ESTIMATES OF THE GAPS BETWEEN CONSECUTIVE EIGENVALUES OF LAPLACIAN

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Abstract. For the eigenvalue problem of the Dirichlet Laplacian on the bounded domain in Euclidean space $\mathbb{R}^n$, we obtain the estimates for the upper bounds of the gap of consecutive eigenvalues which are the best possible in the meaning of the orders of eigenvalues. Therefore, it is reasonable to conjecture that this type estimates also hold for the eigenvalue problem on Riemannian manifolds. In particular, we give some examples in this paper.

1. Introduction

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M with boundary (possible empty). Then the Dirichlet eigenvalue problem of Laplacian on Ω is given by

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is Laplaican on M. It is well known that the spectrum of (1.1) has the real and purely discrete eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty, \tag{1.2}$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity. The corresponding orthonormal basis of real eigenfunctions will be denoted $\{u_j\}_{j=1}^{\infty}$. We go forward under the assumption that $L^2(\Omega)$ represents the real Hilbert space of real-valued L^2 functions on Ω. We put $\lambda_0 = 0$ if $\partial\Omega = \emptyset$.

An important aspect of estimating higher eigenvalues is to obtain as precise as possible the estimate of gaps of consecutive eigenvalues of (1.1). In this regard, we will review some important results on the estimates of eigenvalue problem (1.1).

For the upper bound of the gap of consecutive eigenvalues of (1.1), when Ω is a bounded domain in an 2-dimensional Euclidean space $\mathbb{R}^2$, in 1956, Payne, Pólya and Weinberger (cf.[28] and [29]) proved

$$\lambda_{k+1} - \lambda_k \leq \frac{2}{k}\sum_{i=1}^{k}\lambda_i. \tag{1.3}$$

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Thompson [33], in 1969, extended (1.3) to n-dimensional case and obtained

$$\lambda_{k+1}-\lambda_k \leq \frac{4}{nk}\sum_{i=1}^{k}\lambda_i. \tag{1.4}$$

Hile and Protter [23] improved (1.4) to

$$\sum_{i=1}^{k}\frac{\lambda_i}{\lambda_{k+1}-\lambda_i} \geq \frac{nk}{4}. \tag{1.5}$$

Yang (cf. [37] and more recently [16]) has obtained a sharp inequality

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_{k+1}-\left(1+\frac{4}{n}\right)\lambda_i\right) \leq 0. \tag{1.6}$$

From (1.6), one can infer

$$\lambda_{k+1} \leq \frac{1}{k}\left(1+\frac{4}{n}\right)\sum_{i=1}^{k}\lambda_i. \tag{1.7}$$

The inequalities (1.6) and (1.7) are called Yang's first inequality and second inequality, respectively (cf. [2, 3, 10, 21]). Also we note that Ashbaugh and Benguria gave an optimal estimate for $k = 1$ (cf. [5, 6, 7]). From the Chebyshev's inequality, it is easy to prove the following relations

$$(1.6) \Longrightarrow (1.7) \Longrightarrow (1.5) \Longrightarrow (1.4).$$

From (1.6), Cheng and Yang [14] obtained

$$\lambda_{k+1}-\lambda_k \leq 2\left[\left(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2-\left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\left(\lambda_i-\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)^2\right]^{\frac{1}{2}}. \tag{1.8}$$

Cheng and Yang [16], using their recursive formula, obtained

$$\lambda_{k+1} \leq C_0(n)k^{\frac{2}{n}}\lambda_1, \tag{1.9}$$

where $C_0(n) \leq 1+\frac{4}{n}$ is a constant (see Cheng and Yang's paper [16]). From the Weyl's asymptotic formula (cf. [35]), we know that the upper bound (1.9) of Cheng and Yang is best possible in the meaning of the order on k.

For a complete Riemannian manifold M, from the Nash's theorem [27], there exists an isometric immersion

$$\psi : M \longrightarrow \mathbb{R}^N,$$

where $\mathbb{R}^N$ is a Euclidean space. The mean curvature of the immersion ψ is denoted by H and $|H|$ denotes its norm. Define

$$\Phi = \{\psi \mid \psi \text{ is an isometric immersion from } M \text{ into a Euclidean space}\}.$$

When Ω is a bounded domain of a complete Riemannian manifold M, isometrically immersed into a Euclidean space $\mathbb{R}^N$, the first author and Cheng [12] (cf. [18, 22])obtained

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i+\frac{n^2}{4}H_0^2\right), \tag{1.10}$$

where

$$H_0^2 = \inf_{\psi\in\Phi}\sup_{\Omega}|H|^2. \tag{1.11}$$

Using the recursive formula in Cheng and Yang [16], the first author and Cheng in [12] also deduced

$$\lambda_{k+1} + \frac{n^2}{4}H_0^2 \leq C_0(n)k^{\frac{2}{n}}\left(\lambda_1 + \frac{n^2}{4}H_0^2\right), \tag{1.12}$$

where $H_0^2, C_0(n)$ are given by (1.11) and (1.9) respectively.

From (1.10), we can get the gaps of the consecutive eigenvalues of Laplacian

$$\lambda_{k+1} - \lambda_k \leq 2\left(\left(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{n}{2}H_0^2\right)^2 - \left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\left(\lambda_i - \frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)^2\right)^{\frac{1}{2}}. \tag{1.13}$$

Remark 1.1. When Ω is an n-dimensional compact homogeneous Riemannian manifold, a compact minimal submanifold without boundary and a connected bounded domain in the standard unit sphere $\mathbb{S}^N(1)$, and a connected bounded domain and a compact complex hypersurface without boundary of the complex projective space $\mathbb{CP}^n(4)$ with holomorphic sectional curvature 4, many mathematicians have sutudied the universal inequalities for eigenvalues and the difference of the consecutive eigenvalues (cf. [14, 15, 17, 19, 20, 21, 26, 38, 25, 32, 13]).

Remark 1.2. Another problem is the lower bound of the gap of the first two eigenvalues. In general, there exists the famous fundamental gap conjecture for the Dirichlet eigenvalue problem of the Schrödinger operator(cf. [4, 34, 31, 39, 40] and the references therein). The fundamental gap conjecture was solved by Andrews and Clutterbuck in [1] .

From (1.8) and (1.13), it is not difficult to see that both Yang's estimate for the gap of consecutive eigenvalues of (1.1) implicited in [37] and the estimate from [12] are on the order of $k^{\frac{3}{2n}}$. However, by the calculation of the gap of the consecutive eigenvalues of $\mathbb{S}^n$ with standard metric and the Weyl's asymptotic formula, the order of the upper bound of this gap is $k^{\frac{1}{n}}$. Therefore, we make a conjecture that

Conjecture 1.1. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then for the Dirichlet problem (1.1), the upper bound for the gap of consecutive eigenvalues of Laplacian should be*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega}k^{\frac{1}{n}}, \quad k > 1, \tag{1.14}$$

where $C_{n,\Omega}$ is a constant dependent on Ω itself and the dimension n.

Remark 1.3. The famous Panye-Pólya-Weinberger conjecture (cf.[28, 29, 33, 8, 9]) states that, when $M = \mathbb{R}^n$, for Dirichlet eigenvalue problem (1.1), one should have

$$\frac{\lambda_{k+1}}{\lambda_k} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{\mathbb{B}^n} = \left(\frac{j_{n/2,1}}{j_{n/2-1,1}}\right)^2, \tag{1.15}$$

where $\mathbb{B}^n$ is the n-dimensional unit ball in $\mathbb{R}^n$, and $j_{p,k}$ is the k^{th} positive zero of the Bessel function $J_p(t)$. From the Weyl's asymptotic formula and (1.15), the order of the upper bound of the consecutive eigenvalues of eigenvalue problem (1.1) is $k^{\frac{2}{n}}$. Therefore, Conjecture 1.1 reflects the distribution of eigenvalues from another point of view. From the order of the upper bound of the gap of the consecutive eigenvalues of $\mathbb{S}^n$, the estimate in (1.14) is best possible in the meaning of the order on k.

In the following, the constants $C_{n,\Omega}$ are allowed to be different in different cases.

When Ω is a bounded domain in $\mathbb{R}^n$, for the Dirichlet eigenvalue problem (1.1), we give the affirmative answer to Conjecture 1.1.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in Euclidean space $\mathbb{R}^n$ and λ_k be the k^{th} $(k > 1)$ eigenvalue of the Dirichlet eigenvalue problem (1.1). Then we have*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega}\, k^{\frac{1}{n}}, \tag{1.16}$$

where $C_{n,\Omega} = 4\lambda_1\sqrt{\frac{C_0(n)}{n}}$, $C_0(n)$ is given by (1.9).

It is reasonable to to conjecture that this type estimate also holds on Riemannian manifold. In particular, we give some examples as follows.

Corollary 1.2. *Let $\Omega \subset \mathbb{H}^n(-1)$ be a bounded domain in hyperbolic space $\mathbb{H}^n(-1)$, and λ_k be the k^{th} $(k > 1)$ eigenvalue of the Dirichlet eigenvalue problem (1.1). Then we have*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{\frac{1}{n}}, \tag{1.17}$$

where $C_{n,\Omega}$ depends on Ω and the dimension n, given by

$$C_{n,\Omega} = 4\left[C_0(n)\left(\lambda_1 - \frac{(n-1)^2}{4}\right)\left(\lambda_1 + \frac{n^2}{4}H_0^2\right)\right]^{\frac{1}{2}}, \tag{1.18}$$

$C_0(n)$ and H_0^2 are the same as the ones in (1.12).

Moreover, by the comparison theorem for the distance function in Riemannian manifold, we have

Corollary 1.3. *Let M be an n-dimensional $(n \geq 3)$ simply connected complete noncompact Riemannian manifold with sectional curvature Sec satisfying*

$$-a^2 \leq Sec \leq -b^2,$$

where $0 \leq b \leq a$ are constants. Let $\Omega \subset M$ be a bounded domain of M and λ_k be the k^{th} $(k > 1)$ eigenvalue of (1.1). Then we have

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega}\, k^{\frac{1}{n}} \tag{1.19}$$

where $C_{n,\Omega}$ depends on Ω and the dimension n, given by

$$C_{n,\Omega} = 4\left[C_0(n)\left(\lambda_1 - \frac{(n-1)^2}{4}b^2 + \frac{a^2 - b^2}{4}\right)\left(\lambda_1 + \frac{n^2}{4}H_0^2\right)\right]^{\frac{1}{2}}, \tag{1.20}$$

$C_0(n)$ and H_0^2 are the same as the ones in (1.12).

2. Preliminaries

In this section, we will firstly recall some basic concepts and a theorem of Chapter 10 in [24] and then prove a theorem which will be used in next section.

Define

$$\mathcal{H}^\infty = \left\{ x = (x_j)_{j=1}^\infty \,\middle|\, x_j \in \mathbb{R},\ \left(\sum_{j=1}^\infty x_j^2\right)^{\frac{1}{2}} < +\infty \right\}$$

and

$$\mathcal{H}^2 = \left\{ x = (x_1, x_2) \,\middle|\, x_1, x_2 \in \mathbb{R},\ (x_1^2 + x_2^2)^{\frac{1}{2}} < +\infty \right\}.$$

The inner product $\langle \cdot, \cdot \rangle_\infty$ on $\mathcal{H}^\infty$ is defined by

$$\langle x, y \rangle_\infty = \sum_{j=1}^{\infty} x_j y_j, \quad \forall\, x = (x_j)_{j=1}^{\infty}, \quad y = (y_j)_{j=1}^{\infty}.$$

The inner product $\langle \cdot, \cdot \rangle_2$ on $\mathcal{H}^2$ can be defined in the similar way. Obviously, both $\mathcal{H}^\infty$ and $\mathcal{H}^2$ are Hilbert spaces. The dual space of $\mathcal{H}^2$ is denoted by $\left(\mathcal{H}^2\right)^*$. It is well known that $\left(\mathcal{H}^2\right)^*$ is isomorphic to $\mathcal{H}^2$ itself.

In order to prove our theorem, we need the following Lagrange multiplier theorem for real Banach spaces(see Page 403 in [24] or Page 270 in [41]).

Theorem 2.1. *Let X and Y be real Banach spaces. Assume that $F : x_0 \in U \subset X \longrightarrow \mathbb{R}$ and $\Phi : x_0 \in U \subset X \longrightarrow Y$ are continuously Fréchet differential on an open neighborhood of x_0, where $x_0 \in \Phi^{-1}(0) = \{x \in U | \Phi(x) = 0 \in Y\}$. If $\{\Phi'(x_0)(x) \in Y | x \in X\}$ is closed and x_0 is an extremum (maxima or minima) of F on $\Phi^{-1}(0)$, then there exists $\lambda_0 \in \mathbb{R}$ and a linear functional $y^* \in Y^*$ with*

$$\lambda_0^2 + \|y^*\|^2 \neq 0,$$

such that

$$\lambda_0 F'(x_0) + \left(\Phi'(x_0)\right)^* (y^*) = 0. \tag{2.1}$$

Moreover, if $\{\Phi'(x_0)(x) \in Y | x \in X\} = Y$, then we can take $\lambda_0 = 1$.

Using Theorem 2.1, we have

Theorem 2.2. *Assume that $\{\mu_j\}_{j=1}^{\infty}$ is an nondecreasing sequence, i.e.,*

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots \nearrow \infty,$$

where each μ_i has finite multiplicity m_i which is repeated according to its multiplicity.

Define

$$B = \sum_{j=1}^{\infty} x_j^2 > 0, \tag{2.2}$$

$$A = \sum_{j=1}^{\infty} \mu_j^2 x_j^2, \quad x = (x_j)_{j=1}^{\infty} \in \mathcal{H}^\infty. \tag{2.3}$$

If $x_{m_1} \neq 0$ and $\sum_{j=1}^{\infty} \mu_j x_j^2 < \sqrt{AB}$, under the conditions in (2.2) and (2.3), we have

$$\sum_{j=1}^{\infty} \mu_j x_j^2 \leq \frac{A + \mu_{m_1}\mu_{m_1+1} B}{\mu_{m_1} + \mu_{m_1+1}}. \tag{2.4}$$

Proof. First, assume that $\{\mu_j\}_{j=1}^{\infty}$ is a strictly increasing sequence, i.e.,

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots \nearrow \infty.$$

Suppose

$$F(x) = \sum_{j=1}^{\infty} \mu_j x_j^2,$$

$$\Psi(x) = \left(\sum_{j=1}^{\infty} x_j^2 - B, \sum_{j=1}^{\infty} \mu_j^2 x_j^2 - A \right) \in \mathcal{H}^2,\, x \in \mathcal{H}^\infty.$$

Let $x_0 = (a_j)_{j=1}^\infty$ be an extremum of $F(x)$ on $\Phi^{-1}(0)$. Since $\forall\, h = (h_j)_{j=1}^\infty \in \mathcal{H}^\infty$,

$$\begin{aligned} F'(x_0)h =& 2\sum_{j=1}^\infty \mu_j x_j h_j, \\ \Psi'(x_0)h =& \left(2\sum_{j=1}^\infty x_j h_j,\ 2\sum_{j=1}^\infty \mu_j^2 x_j h_j \right), \end{aligned}$$

and

$$\Psi'(x_0)(\mathcal{H}^\infty) = \mathcal{H}^2,$$

there exists $y^* \in \left(\mathcal{H}^2\right)^*$ such that

$$F'(x_0)h + \left(\Psi'(x_0)\right)^* (y^*)h = 0. \tag{2.5}$$

Since $\mathcal{H}^2 = (\mathcal{H}^2)^*$, we can use some unique vector $(\mu,\, \lambda) \in \mathcal{H}^2$ to rewrite (2.5) as

$$\sum_{j=1}^\infty \mu_j a_j h_j + \mu \sum_{j=1}^\infty a_j h_j + \lambda \sum_{j=1}^\infty \mu_j^2 a_j h_j = 0. \tag{2.6}$$

Choosing

$$h_j = \delta_{jk},\ j = 1,\, 2, \cdots,$$

from (2.6), we obtain a system of equations

$$\mu_k a_k + \mu a_k + \lambda \mu_k^2 a_k = 0,\ k = 1,\, 2, \cdots. \tag{2.7}$$

Since $\{\mu_k\}$ is a strictly increasing sequence, and there are only two varieties μ and λ, there are only two cases of x_0.

Case 1: There is only one $a_k \neq 0$, whether $k = 1$ or not. In this case, the critical value of $F(x)$ is given by

$$F(x_0) = \sqrt{AB},$$

which contradicts to the assumption of the theorem.

Case 2: There are only two components of x_0, saying a_k and a_ℓ(without loss of generality, set $k < \ell$), are nonzero. In this case, we have

$$A = \mu_k^2 a_k^2 + \mu_\ell^2 a_\ell^2, \tag{2.8}$$

$$B = a_k^2 + a_\ell^2. \tag{2.9}$$

From (2.8) and (2.9), we have

$$F(x_0) = \frac{A + \mu_k \mu_\ell B}{\mu_k + \mu_\ell}.$$

Since

$$A = \mu_k^2 a_k^2 + \mu_\ell^2 a_\ell^2 > \mu_k^2 (a_k^2 + a_\ell^2) = \mu_k^2 B,$$

we can deduce

$$\mu_k < \sqrt{\frac{A}{B}}. \tag{2.10}$$

Similarly, we can also obtain

$$\mu_\ell > \sqrt{\frac{A}{B}}. \tag{2.11}$$

Hence, we have

$$F(x_0) - \sqrt{AB} = \frac{B\left(\mu_k - \sqrt{\frac{A}{B}}\right)\left(\mu_\ell - \sqrt{\frac{A}{B}}\right)}{\mu_k + \mu_\ell} < 0. \tag{2.12}$$

Since the $\{\mu_i\}$ is a strictly increasing sequence, for μ_k fixed, from (2.10) and (2.11), we know that the right side of (2.12) is strictly decreasing of μ_l, i.e.

$$\frac{B\left(\mu_k - \sqrt{\frac{A}{B}}\right)\left(\mu_{k+1} - \sqrt{\frac{A}{B}}\right)}{\mu_k + \mu_{k+1}} > \frac{B\left(\mu_k - \sqrt{\frac{A}{B}}\right)\left(\mu_{k+2} - \sqrt{\frac{A}{B}}\right)}{\mu_k + \mu_{k+2}} > \cdots.$$

Hence, we know that

$$\frac{A + \mu_k \mu_{k+1} B}{\mu_k + \mu_{k+1}}, \ k = 1,\, 2, \cdots$$

are local maximal values of $F(x)$.

Since $x_{m_1} = x_1 \neq 0$, k must be equal to $m_1 = 1$ only. Finally, we have the global maximum of $F(x)$

$$\frac{A + \mu_1 \mu_2 B}{\mu_1 + \mu_2}.$$

Second, assume that $\{\mu_j\}_{j=1}^{\infty}$ is an increasing sequence, i.e.,

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots \nearrow \infty,$$

where each μ_i has finite multiplicity m_i which is repeated according to its multiplicity. Replacing (2.8) and (2.9) by

$$\begin{aligned} A =& \mu_k^2 \sum_{j=k}^{m_i+k-1} a_j^2 + \mu_\ell^2 \sum_{j=\ell}^{m_i+\ell-1} a_j^2, \\ B =& \sum_{j=k}^{m_i+k-1} a_j^2 + \sum_{j=\ell}^{m_i+\ell-1} a_j^2, \end{aligned}$$

and following the above steps almost word for word, we can deduce that the local maximal value of $F(x)$ is

$$\frac{A + \mu_{m_k} \mu_{m_k+1} B}{\mu_{m_k} + \mu_{m_k+1}}$$

and

$$\mu_{m_k} < \sqrt{\frac{A}{B}}, \quad \mu_{m_k+1} > \sqrt{\frac{A}{B}}.$$

Since $x_{m_1} \neq 0$, m_k must be equal to m_1 and the local maximal value of $F(x)$ is the global maximum. Since

$$\frac{A + \mu_{m_1} \mu_{m_1+1} B}{\mu_{m_1} + \mu_{m_1+1}} - \sqrt{AB} = \frac{B\left(\mu_{m_1} - \sqrt{\frac{A}{B}}\right)\left(\mu_{m_k+1} - \sqrt{\frac{A}{B}}\right)}{\mu_{m_1} + \mu_{m_1+1}} < 0,$$

we can obtain (2.4). This completes the proof of the theorem. □

3. Proofs of main results

In this section, we will give the proof of Theorem 1.1. In order to prove our main results, we need the following key lemma and related corollaries from Theorem 2.2.

Lemma 3.1. *For the Dirichlet eigenvalue problem (1.1), let u_k be the orthonormal eigenfunction corresponding to the k^{th} eigenvalue λ_k, i.e.,*

$$\begin{cases} \Delta u_k = -\lambda_k u_k, & in\ \Omega, \\ u_k = 0, & on\ \partial\Omega, \\ \int_\Omega u_i u_j = \delta_{ij}. \end{cases}$$

Then for any complex value function $g \in C^3(\Omega) \cap C^2(\overline{\Omega})$ satisfying that gu_i is not the $\mathbb{C}$-linear combination of

$$u_1, \cdots, u_{k+1},$$

and

$$a_{k+1} = \int_\Omega g u_i u_{k+1} \neq 0,$$

$\lambda_i < \lambda_{k+1} < \lambda_{k+2}$, k, $i \in \mathbb{Z}^+$, $i \geq 1$, we have

$$\Big((\lambda_{k+1} - \lambda_i) + (\lambda_{k+2} - \lambda_i)\Big) \int_\Omega |\nabla g|^2 u_i^2 \leq \int_\Omega \Big|2\nabla g \cdot \nabla u_i + u_i \Delta g\Big|^2 + (\lambda_{k+1} - \lambda_i)(\lambda_{k+2} - \lambda_i) \int_\Omega |g u_i|^2. \tag{3.1}$$

Proof. Define

$$\begin{cases} a_{ij} = \int_\Omega g u_i u_j, \\ b_{ij} = \int_\Omega \left(\nabla u_i \cdot \nabla g + \frac{1}{2} u_i \Delta g\right) u_j, \end{cases}$$

where ∇ denotes the gradient operator. Obviously,

$$a_{ij} = a_{ji}. \tag{3.2}$$

Then, from the Stokes' theorem, we get

$$\begin{aligned} \lambda_j a_{ij} =& \int_\Omega g u_i (-\Delta u_j) \\ =& -\int_\Omega (u_i \Delta g + g \Delta u_i + 2\nabla g \cdot \nabla u_i) u_j \\ =& \lambda_i \int_\Omega g u_i u_j - 2\int_\Omega \left(\nabla u_i \cdot \nabla g + \frac{1}{2} u_i \Delta g\right) u_j, \end{aligned}$$

i.e.,

$$2b_{ij} = (\lambda_i - \lambda_j) a_{ij}. \tag{3.3}$$

Again, the Stokes' theorem implies

$$\int_\Omega |\nabla g|^2 u_i^2 = -2\int_\Omega g u_i \left(\nabla \overline{g} \cdot \nabla u_i + \frac{1}{2} u_i \Delta \overline{g}\right). \tag{3.4}$$

Since $\{u_k\}_{k=1}^{\infty}$ consists of a complete system of orthonormal basis of $L^2(\Omega)$, by the definitions of a_{ij} and b_{ij}, from (3.3), (3.4) and the Paseval identity, we obtain

$$\int_\Omega |gu_i|^2 = \sum_{j=1}^{\infty} |a_{ij}|^2, \tag{3.5}$$

$$\begin{aligned}\int_\Omega |\nabla g|^2 u_i^2 =& 2\sum_{j=1}^{\infty} a_{ij}\overline{b_{ij}} \\ =& \sum_{j=1}^{\infty}(\lambda_j-\lambda_i)|a_{ij}|^2,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\int_\Omega |2\nabla \overline{g}\cdot\nabla u_i + u_i\Delta\overline{g}|^2 =& 4\sum_{j=1}^{\infty} |b_{ij}|^2 \\ =& \sum_{j=1}^{\infty}(\lambda_j-\lambda_i)^2|a_{ij}|^2.\end{aligned} \tag{3.7}$$

From the Cauchy-Schwarz inequality, we have

$$\left(\sum_{j=k+1}^{\infty}(\lambda_j-\lambda_i)|a_{ij}|^2\right)^2 \le \sum_{j=k+1}^{\infty}(\lambda_j-\lambda_i)^2|a_{ij}|^2 \sum_{j=k+1}^{\infty}|a_{ij}|^2. \tag{3.8}$$

Combining (3.5), (3.6), (3.7) and (3.8) gives

$$\begin{aligned}&\left(\int_\Omega |\nabla g|^2 u_i^2 - \sum_{j=1}^{k}(\lambda_j-\lambda_i)|a_{ij}|^2\right)^2 \\ \le& \left(\int_\Omega |gu_i|^2 - \sum_{j=1}^{k}|a_{ij}|^2\right)\left(\int_\Omega |2\nabla g\cdot\nabla u_i + u_i\Delta g|^2 - \sum_{j=1}^{k}(\lambda_j-\lambda_i)^2|a_{ij}|^2\right).\end{aligned} \tag{3.9}$$

Define

$$\begin{cases} \widetilde{B}(i) = \int_\Omega |gu_i|^2 - \sum_{j=1}^{k}|a_{ij}|^2 = \sum_{j=k+1}^{\infty}|a_{ij}|^2 > 0, \text{ since } \int_\Omega gu_iu_{k+1}\neq 0, \\ \widetilde{A}(i) = \int_\Omega |2\nabla g\cdot\nabla u_i + u_i\Delta g|^2 - \sum_{j=1}^{k}(\lambda_j-\lambda_i)^2|a_{ij}|^2 = \sum_{j=k+1}^{\infty}(\lambda_j-\lambda_i)^2|a_{ij}|^2 \ge 0, \\ \widetilde{C}(i) = \int_\Omega |\nabla g|^2u_i^2 - \sum_{j=1}^{k}(\lambda_j-\lambda_i)|a_{ij}|^2 = \sum_{j=k+1}^{\infty}(\lambda_j-\lambda_i)|a_{ij}|^2. \end{cases}$$

Since gu_i is not the $\mathbb{C}$-linear combination of

$$u_1, \cdots, u_{k+1},$$

there exist some $\ell > k+1$ such that

$$a_\ell = \int_\Omega gu_iu_\ell \neq 0.$$

Since

$$\lambda_i < \lambda_{k+1} < \lambda_{k+2} \le \lambda_\ell,$$

the vector

$$(|a_{ij}|)_{j=k+1}^{\infty}$$

is not proportional to

$$\left((\lambda_j - \lambda_i)^2 |a_{ij}|\right)_{j=k+1}^{\infty}.$$

From the Cauchy-Schwarz inequality, we have

$$\widetilde{C}(i) < \sqrt{\widetilde{A}(i)\widetilde{B}(i)} \tag{3.10}$$

Since $a_{k+1} \neq 0$, from (3.10) and Theorem 2.2, we have

$$\widetilde{C}(i) \le \frac{\widetilde{A}(i) + (\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)\widetilde{B}(i)}{(\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)}. \tag{3.11}$$

From (3.11), and the definitions of $\widetilde{A}(i)$, $\widetilde{B}(i)$ and $\widetilde{C}(i)$, we obtian

$$\begin{aligned}
&((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)) \int_\Omega |\nabla g|^2 u_i^2 \\
\le& \int_\Omega |2\nabla g \cdot \nabla u_i + u_i \Delta g|^2 + (\lambda_{k+1} - \lambda_i)(\lambda_{k+2} - \lambda_i) \int_\Omega |g u_i|^2 \\
&- \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j)(\lambda_{k+2} - \lambda_j)|a_{ij}|^2 \\
\le& \int_\Omega |2\nabla g \cdot \nabla u_i + u_i \Delta g|^2 + (\lambda_{k+1} - \lambda_i)(\lambda_{k+2} - \lambda_i) \int_\Omega |g u_i|^2.
\end{aligned} \tag{3.12}$$

This finishes the proof of Lemma 3.1. □

From Lemma 3.1, we have

Corollary 3.2. *Under the assumption of Lemma 3.1, for any non-constant real value function $f \in C^3(\Omega) \cap C^2(\overline{\Omega})$, we have*

$$\begin{aligned}
&((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)) \int_\Omega |\nabla f|^2 u_i^2 \\
\le& 2\sqrt{((\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)) \int_\Omega |\nabla f|^4 u^2} + \int_\Omega (2\nabla f \cdot \nabla u_i + u_i \Delta f)^2.
\end{aligned} \tag{3.13}$$

Proof. Taking $g = \exp(\sqrt{-1}\alpha f)$, $\alpha \in \mathbb{R}\backslash\{0\}$ in (3.1) gives

$$\begin{aligned}
&\alpha^2 ((\lambda_{k+1} - \lambda_i) + (\lambda_{k+2} - \lambda_i)) \int_\Omega |\nabla f|^2 u_i^2 \\
\le& \alpha^4 \int_\Omega |\nabla f|^4 u_i^2 + \alpha^2 \int_\Omega |2\nabla f \cdot \nabla u_i + u_i \Delta f|^2 + (\lambda_{k+1} - \lambda_i)(\lambda_{k+2} - \lambda_i),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&((\lambda_{k+1} - \lambda_i) + (\lambda_{k+2} - \lambda_i)) \int_\Omega |\nabla f|^2 u_i^2 \\
\le& \alpha^2 \int_\Omega |\nabla f|^4 u_i^2 + \frac{1}{\alpha^2}(\lambda_{k+1} - \lambda_i)(\lambda_{k+2} - \lambda_i) + \int_\Omega |2\nabla f \cdot \nabla u_i + u_i \Delta f|^2.
\end{aligned} \tag{3.14}$$

Since the inequality (3.14) is valid for any $\alpha \neq 0$ and

$$(\lambda_{k+1} - \lambda_i)(\lambda_{k+2} - \lambda_i) \neq 0, \quad \int_\Omega |\nabla f|^4 u_i^2 \neq 0,$$

we can choose

$$\alpha^2 = \left(\frac{(\lambda_{k+1} - \lambda_i)(\lambda_{k+2} - \lambda_i)}{\int_\Omega |\nabla f|^4 u_i^2} \right)^{\frac{1}{2}}$$

to get (3.13). □

Corollary 3.3. *Under the assumption of Lemma 3.1, for any real value function* $f \in C^3(\Omega) \cap C^2(\overline{\Omega})$ *with* $|\nabla f|^2 = 1$*, we have*

$$(\lambda_{k+2} - \lambda_{k+1})^2 \leq 16 \left(\int_\Omega (\nabla f \cdot \nabla u_i)^2 - \frac{1}{4} \int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2} \int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2 \right) \lambda_{k+2}. \quad (3.15)$$

Furthermore, we have

$$\lambda_{k+2} - \lambda_{k+1} \leq 4 \left(\lambda_i - \frac{1}{4} \int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2} \int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2 \right)^{\frac{1}{2}} \sqrt{\lambda_{k+2}}. \quad (3.16)$$

Proof. From Corollary 3.2 and $|\nabla f|^2 = 1$, we have

$$((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)) - 2\sqrt{(\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)} \leq \int_\Omega (2\nabla f \cdot \nabla u_i + u_i \Delta f)^2,$$

i.e.,

$$\left(\sqrt{\lambda_{k+2} - \lambda_i} - \sqrt{\lambda_{k+1} - \lambda_i} \right)^2 \leq \int_\Omega (2\nabla f \cdot \nabla u_i + u_i \Delta f)^2.$$

By integration by parts, we have

$$\int_\Omega (2\nabla f \cdot \nabla u_i + u_i \Delta f)^2 = 4 \int_\Omega (\nabla f \cdot \nabla u_i)^2 - \int_\Omega (\Delta f)^2 u_i^2 - 2 \int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2.$$

Hence, we have

$$\left(\sqrt{\lambda_{k+2} - \lambda_i} - \sqrt{\lambda_{k+1} - \lambda_i} \right)^2 \leq 4 \int_\Omega (\nabla f \cdot \nabla u_i)^2 - \int_\Omega (\Delta f)^2 u_i^2 - 2 \int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2 \quad (3.17)$$

Multiplying (3.17) by $\left(\sqrt{\lambda_{k+2} - \lambda_i} + \sqrt{\lambda_{k+1} - \lambda_i} \right)^2$ on both sides implies

$$\begin{aligned}(\lambda_{k+2} - \lambda_{k+1})^2 \leq & 4 \left(\int_\Omega (\nabla f \cdot \nabla u_i)^2 - \frac{1}{4} \int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2} \int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2 \right) \\ & \times \left(\sqrt{\lambda_{k+2} - \lambda_i} + \sqrt{\lambda_{k+1} - \lambda_i} \right)^2 \\ \leq & 16 \left(\int_\Omega (\nabla f \cdot \nabla u_i)^2 - \frac{1}{4} \int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2} \int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2 \right) \lambda_{k+2},\end{aligned}$$

which is the inequality (3.15).

Since $|\nabla f|^2 = 1$, from (3.15), the Cauchy-Schwarz inequality and integration by parts, we obtain

$$\begin{aligned}(\lambda_{k+2} - \lambda_{k+1})^2 \leq & 16 \left(\int_\Omega |\nabla u_i|^2 - \frac{1}{4} \int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2} \int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2 \right) \lambda_{k+2} \\ = & 16 \left(\lambda_i - \frac{1}{4} \int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2} \int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2 \right) \lambda_{k+2}.\end{aligned}$$

□

Remark 3.1. If $\lambda_{k+1} = \lambda_{k+2}$, (3.16) also holds trivially. Hence, under the conditions in Corollary 3.3, when $i = 1$, (3.16) holds for any $k \geq 1$.

Proof of Theorem 1.1 . Since the inequality (3.19) always holds for $\lambda_{k+1} = \lambda_{k+2}$, without loss of generality, we assume that $\lambda_{k+1} < \lambda_{k+2}$ in the following discussion.

Let $\{x_1, x_2, \cdots, x_n\}$ be the standard coordinate functions in $\mathbb{R}^n$. Taking

$$i = 1 \text{ and } f = x_\ell,\ \ell = 1, \cdots, n$$

in (3.15) and making summation over ℓ from 1 to n, we have

$$\begin{aligned} n(\lambda_{k+2} - \lambda_{k+1})^2 \leq & 16\lambda_{k+2} \int_\Omega \sum_{\ell=1}^{n} \left(\frac{\partial u_1}{\partial x_\ell} \right)^2 \\ = & 16\lambda_1 \lambda_{k+2}. \end{aligned} \tag{3.18}$$

where we use $|\nabla x_\ell| = 1, \ell = 1, \cdots, n$.

From Theorem 3.1 in [16] (see also (1.9)) and (3.18), we deduce

$$\lambda_{k+2} - \lambda_{k+1} \leq 4\sqrt{\frac{\lambda_1}{n}} \sqrt{\lambda_{k+2}} \leq 4\lambda_1 \sqrt{\frac{C_0(n)}{n}} (k+1)^{\frac{1}{2}} = C_{n,\Omega}(k+1)^{\frac{1}{2}}, \tag{3.19}$$

where $C_{n,\Omega} = 4\lambda_1 \sqrt{\frac{C_0(n)}{n}}$, $C_0(n)$ is given by (1.9). The fact that (3.19) holds for any $k \geq 1$ implies that (1.16) holds for arbitrary $k > 1$. □

Proof of Corollary 1.2. For convenience, we will use the upper half-plane model of the hyperbolic space, i.e.,

$$\mathbb{H}^n(-1) = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n > 0\}$$

with the standard metric

$$\mathrm{d}s^2 = \frac{(\mathrm{d}x_1)^2 + \cdots + (\mathrm{d}x_n)^2}{(x_n)^2}.$$

Taking $r = \log x_n$, we have

$$\mathrm{d}s^2 = (\mathrm{d}r)^2 + \mathrm{e}^{-2r} \sum_{i=1}^{n-1} (\mathrm{d}x_i)^2.$$

Without loss of generality, we assume that $\lambda_{k+1} < \lambda_{k+2}$. Taking $f = r$ and $i = 1$ in (3.16) implies

$$\begin{aligned} \lambda_{k+2} - \lambda_{k+1} \leq & 4 \left(\lambda_1 - \frac{1}{4} \int_\Omega (\Delta r)^2 u_i^2 - \frac{1}{2} \int_\Omega (\nabla(\Delta r) \cdot \nabla r) u_1^2 \right)^{\frac{1}{2}} \sqrt{\lambda_{k+2}} \\ = & 4 \left(\lambda_1 - \frac{(n-1)^2}{4} \right)^{\frac{1}{2}} \sqrt{\lambda_{k+2}}, \end{aligned} \tag{3.20}$$

where $|\nabla r| = 1,\ \Delta r = -(n-1)$ are used.

Combining (1.12) and (3.20) gives

$$\begin{aligned} \lambda_{k+2} - \lambda_{k+1} \leq & 4 \left(\lambda_1 - \frac{(n-1)^2}{4} \right)^{\frac{1}{2}} \sqrt{C_0(n) \left(\lambda_1 + \frac{n^2}{4} H_0^2 \right)} (k+1)^{\frac{1}{n}} \\ = & C_{n,\Omega} (k+1)^{\frac{1}{n}}, \end{aligned} \tag{3.21}$$

where $C_{n,\Omega}$ is defined by (1.18). Noting that (3.21) holds for any $k \geq 1$, we know that (1.17) holds for any $k > 1$. □

4. Proof of Corollary 1.3

Proof of Corollary 1.3. Assume that (M, g) is an n-dimensional complete noncompact Riemannian manifold with sectional curvature Sec satisfying $-a^2 \leq Sec \leq -b^2$, where $0 \leq b \leq a$ are constants. Let Ω be a bounded domain of M. For $p \notin \overline{\Omega}$ fixed point, the distance function $\rho(x)$ is defined by $\rho(x) = \text{distance}(x, p)$. From $|\nabla\rho| = 1$ and Proposition 2.2 of [30], we have

$$\nabla\rho \cdot \nabla(\Delta\rho) = -|\text{Hess }\rho|^2 - \text{Ric}(\nabla\rho, \nabla\rho). \tag{4.1}$$

Assume that $0 \leq h_1 \leq \cdots \leq h_{n-1}$ are the eigenvalues of Hessρ. We have

$$\begin{aligned}
2|\text{Hess}\rho|^2 - (\Delta\rho)^2 =& 2\sum_{i=1}^{n-1} h_i^2 - \left(\sum_{i=1}^{n-1} h_i\right)^2 \\
=& \sum_{i=1}^{n-1} h_i^2 - \sum_{i\neq j} h_i h_j \\
\leq& h_{n-1}^2 + h_1h_2 + \cdots + h_{n-2}h_{n-1} - \sum_{i\neq j} h_i h_j \\
=& h_{n-1}^2 - h_1h_2 - \cdots - h_{n-2}h_{n-1} - \sum_{\substack{i\neq j \\ i,j\leq n-2}} h_i h_j \\
\leq& h_{n-1}^2 - (n-2)^2 h_1^2.
\end{aligned} \tag{4.2}$$

From the Hessian comparison theorem (cf. [36]), we have

$$a\frac{\cosh a\rho}{\sinh a\rho} \geq h_{n-1} \geq \cdots \geq h_1 \geq b\frac{\cosh b\rho}{\sinh b\rho}. \tag{4.3}$$

Since $n \geq 3$ and $\frac{a^2}{\sinh^2 a\rho}$ is a decreasing function of a, from (4.2) and (4.3), we have

$$\begin{aligned}
& 2|\text{Hess}\rho|^2 + 2\text{Ric}(\nabla\rho, \nabla\rho) - (\Delta\rho)^2 \\
\leq& a^2\frac{\cosh^2 a\rho}{\sinh^2 a\rho} - (n-2)^2 b^2 \frac{\cosh^2 b\rho}{\sinh^2 b\rho} - 2(n-1)b^2 \\
=& a^2 + \frac{a^2}{\sinh^2 a\rho} - (n-2)^2 b^2 - (n-2)^2 \frac{b^2}{\sinh^2 b\rho} - 2(n-1)b^2 \\
\leq& -(n-1)^2 b^2 + (a^2 - b^2) + \frac{b^2}{\sinh^2 b\rho} - (n-2)^2 \frac{b^2}{\sinh^2 b\rho} \\
\leq& -(n-1)^2 b^2 + (a^2 - b^2).
\end{aligned} \tag{4.4}$$

Without loss of generality, we assume $\lambda_{k+1} < \lambda_{k+2}$. Taking $f = \rho$ and $i = 1$ in (3.16) gives

$$\lambda_{k+2} - \lambda_{k+1} \leq 4\left(\lambda_1 - \frac{1}{4}\int_\Omega (\Delta\rho)^2 u_1^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta\rho)\cdot\nabla\rho) u_1^2\right)^{\frac{1}{2}} \sqrt{\lambda_{k+2}}. \tag{4.5}$$

From (4.1) and (4.4), we obtain

$$\lambda_1 - \frac{1}{4}\int_\Omega (\Delta\rho)^2 u_1^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta\rho)\cdot\nabla\rho) u_1^2$$

$$
\begin{aligned}
&=\lambda_1 + \frac{1}{4}\int_\Omega \left(2|\mathrm{Hess}\rho|^2 + 2\mathrm{Ric}(\nabla\rho,\, \nabla\rho) - (\Delta\rho)^2\right) u_1^2 \\
&\le \lambda_1 - \frac{(n-1)^2}{4} b^2 + \frac{a^2-b^2}{4}. \qquad (4.6)
\end{aligned}
$$

Combining (1.12), (4.5) and (4.6) implies

$$
\begin{aligned}
\lambda_{k+2} - \lambda_{k+1} \le & 4\left(\lambda_1 - \frac{(n-1)^2}{4}b^2 + \frac{a^2-b^2}{4}\right)^{\frac{1}{2}} \sqrt{C_0(n)\left(\lambda_1 + \frac{n^2}{4}H_0^2\right)}(k+1)^{\frac{1}{n}} \\
\le & C_{n,\Omega}(k+1)^{\frac{1}{n}}, \qquad (4.7)
\end{aligned}
$$

where $C_{n,\Omega}$ is defined by (1.20). Noting that (4.7) holds for any $k \ge 1$, we can conclude that (1.19) holds for any $k > 1$. □

References

[1] Andrews, B.; Clutterbuck, J. *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. **24** (2011), no. 3, 899-916.

[2] Ashbaugh, M. S. *Isoperimetric and universal inequalities for eigenvalues.* Spectral theory and geometry (Edinburgh,1998), 95-139, London Math. Soc. Lecture Note Ser., **273**, Cambridge Univ. Press, Cambridge, 1999.

[3] Ashbaugh, M. S. *The universal eigenvalue bounds of Payne-Pólya-Weinberger, Hile-Prottter, and H. C. Yang*, Proc. Indian Acad. Sci. Math. Sci. **112** (2002), no. 1, 3-30.

[4] Ashbaugh, M. S.; Benguria, R. D. *Optimal lower bound for the gap between the first two eigenvalues of one-dimensional Schrodinger operators with symmetric single-well potentials*, Proc. Amer. Math. Soc. **105** (1989), no. 2, 419-424.

[5] Ashbaugh, M. S.; Benguria, R. D. *Proof of the Payne-Pólya-Weinberger conjecture*, Bull. Amer. Math. Soc. (N. S.), **25** (1991), no. 1, 19-29.

[6] Ashbaugh, M. S.; Benguria, R. D. *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. of Math., **123** (1992), no. 3, 601-628.

[7] Ashbaugh, M. S.; Benguria, R. D. *A second proof of the Payne-Pólya-Weinberger conjecture*, Comm. Math. Phys., **147** (1992), no. 1, 181-190.

[8] Ashbaugh, M. S.; Benguria, R. D. *More bounds on eigenvalue ratios for Dirichlet Laplacians in n dimension*, SIAM J. Math. Anal., **24** (1993), no. 6, 1622-1651.

[9] Ashbaugh, M. S.; Benguria, R. D. *Isoperimetric bounds for higher eigenvalue ratios for the n-dimensional fixed membrane problem*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), no. 6, 977-985.

[10] Ashbaugh, M. S.; Benguria, R. D. *Bounds for ratios of the first, second, and third membrane eigenvalues.* Nonlinear problems in applied mathematics, 30-42, SIAM, Philadelphia, PA, 1996.

[11] Chavel, I. *Eigenvalues in Riemannian Geometry*, including a chapter by Burton Randol. With an appendix by Jozef Dodziuk. Pure and Applied Mathematics, **115**. Academic Press, Inc., Orlando, FL, 1984. xiv+362 pp.

[12] Chen, D.; Cheng, Q.-M. *Extrinsic estimates for estimates for eigenvalues of the Laplacian operator*, J. Math. Soc. Japan, **60** (2008), no. 2, 325-339.

[13] Chen, D.; Zheng, T.; Lu, M. *Eigenvalue estimates on domains in complete noncompact Riemannian manifolds*, Pacific J. Math., **255** (2012), no. 1, 41-54.

[14] Cheng, Q.-M.; Yang, H. *Estimates on eigenvalues of Laplacian*, Math. Ann., **331** (2005), no. 2, 445-460.

[15] Cheng, Q.-M.; Yang, H. *Inequalities for eigenvalues of Laplacian on domains and compact complex hypersurfaces in complex projective spaces* , J. Math. Soc. Japan, **58** (2006), no. 2, 545-561.

[16] Cheng, Q.-M.; Yang, H. *Bounds on eigenvalues of Dirichlet Laplacian*, Math. Ann., **337** (2007), no. 1, 159-175.

[17] Cheng, Q.-M.; Yang, H. *Estimates for eigenvalues on Riemannian manifolds*, J. Differential Equations., **247** (2009), no. 8, 2270-2281.

[18] El Soufi, A.; Harrell, E. M., II; Ilias, S. *Universal inequalities for the eigenvalues of Laplace and Schrödinger operators on submanifolds*, Trans. Amer. Math. Soc. **361** (2009), no. 5, 2337-2350.

[19] Harrell, E. M., II *Some geometric bounds on eigenvalue gaps*, Comm. Partial Differential Equations, **18** (1993), no. 1-2, 179-198.

[20] Harrell, E. M., II; Michel, Patricia L. *Commutator bounds for eigenvalues, with applications to spectral geometry*, Comm. Partial Differential Equations, **19** (1994), no. 11-12, 2037-2055.
[21] Harrell, E. M., II; Stubbe, J. *On trace identities and universal eigenvalue estimates for some partial differential operators*, Trans. Amer. Math. Soc., **349** (1997), no. 5, 1797-1809.
[22] Harrell, E. M., II, *Commutator, eigenvalue gaps and mean curvature in the theory of Schröinger operators*, Comm. Partial Differential Equations, **32** (2007), no. 1-3, 401-413.
[23] Hile, G. N.; Protter, M. H. *Inequalities for eigenvalues of the Laplacian*, Indiana Univ. Math. J., **29** (1980), 523-538.
[24] Kolmogorov, A. N.; Fomin, S. V. *Elements of the Theory of Functions and Functional Analysis*(7ed),(translated by Yurong Duan, Hongshen Zheng and Sixu Guo) Higher Education Press, 2006.
[25] Leung, P.-F. *On the consecutive eigenvalues of the Laplacian of a compact minimal submanifold in a sphere*, J. Aust. Math. Soc., 50 (1991), 409-426.
[26] Li, P. *Eigenvalue estimates on homogeneous manifolds*, Comment. Math. Helv., **55** (1980), 347-363.
[27] Nash, J. *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20-63.
[28] Payne, L. E.; Pólya, G.; Weinberger, H. F. *Sur le quotient de deux fréquences propres consécutives*, C. R. Acad. Sci. Paris, **241** (1955), 917-919.
[29] Payne, L. E.; Pólya, G.; Weinberger, H. F. *On the ratio of consecutive eigenvalues*, J. Math. and Phys., **35** (1956), 289-298.
[30] Schoen, R.; Yau, S. T. *Lectures on Differntial Geometry.* Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu. Translated from the Chinese by Ding and S. Y. Cheng. Preface translated from the Chinese by Kaising Tso. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994. v+235 pp.
[31] Singer, I. M.; Wong, B.; Yau, S. T.; Yau, Stephen S. T. *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola. Norm. Sup. Pisa, CI. Sci. (4), **12** (1985), no. 2, 319-333.
[32] Sun, H.; Cheng, Q.-M.; Yang, H. *Lower order eigenvalues of Dirichlet Laplacian*, Manuscripta math., **125** (2008), no. 2, 139-156.
[33] Thompson, C. J. *On the ratio of consecutive eigenvalues in n-dimensions*, Stud. Appl. Math., **48** (1969), 281-283.
[34] van den Berg, M. *On condensation in the free-boson gas and the spectrum of the Laplacian*, J. Statist. Phys. **31** (1983), no. 3, 623-637.
[35] Weyl, H. *Der Asymptotische Verteilungsgesetz der Eigenwerte Linearer partieller Differential-gleichungen*, Math. Ann. **71** (1912), 441-469.
[36] Wu, H.; Shen, L.; Yu, Y. *Introduction to Riemannian Geometry* (in Chinese), Peking University Press, 1989.
[37] Yang, H. *An estimate of the difference between consecutive eigenvalues*, preprint IC/91/60 of ICTP, Trieste, 1991.
[38] Yang, P. C.; Yau, S. T. *Eigenvalues of the Laplacian of compact Riemannian surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa CI. Sci. (4), **7** (1980), no. 1, 55-63.
[39] Yau, S. T. *Nonlinear analysis in geometry*, Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique], **33**. Série des Conférences de l'Union Mathématique Internationale [Lecture Series of the International Mathematics Union], **8**. L'Enseignement Mathématique, Geneva, 1986. 54 pp.
[40] Yu, Q.; Zhong, J. *Lower bounds of the gap between the first and second eigenvalues of the Schödinger operator*, Trans. Amer. Math. Soc., **294** (1986), no. 1, 341-349.
[41] Zeidler, E. *Applied functional analysis. main principles and their applications*, Applied Mathematical Sciences **109**. Springer-Verlag, New York, 1995. xvi+404 pp.

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