mathcal{K}^*_{\mathcal{N}, \alpha}, k \partial_{\mathcal{N}, \alpha}) \to (k \mathcal{K}^*_{\mathcal{N}, \alpha} / k \mathcal{K}^*_{\mathcal{N}, \alpha} [-1], k \partial_{\mathcal{N}, \alpha}) \), and patching isomorphisms \( k \psi_{\mathcal{N}, \alpha,i} : k \mathcal{K}^*_{\mathcal{N}, \alpha} \to k \mathcal{K}^*_{\mathcal{N}, \alpha} \) for triples \((\mathcal{V}_i; \mathcal{V}_\alpha, \mathcal{V}_\beta)\) with \( \mathcal{V}_i \subset \mathcal{V}_\alpha \) fulfilling all the required conditions). We further
assume that the complex $(\tilde{k}\mathcal{K}_{n,\alpha}^*[u], \tilde{k}\partial_{n,\alpha})$, where
\[
\tilde{k}\partial_{n,\alpha}(\sum_{s=0}^{l} \nu_{s}u^{s}) := \sum_{s} (\tilde{k}\partial_{n,\alpha}\nu_{s})u^{s} + sd\log(q) \wedge \nu_{s}u^{s-1},
\]
satisfies the holomorphic Poincaré Lemma in the sense that for each Stein open subset $U$ and any $\sum_{s} \nu_{s}u^{s} \in \tilde{k}\mathcal{K}_{n,\alpha}^*[u]$ with $\tilde{k}\partial_{n,\alpha}(\nu) = 0$, we have $\sum_{s} \eta_{s}u^{s} \in \tilde{k}\mathcal{K}_{n,\alpha}^*[U]$ satisfying $\tilde{k}\partial_{n,\alpha}(\sum_{s} \eta_{s}u^{s}) = \sum_{s} \nu_{s}u^{s}$ on $U$, and if in addition $\tilde{k}\partial_{n,\alpha}(\eta_{0}) = 0$ in $(0\mathcal{K}_{n,\alpha}/1\mathcal{K}_{n,\alpha})(U)$, then $\sum_{s} \eta_{s}u^{s}$ can be chosen so that $\tilde{k}\partial_{n,\alpha}(\eta_{0}) = 0$ in $(0\mathcal{K}_{n,\alpha}/1\mathcal{K}_{n,\alpha})(U)$.

Assumption 4.15 allows us to construct the total de Rham complex $(\bullet\mathcal{A}_{n}^{\bullet*}, \mathcal{d}_{n} = \bar{\partial}_{n} + \partial_{n} + h_{n})$ as a dg-module over $\bar{\Omega}_{\mathcal{A}_{n}^{1,\dag}}^*$ such that $\|\mathcal{A}^{*} \otimes \bar{\mathcal{R}} (\mathcal{C}[q]/(q^{k+1})) = \tilde{k}\mathcal{A}_{n}^{\bullet*} / \tilde{k}\mathcal{A}_{n}^{\bullet*} = \tilde{k}\mathcal{A}_{n}^{\bullet*}$.

4.3.2. Freeness of the Hodge bundle.

**Lemma 4.16.** Under Assumption 4.15, $H^*(\|A, d)$ is a free $\bar{\mathcal{R}}$ module.

**Proof.** As in [32, p. 404] and the proof of [21, Theorem 4.1], it suffices to show that the map $\tilde{k}\partial_{n} : H^*(\|\mathcal{A}_{n}^{\bullet*}, \mathcal{d}_{n}) \rightarrow H^*\mathcal{A}_{n}^{\bullet*}$, which is induced by the maps $\tilde{k}\partial_{n,\alpha}$’s in Assumption 4.15 is surjective for all $k \in \mathbb{Z}_{\geq 0}$. Following the proof of [21, Theorem 4.1], we consider the complex $(\tilde{k}\mathcal{A}_{n}^{\bullet*}, \mathcal{d}_{n})$ constructed from the complexes $(\tilde{k}\mathcal{K}_{n,\alpha}^*[u], \tilde{k}\partial_{n,\alpha})$’s as in Definition 4.10. There is a natural restriction map $\tilde{k}\partial_{\mathcal{A}_{n}^{\bullet*}} : \tilde{k}\mathcal{A}_{n}^{\bullet*} \rightarrow \|\mathcal{A}_{n}^{\bullet*}$ defined by $\tilde{k}\partial_{\mathcal{A}_{n}^{\bullet*}}(\sum_{s=0}^{l} \eta_{s}u^{s}) = k\partial_{n,\alpha}(\eta_{0})$ on $\tilde{k}\mathcal{K}_{n,\alpha}^*[u]$ for each $\alpha$. Since $\tilde{k}\partial_{\mathcal{A}_{n}^{\bullet*}}$ (and hence the induced map on cohomology) factors through $\tilde{k}\partial_{n}^{\bullet}$, we only need to show that $\tilde{k}\partial_{\mathcal{A}_{n}^{\bullet*}} : H^*(\tilde{k}\mathcal{A}_{n}^{\bullet*}, \mathcal{d}_{n}) \rightarrow H^*(\|\mathcal{A}_{n}^{\bullet*}, \mathcal{d}_{n})$ is an isomorphism.

By gluing the sheaves $\tilde{k}\mathcal{K}_{n,\alpha}^*[u]$’s (resp. $\tilde{k}\mathcal{K}_{n,\alpha}^*[u]$’s) as in Definition 4.15, we can construct the Čech-Thom-Whitney complexes $\tilde{k}\mathcal{C}^*(\mathcal{A}_{n}^{\bullet*}, \hat{g}_{n})$ (resp. $\tilde{k}\mathcal{C}^*(\mathcal{B}_{n}^{\bullet*}, \hat{g}_{n})$) and obtain the exact sequences
\[
0 \rightarrow \tilde{k}\mathcal{A}_{n}^{\bullet*}(\hat{g}_{n}) \rightarrow \tilde{k}\mathcal{C}^{0}(\mathcal{A}_{n}^{\bullet*}, \hat{g}_{n}) \rightarrow \tilde{k}\mathcal{C}^{1}(\mathcal{A}_{n}^{\bullet*}, \hat{g}_{n}) \rightarrow \cdots \rightarrow \tilde{k}\mathcal{C}^{l}(\mathcal{A}_{n}^{\bullet*}, \hat{g}_{n}) \rightarrow \cdots,
\]
\[
0 \rightarrow \tilde{k}\mathcal{A}_{n}^{\bullet*}(\hat{g}_{n}) \rightarrow \tilde{k}\mathcal{C}^{0}(\mathcal{B}_{n}^{\bullet*}, \hat{g}_{n}) \rightarrow \tilde{k}\mathcal{C}^{1}(\mathcal{B}_{n}^{\bullet*}, \hat{g}_{n}) \rightarrow \cdots \rightarrow \tilde{k}\mathcal{C}^{l}(\mathcal{B}_{n}^{\bullet*}, \hat{g}_{n}) \rightarrow \cdots.
\]

Now we have a commutative diagram
\[
\begin{array}{ccc}
\tilde{k}\mathcal{A}_{n}^{\bullet*} & \xrightarrow{k\delta_{-1}} & \tilde{k}\mathcal{B}_{n}^{\bullet*} \\
\|\mathcal{A}_{n}^{\bullet*} & \xrightarrow{0\delta_{-1}} & \|\mathcal{B}_{n}^{\bullet*}
\end{array}
\]

where the horizontal arrows are quasi-isomorphisms. So what we need is to show that $\tilde{k}\partial_{n} : \tilde{k}\mathcal{C}^*(\mathcal{B}_{n}^{\bullet*}, \hat{g}_{n}) \rightarrow \|\mathcal{C}^*(\|\mathcal{A}_{n}^{\bullet*}, \hat{g}_{n})$ is a quasi-isomorphism.

The decreasing filtrations
\[
F_{\geq l}^g(\tilde{k}\mathcal{C}^*(\mathcal{B}_{n}^{\bullet*}, \hat{g}_{n})) := \tilde{k}\mathcal{C}^*(\mathcal{B}_{n}^{\bullet*}, \geq l, \hat{g}_{n}), \quad F_{\geq l}^g(\|\mathcal{C}^*(\|\mathcal{A}_{n}^{\bullet*}, \hat{g}_{n})) := \|\mathcal{C}^*(\|\mathcal{A}_{n}^{\bullet*}, \geq l), \hat{g}_{n})
\]

induce spectral sequences with
\[
E_{0}^{p,q} \left( k\mathcal{C}^*(\mathcal{B}_{n}^{\bullet*}, \hat{g}_{n}) \right) = \bigoplus_{p+l=r} k\mathcal{C}^{p,q}(\mathcal{B}_{n}^{\bullet*}, \hat{g}_{n}), \quad E_{0}^{p,q} \left( \|\mathcal{C}^*(\|\mathcal{A}_{n}^{\bullet*}, \hat{g}_{n}) \right) = \bigoplus_{p+l=r} \|\mathcal{C}^{p,q}(\|\mathcal{A}_{n}^{\bullet*}, \hat{g}_{n})
\]
respectively converging to their cohomologies. Therefore it remains to prove that the map
\[ \tilde{\kappa}_{b_0}: \bigoplus_{p+\ell=x} k\hat{A}^\ell(\mathbb{B}_n^{p,q}, \hat{g}_n) \to \bigoplus_{p+\ell=x} 0\hat{A}^\ell(0,1\mathbb{A}_n^{p,q}, \hat{g}_n) \]
induces an isomorphism on the cohomology of the \( E_0 \)-page for each fixed \( q \).

If we further consider the filtrations \( \bigoplus_{p+\ell=x} k\hat{A}^\ell(\mathbb{B}_n^{p,q}, \hat{g}_n) \) and \( \bigoplus_{p+\ell=x} 0\hat{A}^\ell(0,1\mathbb{A}_n^{p,q}, \hat{g}_n) \), then we only need to show that the induced map
\[ k\hat{A}^\ell(\mathbb{B}_n^{*}, \hat{g}_n) \to 0\hat{A}^\ell(0,1\mathbb{A}_n^{*}, \hat{g}_n) \]
on the corresponding \( E_0 \)-page is a quasi-isomorphism for any fixed \( c \) and \( q \), where \( k\mathbb{B}_n^{p,q} \) is constructed by gluing together the sheaves \( k\mathbb{K}^{p,q}_n[u] \)’s as in Definition 4.5.

Note that the differential on \( \bigoplus_{p\geq 0} \prod_{\alpha_0,\ldots,\alpha_\ell} k\mathbb{B}_n^{p,q} \) is given componentwise by the differential \( k\partial_n : k\mathbb{B}_n^{p,q} \to k\mathbb{B}_n^{p+1,q} \) (where the term \( k\mathbb{B}_n^{p,q} \) is defined as in Definition 4.10 using \( k\mathbb{K}^{p,q}_n[u] \)). Using similar argument as in Lemma 3.27, we can see that the bottom horizontal map \( k\tilde{\kappa}_{b_0} \) in the above diagram is surjective. Finally, the kernel complex of this map is acyclic by the holomorphic Poincaré Lemma in Assumption 4.15 and arguments similar to Lemma 3.27.

5. AN ABSTRACT UNBOUNDEDNESS THEOREM

Theorem 3.34 produces an almost differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra \( (PV^{*,*},\tilde{\partial},\Delta,\wedge) \) (where “almost” means \( (\tilde{\partial}+\Delta)^2 \) is zero only at \( 0 \)-th order), together with an almost de Rham module \( (\mathbb{A}^{*,*},\tilde{\partial},\partial,\wedge) \) and the volume element \( \omega \in \mathbb{A}^{d,0} \). From these we can prove an unobstructedness theorem, using the techniques from [1] 33, 30, 47.

5.1. Solving the Maurer-Cartan equation from dgBV structures. We first introduce some notations, following Barannikov [1]:

**Notations 5.1.** Let \( t \) be a formal variable. We consider the spaces of formal power series or Laurent series in \( t \) or \( t^\frac{1}{2} \) with values in polyvector fields
\[ kPV^{p,q}([t]), kPV^{p,q}([t^\frac{1}{2}]), kPV^{p,q}([t^\frac{1}{2}],[t^{-\frac{1}{2}}]), \]

with a scaling morphism \( l_\ell : kPV^{p,q}([t^\frac{1}{2}]), kPV^{p,q}([t^\frac{1}{2}],[t^{-\frac{1}{2}}]) \to kPV^{p,q}([t^\frac{1}{2}],[t^{-\frac{1}{2}}]) \) induced by \( l_\ell(\varphi) = t^{\frac{d-p-2}{2}} \varphi \) for \( \varphi \in kPV^{p,q}([t^\frac{1}{2}],[t^{-\frac{1}{2}}]) \). We have the identification \( k\tilde{\partial}_\ell := t^{\frac{1}{2}}l_\ell^{-1} \circ k\partial \circ l_\ell = k\tilde{\partial} + t(k\Delta) + t^{-1}(k\partial) \wedge \). Also we consider spaces of formal power series or Laurent series in \( t \) or \( t^\frac{1}{2} \) with values in the relative de Rham module
\[ k\mathbb{A}^{p,q}([t^\frac{1}{2}]), k\mathbb{A}^{p,q}([t^\frac{1}{2}],[t^{-\frac{1}{2}}]), \]

with the rescaling \( l_\ell : k\mathbb{A}^{p,q}([t^\frac{1}{2}],[t^{-\frac{1}{2}}]) \to k\mathbb{A}^{p,q}([t^\frac{1}{2}],[t^{-\frac{1}{2}}]) \) given by \( l_\ell(\alpha) = t^{\frac{d-p-2}{2}} \alpha \) which preserves the filtration on \( k\mathbb{A}^{p,q}([t^\frac{1}{2}],[t^{-\frac{1}{2}}]) \), and gives \( l_\ell(\varphi) \wedge (k\omega) = l_\ell(\varphi \wedge k\omega) \) and \( k\tilde{\partial}_\ell := t^{\frac{1}{2}}l_\ell^{-1} \circ k\partial \circ l_\ell = k\tilde{\partial} + t(k\partial) + t^{-1}(k\partial) \).
For the purpose of constructing log Frobenius structures in the next section, we consider a finite-dimensional graded vector space \( V^\ast \) and the associated graded symmetric algebra \( T := \text{Sym}^\ast(V^\ast) \), equipped with the maximal ideal \( I \) generated by \( V^\ast \). We will abuse notations by using \( m \) and \( I \) again to denote the respective ideals of \( R_T := R \otimes T \), where \( R \) is the coefficient ring introduced in Notations 1.6. We also let \( I := m + I \) be the ideal generated by \( m \otimes T + R \otimes I \) and write \( kR_T := (R_T/I)^{k+1} \). We write \( kPV_T := kPV \otimes C T \otimes R_T (R_T/I)^{k+1} \) and \( kA_T := kA \otimes C T \otimes R_T (R_T/I)^{k+1} \), and let \( kPV_T^\ast[t] \), \( kPV_T^\ast[[t]](t^{-1}) \) and \( kA_T^\ast[[t]](t^{-1}) \) be the complexes of formal series or Laurent series in \( t^2 \) or \( t \) with values in those coefficient rings.

**Remark 5.2.** We can also define the Hodge bundle \( H^\ast(A^\ast, d) \otimes \hat{T} \) over the formal power series ring \( \hat{R}_T := \varprojlim_k R_T \), which is equipped with the Gauss-Manin connection \( \nabla \) defined as in Definition 4.12. Then Lemma 4.16 implies that the Hodge bundle \( H^\ast(A^\ast, d) \otimes \hat{T} \) is free over \( \hat{R}_T \), or equivalently, \( H^\ast(kPV_T^\ast, k \cdot d) \) is free over \( kR_T \) for each \( k \in \mathbb{Z}_{>0} \).

**Definition 5.3.** An element \( k \varphi \in kPV_T^\ast[t] \) with \( k \varphi_1 = 0 \) (mod \( m + I \)) is called a Maurer-Cartan element over \( R_T/I^{k+1} \) if it satisfies the Maurer-Cartan equation

\[
(k \hat{\partial} + t(k\Delta))k \varphi + \frac{1}{2}[k \varphi, k \varphi] + (kI + t(k\eta)) = 0,
\]

or equivalently, \((k \hat{\partial} + t(k\Delta) + [k \varphi, \cdot])^2 = 0\).

Notice that the MC equation (5.1) is also equivalent to \( k \hat{d}(e^{l_0(\varphi)} \omega) = 0 \) \( \Leftrightarrow \) \( k \hat{d}(e^{l_0(\varphi)}) = (k \hat{\partial} + k \Delta + (kI + t(k\eta) \wedge) (e^{l_0(\varphi)}) = 0 \). In order to solve (5.1) using algebraic techniques as in [30], we need Assumption 4.15 which guarantees freeness of the Hodge bundle, as well as a suitable version of the Hodge-to-de Rham degeneracy; recall that these are also the essential conditions to ensure unobstructedness for smoothing of log smooth Calabi-Yau varieties in [32].

Recall from Remark 4.13 that the cohomology \( H^\ast(0A^\ast, 0d) \) computes the hypercohomology \( H^\ast(\Omega^\ast, 0\partial) \), so the Hodge filtration \( F^2P^\ast H^\ast = H^\ast(0A^{2p, *}, 0d) \) (where \( 0d = \hat{\partial} + 0\partial \)) is induced by the filtration \( F^\geq 0(0A^\ast) := 0A^{2p, * \geq 0} \) on the complex \( 0A^\ast, 0d \).

**Assumption 5.4** (Hodge-to-de Rham degeneracy). We assume that the spectral sequence associated to the decreasing filtration \( F^\geq 0(0A^\ast) \) degenerates at the \( E_1 \) term.

Assumption 5.4 is equivalent to the condition that \( H^\ast(0PV[[t]], 0d = \hat{\partial} + t(0\Delta)) \) (or equivalently, that \( H^\ast(0A[[t]], \hat{\partial} + t(0\partial)) \)) is a finite rank free \( \mathbb{C}[[t]] \)-module (cf. [30]).

**Theorem 5.5.** Suppose Assumptions 5.4 and 4.15 hold. Then for any degree 0 element \( \psi \in 0PV[[t]] \otimes (I/I^2) \) with \( (0\hat{\partial} + t(0\Delta)) \psi = 0 \), there exists a Maurer-Cartan element \( k \varphi \in kPV_T^\ast[t] \) over \( R_T/I^{k+1} \) for each \( k \in \mathbb{Z}_{\geq 0} \) such that \( k^{+1} \varphi = k \varphi \) (mod \( I^{k+1} \)) and \( k \varphi = \psi \) (mod \( m + I^2 \)).

**Proof.** We will consider the surjective map \( k^{+1} k^b : k^{+1} P^p(q)[t] \rightarrow kPV^p(q)[t] \) obtained from Corollary 3.28 and inductively solve for \( k \varphi \in kPV_T^\ast[t] \) for each \( k \in \mathbb{Z}_{\geq 0} \) so that \( k^{+1} k^b(k^{+1} \varphi) = k \varphi \), \( k \varphi = \psi \) (mod \( m + I^2 \)) and \( k \varphi \) satisfies the Maurer-Cartan equation (5.1) in \( kPV_T^\ast[t] \).
to zero under $\infty, 1)$. Together with the fact that $(1 + \eta) = 0$ (mod $I$), we see that $[(1 + \eta)]$ represents a cohomology class in $(0^* PV^*, 0^* d) \otimes C(I/T^2)$. Since $\hat{d}(1) = (1 + \eta)$, we deduce that $[1 + \eta] = 0$ in $(1^* PV_T, 1^* d)$. Now applying Lemma 4.16 or Remark 5.2 (freeness of the Hodge bundle) to $(1^*, 1^* d)$ gives the short exact sequence

$$0 \to H^*(0^* PV^*) \otimes C(I/T^2) \to H^*(1^* PV_T) \to H^*(0^* PV^*) \to 0$$

under the identification by the volume element $\omega$. We conclude that the class $[1 + \eta]$ is zero in $H^*(0^* PV^*) \otimes (I/T^2)$ which means that $(1 + \eta) = (\partial + \Delta)(\tilde{\zeta})$ (mod $T^2$) for some $\zeta \in 0^* PV^0 \otimes (I/T^2)$, and we have $(1 + t \eta) = (\partial + t \Delta)(\zeta)$ for some $\zeta \in 0^* PV^0[[t]][t^{-1}] \otimes (I/T^2)$ (here we can take $\zeta$ to be the sum of terms with integral $t$-powers in $t$), and then we can take $\psi$ to be lifting of $\zeta$ in $1^* PV_{T, 0}[1]^0[[t]]$. We further observe the Maurer-Cartan element $^1\varphi$ can be modified by adding any $\xi \in 1^* PV_{T, 0}[1]^0[[t]]$ with $\xi = 0$ (mod $I$) and $1^* d\xi = 0$ (mod $T^2$). Therefore we can always achieve $^1\varphi + \xi = \psi$ (mod $m + I^2$) by choosing a suitable $\xi$ and letting $^1\varphi + \xi$ be the new $^1\varphi$.

Next suppose $^k-1\varphi$ satisfying the Maurer-Cartan equation $\hat{d}(e^{^k-1\varphi}) = 0$ (mod $T^k$) up to order $k - 1$ has been constructed. We take an arbitrary lifting $^k-1\varphi$ in $k^* PV_{T, 0}[1]^0[[t]]$ and let $^kO := k^* d\left( e^{\frac{^k-1\varphi}{t}} \right) \mod T^{k+1}$, with $[^kO]$ represents a cohomology class in $(0^* PV^0[[t]][t^{-1}]) \otimes (T^k/I^{k+1}, 0^* d)$. We again apply Lemma 4.16 to obtain a short exact sequence

$$0 \to H^*(0^* PV^0[[t]][t^{-1}]) \otimes (T^k/I^{k+1}) \to H^*(k^* PV^0[[t]][t^{-1}]) \to H^*(0^* PV^0[[t]][t^{-1}]) \to 0,$$

which forces $[^kO] = 0$ as in the initial case. So we can find $\zeta \in 0^* PV^0[1]^0[0][t] \otimes (T^k/I^{k+1})$ such that $(^0\partial + t(^0\Delta))(\zeta) = l^{-1}(^kO)$ and then set $^k\varphi := ^k-1\varphi + \zeta$ to solve the equation.}

### 5.2. Homotopy between Maurer-Cartan elements for different sets of gluing morphisms.

Theorem 5.5 is proven for a fixed set of compatible gluing morphisms $g = \{^k g_{\alpha \beta}\}$. In this subsection, we study how Maurer-Cartan elements for two different sets of compatible gluing morphisms $g(0) = \{^k g_{\alpha \beta}(0)\}$ and $g(1) = \{^k g_{\alpha \beta}(1)\}$ are related through a fixed homotopy $h = \{^k h_{\alpha \beta}\}$.

We begin by assuming that the data $D = (D_\alpha)\alpha \in \mathcal{C}(0^0(TW^{-1, 1}, h) and $\mathcal{F} = (\mathcal{F}_\alpha)\alpha \in \mathcal{C}(0^0(TW^{0, 0}, h)$ for the construction of the operators $D$ and $\Delta$ in Proposition 3.35 are related to the data $\mathcal{D}_j$ and $\mathcal{F}_j$ for the construction of the operators $\tilde{D}_j$ and $\tilde{\Delta}_j$ in Theorem 3.34 by the relations

$$r_j^*(D) = \mathcal{D}_j, \quad \tilde{r}_j(\mathcal{F}) = \tilde{f}_j$$

for $j = 0, 1$, where $r_j : PV^{*\times}(h) \to PV^{*\times}(g(j))$ is the map introduced in Definition 3.29.

**Notations 5.6.** Similar to Lemma 4.9, we let $\mathcal{L}_\alpha := D_\alpha(D_\alpha) + \frac{1}{2}[D_\alpha, D_\alpha] \otimes C(I/PV_{m-1, 2}^{*}) \otimes C(I/PV_{m+1, 0})$, and $\mathcal{E}_\alpha$ to be $\Delta_\alpha(D_\alpha) + D_\alpha(\mathcal{F}_\alpha) + [D_\alpha, \mathcal{F}_\alpha]$; $(\mathcal{L}_\alpha)\alpha$ and $(\mathcal{E}_\alpha)\alpha$ glue to give global terms $\mathcal{L} \in PV_{m-1, 2}^{*}(h)$ and $\mathcal{E} \in PV_{m+1, 0}^{*}(h)$ respectively.

We set $\tilde{D} := D + \Delta + (\mathcal{L} + \mathcal{E}) \wedge$, which defines an operator acting on $PV^{*}(h)$ (and we will use $\tilde{D}$ to denote the corresponding operator acting on $^k PV^{*}(h)$). We have $\tilde{D}^2 = 0$ as in Proposition 4.11.

We also introduce a scaling $l_t : PV^p_q(h)[[t]][t^{-1}] \to PV^p_q(h)[[t]][t^{-1}]$ defined by $l_t(\varphi) = t^{\frac{2-p-2}{k}} \varphi$ for $\varphi \in PV^p_q(h)$. Then we have the identity $^k\tilde{D}_t := t^{2k}l^{-1}_t \circ \tilde{D} \circ l_t = ^kD + t(^k\Delta) + t^{-1}(^kL + t^k(\mathcal{E}) \wedge)$ as in Notations 5.1.
Similar to Notation 5.1, we consider the complex $k \PV_T^*(h)[[t]]$ (or formal power series or Laurent series in $t$ or $t^{\frac{1}{2}}$) for any graded ring $T = \mathbb{C}[V^*]$.

**Lemma 5.7.** The natural restriction map $r_j^* : (k \PV^*(h), k \mathcal{D}^*) \to (k \PV^*(g(j)), k \mathcal{D})$ is a quasi-isomorphism for $j = 0, 1$ and all $k \in \mathbb{Z}_{\geq 0}$.

**Proof.** We will only give a proof of the case $j = 0$ because the other case is similar. We first consider the following diagram

$$
\begin{array}{cccccc}
0 & \to & k^{-1}\PV^*(h) \otimes_{\mathbb{C}} (m/m^2) & \xrightarrow{r_0^*} & k\PV^*(h) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & k^{-1}\PV^*(g(0)) \otimes_{\mathbb{C}} (m/m^2) & \xrightarrow{r_0^*} & k\PV^*(g(0)) & \to & 0
\end{array}
$$

with exact horizontal rows. By passing to the corresponding long exact sequence, we see that it suffices to prove that $r_0^* : (0 \PV^*(h), 0\mathcal{D} + 0\Delta) \to (0 \PV^*(g(0)), 0\mathcal{D} + 0\Delta)$ is a quasi-isomorphism. In this case we have $0h = id (\mod m)$ and $0g(0) = id (\mod m)$, from which we deduce that $0\PV^*(h) = A^*(\Delta) \otimes_{\mathbb{C}} 0\PV^*(g(0))$ in which the operators are related by $0\mathcal{D} + 0\Delta = 0\mathcal{D} + 0\Delta + d_s$, where $s$ is the coordinate function on the 1-simplex $\Delta$ and $d_s$ is the usual de Rham differential acting on $A^*(\Delta)$. The quasi-isomorphism is then obtained using the homotopy operator constructed by integration $\int_0^s$ along the 1-simplex. Details are left to the readers.

The following proposition relates Maurer-Cartan elements $\varphi$ of $\PV_T^*(g(0))[[t]]$ and those of $\PV_T^*(h)[[t]]$.

**Proposition 5.8.** Given any Maurer-Cartan element $k \varphi \in k \PV_T^0(g(0))[[t]]$ as in Theorem 5.5, there exists a lifting $k \tilde{\varphi} \in k \PV_T^0(h)[[t]]$ which is a Maurer-Cartan element for each $k$ such that $k^{+1} \varphi = k^{\varphi} (\mod k^{+1})$ and $r_0^*(k \varphi) = k \varphi_0$. If there are two liftings $(k \varphi)_k$ and $(k \psi)_k$ of $(k \varphi_0)_k$, then there exists a gauge element $k \eta \in k \PV_{-1}^1(h)[[t]]$ for each $k$ such that $r_0^*(k \eta) = 0$, $k^{+1} \varphi = k \eta (\mod k^{+1})$ and $e^{k \eta} \ast k \varphi = k \psi$.

**Proof.** We construct $k \tilde{\varphi}$ by induction on $k$. Given a Maurer-Cartan element $k^{-1} \varphi \in k \PV_T^0(h)[[t]]$ such that $r_0^*(k^{-1} \varphi) = k^{-1} \varphi_0$, our goal is to construct a lifting $k \tilde{\varphi}$ of $k^{-1} \varphi$ with $r_0^*(k \tilde{\varphi}) = k \varphi_0$.

By surjectivity of $k^{k-1} : k \PV_T(h)[[t]] \to k^{-1} \PV_T(h)[[t]]$, we get a lifting $k^{-1} \tilde{\varphi}$ of $k^{-1} \varphi$. By surjectivity of $r_0^* : k \PV_T^0(h)[[t]] \to PV_0^0(g(0))[[t]]$ for any $k$ from Lemma 3.30 we further obtain a lifting $\eta$ of $k \varphi_0 - r_0^*(k^{-1} \varphi)$ such that $\eta = 0 (\mod k)$ in $k \PV_T^0(h)[[t]]$. Then we set $k \tilde{\varphi} := k^{-1} \varphi + \eta$ so that $r_0^*(k \tilde{\varphi}) = k \varphi_0$. Similar to the proof of Theorem 5.5, we define the obstruction class

$$
k^{+1} : k \mathcal{D}_t(\varphi / t) = (k \mathcal{D} + t(k \Delta))^{k \tilde{\varphi} / t} + \frac{1}{2}[k \varphi, k \tilde{\varphi}] + (k \mathcal{E} + t(k \mathcal{E}))
$$

in $k \PV_T^1[[t]]$ which satisfies $k^{-1} \varphi(k^{+1} \varphi) = 0$ and $k^{+1} \mathcal{D}_t(k^{+1} \varphi) = 0$.

Considering the short exact sequences

$$
\begin{array}{cccccc}
0 & \to & k^{-1} \PV_T^*(h)[[t]] & \xrightarrow{r_0^*} & k \PV_T^*(g(0))[[t]] & \to & 0, \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & k^{-1} \PV_T^*(h)[[t]] & \xrightarrow{r_0^*} & k \PV_T^*(g(0))[[t]] & \to & 0
\end{array}
$$

we see that $k \tilde{\varphi}$ is uniquely determined. Details are left to the readers.
and observing that \((\mathcal{K}^*, k\mathcal{D}_t)\) is acyclic, we conclude that \(k\mathcal{O} \in \mathcal{K}^1\). Hence we can find \(\zeta \in \mathcal{K}^0\) such that \(k\mathcal{D}_t(\zeta) = k\mathcal{O}\) and \(\zeta = 0 \pmod{\mathcal{I}^k}\). Then \(k\varphi := k\varphi + \zeta\) is the desired lifting of \(k^{-1}\varphi\).

The gauge \((k\varphi)\) can be constructed by a similar inductive process. Given \(k^{-1}\varphi\), we need to construct a lifting \(k\varphi \in kPV_{\mathcal{T}}^{-1}(h)[[t]]\) which serves as a homotopy from \(k\varphi\) to \(k\psi\). Again we take a lifting \(k\varphi\) satisfying \(k^{-1}\varphi(k\varphi) = k^{-1}\varphi\) and \(r_0^\varphi(k\varphi) = 0\), and consider the obstruction

\[
\| 0 := k\psi - \exp([k\varphi, \cdot])\]k\varphi + \frac{1}{[k\varphi, \cdot]}(t^kD + t^k\Delta)(k\varphi),
\]

which satisfies \(k^{-1}\varphi(\| 0) = 0\) and \(r_0^\varphi(\| 0) = 0\). We can find \(\zeta \in 0 PV_{\mathcal{T}}^{-1}[[t]] \otimes (\mathcal{I}^k/\mathcal{I}^{k-1})\) with \(r_0^\varphi(\zeta) = 0\) such that \(-\langle 0 D + t^k\Delta \rangle \zeta = \| 0\) and letting \(k\varphi := k\varphi + \zeta\) gives the desired gauge element. \(\square\)

Given a homotopy \(h\), we define a map \(F_h\) from the set of Maurer-Cartan elements modulo gauge equivalence with respect to \(g(0)\) to that with respect to \(g(1)\) by \(F_h((k\varphi_0)_k) := (\varphi_1^* (k\varphi))_k\) with \(k\varphi \in kPV_{\mathcal{T}}^0[[t]]\). Proposition 5.8 says that this map is well-defined, and its inverse \(F_h^{-1}\) is given by reversing the roles of \(g(0)\) and \(g(1)\), so \(F_h\) is a bijection.

Next we consider the situation where we have a fixed set of compatible gluing morphisms \(g = \{k\varphi_{\alpha\beta}\}\) but the complex \(kPV^*\) is equipped with two different choices of operators \(k\varphi\) and \(k\varphi^\Delta\), whose differences are captured by elements \(\varphi_1 \in PV^{-1}(g)\) and \(\varphi_2 \in PV^{0,0}(g)\), as in Theorem 3.34. We write \(\varphi = \varphi_1 + \varphi_2\) and consider the complex \(\mathcal{A}^*(\mathbf{A}_1) \otimes_{\mathcal{C}} kPV^*\) equipped with the differential

\[
k\mathcal{D} := k\mathcal{D} + d_{\mathbf{A}_1} + t_1[\varphi, \cdot] + (t_1(k\varphi + k\varphi^\Delta)\varphi + t_2^2[\varphi, \varphi]) \wedge + (dt_1 \wedge \varphi) \wedge,
\]

where \(d_{\mathbf{A}_1}\) is the de Rham differential for \(\mathcal{A}^*(\mathbf{A}_1)\). We let \(k\mathcal{O}_{\mathbf{A}_1, \varphi} := (t_1(k\varphi + k\varphi^\Delta)\varphi + t_2^2[\varphi, \varphi]) + (t_1 \wedge \varphi)\) and compute

\[
(k\mathcal{D})^2 = (k\varphi + k\varphi^\Delta + t_1[\varphi, \cdot])^2 - \| 0_{\mathbf{A}_1, \varphi} \wedge + dt_1 \wedge \frac{\varphi}{\partial t_1} (k\mathcal{O}_{\mathbf{A}_1, \varphi}) \wedge - dt_1 \wedge (k\varphi + k\varphi^\Delta)(\varphi + t_2[\varphi, \varphi]) \wedge
\]

\[\| 0_{\mathbf{A}_1, \varphi} \wedge - \| 0_{\mathbf{A}_1, \varphi} \wedge + dt_1 \wedge (k\varphi + k\varphi^\Delta)(\varphi + t_2[\varphi, \varphi]) \wedge - dt_1 \wedge (k\varphi + k\varphi^\Delta)(\varphi + t_2[\varphi, \varphi]) \wedge
\]

\[= 0.
\]

Repeating the argument in this subsection but replacing \((kPV^*(h), k\mathcal{D})\) by \((\mathcal{A}^* \otimes_{\mathcal{C}} kPV^*, k\mathcal{D})\) and arguing as in the proof of Proposition 5.8 yields the following:

**Proposition 5.9.** Given any Maurer-Cartan element \(k\varphi_0 \in kPV_T^0[[t]]\) with respect to the operators \(k\varphi\) and \(k\varphi^\Delta\) as in Theorem 5.5, there exists a lifting \(k\varphi \in \mathcal{A}^*(\mathbf{A}_1) \otimes kPV_T^0[[t]]\) which is a Maurer-Cartan element with respect to the operators \((k\varphi + d_{\mathbf{A}_1} + t_1[\varphi, \cdot])\) and \((k\varphi^\Delta + [\varphi, \cdot])\) (meaning that \((k\varphi + d_{\mathbf{A}_1} + t_1[\varphi, \cdot]) + t_2[\varphi, \cdot]) = 0\) for each \(k\) satisfying \(k^{-1}\varphi = k\varphi (mod \mathcal{I}^{k+1})\) and \(r_0^k(k\varphi) = k\varphi_0\). If there are two liftings \((k\varphi)_k\) and \((\psi)_k\) of \((k\varphi_0)_k\), then there exists a gauge element \(k\varphi \in \mathcal{A}^*(\mathbf{A}_1) \otimes kPV_T^0[[t]]\) for each \(k\) such that \(r_0^k(k\varphi) = 0\), \(k^{-1}\varphi = k\varphi (mod \mathcal{I}^{k+1})\) and \(e^{k\varphi}\).
5.3. From a Maurer-Cartan element to geometric Čech gluing. In this subsection, we show that a Maurer-Cartan (MC) element $\varphi = (k \varphi)_{k \in \mathbb{Z}_{\geq 0}}$ as defined in Definition 5.3 contains the data for gluing the sheaves $k\mathcal{G}^*_\alpha$’s consistently.

We fix a set of gluing morphisms $g = (k_{\alpha \beta})$ and consider a MC element $\varphi = (k \varphi)_{k \in \mathbb{Z}_{\geq 0}}$ (where we take $T = \mathbb{C}$) obtained in Theorem 5.5. Setting $t = 0$, we have the element $k \bar{\varphi} := k \varphi|_{t=0}$ which satisfies the following extended MC equation (5.2).

**Definition 5.10.** An element $k \bar{\varphi} \in k PV^0$ is said to be a Maurer-Cartan element in $k PV^*$ if it satisfies the extended Maurer-Cartan equation:

\[
(k \bar{\varphi} + 1[k \varphi, k \bar{\varphi}] + k I = 0.
\]

Note that $(k PV^{-1,*}[−1], k \bar{\varphi}, [•, •])$ forms a dgLa, and an element $k \bar{\psi} \in k PV^{-1,1}$ is called a classical Maurer-Cartan element if it satisfies (5.2).

**Lemma 5.11.** In the proof of Theorem 5.5, the Maurer-Cartan element $k \varphi = k \phi_0 + k \phi_1 t^1 + \cdots + k \phi_p t^p + \cdots \in k PV^0[[t]]$, where $k \phi_0 = k \psi_0 + k \psi_1 + \cdots + k \psi_d$ with $k \psi_i \in k PV^{-i,i}$, can be constructed so that $k \psi_0 = 0$. In particular, $k \psi_1 \in k PV^{-1,1}$ is a classical Maurer-Cartan element.

**Proof.** We prove by induction on $k$. Recall from the initial step of the inductive proof of Theorem 5.5 that $1 \varphi \in 1 PV^0[[t]]$ was constructed so that $(1 \bar{\varphi} + t(1 \Delta))(1 \varphi) = t I + t \eta_1 \eta$. As $1 \eta = 1 PV^{-1,2}$ and $1 \eta \in 1 PV^{0,1}$, we have $1 \bar{\varphi} = 1 \eta = 0$. Also, we know $1 \Delta(1 \varphi) = 0$ by degree reasons, so we obtain the equation $(1 \bar{\varphi} + t(1 \Delta))(1 \varphi - 1 \psi_0) = t I + t \eta$. Hence we can replace $1 \varphi$ by $1 \varphi - 1 \psi_0$ in the construction so that the desired condition is satisfied.

For the induction step, suppose that $k-1 \varphi = k-1 \phi_0 + k-1 \phi_1 t + \cdots \in k-1 PV^0$ with $k-1 \psi_0 = 0$ has been constructed. Again recall from the construction in Theorem 5.5 that we have solved the equation

\[
(k \bar{\varphi} + t(k \Delta))(k \eta) = k \bar{d}_t (te^{-1} \varphi / t)
\]

for $k \eta \in k PV^0[[t]]$. We are only interested in the coefficient of $t^0$ of the component lying in $k PV^{0,1}$ on the RHS of the above equation, which we denote as $[k \bar{d}_t (te^{-1} \varphi / t)]_0$. By writing $k-1 \phi_0 = k-1 \psi_1 + \cdots + k-1 \psi_d$ using the induction hypothesis, we have

\[
[k \bar{d}_t (t^{-1} \varphi / t)]_0 = \left[\left((k \bar{\varphi}(k-1 \varphi) + t(k \Delta)(k-1 \varphi) + \frac{1}{2}[k-1 \varphi, k-1 \varphi] + k I + t \eta) \wedge (exp(k-1 \varphi / t))\right)_0
\]

Therefore by writing $k \eta = \zeta_0 + \zeta_0 t^1 + \cdots$ and $\zeta_0 = \xi_0 + \cdots + \xi_d$ with $\xi_i \in k PV^{-i,i}$, we conclude that $k \bar{\varphi}(\zeta_0) = 0$ and hence $(k \bar{\varphi} + t(k \Delta))(\zeta_0) = 0$. As a result, if we replace $k \eta$ by $k \xi - \xi_0$ in the construction, we get the desired element $k \varphi$ for the induction step.

The second statement follows from the first because $k \phi_0 = k \psi_1 + \cdots + k \psi_d$ satisfies the extended MC equation (5.2). Then by degree reasons, we conclude that $k \bar{\varphi}(k \psi_1) + \frac{1}{2}[k \psi_1, k \psi_1] + k I = 0$. □

In view of Lemma 5.11, we restrict our attention to the dgLa $(k PV^{-1,*}[−1])$ and a classical Maurer-Cartan element $k \psi \in k PV^{-1,1}$. We write $k \psi = (k \psi_\alpha)_{\alpha}$ where $k \psi_\alpha \in k TW^{p,0}_{\alpha,0}$ with regard to the Čech-Thom-Whitney complexes in Definition 3.25.

Since $V_\alpha$ is Stein and $k \mathcal{G}^*_\alpha$ is a coherent sheaf over $V_\alpha$, we have $H^0(k TW^{p,0}_{\alpha,0}[p], k \bar{\partial}_\alpha) = 0$ for any $p$ (here $[p]$ is the degree shift so that $k TW^{p,0}_{\alpha,0}$ is at degree $0$). In particular, the operator
$k\tilde{\partial}_\alpha + [ k\partial_{\alpha}, ] + [ k\psi_{\alpha}, ]$ is gauge equivalent to $k\tilde{\partial}_\alpha$ via a gauge element $k\partial_{\alpha} \in kTW_{\alpha,0}^{-1}$. As $k^{+1}\psi_{\alpha} = k\psi_{\alpha} \text{ (mod } m^{k+1})$, we can further construct $k\partial_{\alpha}$ via induction on $k$ so that $k^{+1}\partial_{\alpha} = k\partial_{\alpha} \text{ (mod } m^{k+1})$.

Given any open subset $W \subset V_\alpha$, we use the restrictions $k\partial_{\alpha}|_W \in kTW^{-1,0}(kG_\alpha|_W)$, $k\partial_{\beta} \in kTW^{-1,0}(kG_\beta|_W)$ to define an isomorphism $k\mathbf{g}_{\alpha\beta} : kTW^{*,*}(kG_\alpha|_W) \to kTW^{*,*}(kG_\beta|_W)$ which fits into the following commutative diagram

$$
\begin{array}{ccc}
 kTW^{*,*}(kG_\alpha|_W) & \xrightarrow{k\mathbf{g}_{\alpha\beta}} & kTW^{*,*}(kG_\beta|_W) \\
 \exp([k\partial_{\alpha},]) & \downarrow \exp([k\partial_{\beta},]) & \\
(k^{k\alpha}|_W, \tilde{k\partial}_{\alpha}) & \xrightarrow{k\mathbf{g}_{\alpha\beta}} & (k^{k\beta}|_W, \tilde{k\partial}_{\beta})
\end{array}
$$

here we emphasis that $k\mathbf{g}_{\alpha\beta}$ identifies the differentials $k\tilde{\partial}_{\alpha}$ and $k\tilde{\partial}_{\beta}$.

Note that there is an identification $k\mathbf{g}_{\alpha\beta}^*(W) = H^0(kTW^{*,*}(kG_\alpha|_W)[p], k\tilde{\partial}_{\alpha})$, enabling us to treat $k\mathbf{g}_{\alpha\beta} : kG_\alpha^*(W) \to kG_\beta^*(W)$ as a graded Lie algebra isomorphism. These isomorphisms can then be put together to give an isomorphism of sheaves of graded Lie algebras $k\mathbf{g}_{\alpha\beta} : kG_\alpha^*|_{V_{\alpha\beta}} \to kG_\beta^*|_{V_{\alpha\beta}}$. Furthermore, the cocycle condition for the gluing morphisms $k\mathbf{g}_{\alpha\beta}$ (see Definition 3.17) implies the cocycle condition $k\mathbf{g}_{\gamma\alpha} \circ k\mathbf{g}_{\beta\gamma} \circ k\mathbf{g}_{\alpha\beta} = \text{id}$.

**Definition 5.12.** A set of $k$-th order geometric gluing morphisms $k\mathbf{g}$ consists of, for any pair $V_\alpha, V_\beta \in \mathcal{V}$, an isomorphism of sheaves of graded Lie algebras $k\mathbf{g}_{\alpha\beta} : kG_\alpha^*|_{V_{\alpha\beta}} \to kG_\beta^*|_{V_{\alpha\beta}}$ satisfying $k\mathbf{g}_{\alpha\beta} = \text{id} \text{ (mod } m)$, and the cocycle condition $k\mathbf{g}_{\gamma\alpha} \circ k\mathbf{g}_{\beta\gamma} \circ k\mathbf{g}_{\alpha\beta} = \text{id}$. Two such sets of $k$-th order geometric gluing morphisms $k\mathbf{g}$ and $k\mathbf{h}$ are said to be equivalent if there exists a set of isomorphisms of sheaves of graded Lie algebras $k\mathbf{a}_{\alpha} : kG_\alpha^* \to kG_\alpha^*$ with $k\mathbf{a}_{\alpha} = \text{id} \text{ (mod } m)$ fitting into the following commutative diagram

$$
\begin{array}{ccc}
 kG_\alpha^*|_{V_{\alpha\beta}} & \xrightarrow{k\mathbf{g}_{\alpha\beta}} & kG_\beta^*|_{V_{\alpha\beta}} \\
 k\mathbf{a}_{\alpha} & \downarrow k\mathbf{h}_{\alpha\beta} & k\mathbf{a}_{\beta} \\
 kG_\alpha^*|_{V_{\alpha\beta}} & \xrightarrow{k\mathbf{a}_{\beta}} & kG_\beta^*|_{V_{\alpha\beta}}
\end{array}
$$

If we have two classical Maurer-Cartan elements $k\psi$ and $k\tilde{\psi}$ which are gauge equivalent via $k\theta = (k\theta_{\alpha})_{\alpha}$, then we can construct as isomorphism $\exp(-[k\tilde{\partial}_{\alpha},]) \circ \exp([k\theta_{\alpha},]) \circ \exp([k\partial_{\alpha},]) : (kTW_{\alpha,0}^{*,*}, k\tilde{\partial}_{\alpha}) \to (kTW_{\alpha,0}^{*,*}, k\partial_{\alpha})$ inducing an isomorphism $k\mathbf{a}_{\alpha} : kG_\alpha^*(V_{\alpha}) \to kG_\alpha^*(V_{\alpha})$ by taking $H^0(kTW_{\alpha,0}^{*,*}, k\partial_{\alpha})$, so that the two sets of $k$-th order geometric gluing morphisms $k\mathbf{g}$ and $k\mathbf{h}$ associated to $k\psi$ and $k\tilde{\psi}$ respectively are equivalent via $k\mathbf{a} = (k\mathbf{a}_{\alpha})_{\alpha}$. This gives the following:

**Proposition 5.13.** Given classical Maurer-Cartan elements $k\psi \in kPV^{-1,1}$ such that $k^{+1}\psi = k\psi \text{ (mod } m^{k+1})$ and $k\tilde{\psi} = 0 \text{ (mod } m)$, there exists an associated set of geometric gluing morphisms $k\mathbf{g}$ for each $k$ satisfying $k^{+1}\mathbf{g} = k\mathbf{g}$ (mod $m^{k+1}$). For two classical Maurer-Cartan elements $k\psi$ and $k\tilde{\psi}$ which are gauge equivalent via $k\theta$ such that $k^{+1}\theta = k\theta \text{ (mod } m^{k+1})$, there exists an equivalence $k\mathbf{a}$, satisfying $k^{+1}\mathbf{a} = k\mathbf{a} \text{ (mod } m^{k+1})$ between the associated geometric gluing data $k\mathbf{g}$ and $k\mathbf{h}$.

Lemma 5.11 together with Proposition 5.13 produces a geometric Čech gluing of the sheaves $kG_\alpha^*$’s, unique up to equivalence, from a gauge equivalence class of the MC elements obtained in Theorem 5.5.
6. Abstract semi-infinite variation of Hodge structures

In this section, we apply techniques of [1, 30, 37] in our abstract framework under suitable Assumptions 6.12 and 6.17 to give a construction of logarithmic Frobenius manifolds introduced in [43].

6.1. Logarithmic semi-infinite variation of Hodge structure.

Notations 6.1. Following the previous Notation 5.1, we let \( \hat{R}_T := \lim_{\to} k R_T \) to be the completion of \( R \otimes \mathbb{T} \), and we abuse the notation and use \( m, l, I \) and \( I \) to stand for the corresponding ideals in the completion \( \hat{R}_T \). Similar to the Notation 4.6, we have a monoid homomorphism \( Q \rightarrow k R_T \) by sending \( m \rightarrow q^m \) equipping \( k R_T \) with a structure of (graded) Log ring. We also denote the corresponding formal germ of Log space by \( k S^1_T \).

We define the module of Log differential
\[
^k \Omega^l_{S^1_T} := k R_T \otimes_{\mathbb{C}} \text{Sym}^l \left( (K_{\mathbb{C}} \otimes \mathbb{V})[-1] \right)
\]
where \( \text{Sym}^l \) refers to the graded symmetric product. For \( ^k \Omega^1_{S^1_T} = k R_T \otimes_{\mathbb{C}} (K_{\mathbb{C}} \otimes \mathbb{V})[-1] \), we will write \( d \log q^m \) to stand for the element corresponding to \( m \in K_{\mathbb{C}} \) and \( dz \) for an element corresponding to \( z \in \mathbb{V} \). Then we have a de Rham differential \( d : k R_T \rightarrow ^k \Omega^1_{S^1_T} \), satisfying the graded Leibniz rule, the relation \( d(q^m) = q^m d(\log q^m) \) for \( m \in Q \), and \( d(z) = dz \) for \( z \in \mathbb{V} \).

We can also let \( k \Theta_{S^1_T} := k R_T \otimes_{\mathbb{C}} (K_{\mathbb{C}} \otimes \mathbb{V})[1] \) to be the space of Log derivation, equipped with a Lie bracket \([\cdot, \cdot]\) and a natural pairing between \( X \in k \Theta_{S^1_T} \) and \( \alpha \in ^k \Omega^1_{S^1_T} \). Similarly, we can talk about \( \hat{\Omega}_S^* \) and \( \hat{\Theta}_{S^1_T}^* \) by taking inverse limit.

Definition 6.2. A data of Log semi-infinite variation of Hodge structure (abbrev. \( \otimes \)-LVHS) over \( \hat{R}_T \) consists of \( (k^H, k \nabla, k\langle \cdot, \cdot \rangle) \) for each \( k \in \mathbb{Z}_{\geq 0} \) with \( k^1 \nabla \) and \( k \langle \cdot, \cdot \rangle \) for \( k \geq 1 \) where

1. \( k^H = k^H^* \) is a graded free \( k R_T[[t]] \) module called the (sections of) Hodge bundle, with the a frame of degree in \( 0 \leq \ast \leq 2d \) which is compatible with \( k^1 \nabla^l \)'s;
2. \( k \nabla \) is the Gauss-Manin (partial) connection compatible with \( k^1 \nabla^l \)'s which is of the form

\[
(6.1) \quad \nabla : k^H \rightarrow \frac{1}{t} \left( ^k \Omega^1_{S^1_T} \right) \otimes (k R_T) k^H;
\]

3. \( k \langle \cdot, \cdot \rangle : k^H \times k^H \rightarrow k R_T[[t]][-2d] \) is a degree preserving pairing compatible with \( k^1 \nabla^l \)'s.

We will omit the dependence on \( k \) for \( k \nabla \) and \( k \langle \cdot, \cdot \rangle \) and simply write \( \nabla \) and \( \langle \cdot, \cdot \rangle \) if there is no confusion. This set of data should satisfy

1. \( \langle s_1, s_2 \rangle(t) = \langle -1 \rangle |s_1| |s_2| \langle s_1, s_2 \rangle(-t) \), where \( |s_i| \) is the degree of the homogeneous element \( s_i \);
2. \( \langle f(t) s_1, s_2 \rangle = \langle -1 \rangle |s_1| [f] \langle s_1, f(-t) s_2 \rangle = f(t) \langle s_1, s_2 \rangle \) for \( s_i \in k^H \) and \( f(t) \in k R_T[[t]] \);
3. \( \nabla_X \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle -1 \rangle |s_2| \langle X, \nabla_X s_2 \rangle \) for \( X \in k \Theta_{S^1_T} \);
4. the induced pairing \( g(\cdot, \cdot) : (k^H/t^k H) \times (k^H/t^k H) \rightarrow k R_T[-2d] \) being non-degenerated.

\( ^1\)Here we abuse the notation by writing \( q^m \) to stand for \( q^m \otimes 1 \).
\( ^2\)Here \( f(-t) \in k R_T[[t]] \) is the element obtained from \( f(t) \) by substituting \( t \) with \( -t \).
Definition 6.3. For a $\frac{\infty}{2}$-LVHS $(k\mathcal{H}^*, \nabla, \langle \cdot, \cdot \rangle)$, a grading structure is an extension of the Gauss-Manin connection $\nabla$ along $t$-coordinate compatible with $k, l$'s as
$$\nabla_{\frac{\partial}{\partial t}} : k\mathcal{H} \to t^{-1}(k\mathcal{H})$$
such that we have $[\nabla_X, \nabla_{\frac{\partial}{\partial t}}] = 0$, i.e. it is a flat connection on $kS_T^+ \times (\mathbb{C}, 0)$. We further require the pairing $\langle \cdot, \cdot \rangle$ is flat with respect to $\nabla_{\frac{\partial}{\partial t}}$ in the sense that $t\frac{\partial}{\partial t}(s_1, s_2) = (\nabla_{\frac{\partial}{\partial t}} s_1, s_2) + (s_1, \nabla_{\frac{\partial}{\partial t}} s_2)$.

Notations 6.4. We let $k\mathcal{H}_\pm := k\mathcal{H} \otimes \mathbb{C}[t][t^{-1}]$ to be a module over $kR_T[[t]][t^{-1}]$ with the natural submodule $k\mathcal{H}_+ := k\mathcal{H} \subset k\mathcal{H}_+$ which is closed under multiplication by $kR_T[[t]]$. There is a natural symplectic structure $k\mathbf{w}(\cdot, \cdot) : k\mathcal{H}_+ \times k\mathcal{H}_- \to kR_T$ defined by $k\mathbf{w}(\alpha, \beta) = \text{Res}_{t=0}(\alpha, \beta)dt$.

We will let $\mathcal{H}_+ := \lim_{\leftarrow k} k\mathcal{H}_+$ and $\mathcal{H}_- := \lim_{\leftarrow k} k\mathcal{H}_+$ which is a module over $R_T[[t]][t^{-1}] := \lim_{\leftarrow k} kR_T[[t]][t^{-1}]$, with a natural $R_T[[t]]$ submodule $\mathcal{H}$, which is equipped with the symplectic structure $\mathbf{w} := \lim_{\leftarrow k} k\mathbf{w}$. We will also write $R_T[t^{-1}] := \lim_{\leftarrow k} kR_T[t^{-1}]$.

Definition 6.5. An opposite filtration is a choice of $kR_T[t^{-1}]$ submodule $k\mathcal{H}_- \subset k\mathcal{H}_+$ for each $k$ compatible with $k, l$'s such that it satisfies for each $k$

1. $k\mathcal{H}_+ \oplus k\mathcal{H}_- = k\mathcal{H}_+$;
2. $k\mathcal{H}_-$ being preserved by $\nabla_X$ for any $X \in k\Theta S_T^+$;
3. $k\mathcal{H}_-$ being isotropic with respect to the symplectic structure $k\mathbf{w}(\cdot, \cdot)$;
4. $k\mathcal{H}_-$ being preserved by $\nabla_{\frac{\partial}{\partial t}}$

We will also write $\mathcal{H}_- := \lim_{\leftarrow k} k\mathcal{H}_-$.

Similar to [18] and [36], with an opposite filtration $\mathcal{H}_-$ we have a natural isomorphism

$$k\mathcal{H}_+/t(k\mathcal{H}_+) \cong k\mathcal{H}_+ \cap t(k\mathcal{H}_-) \cong t(k\mathcal{H}_-)/k\mathcal{H}_-,$$

for each $k$ and giving identification

$$\tau_+ : k\mathcal{H}_+ \cap t(k\mathcal{H}_-) \otimes_{\mathbb{C}} \mathbb{C}[t] \to k\mathcal{H}_+, \quad (6.3)$$

$$\tau_- : k\mathcal{H}_+ \cap t(k\mathcal{H}_-) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1}] \to t(k\mathcal{H}_-), \quad (6.4)$$

Using the arguments from [36] we learn that we have $\langle k\mathcal{H}_+ \cap t(k\mathcal{H}_-), k\mathcal{H}_+ \cap t(k\mathcal{H}_-) \rangle \in kR_T$ and $\langle k\mathcal{H}_-, k\mathcal{H}_- \rangle \in kR_T[t^{-1}]t^{-2}$ for the pairing $\langle \cdot, \cdot \rangle$ in Definition 6.2.

Morally a choice of opposite filtration $\mathcal{H}_-$ give rise to a (trivial) bundle $\mathbf{H}$ of $\hat{S}_T^+ \times \mathbb{P}^1$ where $t$ in the coordinate along $\mathbb{P}^1$ as follows. We take the sections of germ of $k\mathbf{H}$ near $kS_T^+ \times (\mathbb{C}, 0)$ to be $k\mathcal{H}_+$, and that of $k\mathbf{H}$ over $kS_T^+ \times (\mathbb{P}^1 \setminus \{0\})$ to be $t(k\mathcal{H}_-)$. The identifications in above equations (6.3) and (6.4) give a trivialization of the bundle $k\mathbf{H}$ over $kS_T^+ \times \mathbb{P}^1$ whose global sections being $k\mathcal{H}_+ \cap t(k\mathcal{H}_-)$. The pairing $\langle \cdot, \cdot \rangle$ can be extended to $\hat{S}_T^+ \times \mathbb{P}^1$ using the trivialization in equation (6.4), and the extended connection $\nabla$ gives a flat connection on $\hat{S}_T^+ \times \mathbb{P}^1$ preserving the pairing $\langle \cdot, \cdot \rangle$ with order 2 irregular singularity at $t = 0$ and order 1 regular pole at $t = \infty$ (besides those coming from the Log-structure on $\hat{S}^1$). The extended pairing and the extended connection give a structure known as $(\log\text{-}\text{trTLEP}(w))-\text{structure}$ as in [43]. Finally, we have the definition of a miniversal element from [43] (or a primitive form from [36]).

\footnote{Here we abuse the notation and use $\langle \cdot, \cdot \rangle$ as pairing on $k\mathcal{H}_+$ for each $k$.}
Definition 6.6. A miniversal section \( \xi = (k\xi)_{k \geq 0} \) is an element \( k\xi \in \mathcal{H}_+ \cap t(k\mathcal{H}_-) \) such that

1. \( k+1\xi \equiv k\xi \mod t(T) \);
2. \( \nabla_X k\xi = 0 \) for each \( k \) which restricted to \( t(k\mathcal{H}_-)/k\mathcal{H}_- \);
3. \( \nabla_{t\partial} k\xi \equiv r(k\xi) \) for each \( k \) which restricted to \( t(k\mathcal{H}_-)/k\mathcal{H}_- \), with the same \( r \in \mathbb{C} \);
4. The Kodaira-Spencer map \( KS : k\Theta S_{t^*} \to k\mathcal{H}_+ / t(k\mathcal{H}_+) \) given by \( KS(X) = t\nabla_X \xi \mod (t(k\mathcal{H}_+)) \) is a bundle isomorphism for each \( k \).

Making use of Proposition 1.11. from [43], a choice of opposite filtration \( \mathcal{H}_- \) and a miniversal element \( \xi \) will give a (germ of) Logarithmic Frobenius manifold.

6.2. Construction of \( \mathcal{H}_\tau \)-VHS from polyvector field. Following [20, 30, 37], we give a construction of \( \mathcal{H}_\tau \)-VHS using our deformation theory via the dgBV \( kPV_T[[t]] \) as in Notation 5.1.

Condition 6.7. Recall that we have the \( 0^\text{th} \)-order Kodaira-Spencer map \( 0^\nabla([0, \omega]) : k\mathcal{K}_0 \to F^{\geq d-1}H^0 \) after Proposition 2.12 we further assume that the induced map \( 0^\nabla([0, \omega]) : k\mathcal{K}_0 \to F^{\geq d-1}H^0 / F^{d-1}H^0 \) is injective. Furthermore, we fix our choice of the graded vector space \( V^* := Gr_F(H^*)/\text{Im}(0^\nabla([0, \omega])) \).

Notations 6.8. From the Lemma 4.16 or Remark 5.2 we define the relative de Rham complex with coefficient in \( T \) as \( k\mathcal{A}_T^* := (\bigwedge^p A \otimes C T) \otimes R_T(R_T/T^{k+1}) \) and consider \( H^*(k\mathcal{A}_T)([t^\frac{1}{2}][t^{-\frac{1}{2}}]) \) for each \( k \) which is a locally free \( kR_T[[t^\frac{1}{2}][t^{-\frac{1}{2}}]] \) as in the Remark 5.2.

Since the ring \( T \) is itself graded, for an element \( \varphi \in (kPV^p_q \otimes T)/T^{k+1} \subset kPV_T (\alpha \in (kA^{p,q} \otimes T)/T^{k+1} \subset k\mathcal{A}_T \text{ resp.}) \), we call the index of \( \varphi \) (\( \alpha \) resp.) to be \( p+q \), which is denoted by \( \tilde{\varphi} \) (\( \tilde{\alpha} \) resp.).

6.2.1. Construction of \( \mathcal{H}_+ \).

Definition 6.9. We consider the scaling morphism

\[ l_t : H^*(kPV_T[[t]][t^{-1}], k\mathcal{d}) \to H^*(k\mathcal{A}_T, k\mathcal{d})([t^\frac{1}{2}][t^{-\frac{1}{2}}]), \]

and define the (sections of) the Hodge bundle over \( 0S^\dagger \) to be \( 0\mathcal{H}_+ := l_t(H^*(0PV[[t]], 0\mathcal{d})) \) as a submodule of \( 0R_T[[t]] = \mathbb{C}[[t]] \).

We further take

\[ k\mathcal{H}_\pm := \text{Im}(l_t)[t^{-1}] = \bigoplus (H^{d+ev}(k\mathcal{A}, k\mathcal{d})([t])[[t^{-1}]] \oplus (H^{d+odd}(k\mathcal{A}, k\mathcal{d})([t])[[t^{-1}]]t^\frac{1}{2}) \]

\((\mathcal{H}_\pm := \lim_k k\mathcal{H}_\pm \text{ resp.}) \) as a module over \( kR_T[[t]][t^{-1}] \) (\( R_T[[t]][t^{-1}] \) resp.).

With a Maurer-Cartan element \( \varphi = (k\varphi)_k \) as in Theorem 5.3, we notice that the cohomology \( H^*(kPV_T[[t]], k\partial + t(k\Delta) + (k\varphi, \cdot)) \) is again a free module over \( kR_T[[t]] \). Therefore, we define

\[ k\mathcal{H}_+ := \{ (l_t(\alpha) \wedge c^{l_t(k\varphi)} ) \cdot t \omega \mid \alpha \in H^*(kPV_T[[t]], k\partial + t(k\Delta) + (k\varphi, \cdot)) \}

\] to be the \( kR_T[[t]] \) submodule of \( k\mathcal{H}_\pm \). Similarly, we will let \( \mathcal{H}_+ := \lim_k k\mathcal{H}_+ \) to be the \( \tilde{R}_T[[t]] \) module.

Remark 6.10. We notice that we have \( 0H^{d+ev}_+ = \bigoplus_{r=0}^d (F^{r} \cap H^{d+ev}(0A, 0\mathcal{d})) \mathbb{C}[[t]]t^{-r-d+1} \subset H^{d+ev}(0A, 0\mathcal{d})([t])[[t^{-1}]] \) and \( 0\mathcal{H}^{d+odd}_+ = \bigoplus_{r=0}^{d-1} (F^{r+\frac{1}{2}} \cap H^{d+odd}(0A, 0\mathcal{d})) \mathbb{C}[[t]]t^{-r-d+1} \) related to the Hodge filtration given in Definition 2.10, and hence we have \( 0\mathcal{H}/t(0\mathcal{H}) = Gr_F(H^*(0A)) \) as vector space.
Lemma 6.11. The $k R_T[[t]]$ submodule $k \mathcal{H}_+$ is preserved under the operation $t (k \nabla_X)$ for any $X \in k \Theta_{ST}^1$. Therefore, we have $\nabla : k \mathcal{H}_+ \to \frac{1}{t} (k \Omega_{ST}^1) \otimes (k R_T)^k \mathcal{H}_+$.

Proof. It suffices to prove the first statement of the above Lemma 6.11. We begin by consider the case $\alpha = 1$ and restricting the Maurer-Cartan element $k \varphi$ to the coefficient ring $k R$ (because the extra coefficient $T$ is not involved in the differential $k d$ for defining the Gauss-Manin connection in the Definition 4.12).

We notice that $l_t(1) \otimes e^{k \varphi} \varphi_{ij} (k \omega) = l_t(e^{k \varphi} / t) \varphi_{ij} (k \omega)$. We take a lifting $w \in \frac{1}{2} A^{d,0} / \frac{1}{2} A^{d,0}$ for the element $k \omega$ for computing the connection $k \nabla$ via the sequence (4.3). Direct computation shows $k d (l_t(e^{k \varphi} / t) \omega) = t^{-\frac{1}{2}} l_t \left( \sum_i d \log \kappa_{k,m_i} \otimes e^{k \varphi} / t (\psi_i \otimes k \omega) \right) = \sum_i d \log \kappa_{k,m_i} \otimes t^{-1} (l_t(\psi_i) \otimes e^{k \varphi}) \otimes k \omega$, for some $m_i \in K$ and $\psi_i \in k P V^*[[t]]$. Since we have $(l_t(\psi_i) \otimes e^{k \varphi}) \otimes k \omega \in k \mathcal{H}_+$, we obtain that $k \nabla (\psi_i \otimes e^{k \varphi}) \otimes k \omega \in \frac{1}{t} (k \Omega_{ST}^1) \otimes (k R_T)^k \mathcal{H}_+$.

For the case of $(l_t(\alpha) \otimes e^{k \varphi}) \otimes k \omega$, we may simply introduce a formal parameter $\epsilon$ of degree $-|\alpha|$ such that $\epsilon^2 = 0$, and repeat the above argument for the Maurer-Cartan element $k \varphi + \epsilon \alpha$ over the ring $k R[\epsilon] / (\epsilon^2)$.

6.2.2. Construction of $\mathcal{H}_-$. With the 0th-order Gauss-Manin connection defined in (2.11) for every element $\nu \in K^v$ we have an endomorphism $N_{\nu} := 0 \nabla_{\nu} : H^*(0||A, 0, d) \to H^*(0||A, 0, d)$. From the flatness of the Gauss-Manin connection from Proposition 4.14 we obtain that $N_{\nu_1 \nu_2} = N_{\nu_2} N_{\nu_1}$ being commuting operators. We further assume that these commuting operators $N_{\nu}$'s are nilpotent and preserving the same opposite filtration as in the following Assumption 6.12.

Assumption 6.12. We assume there is an increasing filtration $W \leq_* H^* (0||A, 0, d)$

$$\{0\} \subset W_{\leq 0} \subset \cdots \subset W_{\leq r} \subset \cdots \subset W_{d} = H^* (0||A, 0, d)$$

for $2r \in \mathbb{Z}_{\geq 0}$, which is preserved by the 0th-order Gauss-Manin connection in the sense that we have $N_{\nu} W_{\leq r} \subset W_{\leq r-1}$ for any $\nu \in K^v$. Furthermore, we require that it is an opposite filtration to the Hodge filtration $F_{2r}$ in the sense that we have $F_{2r} \oplus W_{\leq r-\frac{1}{2}} = H^* (0||A, 0, d)$.

Lemma 6.13. Under the above Assumption 6.13 there exist a index (introduced in Notation 6.8) and degree preserving trivialization

$$\varpi : H^* (0||A, 0, d) \otimes_{C} \hat{R}_T \to H^* (0||A_T, d)$$

identifying the connection form of Gauss-Manin connection $\nabla$ with the nilpotent operator $N$, i.e. for any $\nu \in K^v$ we have $\nabla_{\nu} (s \otimes 1) = N_{\nu} (s \otimes 1)$ for $s \in H^* (0||A, 0, d)$.

Proof. Since the extra coefficient ring $T$ does not couple with the differential $d$, we only have to construct inductively a trivialization $k \varpi : H^* (0||A, 0, d) \otimes_{C} k R \to H^* (0||A, 0, d)$ for every $k$ such that $k+1 \varpi = k \varpi$ (mod $m^{k+1}$) and identifying $k \nabla$ with the nilpotent operator $N$.

We prove the induction step by assuming that $k-1 \varpi$ is constructed, and we have to construct its lifting $k \varpi$. We first choose an arbitrary lifting $\tilde{k} \varpi$ and a filtered basis $e_1, \ldots, e_m$ of the finite dimensional vector space $H^* (0||A, 0, d)$ in a sense that it is constructed by lifting a basis of the associated quotient $\text{Gr}_W (H^* (0||A, 0, d))$. We will also write $\tilde{e}_i$ to stand for $k \tilde{k} \varpi (e_i \otimes 1)$. With respect

\(14\) Notice that we use the index $W_{< r}$ by half integer $r$ following \(\|\), and multiplying by 2 we obtain the usual index $W_{< 0} \subset \cdots \subset W_{< r} \subset \cdots \subset W_{< 2d}$ with $r \in \mathbb{Z}$. 

to the frame $\tilde{e}_i$'s of $H^*([A, k])$, we define a connection $\tilde{\nabla}$ with connection form given by $\tilde{\nabla}_\nu(\tilde{e}_i) = \sum_j(N_\nu)_i^j(e_j)$ for $\nu \in K_C^* \cap \mathbb{C}$ where $(N_\nu)_i^j$'s are the matrix coefficients of the operator $N_\nu$ with respect to the basis $e_i$'s. We may also treat $N = (N_\nu)_i^j$ as $K_C$-valued endomorphisms on $H^*([0, A, 0])$.

From the induction hypothesis we notice that $k\nabla - \tilde{\nabla} = 0 \mod (m^k)$ and therefore we have $(k\nabla - \tilde{\nabla})(\tilde{e}_i) = \sum_j \alpha_{mi}^j e_j q^m \in (\Omega^1_{\mathbb{C}} \otimes \mathbb{C}) H^r(\mathbb{C} A) \otimes \mathbb{C}(m^k/m^{k+1})$. From the flatness of both $k\nabla$ and $\tilde{\nabla}$ we notice that $(d \log q^m) \wedge \alpha_{mi}^j = 0$ and hence $\alpha_{mi}^j = \alpha_{mi}^j d \log q^m$ for some constant $\alpha_{mi}^j \in \mathbb{C}$ for every $m$ and $j$. We will use $c_m$ to denote the endomorphism on $H^r([0, A, 0])$ whose matrix coefficients given by $c_m = (\alpha_{mi}^j)$ with respect to the basis $e_i$'s.

As a result if we define a new frame $\tilde{e}_i^{(0)} := \tilde{e}_i - \sum_{m,j} c_m^j e_j q^m$ and a new connection $\tilde{\nabla}^{(0)}$ by letting $\tilde{\nabla}_\nu^{(0)}(\tilde{e}_i^{(0)}) = \sum_j(N_\nu)_i^j(\tilde{e}_j^{(0)})$, we have

$$(k\nabla - \tilde{\nabla}^{(0)})(\tilde{e}_i^{(0)}) = \sum_{j,m}[c_m, N_i]_j^m (e_j)q^m$$

where $[c_m, N]$ being their usual Lie bracket with its $K_C$-valued matrix coefficient given by $([c_m, N])_i^j = \alpha_{mi}^j N_i^j - \alpha_{mi}^j N_i^j$. Once again using both connection $k\nabla$ and $\tilde{\nabla}^{(0)}$ being flat we have some constant $c_m^{(1),j}$ such that $[c_m, N]_j^m = c_m^{(1),j} d \log q^m$. Taking an element $\nu \in K_C^* \cap \mathbb{C}$ such that $(m, \nu) \neq 0$ we obtain $c_m^{(1)} = \frac{1}{(m, \nu)} [c_m(N_\nu)_i^j]$. Once again if we define a new frame $\tilde{e}_i^{(1)} := \tilde{e}_i^{(0)} - \sum_{j,m} c_m^{(1),j} e_j q^m$ and a new connection $\nabla^{(1)}(\tilde{e}_i^{(1)}) := \sum_j N_j^i e_j^{(1)}$, we will find $c_m^{(2)} = \frac{1}{(m, \nu)} [c_m^{(1)}(N_\nu)_i^j] = \frac{1}{(m, \nu)(m, \nu_\nu)} [c_m(N_\nu)_i^j, N_{\nu\nu}]$ such that $c_m^{(2),d} = \frac{1}{(m, \nu)} [c_m^{(1)}(N_\nu)_i^j, N_{\nu\nu}]$.

By repeating the argument we will obtain a frame $\tilde{e}_i^{(2d),\tilde{d}}$ such that if we let $\nabla^{(2d)}(\tilde{e}_i^{(2d)}) = \sum_j N_j^i e_j^{(2d)}$, we will have $c_m^{(2d+1)} = \frac{1}{(m, \nu)} [c_m^{(2d+1)}(N_\nu)_i^j]^{2d+1} = 0$. As a result if we let $k\nabla(\tilde{e}_i \otimes 1) = \tilde{e}_i^{(2d)}$ we have the desired trivialization for the Hodge bundle.

From the Assumption 6.12 we notice that we can take a filtered basis $(e_r)_0 \leq r \leq 2d$ of the vector space $H^r([0, A, 0])$ such that $e_r \in W_{\leq r} \cap (H^{d+ev}_{\leq r}([0, A])$ if $r \in \mathbb{Z}$, $e_r \in W_{\leq r} \cap (H^{d+odd}_{\leq r}([0, A])$ if $r \in \mathbb{Z} + \frac{1}{2}$ and $(e_r)_0 \leq r \leq m_r$ form a basis of $W_{\leq r} / W_{\leq r-\frac{1}{2}}$.

**Definition 6.14.** Taking the trivialization $\pi$ in the above Lemma 6.13, we let $\epsilon_{r,i} := \pi(e_r \otimes 1)$ to be the section of the Hodge bundle $H^*(0_A)$. The collections $(\epsilon_{r,i})$'s will be called the set of elementary sections (motivated from Deigne’s canonical extension [7]) which forms a frame of the Hodge bundle.

**Lemma 6.15.** If we let

$$0\mathcal{H}^{d+ev} : = \bigoplus_{r=0}^d (W_{\leq r} \cap H^{d+ev}_{\leq r}([0, A])) \subset [t^{-1}]^{t-r-d-2} \subset 0\mathcal{H}^{d+ev},$$

$$0\mathcal{H}^{d+odd} : = \bigoplus_{r=0}^{d-1} (W_{\leq \frac{r}{2}+1} \cap H^{d+odd}_{\leq \frac{r}{2}+1}([0, A])) \subset [t^{-1}]^{t-(r+\frac{1}{2})+d-2} \subset 0\mathcal{H}^{d+odd},$$

15Here $d$ is the length of the weighted filtration $W_{\leq r}$ in the above Assumption 6.12.
16We notice that the index of $e_r \in V_\bullet$ is the same as that of $e_r \in V_\bullet$. 
and $0\mathcal{H}_- := \bigoplus_{d \geq 0} \mathcal{H}_-^{d,e} \oplus \bigoplus_{d \leq 0} \mathcal{H}_-^{d,\text{odd}}$ to be the $\mathbb{C}[t^{-1}]$ submodule of $0\mathcal{H}_\pm$, then there exist a unique free $R_T[1^{-1}]$ submodule $\mathcal{H}_- = \varinjlim k\mathcal{H}_-$ of $\mathcal{H}_\pm$ which is preserved by $\nabla_X$ for any $X \in \tilde{X}_{\mathcal{H}}$, and satisfying $k^{1+1}\mathcal{H}_- = k\mathcal{H}_-$ (mod $T^{k+1}$).

Proof. For the existence we take a set of elementary sections $(e_{r,i})_{0 \leq 2r \leq 2l}$ as in the previous Definition [6.14]. We can take the free submodules $^k\mathcal{V}^{d,e}_{\leq r} = \bigoplus_{t \leq r} \bigoplus_{0 \leq i \leq m_r} ^kR_T \cdot e_{t,i}$, $^k\mathcal{V}^{d,\text{odd}}_{\leq r+\frac{1}{2}} = \bigoplus_{t \leq r+\frac{1}{2}} \bigoplus_{0 \leq i \leq m_{r+\frac{1}{2}}} ^kR_T \cdot e_{t+\frac{1}{2},i}$ of $H^*(k\mathcal{T}_-^1, \mathcal{A}_-^1, \mathcal{D})$ and let

$$^{k}\mathcal{H}_- := \bigoplus_{0 \leq r \leq d} ^k\mathcal{V}^{d,e}_{\leq r} \mathbb{C}[t^{-1}]|^{t-r+d-2} \oplus \bigoplus_{0 \leq r \leq d-1} ^k\mathcal{V}^{d,\text{odd}}_{\leq r+\frac{1}{2}} \mathbb{C}[t^{-1}]|^{t-(r+\frac{1}{2})+d-2}$$

to be the desired $\nabla_X$ invariant subspace.

Once again, for the uniqueness part we only have to argument the uniqueness of $^kR_T[1^{-1}]$ submodule $^k\mathcal{H}_-$ of $^k\mathcal{H}_\pm$ for each $k$ by induction. Once again since the coefficient ring $T$ does not couple with the differential $\mathcal{D}$ we only have to consider the corresponding statement for Hodge bundle over $^k\mathcal{T}$. For the induction step we assume that there is another increasing filtration $^{k}_{\sim} \mathcal{W}_{\leq r}$ of $H^*(\mathbb{C}, \mathcal{A}_-, \mathcal{D})$ satisfying the desired properties such that they agree when passing to $H^*(k\mathcal{T}_-^1, \mathcal{A}_-^1, \mathcal{D})$. We should prove that $^k\mathcal{W}_{\leq r} \subset ^k\mathcal{V}_{\leq r}$ for each $r$.

We take $r$ to be a half integer such that $^{k}_{\sim} \mathcal{W}_{\leq r} \neq 0$, using the proof as in the above Lemma [6.13] we obtain a trivialization $^{0}_{\sim} \mathcal{W}_{\leq r} \otimes \mathbb{C}^k R \rightarrow ^{k}_{\sim} \mathcal{W}_{\leq r}$ with frames $e_{r,i}$'s which identifies $^k\nabla$ with $N$, and in particular we must have $r \geq 0$. We let $l_r \geq r$ to be the minimum half integer such that $^{k}_{\sim} \mathcal{W}_{\leq r} \subset ^k\mathcal{W}_{\leq l_r}$, and we take the frames $(e_{r,i})_{0 \leq 2r \leq 2l_r}$'s for the submodule $^k\mathcal{W}_{\leq l_r}$. Therefore we can write

$$\tilde{e}_{r,i} = \sum_{0 \leq 2r \leq r} f_{l_r,i} e_{r,i} + \sum_{2r+1 \leq 2l \leq 2l_r} f_{l_r,i} e_{r,i}$$

for some $f_{l_r,i} \in k\mathcal{R}$ with $f_{l_r,i} = 0 \mod m^k$ for $r + \frac{1}{2} \leq l$.

Starting with $r = 0$ we assume on the contrary that $l_0 > 0$, and therefore we notice that $\tilde{e}_{0,i} = \sum_{0 \leq i \leq m_0} f_{0,i} e_{0,i} + \sum_{1 \leq l \leq 2l_0} f_{l_0,i} e_{r,i}$ and applying the connection $^k\nabla$ we obtain

$$0 = \sum_{0 \leq i \leq m_0} \partial(f_{0,i}) e_{0,i} + \sum_{1 \leq l \leq 2l_0} \partial(f_{l_0,i}) e_{r,i} + \sum_{1 \leq l \leq 2l_0} f_{l_0,i} N(e_{r,i}).$$

Passing to the quotient $^k\mathcal{W}_{\leq l_0} / ^k\mathcal{W}_{\leq l_0 - \frac{1}{2}}$ we have $0 = \sum_{1 \leq l \leq 2l_0} \partial(f_{l_0,i}) e_{r,i}$ which implies $f_{l_0,i} = 0$ for $l \geq 1$ and hence $t_0 = 0$. By induction on $r$ we have $^{k}_{\sim} \mathcal{W}_{\leq r - \frac{1}{2}} \subset ^k\mathcal{W}_{\leq r - \frac{1}{2}}$ by the induction hypothesis. We assume on the contrary that $l_r > r$ and we consider

$$N(\tilde{e}_{r,i}) = \sum_{0 \leq 2r \leq 2l_r} ^k\nabla(f_{l_r,i} e_{r,i}) + \sum_{2r+1 \leq 2l_r} \partial(f_{l_r,i}) e_{r,i} + \sum_{2r+1 \leq 2l_r} f_{l_r,i} N(e_{r,i}),$$

which gives $0 = \sum_{2r+1 \leq 2l_r} \partial(f_{l_r,i}) e_{r,i}$ by passing to the quotient $^k\mathcal{W}_{\leq l_r} / ^k\mathcal{W}_{\leq l_r - \frac{1}{2}}$ and therefore we obtain $f_{l_r,i} = 0$ for $l \geq r + \frac{1}{2}$. This gives a contradiction $l_r = r$ and hence completes the induction step and the proof of the Lemma [6.15].
Remark 6.16. The above Lemma 6.15 said that the opposite filtration $\mathcal{H}_-$ is determined uniquely by $^0\mathcal{H}_-$ which is given by the opposite filtration in Assumption 6.12. When apply to the case of maximally degenerated log Calabi-Yau studied by Gross-Siebert \cite{Gr-Si}, this filtration is the weighted filtration determined by the nilpotent operators $N_\nu$ for any $\nu \in \text{int}(\mathbb{Q}_{\geq 0}^d) \cap \mathbb{K}^\vee$ and is canonical associated to the maximally degenerated log Calabi-Yau.

6.2.3. Construction of Pairing $(\cdot, \cdot)$. The next Assumption 6.17 is concerning the existence of pairing structure $(\cdot, \cdot)$ stated in the above Definition 6.2.

Assumption 6.17. We assume that all the nontrivial cohomology group $H^*(\| A, 0 \mathbf{d})$ are in degree $0 \leq \ast \leq 2d$ and $H^{p>d}(\| A, 0 \mathbf{d}) = 0$ for all $0 \leq p \leq d$. We further assume that there is a trace map $\text{tr} : H^{d,d}(\| A, 0 \mathbf{d}) = H^{2d}(\| A, 0 \mathbf{d}) \to \mathbb{C}$\textsuperscript{17}. We define a pairing $0\mathbf{p}(\cdot, \cdot)$ on $H^*(\| A, 0 \mathbf{d})$ by taking

$$0\mathbf{p}(\alpha, \beta) := \text{tr} (\alpha \land \beta)$$

for $\alpha, \beta \in H^*(\| A, 0 \mathbf{d})$. We further assume that the trace map $\text{tr}$ and the corresponding pairing $0\mathbf{p}(\cdot, \cdot)$, when descended to $\text{Gr}_F(H^*(\| A))$ (notice that $F_{\geq \ast}$ being isotropic with respect to the pairing $0\mathbf{p}(\cdot, \cdot)$, i.e. we have $0\mathbf{p}(F_{\geq r}, F_{\geq s}) = 0$ when $r + s > d$ from its definition) being non-degenerated.

We have $\text{tr}(N_\nu(\alpha)) = 0$ for any $\alpha \in H^*(\| A)$ and $\nu \in \mathbb{K}^\vee$, and since the filtration $W_{\leq \bullet}$ is opposite to $F_{\geq \bullet}$, we have $W_{\leq \bullet}$ being isotropic with respect to the pairing $0\mathbf{p}(\cdot, \cdot)$, i.e. we have $0\mathbf{p}(W_{\leq s}, W_{\leq r}) = 0$ when $r + s < d$.

Lemma 6.18. If we take the truncation $^k\tau (A)^{\ast, \ast} : = k\| A^\ast, \leq d$ of the complex $^k\| A^\ast, \ast, k \mathbf{d}$ as a quotient complex, then we have the natural map between the cohomology group $H^*(\| A, k \mathbf{d}) \to H^*(^k\tau (A), k \mathbf{d})$ is an isomorphism for all $k$ under the above Assumption 6.17.

Proof. We consider the exact sequence of complexes $0 \to k\| A^\ast, > d \to k\| A^\ast, \ast \to ^k\tau (A)^{\ast, \ast} \to 0$ and we need to show that $k\| A^\ast, > d$ is acyclic. Using exactly the same argument as in \textsuperscript{4.3.2} we have the cohomology $H^*(k\| A^\ast, > d, k \mathbf{d})$ is a free $KR$ module. By considering $k = 0$ we notice that $H^*(\| A^\ast, > d, 0 \mathbf{d}) = 0$ from the above Assumption 6.17 on the vanishing of cohomology group $H^{p,d}(\| A, 0 \mathbf{d}) = 0$ using a standard zig-zag argument, and therefore we conclude that $H^*(k\| A^\ast, > d, k \mathbf{d}) = 0$.

The above Lemma 6.18 allows us to work in the cohomology of truncated complex $H^*(^k\tau (A)^{\ast,}, k \mathbf{d})$ for defining the pairing $(\cdot, \cdot)$.

Definition 6.19. Using the elementary sections $e_{r;i}$’s in Definition 6.14, we extend the trace map $\text{tr}$ to the Hodge bundle $H^*(^k\tau (A)[(t\frac{1}{2})](t^{-\frac{1}{2}})]$, which is denoted by $\text{tr}$ by extending linearly via the formulae

$$\text{tr}(f e_{r;i}) := f \text{tr}(e_{r;i})$$

for $f \in kR_T[(t\frac{1}{2})](t^{-\frac{1}{2}})]$.

We also extend the pairing $0\mathbf{p}$ to a pairing $\mathbf{p}$ on $H^*(^k\tau (A)[(t\frac{1}{2})](t^{-\frac{1}{2}})]$ by extending linearly the formula

$$\mathbf{p}(f(t) e_{r;i}, g(t) e_{i;j}) := (-1)^{|e_{r;i}|} f(t) g(t) 0\mathbf{p}(e_{r;i}, e_{i;j}),$$

for $f(t), g(t) \in kR_T[(t\frac{1}{2})](t^{-\frac{1}{2}})]$.

\textsuperscript{17}We will let $\text{tr} : H^*(\| A, 0 \mathbf{d}) \to \mathbb{C}$ to denote the map which is trivial on $H^{< 2d}(\| A, 0 \mathbf{d})$. 

We further define a pairing $\langle \cdot, \cdot \rangle$ on $k\mathcal{H}_\pm$ (motivated by [1]) by the formula
\[
\langle l_t(\alpha(t)), l_t(\beta(t)) \rangle := (-1)^{-1-d|\beta|+\frac{1}{2}d^2}p(l_t(\alpha(t)), l_{-t}(\beta(-t))),
\]
for $l_t(\alpha(t)), l_t(\beta(t)) \in k\mathcal{H}_\pm$.

**Lemma 6.20.** There is an identity $p(\alpha, \beta) = \text{tr}(\alpha \wedge \beta)$ identifying the pairing and the trace map defined in the above Definition 6.19, and we further have the flatness of the pairing $p$ as
\[
X(p(\alpha, \beta)) = p(\nabla_X \alpha, \beta) + p(\alpha, \nabla_X \beta)
\]
for any $\alpha, \beta \in H^*(k\mathcal{A})$, and $X \in k\Theta_s^\tau$.

**Proof.** Using the short exact sequence $0 \rightarrow K_C \otimes_C kA^*[-1] \rightarrow kA^*/k^2A^* \rightarrow kA^* \rightarrow 0$ defining the $k^{th}$ order Gauss-Manin connection $k\nabla$ we have the flatness of product
\[
k^{\nabla}(\alpha \wedge \beta) = (k^{\nabla} \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (k^{\nabla} \beta).
\]

We first show the identity $p(\alpha, \beta) = \text{tr}(\alpha \wedge \beta)$. We choose a basis $e_{ri}$'s of $H^*(0\mathcal{A}, 0\mathcal{d})$ and the corresponding elementary sections $e_{ri}$'s as in Definition 6.14. We have to show that there is a relation $e_{ri} \wedge e_{lj} = \sum s_{k} e_{r_i,l_i-j} e_{s_k}$ for some constant $e_{r_i,l_i,j} \in \mathbb{C}$. This can be done by induction on order $k$, followed by an induction on the lexicographical order on $(r,l)<(r',l')$ if $r>r'$, or $r=r'$ and $l<l'$ for each fixed $k$. We fix $r, l, i, j$ and consider the product $e_{ri} \wedge e_{lj}$ and suppose the above relation is true for any $(r', l') < (r, l)$. Assuming that we have $e_{ri} \wedge e_{lj} = \sum s_{k} e_{s_k} k e_{s_k} k q^m$, taking the Gauss-Manin connection we have $\nabla_{\nu}(e_{ri} \wedge e_{lj}) = (N_{\nu} e_{ri}) \wedge e_{lj} + e_{ri} \wedge (N_{\nu} e_{lj})$, and by our induction hypothesis it forces $\sum s_{k} \sum q^m e_{s_k} k q^m = 0$. As a result we have $p(e_{ri}, e_{lj}) = \text{tr}(e_{ri} \wedge e_{lj})$ and the general relation $p(\alpha, \beta) = \text{tr}(\alpha \wedge \beta)$ follows. Flatness of the pairing $p$ follows. □

**Lemma 6.21.** The above pairing in Definition 6.19 satisfies $\langle s_1, s_2 \rangle \in kR_T[[t]]$ for any $s_1, s_2 \in k\mathcal{H}_+$, and $\langle s_1, s_2 \rangle \in kR_T[t^{-1}]$ for any $s_1, s_2 \in k\mathcal{H}_-$. Furthermore, the pairing when restricted as $g(\cdot, \cdot) : k\mathcal{H}_+/t(k\mathcal{H}_+) \times k\mathcal{H}_+/t(k\mathcal{H}_+) \rightarrow kR_T[-2d]$ is non-degenerated.

**Proof.** We notice that the statement for $k\mathcal{H}_-$ follows from Property (2) of the above Assumption 6.17 and the Definition 6.19 using elementary sections. Therefore we only have to prove it for $k\mathcal{H}_+$.

For the statement on $k\mathcal{H}_+$ we will work in the truncated complex $k\mathcal{H}^r(\mathcal{A})_r^\tau[[t^{\frac{1}{2}}]]t^{-\frac{1}{2}}$, $k\mathcal{d}$ introduced in Lemma 6.18. First we notice that for two cohomology class of $H^r(k\mathcal{H}(\mathcal{A})_r^\tau)[[t^{\frac{1}{2}}]]t^{-\frac{1}{2}}$ represented by $l_t(\alpha), l_t(\beta)$, $\omega \in k\mathcal{H}(\mathcal{A})_r^\tau$ for some element $\alpha, \beta \in kPV^\tau_\tau[[t]][t^{-1}]$ with a fixed index $\bar{\alpha} = \bar{l}_1$ and $\bar{\beta} = \bar{l}_2$ (recalled that the index is introduced in Notation 6.8), we have the following formula from [1] Proposition 5.9.4. in the truncated complex $H^r(k\mathcal{H}(\mathcal{A})_r^\tau)[[t^{\frac{1}{2}}]]t^{-\frac{1}{2}}$
\[
(5.5) \quad (-1)^{-1-d|\beta|+\frac{1}{2}d^2}(l_t(\alpha(t)) \wedge l_{-t}(\beta(-t)) \wedge k^{\mathcal{H}}) = \left( l_t(\alpha(t)) \wedge l_{-t}(\beta(-t)) \right) \wedge k^{\mathcal{H}} \wedge k^{\mathcal{W}}.
\]

Once the above formula holds, it follows from the same argument as in [1] Proposition 5.9.4. that the R.H.S. lying in $kR_T[[t]]$.

For the non-degeneracy it suffices to consider the pairing for $0\mathcal{H}_+$, which follows from non-degeneracy condition in the Assumption 6.17.

---

\textsuperscript{18}We recall from Notation 6.8 that $\bar{\beta}$ is the index of $\beta \in kPV^\tau_\tau[[t]][t^{-1}]$.

\textsuperscript{19}Notice that it does not hold in the full complex and it is the reason of introducing the truncated complex.
The above constructions of \((\mathcal{H}_+, \nabla, \langle \cdot, \cdot \rangle)\) gives a \(\infty\)-LVHS together with an opposite filtration \(\mathcal{H}_-\) satisfying (1) – (3) in Definition 6.5, and it remains to construct the grading structure.

6.2.4. Construction of grading structure.

**Definition 6.22.** For each \(k\), we define the extended connection \(\nabla_{\partial_f^{\mathcal{H}}}\) acting on \(H^*(k\mathcal{A}_T, k\mathcal{d})\) by the rule that \(\nabla_{\partial_f^{\mathcal{H}}} (s) = 2^{d-2} s \) for \(s \in H^*(k\mathcal{A}_T, k\mathcal{d})\) and \(\nabla_{\partial_f^{\mathcal{H}}} (f) = \partial^\mathcal{H}(f) + f(\nabla_{\partial_f^{\mathcal{H}}} (s))\).

**Proposition 6.23.** The extended connection \(\nabla\) is a flat connection acting on \(k\mathcal{H}_\pm\), i.e. we have \([\nabla_{\partial_f^{\mathcal{H}}}, \nabla_X] = 0\) for any \(X \in k\Theta S^T\). The submodule \(k\mathcal{H}_-\) is preserved by \(\nabla_{\partial_f^{\mathcal{H}}}\) and we have \(\nabla_{\partial_f^{\mathcal{H}}} (\mathcal{H}_+) \subset t^{-1}(k\mathcal{H}_+),\) and the pairing \(\langle \cdot, \cdot \rangle\) is flat with respect to \(\nabla_{\partial_f^{\mathcal{H}}}\).

**Proof.** Beside the property for \(k\mathcal{H}_+\), the others simply follows from its definitions. Suppose we take \(\alpha \in kPV_T[[t]]\) and consider \((l_t(\alpha) \wedge e^{l_t(\varphi)}) \cdot f \omega\). Therefore we have

\[
\nabla_{\partial_f^{\mathcal{H}}} (l_t(\alpha) \wedge e^{l_t(\varphi)}) \cdot f \omega = ((\nabla_{\partial_f^{\mathcal{H}}} l_t(\alpha)) \wedge e^{l_t(\varphi)} + l_t(\alpha) \wedge (\nabla_{\partial_f^{\mathcal{H}}} l_t(\varphi)) \wedge e^{l_t(\varphi)}) \cdot f \omega.
\]

Since we can write both \(\nabla_{\partial_f^{\mathcal{H}}} l_t(\alpha) = l_t(\beta)\) and \(\nabla_{\partial_f^{\mathcal{H}}} l_t(\varphi) = l_t(\gamma)\) for some \(\beta, \gamma \in kPV_T[[t]]\), therefore we can rewrite

\[
\nabla_{\partial_f^{\mathcal{H}}} (l_t(\alpha) \wedge e^{l_t(\varphi)}) \cdot f \omega = \left( (l_t(\beta) + t^{-1} l_t(\alpha + \gamma)) e^{l_t(\varphi)} \right) \cdot f \omega.
\]

we obtain the desired result. \(\square\)

6.3. Construction of miniversal section.

**Notations 6.24.** Notice that we have the cohomology class \([0, \omega] \in \mathcal{F}^{d-1} \cap W_{d, d}\), and we let \(k\mu\) to be the extension of the cohomology classes \([0, \omega] \in (t^0(\mathcal{H}_+) \cap t^2(\mathcal{H}_-))\) by first expressing it as a summation of filtered basis \([0, \omega] = \sum_{r, i} c_{r, i} e_{r, i}\), and then extend it by elementary sections in Definition 6.14 to \(t^2(k\mathcal{H}_-)\) using the formula \(k\psi = \sum_{r, i} c_{r, i} e_{r, i}\) for each \(k\).

**Notations 6.25.** By our choice of the graded vector space \(\mathbb{V}^* = \frac{Gr_T(H^*(\mathcal{A}))}{\text{Tor}^\mathcal{V}(H^*(\mathcal{A}))}\) in Condition 6.7, we further make a choice of degree 0 element \(\psi \in 0PV_T[[t]] \otimes \mathbb{V}^*\) appearing in Theorem 5.3 such that its cohomology class \([l_t(\psi)]\otimes t^0(\mathcal{H}_+) \otimes \mathbb{V}^* = Gr_T(H^*(\mathcal{A})) \otimes \mathbb{V}^*\) maps to the identity element \(\text{id} \in \mathbb{V} \otimes \mathbb{V}^*\) under the natural quotient \(Gr_T(H^*(\mathcal{A})) \otimes \mathbb{V}^* \to \mathbb{V} \otimes \mathbb{V}^*\).

**Definition 6.26.** For the chosen \(\psi\) described above, with the corresponding Maurer-Cartan element \(\varphi = (k \varphi)_k\) constructed in Theorem 5.5, \((t^{-1} e^{l_t(\varphi)} \cdot k \omega)_k\) is called a miniversal section (or primitive form) if it further satisfy

\[
t^{-1} e^{l_t(\varphi)} \cdot k \omega - k \mu \in k\mathcal{H}_-
\]

for each \(k\), where \(k \omega\) is the element we constructed in Proposition 4.8.

**Proposition 6.27.** With a Maurer-Cartan element \(\varphi = (k \varphi)_k\) as in Theorem 5.5, we can modify \(\varphi\) to \(\varphi + \zeta\) by some \(\zeta = (k \zeta)_k\) in \(\lim_{\mathcal{H}_+} kPV_T[[t]]\) to get a miniversal section. Furthermore, \(\mathcal{H}_+\) is unchanged under this modification.

**Proof.** The proof of this Proposition 6.27 is a refinement of that of Theorem 5.5 using same argument as in [1] Theorem 1, and we refer readers to [1] for it. \(\square\)

To conclude this section, we have the following Theorem 6.28

\[20\text{Here we abuse the notation and still use } \nabla \text{ for the extended connection.}\]
Theorem 6.28. The triple \((k\mathcal{H}_+, \nabla, \langle \cdot, \cdot \rangle)\) is a \(\frac{\partial}{\partial s}\) LVHS, with \(k\mathcal{H}_-\) being an opposite filtration to it. Furthermore, the element \(k\xi := t^{-1}e^{t(k\varphi)}(k\omega)\) constructed in the above Proposition 6.27 is a miniversal section in the sense of 6.26.

Proof. It remains to check the condition that \(\xi\) is a miniversal section, and we write \(\xi = \lim_{k} k\xi\) and prove the condition for each \(k\). We have \(k\xi \in k\mathcal{H}_+ \cap t(k\mathcal{H}_-)\) from its construction, and the fact that \(k\xi = t^{-1}(k\mu)\) in \(t(k\mathcal{H}_-)/k\mathcal{H}_-\). Therefore, we have \(\nabla_{\nu}(k\xi) = t^{-1}(\nabla_{\nu})(k\mu) = t^{-1}N_{t}(k\mu) \in k\mathcal{H}_-\) for any \(\nu \in K_{\xi}\). We have computed the action of \(\nabla_{t\omega}^{kl}\) in the proof of the above Proposition 6.23, and the formula gives \(\nabla_{t\omega}^{kl}(t^{-1}e^{t(k\varphi)}) \in (1 - d)e^{t(k\varphi)} + (k\mathcal{H}_-)\) and therefore we have \(\nabla_{t\omega}^{kl}(k\xi) = (1 - d)(k\xi)\) in \((k\mathcal{H}_-)/k\mathcal{H}_-\). Finally, to check the isomorphism of the Kodaira-Spencer map we only have to show this for \(0\mathcal{S}_T^\dagger\), which follows from our choice of the initial condition \(\psi\) for solving the Maurer-Cartan equation (5.1) in Theorem 5.5. □

Remark 6.29. Following [13, 37], we can define the semi-infinite period map \(\Phi : \hat{\mathcal{S}}_T^\dagger \to t\mathcal{H}_-/\mathcal{H}_-\) by taking \(\Phi(s) := [e^{t\varphi(s,t)}]_{\mathcal{S}_T} \omega - \mu\). In the case of maximally degenerated log Calabi-Yau varieties studied by [21], this will be the canonical coordinate for the complex moduli.

7. Abstract deformation data from Log smooth Calabi-Yau varieties

We indicate how to apply our result to the case for log smooth Calabi-Yau varieties studied by Kawamata-Namikawa in [32].

Notations 7.1. Following [32], we take a projective \(d\)-dimensional simple normal crossing variety \((X, \mathcal{O}_X)\). We let \(Q = \mathbb{N}^s\) and write \(R = \mathbb{C}[t_1, \ldots, t_s]\), where \(s\) is the number of connected component of \(D = \bigcup_{i=1}^s D_i = \text{Sing}(X)\). There is a log structure on \(X\) over the \(Q\)-log point \(0\mathcal{S}_T^\dagger\) making it a log smooth variety \(X^\dagger\) over \(0\mathcal{S}_T^\dagger\). We further require it to be log Calabi-Yau.

7.1. 0\(^{th}\) order deformation data. Parallel to Definition 2.9, we have the following.

Definition 7.2. We define

1. the sheaf of BV algebra \(0\mathcal{G}^* = \bigwedge^{-*} T_{X^\dagger/0\mathcal{S}_T^\dagger}(\log)\) to be the analytic sheaf of relative polyvector fields, with natural product structure;
2. the sheaf of de Rham module to be the sheaf of log differential \(0\mathcal{K}^* := \Omega_{X^\dagger/\mathbb{C}}(\log)\) which is a locally free sheaf (in particular coherent) of dga, with a natural dga structure over \(0\Omega_{S^\dagger}^*\) inducing the filtration as in Definition 2.9;
3. the volume element \(0\omega\) via the trivialization \(\Omega_{X^\dagger/0\mathcal{S}_T^\dagger}(\log) \cong \mathcal{O}_X\) from the Calabi-Yau condition;
4. the isomorphism of sheaf of BV module \(0\sigma : 0\Omega_{S^\dagger}^* \otimes_{\mathcal{O}} (0\mathcal{K}^*/0\mathcal{K}^*[-r]) \to 0\mathcal{K}^*/r^10\mathcal{K}^*\) via taking wedge product in \(\Omega_{X^\dagger/\mathbb{C}}^*(\log)\);

\(0\mathcal{G}^*\) is equipped with the BV operator defined by \(0\Delta(\varphi) \cdot 0\omega := 0\partial(\varphi \cdot 0\omega)\).

7.2. Higher order deformation data. By the discussion in [32], every point \(\tilde{x} \in X^\dagger\) is covered by Log chart \(V\) which is biholomorphic to an open neighborhood of \((0, \ldots, 0)\) in \(\{z_0 \cdots z_r = 0 \mid (z_0, \ldots, z_d) \in \mathbb{C}^{d+1}\}\). From the notion of Log deformation in [28], we have a smoothing \(V^\dagger\) of \(V\) given by a neighborhood of \((0, \ldots, 0)\) in \(\{z_0 \cdots z_r = s_1 \mid (z_0, \ldots, z_d) \in \mathbb{C}^{d+1}\}\) if \(V \cap D_i \neq \emptyset\). We choose a covering \(V = \{V_\alpha\}_\alpha\) of \(X\) with such log charts together with a local smoothing \(V^\dagger_\alpha\)'s, and we further require that each \(V_\alpha\) being Stein. We let \(kV^\dagger\) to be the \(k\)th order thickening of the local model \(V_\alpha\).
Definition 7.3. We define

(1) the sheaf of \(k\)th-order polyvector field to be \(k\mathcal{G}_*^\alpha = \bigwedge^{-*} T_{(k\mathbf{v}_1^\alpha, k\mathbb{S}^1)}(\log)\) with natural product structure;

(2) the \(k\)th-order de Rham differential to be \(k\mathcal{K}_*^\alpha = \Omega_*(k\mathbf{v}_1^\alpha, k\mathbb{C})\) (log);

(3) the lifting of volume form \(\omega_\alpha\) of \(\omega\) as element in \(\Omega_*^{k\text{th}}(\mathbf{v}_1^\alpha, k\mathbb{S}^1)\) (log), and take \(k\omega_\alpha = \omega_\alpha \pmod{\mathfrak{m}^{k+1}}\);

(4) the isomorphism of sheaf of BV module \(k\mathbf{\sigma}\) induced by taking wedge product as above Definition 7.2.

The BV operator \(k\Delta_\alpha\) on \(k\mathcal{G}_*^\alpha\) is induced by the volume form \(k\omega_\alpha\).

Finally, the higher order patching data \(k\psi_{\alpha\beta,i}\)'s in Definition 2.15 for each Stein open subset \(U_i \subset V_{\alpha\beta}\) is obtained as follows. By using [32] Theorem 2.1., we have the two log deformation \(k\mathbf{v}_1\) and \(k\mathbf{v}\) being isomorphic on \(U_i\) via \(k\psi_{\alpha\beta,i}\) and hence inducing the corresponding isomorphism \(k\psi_{\alpha\beta,i}\) of sheaves. The existence of log vector fields \(k\mathfrak{p}_{\alpha\beta,i}\)'s, \(k\mathfrak{o}_{\alpha\beta\gamma,i}\)'s and \(k\mathfrak{b}_{\alpha\beta,i}\)'s come from the fact that any automorphism of log deformation over \(U_i\) or \(U_{ij}\) comes from exponential action of vector fields. The difference between volume element is compared by \(k\psi_{\alpha\beta,i}(k\omega_\beta) = \exp(k\mathfrak{w}_{\alpha\beta,i}(k\omega_\alpha)\mathfrak{w}_\beta)\) for some holomorphic function \(k\mathfrak{w}_{\alpha\beta,i}\)'s.

The Assumption 4.1.5 can be achieved by simply taking base change of the family \(\pi: \mathcal{V}_1^\dagger \to k\mathbb{S}^1\) with \(I_{\alpha}^\dagger: \mathbb{A} \to \mathbb{S}^1\), and checking the holomorphic Poincaré Lemma for each \(k\mathcal{K}_*^{\alpha}\) via direct computation.

7.3. Hodge theoretic data.

7.3.1. Hodge-de Rham degeneracy. In [32] p.406 in the proof of Lemma 4.1.], a cohomological mixed Hodge complex of sheaves \((A_\mathcal{C}, (A_\mathcal{C}^{\ast}, W), (A_\mathcal{C}^\ast, W, F))\) is constructed (in the sense of [32] Definition 3.13.), where \(A_\mathcal{C} = \bigoplus_{p+q-k} A^{p,q} : = \bigoplus_{p+q-k} Q^{p+q+1}(\log)/W_q Q^{p+q+1}(\log)\) (\(W_q\) refers to the subsheaf with at most \(q\) log poles), \(F^{\geq r} A_\mathcal{C}^\ast = \bigoplus p \bigoplus q \geq r A^{p,q} \) and \(W_{\leq r} A_\mathcal{C}^\ast = W_{r+2p+1} Q^{p+q+1}(\log)/W_q Q^{p+q+1}(\log)\).

There is a natural quasi-isomorphism \(\mu : (\Omega_{\mathcal{X}_1}^*/\mathcal{S}_1^1)(\log), F^{\geq r} := \Omega_{\mathcal{X}_1}^*/\mathcal{S}_1^1(\log) \to (A_\mathcal{C}, F)\) preserving the Hodge filtration \(F\) on the complexes. We apply [42] Theorem 3.18 to obtain a mixed Hodge structure \((\mathcal{H}_{\mathcal{C}}^\ast, (\mathcal{H}_{\mathcal{C}}^\ast, W), (\mathcal{H}_{\mathcal{C}}^\ast, W, F))\), and the Hodge-de Rham degeneracy assumption 5.4 as in [42] Proof of Lemma 4.1.]. The injectivity of the Kodaira-Spencer map \(0\mathcal{\nabla}^\ast(0\mathbf{\omega})\) as in Condition 6.1. can also be easily verified using the cohomological mixed Hodge complex \((A_\mathcal{C}, F, W)\).

7.3.2. Opposite filtration. On each \(\mathcal{H}_{\mathcal{C}}^\ast\), we have the Deligne’s splitting \(\mathcal{H}_{\mathcal{C}}^\ast = \bigoplus_{s,t} I^{s,t}\) such that \(W_{\leq r} = \bigoplus_{s+\leq t} I^{s,t}\) and \(F^{\geq r} = \bigoplus_{s\geq r} I^{s,t}\). Since the nilpotent operator \(N_\nu\) is defined over \(Q\) and satisfying \(N_\nu W_{\leq r} \subset W_{r-2}\), we deduce that \(N_\nu I^{s,t} \subset I^{r-1,t-1}\). Since the Hodge filtration \(F^{\geq r} \mathcal{H}_{\mathcal{C}}^\ast\) in Definition 2.10 are related to \(F^{\geq r} \mathcal{H}_{\mathcal{C}}^\ast\) by a shifting \(F^{\geq r} I^{s,t} = F^{\geq r}\), we let \(W_{\leq r} I^{s,t} : = \bigoplus_{s\leq r} I^{s,t}\) and \(W_{\leq r}(\mathcal{H}_{\mathcal{C}}^\ast) : = \bigoplus_{s\leq r} W_{\leq r}(\mathcal{H}_{\mathcal{C}}^\ast)\) to obtain the opposite filtration satisfying Assumption 6.12.

7.3.3. Trace pairing. For the Assumption 6.17 we use the trace map \(\text{tr}\) defined in [15] Definition 7.11. which induce a pairing \(0\mathbf{p} : \Omega_{\mathcal{X}_1}^*/\mathcal{S}_1^1(\log) \otimes \Omega_{\mathcal{X}_1}^*/\mathcal{S}_1^1(\log) \to \mathbb{C}\) by product structure on \(\Omega_{\mathcal{X}_1}^*/\mathcal{S}_1^1(\log)\), which is denoted by \(Q_{\mathbf{K}}\) in [15] Definition 7.13.]. The pairing is compatible with the weighted filtration \(W_{\leq r}\) on \(\mathcal{H}_{\mathcal{C}}^\ast\) by [15] Lemma 7.18.]. Furthermore, non-degeneracy
of $0p$ follows from that of the induced pairing $\langle \cdot, \cdot \rangle$ defined on $L_C = \text{GrW}(H^*_{\log}(\Omega^*_{X/\partial S^1}(\log)))$ defined in [15, Definition 8.10.] which is a consequence of [15, Theorem 8.11.] (it use projectivity of $X$).

As a result, we use Theorem 5.5, Proposition 5.13 and Theorem 6.28 to obtain the following.

**Proposition 7.4.** The complex analytic space $(X, \mathcal{O}_X)$ is smoothable, i.e. there exist $k$th-order thickening $(kX, k\mathcal{O})$ over $kS^1$ locally modeled on $kV_\alpha$'s (which is log smooth), and is compatible for each $k$'s. Furthermore, there is a structure of log Frobenius manifold on the formal extended moduli $\hat{S}^1_T$ of complex structure near $(X, \mathcal{O}_X)$.

8. Abstract deformation data from Gross-Siebert program

We illustrate how to extract the abstract deformation data needed for §8 from [20, 21].

**Notations 8.1.** We should always take the algebraically closed field $k$ appearing in [20] to be $\mathbb{C}$. We begin with taking an $d$-dimensional integral affine manifold $B$ with holonomy in $\mathbb{Z}^d \times SL_d(\mathbb{Z})$, and codimension 2 singularities $\Delta$ as in [20, Definition 1.15], with a toric polyhedral decomposition $\mathcal{P}$ of $B$ into lattice polytopes as in [20, Definition 1.22].

For simplicity, we take $Q = \mathbb{N}$ following [20]. We should also fix a lifted gluing data $s$ as in Definition [20, Definition 5.1] for the pair $(B, \mathcal{P})$.

**Assumption 8.2.** We assume $(B, \mathcal{P})$ is positive and simple, together with the condition appearing in [21] Theorem 3.21] that convex hull of the monodromy polytopes are standard simplex [21]. We further assume $H^1(B, \mathcal{C}) = H^2(B, \mathcal{C}) = 0$ for applying Serre’s GAGA [15] for projective varieties.

**Definition 8.3.** Given a data $(B, \mathcal{P}, s)$, we should take $(X, \mathcal{O}_X)$ to be the $d$-dimensional complex analytic space constructed from taking the analytifications of the log scheme $X_0(B, \mathcal{P}, s)$ constructed in [20, Theorem 5.2.]. It is further equipped with a log structure over the $Q$-Log point $\mathbf{0}^*_{S^1}$. We denote the log-space by $X^\dagger$ if we want to emphasize the log-structure. We let $Z \subset X$ to be the codimension 2 singular locus of the log-structure (i.e. $X^\dagger$ is log-smooth off $Z$), and write $j : X \setminus Z \to X$ as in [21]. We notice that $X_0(B, \mathcal{P}, s)$ is projective.

8.1. Construction of $0th$-order deformation data. Parallel to Definition 2.9, we have the following Definition 8.4.

**Definition 8.4.** We define (We follow the notation from [20] here.)

1. the sheaf of BV algebra $0G^* = j_*(\Lambda^{\bullet} \Theta_{X/\partial S^1})$ to be the push forward of the analytic sheaf of relative polyvector fields, with the natural wedge product;
2. the sheaf of filtered de Rham module to be the push forward of the analytic sheaf of de Rham differential $0K^* := j_*(\Omega^*_{X/\mathcal{C}})$, equipped with the de Rham differential $0\partial = \partial$ as in [21, §3.2.]. It is naturally a dg module over $0\Omega^*_{S^1}$ (introduced in Notation 1.6) inducing the filtration as in Definition 2.9;
3. the volume element $0\omega$ via the trivialization $j_*(\Omega^d_{X/\partial S^1}) \cong \mathcal{O}_X$ by [21, Theorem 3.23];
4. the morphism of sheaf of BV module $0\sigma^{-1} : 0\Omega^*_0 \otimes \mathcal{C}(0\mathcal{K}^*/[0\mathcal{K}^*[-r]]) \to 0\mathcal{K}^*/[0\mathcal{K}^*]$ via taking wedge product in $j_*(\Omega^*_{X/\mathcal{C}})$.

We define the BV operator by $0\Delta(\varphi) \cdot 0\omega := 0\partial(\varphi \cdot 0\omega)$.

We need this assumption to get the Hodge-de Rham degeneracy using results from [21].
In order for the above Definition 8.4 we need to show that $^0G^*$ and $^0K^*$ are coherent, and morphism $^0\sigma$ is an isomorphism and hence we have the identification $\tau^0K^* = \tau^0K^*/\tau^0K^* = j_*(\Omega^*_{X^1/\mathcal{O}S^1})$ using the notation in Definition 2.9. We briefly indicate how to obtain statements from [21].

**Notations 8.5.** Following [21] Construction 2.1., we take monoid $\mathcal{Q}, \mathcal{P}$ and the corresponding toric varieties $\mathcal{V} = \text{Spec}(\mathcal{C}[\mathcal{P}])$ and $\mathcal{V} = \text{Spec}(\mathcal{C}[\mathcal{Q}])$, with the corresponding analytic space $\mathcal{V} = (\mathcal{V})^\text{an}$ and $\mathcal{V} = (\mathcal{V})^\text{an}$ respectively. $\mathcal{V}$ is equipped with the divisorial log-structure induced from toric divisor $\mathcal{V}$, and $\mathcal{V}$ is equipped with the pull back of the log-structure from $\mathcal{V}$. [21] Theorem 2.6.] shows that for every geometric points $\tilde{x} \in X_0(B, \mathcal{P}, s)$, there is an étale neighborhood $W$ of $\tilde{x}$ which can be identified with an étale neighborhood of $\mathcal{V}$ as a log scheme in the sense that

Taking analyticity we take $V \subset X$ to be homeomorphic image of an open subset $W \subset \mathcal{W}$.  

To check the above statements we work on locally $\mathcal{V}$. As in the [21] Proof of Proposition 1.12., we let $\tilde{\mathcal{V}}$ to be the log-scheme equipped with the smooth divisorial log-structure coming from boundary toric divisor $\text{Spec}(\mathcal{C}[\partial \mathcal{P}])$, and therefore we have $j_*\Omega^\text{alg,}*_{\mathcal{V}/\mathcal{C}} \hookrightarrow j_*(\Omega^\text{alg,}*_{\mathcal{V}/\mathcal{C}})\mid_{V \cap Z} = j_*(\Omega^\text{alg,}*_{\mathcal{V}/\mathcal{C}})$ (where the notation $\Omega^\text{alg,}*_{\mathcal{V}/\mathcal{C}}$ refers to algebraic sheaves). From these we obtain the identification $[j_*\Omega^\text{alg,}*_{\mathcal{V}/\mathcal{C}}]^\text{an} = j_*\Omega^\text{alg,}*_{\mathcal{V}/\mathcal{C}}$ which further globalize to give $[j_*\Omega^\text{alg,}*_{X_0(B, \mathcal{P}, s)/\mathcal{C}}]^\text{an} = j_*\Omega^\text{alg,}*_{X_0(B, \mathcal{P}, s)/\mathcal{C}}$ and similarly $[j_*\Omega^\text{alg,}*_{X_0(B, \mathcal{P}, s)/\mathcal{C}}]^\text{an} = j_*\Omega^\text{alg,}*_{X_0(B, \mathcal{P}, s)/\mathcal{C}}$.  

From the above we notice $j_*\Omega^\text{alg,}*_{X_0(B, \mathcal{P}, s)/\mathcal{C}}$ and $j_*\Omega^\text{alg,}*_{X_0(B, \mathcal{P}, s)/\mathcal{C}}$ are coherent sheaves, implying $^0G^*$ and $^0K^*$ being coherent sheaves through the analytification functor. Taking analytification of the exact sequence

$$0 \to ^0\Omega^1_{S^1} \otimes \mathcal{C} j_*\Omega^\text{alg,}*_{X_0(B, \mathcal{P}, s)/\mathcal{C}}[-1] \to j_*\Omega^\text{alg,}*_{X_0(B, \mathcal{P}, s)/\mathcal{C}} \to j_*\Omega^\text{alg,}*_{X_0(B, \mathcal{P}, s)/\mathcal{C}} \to 0$$

in [21] 4-th line in proof of Theorem 5.1., we obtain the identification $^0K^* = \tau^0K^*/\tau^0K^* = j_*(\Omega^*_{X^1/\mathcal{O}S^1})$.

### 8.2. Higher order deformation data.

From Notation 8.5 we notice that at every point $\tilde{x} \in X$, there is an analytic neighborhood $V$ together with a log space $V^\dagger$ and a log morphism $\pi : V^\dagger \to S^\dagger$, obtain from analytification of the as in Notation 8.5 such that the diagram

$$\begin{array}{ccc}
V & \xrightarrow{\pi} & V^\dagger \\
\downarrow & & \downarrow \\
0 \times S^\dagger & \xrightarrow{j} & V^\dagger
\end{array}$$

is a fiber product of log space. We fix a set of open covering $\mathcal{V}$ by Stein open subset $V_\alpha$'s with local thickening $V_\alpha^\dagger$'s given by as above, and write $^kV_\alpha^{\dagger}$ for the $k$th-order thickening over $^kS^\dagger$. We define higher order thickening data parallel to Definition 2.13 and 2.17. We abuse the notation and write $j : V_\alpha \setminus Z \to V_\alpha$.

**Definition 8.6.** For each $k \in \mathbb{Z}_{\geq 0}$, we define

1. the sheaf of BV algebra $^kG^*_\alpha = j_*(\Lambda^{-\sigma} \Theta^kV_\alpha^{\dagger})$ to be the polyvector fields on $^kV_\alpha^{\dagger}$;
(2) sheaf of filtered de Rham module on $V_\alpha$ to be given by $^k\mathcal{K}_\alpha := j^*(\Omega^*_{(V_\alpha^!/(\mathbb{C})})^{\mathbb{Z}})$ equipped with the de Rham differential $^k\partial_\alpha = \partial$ which is naturally a dg module over $^k\Omega^*_{S^\dagger}$;
(3) a local lifting $\omega_\alpha \in j^*(\Omega^d_{V_\alpha^!/(\mathbb{C})})$ of the volume element $^0\omega$, and we let $^k\omega_\alpha = \omega_\alpha \pmod{m^{k+1}}$;
(4) a morphism of BV module $^k\tau_\sigma^{-1} : ^k\Omega^r_{S^\dagger} \otimes (^kR) ^k\mathcal{K}_\alpha / ^k\mathcal{K}_\alpha[-r] \to ^k\mathcal{K}_\alpha / ^r\mathcal{K}_\alpha$ given by taking wedge product;
(5) the BV operator defined by $^k\Delta_\alpha(\varphi) = ^k\partial_\alpha(\varphi \cdot ^k\omega)$;
(6) for both $^k\mathcal{K}_\alpha$’s and $^k\mathcal{G}_\alpha$’s, the natural restriction map $^k\mathcal{K}_\alpha$ is given by the isomorphism of space as $^k\mathcal{V}_\alpha^! = ^{k+1}\mathcal{V}_\alpha^! \times _{(k+1)S^\dagger} kS^\dagger$.

Similar to the $0^{th}$-order case, we need to show that $^k\mathcal{G}_\alpha$’s and $^k\mathcal{K}_\alpha$ are coherent sheaves which are free over $^kR$, and $^k\tau_\sigma$ being an isomorphism which induced an identification $^k\mathcal{K}_\alpha = j^*(\Omega^*_{(V_\alpha^!/(\mathbb{C})}))$. These checking can be done using [21 Proposition 1.12. and Corollary 1.13.], with similar argument as above [8.1]

8.2.1. Higher order patching data. To obtain the patching data we need to take suitable analytification of statement from [21]. Given $\bar{x} \in V_{\alpha\beta}$, we consider the following diagram of étale neighborhood

$$
\begin{array}{ccc}
W_\alpha \times X_0 & W_\beta \\
\downarrow \pi_\alpha & \downarrow \pi_\beta \\
W_\alpha & W_\beta \\
\end{array}
$$

where $X_0 = X_0(B, \mathcal{P}, s)$, and $^kV_\alpha$ (resp. $^kV_\beta$) are $k^{th}$-order neighborhood of $V_\alpha$ (resp. $V_\beta$). Using [21 Lemma 2.15.] by further passing to étale cover $W_{\alpha\beta}$ of $W_\alpha \times X_0 W_\beta$, we have an isomorphism

$$
^k\Xi_{\alpha\beta,i} : W_{\alpha\beta} \times V_\alpha \to W_{\alpha\beta} \times V_\beta
$$

Taking analytification we can take $U_i \ni \bar{x}$ to be homeomorphic image of some open subset $U_i \subset W_{\alpha\beta}$.

**Definition 8.7.** We define $^k\Psi_{\alpha\beta,i}$ induced from restriction of the analytification of $^k\Xi_{\alpha\beta,i}$ on $U_i$ as

$$
\begin{array}{ccc}
^k\mathcal{V}_\alpha^! |_{U_i} & ^k\Psi_{\alpha\beta,i} & ^k\mathcal{V}_\beta^! |_{U_i} \\
\downarrow \pi_\alpha & \downarrow \pi_\beta & \\
^kS^\dagger & k \mathcal{S}^\dagger & \\
\end{array}
$$

We take the isomorphism of sheaves $^k\psi_{\alpha\beta,i} : j^*(\land^* \Theta(\gamma_{\mathcal{V}_\alpha^!/(k\mathcal{S}^\dagger)})(U_i) \to j^*(\land^* \Theta(\gamma_{\mathcal{V}_\beta^!/(k\mathcal{S}^\dagger)})(U_i)$ appearing in Definition 2.15 to be that induced by $^k\Psi_{\alpha\beta,i}$, and similar for $^k\psi_{\alpha\beta,i}$ in Definition 2.18.

The existence of the vector field $^{k}\nu_{\alpha\beta,ij}$, $^{k}\nu_{\alpha\beta,ij}$ and $^{k}\omega_{\alpha\beta,\gamma,i}$ in Definition 2.15 follows from the fact that any log automorphism of the space $^k\mathcal{V}_\alpha^! |_{U_i}$ (or $^k\mathcal{V}_\alpha^! |_{U_{ij}}$) fixing $X|_{U_i}$ (or $X|_{U_{ij}}$ resp.) is obtain by exponentiating the action of a vector field in $\Theta(\gamma_{\mathcal{V}_\alpha^!/(k\mathcal{S}^\dagger)}(U_i)$ (or $\Theta(\gamma_{\mathcal{V}_\alpha^!/(k\mathcal{S}^\dagger)}(U_{ij})$ resp.) again by using analytic version of [21 Theorem 2.11.]. The element $^k\omega_{\alpha\beta,ij}$ in Definition 2.18 indeed measure the different between the two volume elements $^k\psi_{\alpha\beta,i}(^k\omega_{\beta}) = \exp(^k\omega_{\alpha\beta,\gamma,i})^k\omega_{\alpha}$.

---

22Notice that it is the space of log de Rham differential.
8.2.2. Criteria for freeness for Hodge bundle. For Assumption 4.15, which is needed for proving the freeness of the Hodge bundle in 3.1.2, we notice that by taking $Q = \mathbb{N}$ we are already in the situation of a 1-parameter family. The holomorphic Poincaré Lemma in Assumption 4.15 is proven by taking the analytification of the results from 21, Proof of Theorem 4.1.]

8.3. Hodge theoretic data. With the Hodge-filtration as in Definition 2.10, the Hodge-de Rham degeneracy assumption 5.4 required for proving unobstructedness using our Theorem 5.5 is given by applying Serre’s GAGA to 21, Theorem 3.26] using the same argument as in proof of Grothendieck’s algebraic de Rham theorem, once we have the relation $j_{s}(\Omega^{alg,∗}_{X_{0}/(\mathbb{P},\mathbb{S})}/\mathbb{S}) = j_{s}(\Omega^{∗}_{X_{1}/\mathbb{S}}/\mathbb{S})$.

Applying our Theorem 5.5 and Proposition 5.13, we give alternative proof the following unobstructedness result in 22, under Assumption 8.2, in complex analytic setting.

**Proposition 8.8.** The complex analytic space $(X,\mathcal{O}_{X})$ is smoothable, i.e., there exist $k$th-order thickening $(kX, k\mathcal{O})$ over $k\mathbb{S}^\dagger$ locally modeled on $k\mathbb{V}_{\alpha}$'s, and is compatible for each $k$’s.

8.4. F-manifold structure near LCSL.

8.4.1. The universal monoid $Q$. We show how to apply our Theorem 6.28 to the Gross-Siebert program, and we should only indicate the necessary changes needed. First of all, we consider $(\mathcal{B}, \mathcal{P})$ as in Notation 8.1 work in the cone picture as in 22 instead of fan picture as earlier in this 8, and use the notion of integral piecewise affine function on $\mathcal{B}$ as described before 22, Remark 1.15. We let $\text{MPA}(\mathcal{B}, \mathbb{Z}_{+})$ to be the monoid of convex integral piecewise affine function on $\mathcal{B}$, take $Q = \text{Hom}(\text{MPA}(\mathcal{B}, \mathbb{Z}_{+}), \mathbb{Z}_{+})$ to be universal monoid and consider the universal strictly convex integral piecewise affine function $\varphi : \mathcal{B} \to Q$ as in 19, equation A.2 (it is denoted by $\tilde{\varphi}$ there). Since we work in the cone picture, we fix an open gluing data $s$ as in 22, Definition 1.18, and replace monodromy polytopes in Assumption 8.2 by dual monodromy polytopes associated to each $\tau \in \mathcal{P}$.

8.4.2. Construction of $X^\dagger = X_{0}(\mathcal{B}, \mathcal{P}, s, \varphi)^\dagger$. We take an element $n \in \text{int}_{re}(Q^{∨}_{\mathbb{R}}) \cap K^{∨}$ and define a strictly convex piecewise affine function $\varphi_{n} : \mathcal{B} \to \mathbb{R}$. The cone picture construction described in 22, Construction 1.17 gives a log scheme $X_{0}^\dagger = X_{0}(\mathcal{B}, \mathcal{P}, s, \varphi_{n})^\dagger$ over $\mathbb{C}^{\dagger}$ (here $\mathbb{C}^{\dagger}$ is the standard $\mathbb{Z}_{+}$-log point) which is log smooth away from a codimension 2 locus $i : Z \to X$. 19, Construction A.6] will give a log scheme $X^\dagger$ (with same underlying scheme as $X_{0}^\dagger$) over $\mathbb{S}^{\dagger}$ of $\mathbb{V}_{\alpha}$'s. Definition 8.4 can be carried through.

8.4.3. Local model on thickening of $X^\dagger$. For each $\tau \in \mathcal{P}$ we let $\mathcal{Q}_{\tau}$ be the normal lattice as defined in 20, Definition 1.33, we write $\Sigma_{\tau}$ to be the normal fan of $\tau$ defined in 20, Definition 1.35, on $\mathcal{Q}_{\tau,\mathbb{R}}$, with the strictly convex piecewise linear function $\varphi : |\Sigma_{\tau}| = \mathcal{Q}_{\tau,\mathbb{R}} \to Q^{gp}_{\mathbb{R}}$ induced by $\varphi$. We let $\Delta_{1}, \ldots, \Delta_{r}$ be the dual monodromy polytopes associated to $\tau$ defined in 20, Definition 1.58, 1.60, and let $\psi_{i}(m) := -\inf\{\langle m, n \rangle | m \in \mathcal{Q}_{\tau,\mathbb{R}}, n \in \Delta_{i}\}$ be the integral piecewise linear function on $\mathcal{Q}_{\tau,\mathbb{R}}$.

We define monoid $P_{\tau}$ and $\mathcal{Q}_{\tau}$ given by

\[ P_{\tau} = \{(m, a_{0}, a_{1}, \ldots, a_{r}) | m \in \mathcal{Q}_{\tau}, a_{0} \in Q^{gp}, a_{i} \in \mathbb{Z}, a_{0} - \varphi(m) \in Q, a_{i} - \psi_{i}(m) \geq 0 \text{ for } 0 \leq i \leq r\}, \]

\[ \mathcal{Q}_{\tau} = \{(m, a_{0}, a_{1}, \ldots, a_{r}) | m \in \mathcal{Q}_{\tau}, a_{0} \in Q^{gp}, a_{i} \in \mathbb{Z}, a_{0} = \varphi(m)\} \cup \{\infty\}, \]

where the monoid structure on $\mathcal{Q}_{\tau}$ is given as in 21, p. 22 in Construction 2.1.]. We also let $\nabla_{\tau} = \text{Spec}(\mathbb{C}[P_{\tau}])$ with a natural family $\pi : \nabla_{\tau} \to \text{Spec}(\mathbb{C}[\mathcal{Q}_{\tau}]) = S^{\dagger}$, $\mathcal{V}_{\tau} = \pi^{-1}(0) = \text{Spec}(\mathbb{C}[\mathcal{Q}_{\tau}])$ and $k\mathbb{V}_{\tau} = \pi^{-1}(kS^{\dagger})$ be $k$-th order thickening of $\mathcal{V}_{\tau}$ in $\nabla_{\tau}$.

For $i = 1, \ldots, r$ and a vertex $v \in \Delta_{i}$, we define a submonoid $D_{i,v} := w_{i,v}^{\dagger} \cap P_{\tau}$, where $w_{i,v} = v + e_{i}^{∨}$, and let $D_{i,v}$ be the corresponding toric divisor of $\nabla_{\tau}$. We rewrite $\{w_{1}, \ldots, w_{r}\}$ to be the collection of
$w_i, v_i$'s, with corresponding $D_j$ and $D_j^\tau$. We let $v_1, \ldots, v_l$ to be the generators of 1-dimensional cones in the dual cone $P_\tau^\vee$ other than $w_j$'s, with corresponding toric divisor $V_j$'s. Writing $V = \bigcup_j V_j$ we equipped $V_\tau$ with the divisorsial log structure induced by divisor $V \hookrightarrow \mathbb{V}_\tau$, denoted by $V_\tau^\dagger$. We pull back the log structure from $\mathbb{V}_\tau$ to give the log schemes $V_\tau^\dagger$ and $kV_\tau^\dagger$. [21 Theorem 2.6.1] holds for this setting as described in Notation 8.5 by taking $P = P_\tau$ and $Q = Q_\tau$ for some $\tau \in P$.

Taking analytification as in 8.2 of the log schemes $V_\tau^\dagger$ and $kV_\tau^\dagger$ we obtain log analytic schemes $V_\tau^\dagger$ and $kV_\tau^\dagger$ respectively, and Definition 8.6 can be carried through. We show that $^{k}\mathcal{G}_\alpha$'s and $^{k}\mathcal{K}_\alpha$ are coherent sheaves which are also sheaves of free module over $^{k}R$, and $^{k}\sigma^{-1}$ being an isomorphism by using the following variant of [21 Proposition 1.12. and Corollary 1.13.]

**Proposition 8.9.** Let $Z := V_\tau \cap D_{\text{sing}} \hookrightarrow |^{k}V_\tau| = |V_\tau|$ be the inclusion, we have the following decomposition into $P_\tau$-homogeneous pieces as

$$
\Gamma(V_\tau \setminus Z, \Omega^{r}_{(V_\tau^\dagger)} = \bigoplus_{p \in P_\tau \setminus P_\tau+kQ^+} z^p \wedge \left( \bigcap_{\{j \mid p \in P_\tau \}} D_{j}^{gp} \otimes_{Z} \mathbb{C} \right),
$$

$$
\Gamma(V_\tau \setminus Z, \Omega^{r}_{(kV_\tau^\dagger/\mathcal{S}^\dagger)} = \bigoplus_{p \in P_\tau \setminus P_\tau+kQ^+} z^p \wedge \left( \bigcap_{\{j \mid p \in P_\tau \}} (D_{j}^{gp}/Q^{gp}) \otimes_{Z} \mathbb{C} \right).
$$

The construction of higher order patching data described in 8.2.1 can be carried through, because the divisorial deformation over $^0\mathcal{S}^\dagger$ can be defined as in [21 Definition 2.7.], and [21 Theorem 2.11. and Lemma 2.15.] holds accordingly with the local models $V_\tau$'s.

8.4.4. **Hodge theoretic data.** The weighted filtration in Assumption 6.12 is taken to be the filtration described in [21 Remark 5.7.], which is opposite to the Hodge-filtration and preserved by $N_\nu$'s. The trace map $\text{tr}$ in Assumption 6.17 will be defined via the isomorphism $\text{tr} : \mathcal{H}^d(X, j_*(\Omega^{r}_{X(0,S^\dagger)})) \cong \mathcal{H}^d(X, j_*(\Omega^{r}_{X(0,S^\dagger)})) \cong \mathbb{C}$. We conjecture that the induced pairing $^0\mathcal{P}$ is non-degenerated.

**Proposition 8.10.** There is a structure of log $F$-manifold on the formal extended moduli $\hat{S}_t^\dagger$ of complex structure near $(X, \mathcal{O}_X)$. If the pairing $^0\mathcal{P}$ is non-degenerated, it can be further equipped with a pairing to give a Frobenius manifold structure on $\hat{S}_t^\dagger$.

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