CRITICAL EXPONENTS OF INDUCED DIRICHLET FORMS ON SELF-SIMILAR SETS

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Abstract

In [KLW], we studied certain random walks on the hyperbolic graphs X associated with the self-similar sets K, and showed that the discrete energy $\mathcal{E}_X$ on X has an induced energy form $\mathcal{E}_K$ on K. The domain of $\mathcal{E}_K$ is a Besov space $\Lambda_{2,2}^{\alpha,\beta/2}$ where α is the Hausdorff dimension of K and β is a parameter determined by the "return ratio" of the random walk. In this paper, we consider the functional relationship of $\mathcal{E}_X$ and $\mathcal{E}_K$. In particular, we investigate the critical exponents of the β in the domain $\Lambda_{2,2}^{\alpha,\beta/2}$ in order for $\mathcal{E}_K$ to be a Dirichlet form. We provide some criteria to determine the critical exponents through the effective resistance of the random walk on X, and make use of certain electrical network techniques to calculate the exponents for some concrete examples.

Contents

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1 Introduction

The theory of Dirichlet forms on a metric measure space was originated in the seminal work of Beurling and Deny [BeD] as a generalization of the Laplacian on $\mathbb{R}^d$. Motivated by the development of fractals, in the past three decades, there has been extensive study and development of this form from analytic and probabilistic points of view. In [Jo], Jonsson studied the local regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on the Sierpinski gasket K. He showed that the domain $\mathcal{D}_K$ is the Besov space $\Lambda_{2,\infty}^{\alpha,\beta/2}$ where $\alpha = \log 3/\log 2$ and $\beta = \log 5/\log 2$. This consideration was extended by Pietruska-Paluba to nested fractals and d-sets [P1, P2]. It has also been studied amply in the general setting of metric measure space together with the heat kernel estimates (e.g., [GH1, GH2, GHL1, GHL2, GHL4]). In another direction, Stós [St] investigated the Dirichlet forms on a d-set K as a stable process subordinate to the Brownian motion on K. He showed that $\mathcal{E}_K^{(\beta)}, 0 < \beta < 2$, is non-local, and the domain is another type of Besov space $\Lambda_{2,2}^{\alpha,\beta/2}$. For the associated stable processes, the heat kernel was studied in detail by Chen and Kumagai [CK] on a d-set with $0 < \beta < 2$. Recently, there is a great interest to study non-local regular Dirichlet forms and the associated heat kernels of the jump processes on metric measure spaces (e.g., [CKW1, CKW2, GHL3, GHH1, GHH2, HK]).

Let K be a compact subset in $\mathbb{R}^d$ and ν be an associated measure satisfying $\nu(B(x,r)) \asymp r^\alpha$ for any ball $B(x,r)$ with center at $x \in K$ and radius $r > 0$ (by $f \asymp g$, we mean f and g are positive functions, and $C^{-1}g \le f \le Cg$ for some $C > 0$). We define the Besov space $\Lambda_{2,2}^{\alpha,\beta/2}$ to be the Banach space contained in $L^2(K,\nu)$ with norm defined by

$$\|u\|_{\Lambda_{2,2}^{\alpha,\beta/2}} = \|u\|_{L^2} + \Big(\iint_{K\times K} \frac{|u(\xi)-u(\eta)|^2}{|\xi-\eta|^{\alpha+\beta}} d\nu(\xi)d\nu(\eta)\Big)^{1/2}. \tag{1.1}$$

(Note that $\nu \times \nu$ vanishes on the diagonal.) It is easy to see that $\Lambda_{2,2}^{\alpha,\beta/2}, \beta > 0$ is decreasing with increasing β, and it can be trivial for sufficiently large β. We define the *critical exponent* β^* of K by

$$\beta^* = \sup\{\beta > 0 : \Lambda_{2,2}^{\alpha,\beta/2} \text{ contains nonconstant functions}\}.$$

The critical exponent plays an important role in the study of the Laplacian and Brownian motion on fractals. It is where a local regular Dirichlet form (i.e., the Laplacian) is expected to occur, and the domain is $\Lambda_{2,\infty}^{\alpha,\beta^*/2}$. The β^* is referred to as the "walk dimension" of the underlying set K as it is the scaling exponent for the space-time relation of the diffusive process: $E_x(|X_t - x|^2) \approx t^{2/\beta^*}$. For the special cases, we know that for $\mathbb{R}^d, \beta^* = 2$, and for the d-dimensional Sierpinski gasket, $\beta^* = \log(d+3)/\log 2$ [Jo] (for planar SG, $\beta^* = \log 5/\log 2 \approx 2.322$), and for the Sierpinski carpet, it was estimated that $\beta^* \approx 2.031$ [B, BB]. In [GHL1], it was proved that $2 \le \beta^* \le \alpha + 1$ under the assumption that a sub-Gaussian heat kernel exists together with a chain condition. Despite the many developments, it is still an open question whether a Laplacian will exist on some more general fractal sets.

In [KLW], we studied non-local Dirichlet forms with another approach. For a self-similar set K with the open set condition (OSC), there is a hyperbolic graph $(X, \mathfrak{E})$ (augmented tree) on the symbolic space of K, and the hyperbolic boundary is Hölder equivalent to K [Ka,LW1,LW2]. We introduced a class of transient reversible random walks with return ratio $\lambda \in (0,1)$, called λ-natural random walk (λ-NRW) on $(X, \mathfrak{E})$ with conductance $c(\mathbf{x}, \mathbf{y})$ depending on λ (see Section 2). We showed that the Martin boundary, the hyperbolic boundary and K are homeomorphic, and the hitting distribution ν is the normalized α-Hausdorff measure where α is the dimension of K. Moreover, by using a theory of Silverstein [Si], it was proved that the graph energy

$$\mathcal{E}_X^{(\lambda)}[f] = \frac{1}{2}\sum\nolimits_{\mathbf{x},\mathbf{y}\in X} c(\mathbf{x},\mathbf{y})|f(\mathbf{x}) - f(\mathbf{y})|^2 \tag{1.2}$$

defined by the random walk induces a positive definite bilinear form on K,

$$\mathcal{E}_K^{(\beta)}(u,v) \asymp \iint_{K\times K} \frac{(u(\xi)-u(\eta))(v(\xi)-v(\eta))}{|\xi-\eta|^{\alpha+\beta}} d\nu(\xi)d\nu(\eta),$$

with $\beta = \log\lambda/\log r$, where r is the minimum of the contraction ratios of the IFS maps. Clearly the domain $\mathcal{D}_K^{(\beta)} = \{u \in L^2(K,\mu) : \mathcal{E}_K^{(\beta)}[u] < \infty\}$ is the Besov space $\Lambda_{2,2}^{\alpha,\beta/2}$.

In this paper we continue the investigation of the above induced bilinear functional $\mathcal{E}_K^{(\beta)}$. Our investigation has two main focuses, namely, to establish the functional relationship of the discrete energy $\mathcal{E}_X^{(\lambda)}$ and the induced $\mathcal{E}_K^{(\beta)}$, then use it to study the critical exponents of $\mathcal{D}_K^{(\beta)}$ (i.e., $\Lambda_{2,2}^{\alpha,\beta/2}$). Let $\mathcal{D}_X^{(\lambda)}$ be the domain of $\mathcal{E}_X^{(\lambda)}$, and let $\mathcal{DH}_X^{(\lambda)}$ denote the class of harmonic functions in $\mathcal{D}_K^{(\beta)}$. For $u \in \mathcal{D}_K^{(\beta)}$, we use Hu to denote the Poisson integral of u on X, and for $f \in \mathcal{D}_X^{(\lambda)}$, let $\mathrm{Tr}f(\xi) = \lim_{\mathbf{x}_n\to\xi} f(\mathbf{x}_n)$. By imposing a norm on $\mathcal{D}_X^{(\lambda)}$, we prove (Theorem 3.5, Corollary 3.6)

Theorem 1.1. *Suppose K is a self-similar set and assume that the OSC holds. Then for a λ-NRW with $\lambda \in (0, r^\alpha)$, we have $\mathcal{D}_K^{(\beta)} = \mathrm{Tr}(\mathcal{DH}_X^{(\lambda)})$. Moreover,* Tr *is a Banach space isomorphism on $\mathcal{DH}_X^{(\lambda)}$, and $\mathrm{Tr}^{-1} = H$.*

The condition $\lambda \in (0, r^\alpha)$ in Theorem 1.1 will be used throughout the paper. It implies that $\beta > \alpha$, and functions in $\mathcal{D}_K^{(\beta)}$ are Hölder continuous (Proposition 2.5);

moreover, the convergence rate $(\lambda/r^\alpha)^n$ is essential when we consider functions in $\mathcal{D}_X$ that tend to the boundary K.

To consider the critical exponent of $\mathcal{D}_K^{(\beta)}$, we introduce some finer classification of the domains. We let

$$\begin{aligned}
\beta_1^* &:= \sup\{\beta > 0 : \mathcal{D}_K^{(\beta)} \cap C(K) \text{ is dense in } C(K)\},\\
\beta_2^* &:= \sup\{\beta > 0 : \dim \mathcal{D}_K^{(\beta)} = \infty\}\\
\beta_3^* &:= \sup\{\beta > 0 : \mathcal{D}_K^{(\beta)} \text{ contains nonconstant functions}\},
\end{aligned}$$

Clearly we have $2 \le \beta_1^* \le \beta_2^* \le \beta_3^* \le \infty$, and $\beta_3^* = \beta^*$ for the β^* defined above. In the standard cases, these three exponents are equal, but there are also examples that they are different [GuL]. It is one of the main purposes of this paper to discuss these exponents and to provide some criteria to determine them. Our approach relies on the effective resistance. We use $R^{(\lambda)}(\xi,\eta)$ to denote the effective resistance for $\xi, \eta \in K$ (see Section 4), and note that the infinite word i^∞ of $\{S_i\}_{i=1}^N$ will represent an element in K.

Theorem 1.2. *With the assumptions as in Theorem 1.1, then $\mathcal{D}_K^{(\lambda)}$ consists of only constant functions if and only if $R^{(\lambda)}(i^\infty, j^\infty) = 0$ for all $i, j = 1, \cdots, N$.*

Consequently, if we let $\lambda_3^* = \sup\{\lambda > 0 :\ R^{(\lambda)}(i^\infty, j^\infty) = 0\ \ \forall\ i, j = 1, \cdots, N\}$, then $\beta_3^* = \log \lambda_3^* / \log r$ (Thereom 5.4). Moreover if $\beta_3^* > \alpha$ and K is connected, we have $\beta_2^* = \beta_3^*$ (Theorem 5.6). For β_1^*, we give a result on the post critically-finite (p.c.f.) sets [Ki1]. We let V_0 denote the "boundary" of K.

Theorem 1.3. *If in addition, K is a p.c.f. set and satisfies another mild geometric condition (see Theorem 5.9). Then if*

$$R^{(\lambda-\epsilon)}(\xi,\eta) > 0, \qquad \forall\ \xi \neq \eta \in V_0,$$

for some $0 < \epsilon < \lambda$, then $\mathcal{D}_K^{(\beta)}$ is dense in $C(K)$ with supremum norm.

Consequently, if $\lambda_1^* := \inf\{\lambda > 0 : R^{(\lambda)}(\xi,\eta) > 0,\ \forall\, \xi \neq \eta \in V_0\} \in (0, r^\alpha)$, then $\beta_1^* = \log \lambda_1^* / \log r$.

A challenging task is to determine the effective resistance $R^{(\lambda)}(i^\infty, j^\infty)$ (or $R^{(\lambda-\epsilon)}(\xi,\eta)$ for $\xi, \eta \in V_0$) to be $= 0$ or > 0 in the above theorems. For this we make use of the basic tools in the electrical network theory (series and parallel laws, Δ-Y transform, as well as cutting and shorting) to provide a device for such estimation, which is applied to some special cases as examples.

For the organization of the paper, in Section 2, we summarize the needed results from [KLW]. In Section 3, we prove some basic results on the extension of functions on $\mathcal{D}_K^{(\beta)}$ be the Poisson integral, and the limit of function in $\mathcal{D}_X^{(\lambda)}$, and prove Theorem 1.1. We define and justify the limiting effective resistance in Section 4, and prove Theorems 1.2 and 1.3 in Section 5. In Section 6, we make use of the electrical techniques to give some implementations of the theorems by some examples. Some remarks and open problems are provided in Section 7.

2 Preliminaries

We will give a brief summary of the background results in [KLW] for the convenience of the reader, and all the unexplained notations can be found there. Let $\{S_i\}_{i=1}^N$, $N \geq 2$, be an *iterated function system* (IFS) of contractive similitudes on $\mathbb{R}^d$ with contraction ratios $\{r_i\}_{i=1}^N$, and let K be the *self-similar set*. Let Σ^* be the symbolic space of K. Let $r = \min\{r_i : i = 1, \cdots, N\}$. For $n \geq 1$, define

$$\mathcal{J}_n = \{\mathbf{x} = i_1 \cdots i_k \in \Sigma^* : r_{\mathbf{x}} \leq r^n < r_{i_1 \cdots i_{k-1}}\}, \tag{2.1}$$

and $\mathcal{J}_0 = \{\vartheta\}$ by convention. Consider the *modified symbolic space* $X = \bigcup_{n=0}^{\infty} \mathcal{J}_n$, which has a tree structure with an edge set $\mathfrak{E}_v$. The tree structure can be strengthened to a hyperbolic graph by adding more horizontal edges on the tree according to the neighboring cells on each level n [Ka, LW1, LW2]. According to [LW2], we define

$$\mathfrak{E}_h = \{(\mathbf{x}, \mathbf{y}) \in \mathcal{J}_n \times \mathcal{J}_n : \mathrm{dist}(S_{\mathbf{x}}(K), S_{\mathbf{y}}(K)) \leq \kappa r^n, n \geq 0\},$$

where $\kappa > 0$ is arbitrary but fixed. Let $\mathfrak{E} = \mathfrak{E}_v \cup \mathfrak{E}_h$, and call it an *augmented tree*. It was shown that $(X, \mathfrak{E})$ is a hyperbolic graph in the sense of Gromov [Wo1]. In this case, the horizontal geodesic is uniformly bounded, and for any $\mathbf{x}, \mathbf{y} \in X$, the canonical geodesic consists of three segments $[\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{y}]$, where $[\mathbf{x}, \mathbf{u}], [\mathbf{v}, \mathbf{y}]$ are vertical paths in $\mathfrak{E}_v$, $[\mathbf{u}, \mathbf{v}]$ is a horizontal geodesic in $\mathcal{J}_\ell$ with the smallest ℓ. Using this geodesic, the *Gromov product* $(\mathbf{x}|\mathbf{y}) = \ell - h/2$, where h is the length of $[\mathbf{u}, \mathbf{v}]$. We define a visual metric $\rho_a(\mathbf{x}, \mathbf{y}) = e^{-a(\mathbf{x}|\mathbf{y})}$ on X for some $a > 0$. Let $\widehat{X}_H$ be the completion, and define the *hyperbolic boundary* $\partial_H X = \widehat{X}_H \setminus X$. Then $\partial_H X$ is a compact metric space.

A *geodesic ray* $\{\mathbf{x}_n\}_n$ is a sequence of words with $\mathbf{x}_n = i_1 i_2 \cdots i_n \in \mathcal{J}_n$. If $\xi \in \partial_H X$ has a canonical representation $i_1 i_2 \cdots \in \Sigma^\infty$, then $\{\mathbf{x}_n\}_n$ converges to ξ, and $\xi \in S_{\mathbf{x}_n}(K)$ for all n. It follows that for any other geodesic ray $\{\mathbf{y}_n\}_n$ converging to ξ, we have $\mathbf{x}_n \sim_h \mathbf{y}_n$. In the sequel, we will make use of the geodesic rays frequently to relate functions on X and K. We call the sequence $\{\kappa_n\}_{n=1}^\infty$ a

κ*-sequence* if each κ_n is a selection map from K to $\mathcal{J}_n$, such that for each $\xi \in K$, $(\kappa_n(\xi))_{n=1}^{\infty}$ is a geodesic ray converging to ξ. It follows from the above that

Lemma 2.1. *For any IFS $\{S_i\}_{i=1}^N$, let $(X, \mathfrak{E})$ be the hyperbolic graph as defined above. Let E be a closed subset K. Then for any two κ-sequences $\{\kappa_n\}_{n=1}^{\infty}$ and $\{\kappa'_n\}_{n=1}^{\infty}$, we have*

$$\kappa'_n(E) \subset \{\mathbf{x} \in \mathcal{J}_n : d(\mathbf{x}, \kappa_n(E)) \leq 1\} \quad \textit{for each } n,$$

where $d(\cdot, \cdot)$ is the graph metric on $(X, \mathfrak{E})$.

Theorem 2.2. [LW2] *For any IFS $\{S_i\}_{i=1}^N$, let $(X, \mathfrak{E})$ be the hyperbolic graph as defined above. Then the hyperbolic boundary is Hölder equivalent to the self-similar set K, i.e., for the canonical map $\iota : \partial_H X \longrightarrow K$,*

$$\rho_a(\xi, \eta)(= e^{-a(\mathbf{x}|\mathbf{y})}) \asymp |\iota(\xi) - \iota(\eta)|^{-a/\log r}.$$

Throughout, we will always assume that the IFS $\{S_i\}_{i=1}^N$ satisfies the *open set condition* (OSC) [Fa]. In this case, the self-similar set K has Hausdorff dimension α which is uniquely determined by $\sum_{i=1}^N r_i^\alpha = 1$.

In [KLW], we introduced a class of reversible random walks on the augmented tree $(X, \mathfrak{E})$: for $\lambda \in (0, 1)$, we set the *conductance* $c : \mathfrak{E} \to (0, \infty)$ such that

$$c(\mathbf{x}, \mathbf{x}^-) = r_{\mathbf{x}}^\alpha \lambda^{-|\mathbf{x}|}, \quad \text{and} \quad c(\mathbf{x}, \mathbf{y}) \asymp r_{\mathbf{x}}^\alpha \lambda^{-|\mathbf{x}|}, \quad \mathbf{x} \sim_h \mathbf{y} \in X \setminus \{\vartheta\}, \tag{2.2}$$

where $\mathbf{x}^-$ is the parent of x, $r_{\mathbf{x}} := r_{i_1} \cdots r_{i_m}$ for $\mathbf{x} = i_1 \cdots i_m$. We define the *natural random walk* with return ratio $\lambda \in (0, 1)$ (λ-NRW) to be the Markov chain $\{Z_n\}_{n=0}^\infty$ on X with transition probability $P(\mathbf{x}, \mathbf{y}) = c(\mathbf{x}, \mathbf{y})/m(\mathbf{x})$ if $\mathbf{x} \sim \mathbf{y}$, and 0 otherwise, where $m(\mathbf{x}) = \sum_{\mathbf{y}:\mathbf{x}\sim\mathbf{y}} c(\mathbf{x}, \mathbf{y})$ is the *total conductance* at $\mathbf{x} \in X$. Note that the chain moves forward and backward with a ratio $\lambda \in (0, 1)$; hence $\{Z_n\}_{n=0}^\infty$ is transient. Let $\mathcal{M}$ denote the Martin boundary, and let Z_∞ be the pointwise limit of $\{Z_n\}_{n=0}^\infty$, starting from ϑ.

Theorem 2.3. [KLW] *Let $\{S_i\}_{i=1}^N$ be an IFS satisfying the open set condition, and let $\{Z_n\}_{n=0}^\infty$ be a λ-NRW. Then*

(i) the distribution ν of Z_∞ on $\mathcal{M}$ is the α-Hausdorff measure;

(ii) the Martin kernel $K(\mathbf{x}, \xi) \asymp \lambda^{|\mathbf{x}|-(\mathbf{x}|\xi)} r^{-\alpha(\mathbf{x}|\xi)}$;

(iii) the Martin boundary $\mathcal{M}$, the hyperbolic boundary $\partial_H X$, and the self-similar set are all homeomorphic.

It follows from part (iii) that we can carry Doob's discrete potential theory onto the self-similar set K. We denote the space of *harmonic* functions (w.r.t. P) on X by $\mathcal{H}(X) = \{f \in \ell(X) : Pf = f\}$, where $\ell(X)$ is the collection of real functions on X, and $Pf(\mathbf{x}) = \sum_{\mathbf{y}\in X} P(\mathbf{x},\mathbf{y})f(\mathbf{y})$. The Poisson integral for $u \in L^1(K,\nu)$ is

$$Hu(\cdot) = \int_K K(\cdot,\xi)u(\xi)d\nu(\xi) \in \mathcal{H}(X). \tag{2.3}$$

The *graph energy* of $f \in \ell(X)$ is given by

$$\mathcal{E}_X[f] = \frac{1}{2}\sum_{\mathbf{x},\mathbf{y}\in X:\mathbf{x}\sim\mathbf{y}} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y}))^2, \tag{2.4}$$

and the domain of $\mathcal{E}_X$ is $\mathcal{D}_X = \{f \in \ell(X) : \mathcal{E}_X[f] < \infty\}$. Using Silverstein's theorem [Si], together with Theorem 2.3 (iii), we obtain an induced quadratic form on K.

Theorem 2.4. [KLW] *Under the assumptions in Theorem 2.3, the graph energy in* (2.4) *induces an energy form* $\mathcal{E}_K[u]$ *given by*

$$\mathcal{E}_K[u] := \mathcal{E}_X[Hu] = \frac{m(\vartheta)}{2}\iint_{K\times K}|u(\xi)-u(\eta)|^2\Theta(\xi,\eta)d\nu(\xi)d\nu(\eta), \quad u \in L^2(K,\nu), \tag{2.5}$$

where $\Theta(\xi,\eta) \asymp (\lambda r^\alpha)^{-(\xi|\eta)} \asymp |\xi-\eta|^{-(\alpha+\beta)}$ *(Naim kernel) with* $\beta = \frac{\log\lambda}{\log r}$.

By definition, the domain of $\mathcal{E}_K$ is $\mathcal{D}_K = \{u \in L^2(K,\nu) : Hu \in \mathcal{D}_X\}$. It follows from the above that $\mathcal{D}_K$ is the Besov space $\Lambda_{2,2}^{\alpha,\beta/2}$. If we define $\|u\|^2_{\mathcal{E}_K} = \mathcal{E}_K[u] + \|u\|^2_{L^2(K,\nu)}$, then $(\mathcal{D}_K, \|\cdot\|_{\mathcal{E}_K})$ is a Banach space, and is equivalent to $\Lambda_{2,2}^{\alpha,\beta/2}$. For $\gamma > 0$, we let

$$C^\gamma(K) = \{u \in C(K) : \|u\|_{C^\gamma} := \|u\|_\infty + \operatorname{esssup}_{\xi,\eta\in K}\frac{|u(\xi)-u(\eta)|}{|\xi-\eta|^\gamma} < \infty\} \tag{2.6}$$

denote the Hölder space. We will use the following result frequently. It was proved in [GHL1] (the assumption of heat kernel stated there is not needed in the proof) that

Proposition 2.5. *If* $\beta > \alpha$, *then for all* $u \in L^2(K,\nu)$,

$$\|u\|_{C^\gamma} \le C\|u\|_{\Lambda_{2,2}^{\alpha,\beta/2}} \tag{2.7}$$

with $\gamma = (\beta-\alpha)/2$. *Consequently,* $\Lambda_{2,2}^{\alpha,\beta/2} \hookrightarrow C^\gamma$ *is an imbedding.*

It follows that for $\alpha < \beta < \beta_1^*$, $\mathcal{D}_K \cap C(K) = \mathcal{D}_K$ is trivially dense in $\mathcal{D}_K$ under the norm $\|\cdot\|_{\mathcal{E}_K}$, and in $C(K)$ under the supremum norm. This implies that $(\mathcal{E}_K, \mathcal{D}_K)$ is a non-local regular Dirichlet form.

3 Harmonic functions and trace functions

In this section, we will set up a natural relation between harmonic functions on X with finite graph energy and continuous functions on K with finite induced energy (Theorem 3.5). First we use Theorem 2.3(ii) to provide a "uniform tail estimate" of the Martin kernel. As in [KLW, Section 5], we introduce a projection $\iota : X \to K$ by selecting $\iota(\mathbf{x}) \in S_{\mathbf{x}}(O \cap K)$ arbitrarily, where O is an open set in the OSC satisfying $O \cap K \neq \emptyset$.

Proposition 3.1. *Let $\{S_i\}_{i=1}^N$ be an IFS satisfying the OSC, and let $\{Z_n\}_{n=0}^\infty$ be a λ-NRW on the augmented tree $(X, \mathfrak{E})$. Then for any $\epsilon, \delta > 0$, there exists a positive integer n_0 such that for any $\mathbf{x} \in X$ and $|\mathbf{x}| \geq n_0$, $K(\mathbf{x}, \xi) \leq \varepsilon$ holds for any $\xi \in K \setminus B(\iota(\mathbf{x}), \delta)$.*

Proof. It follows from Theorem 2.3(ii) that

$$K(\mathbf{x}, \xi) \leq C_1 \lambda^{|\mathbf{x}|} (\lambda r^\alpha)^{-(\mathbf{x}|\xi)}, \quad \mathbf{x} \in X,\ \xi \in K.$$

Note that $(\mathbf{x}|\xi) \leq (\iota(\mathbf{x})|\xi)$ by [KLW, Lemma 3.9(ii)]. Hence for $\xi \in K \setminus B(\iota(\mathbf{x}), \delta)$,

$$r^{-(\mathbf{x}|\xi)} \leq r^{-(\iota(\mathbf{x})|\xi)} \leq C_2 |\iota(\mathbf{x}) - \xi|^{-1} \leq C_2 \delta^{-1},$$

(the second inequality follows from Theorem 2.2 (see also [LW2, Theorem 1.2]). Hence for $\varepsilon > 0$, we can pick positive integer n_0 such that the last inequality in the following holds:

$$K(\mathbf{x}, \xi) \leq C_1 \lambda^{n_0} r^{-(\alpha + \log \lambda / \log r)(\mathbf{x}|\xi)} \leq C_1 \lambda^{n_0} (C_2 \delta^{-1})^{\alpha + \log \lambda / \log r} \leq \varepsilon.$$

$\square$

Let $\nu_{\mathbf{x}}$, $\mathbf{x} \in X$, denote the hitting distribution of Z_∞ on K, starting from $\mathbf{x}$. As $K(\mathbf{x}, \cdot) = d\nu_{\mathbf{x}}/d\nu$, the above result shows that the mass of the distribution $\nu_{\mathbf{x}}$ will concentrate around $\iota(\mathbf{x})$ (equivalently, $S_{\mathbf{x}}(K)$) as $|\mathbf{x}| \to \infty$. We have a Fatou-type theorem as a corollary.

Corollary 3.2. *Suppose $\{S_i\}_{i=1}^N$ satisfies OSC, and let $\{Z_n\}_{n=0}^\infty$ be a λ-NRW on the augmented tree $(X, \mathfrak{E})$. Then for $u \in C(K)$ and $\varepsilon > 0$, there exists a positive integer n_0 such that*

$$|Hu(\mathbf{x}) - u(\xi)| \leq \varepsilon \tag{3.1}$$

whenever $|\mathbf{x}| \geq n_0$ and $\xi \in S_{\mathbf{x}}(K)$. In particular, $\lim_{n\to\infty} Hu(\mathbf{x}_n) = u(\xi)$ uniformly for $\xi \in K$, where $\{\mathbf{x}_n\}_n$ is a geodesic ray converging to ξ.

Proof. Since u is continuous on the compact set K, u is bounded and uniformly continuous. We let $\sup_{\xi\in K}|u(\xi)| = M_0 < \infty$ and choose $\delta > 0$ such that $|u(\xi) - u(\eta)| < \varepsilon/3$ whenever $|\xi-\eta| < \delta$ on K. Furthermore, by Proposition 3.1, we choose n_0 such that both $\mathrm{diam}(S_{\mathbf{x}}(K)) \le \delta$ and $K(\mathbf{x},\xi) \le \frac{\varepsilon}{6M_0}$ hold for any $\mathbf{x} \in X$ with $|\mathbf{x}| \ge n_0$ and $\xi \in K \setminus B(\iota(\mathbf{x}),\delta)$. Then for $|\mathbf{x}| \ge n_0$, by using the usual technique of splitting the following integral on K into $K \cap B(\iota(\mathbf{x}),\delta)$ and $K \setminus B(\iota(\mathbf{x}),\delta)$, we can show that

$$|Hu(\mathbf{x}) - u(\iota(\mathbf{x}))| \le \int_K |K(\mathbf{x},\eta)(u(\eta) - u(\iota(\mathbf{x})))| d\nu(\eta) \le \varepsilon$$

Hence for $\xi \in S_{\mathbf{x}}(K)$,

$$|Hu(\mathbf{x}) - u(\xi)| \le |Hu(\mathbf{x}) - u(\iota(\mathbf{x}))| + |u(\iota(\mathbf{x})) - u(\xi)| \le \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

and (3.1) holds. For the last statement, let $\{\mathbf{x}_n\}_n$ be a geodesic ray converging to ξ, then $\mathbf{x}_n = i_1 \cdots i_n$, and this $i_1 i_2 \cdots \in \Sigma^\infty$ is a representation of some ξ' with $\xi' \in S_{\mathbf{x}_n}(K)$, and $\xi' = \xi$ in $\partial_H X$. Hence by (3.1), we have $\lim_{n\to\infty} Hu(\mathbf{x}_n) = u(\xi') = u(\xi)$, and the convergence is uniform on ξ. □

In the rest of this section, we assume that the λ-NRW has a return ratio $\lambda \in (0, r^\alpha)$. Then $\beta = \log\lambda/\log r > \alpha$, and Proposition 2.5 applies.

Lemma 3.3. *Suppose $\{S_i\}_{i=1}^N$ satisfies OSC, and let $\{Z_n\}_{n=0}^\infty$ be a λ-NRW on the augmented tree $(X, \mathfrak{E})$ with $\lambda \in (0, r^\alpha)$. Then for $f \in \mathcal{D}_X$,*

(i) there exists $C > 0$ (depend on f) such that for any geodesic ray $\{\mathbf{x}_n\}_n$,

$$|f(\mathbf{x}_{n+1}) - f(\mathbf{x}_n)| \le C(\lambda/r^\alpha)^{n/2},$$

and hence $\lim\limits_{n\to\infty} f(\mathbf{x}_n)$ exists;

(ii) for two equivalent geodesic rays $(\mathbf{x}_n)_n$ and $(\mathbf{y}_n)_n$, $\lim\limits_{n\to\infty} f(\mathbf{x}_n) = \lim\limits_{n\to\infty} f(\mathbf{y}_n)$.

Proof. (i) Let $\tau = \lambda/r^\alpha < 1$. For a geodesic ray $(\mathbf{x}_n)_n$, since

$$|f(\mathbf{x}_{n+1}) - f(\mathbf{x}_n)| \le \sqrt{\frac{\mathcal{E}_X[f]}{c(\mathbf{x}_{n+1}, \mathbf{x}_n)}} \le C(\lambda/r^\alpha)^{n/2} = C\tau^{n/2},$$

hence the sequence $(f(\mathbf{x}_n))_n$ converges in an exponential rate.

(ii) For two equivalent geodesic rays $(\mathbf{x}_n)_n$ and $(\mathbf{y}_n)_n$ that converge to the same ξ, if they are distinct, then $\mathbf{x}_n \sim_h \mathbf{y}_n$ for all n (or by Lemma 2.1). Then

$$|f(\mathbf{x}_n) - f(\mathbf{y}_n)| \le \sqrt{\frac{\mathcal{E}_X[f]}{c(\mathbf{x}_n, \mathbf{y}_n)}} \le C'\tau^{n/2}, \tag{3.2}$$

which tends to 0 as $n \to \infty$. Hence the two limits are equal. □

With the assumption as in Lemma 3.3, we can define a linear map $\mathrm{Tr} : \mathcal{D}_X \to \ell(K)$ (called a *trace map*) by

$$(\mathrm{Tr} f)(\xi) = \lim_{n\to\infty} f(\mathbf{x}_n), \qquad \xi \in K, \tag{3.3}$$

where $(\mathbf{x}_n)_n$ is a geodesic ray that converges to ξ. We call $\mathrm{Tr} f$ the *trace function* of f. By Lemma 3.3(ii), the limit in (3.3) is "uniform" in the sense as follows.

Proposition 3.4. *Suppose $\{S_i\}_{i=1}^N$ satisfies OSC, and let $\{Z_n\}_{n=0}^\infty$ be a λ-NRW with ratio $\lambda \in (0, r^\alpha)$ on the augmented tree $(X, \mathfrak{E})$. Then the limit in (3.3) is uniform, in the sense that for $f \in \mathcal{D}_X$ and $\varepsilon > 0$, there exists a positive integer n_0 such that*

$$|f(\mathbf{x}) - \mathrm{Tr} f(\xi)| \le \varepsilon \tag{3.4}$$

whenever $|\mathbf{x}| \ge n_0$ and $\xi \in S_{\mathbf{x}}(K)$. Moreover, $\mathrm{Tr} f$ is continuous on K.

Proof. The first part follows from Lemma 3.3. To prove the second part, for $\varepsilon > 0$, by (3.4), we can pick a positive integer n_0 such that $|f(\mathbf{x}) - \mathrm{Tr} f(\xi)| < \varepsilon/3$ whenever $|\mathbf{x}| \ge n_0$ and $\xi \in S_{\mathbf{x}}(K)$. Since the length of the horizontal geodesics in $(X, \mathfrak{E})$ is uniformly bounded [LW1, LW2], say by M. Also let C be a constant such that $c(\mathbf{x}, \mathbf{y}) \ge C^{-1}(r^\alpha/\lambda)^{|\mathbf{x}|}$ for all $\mathbf{x} \sim_h \mathbf{y}$. By assumption $\tau := \lambda/r^\alpha < 1$. We choose $n_1 \ge n_0$ such that $M\sqrt{C\mathcal{E}_X[f]\tau^{n_1}} < \varepsilon/3$.

As $|\xi - \eta| \asymp r^{(\xi|\eta)}$ (Theorem 2.2), we can pick $\delta > 0$ such that $(\xi|\eta) \ge n_1$ whenever $|\xi - \eta| < \delta$. Now for $\xi, \eta \in K$ with $|\xi - \eta| < \delta$, consider a canonical geodesic $[\xi, \mathbf{u}, \mathbf{v}, \eta]$ with horizontal geodesic $\pi(\mathbf{u}, \mathbf{v}) = [\mathbf{u} = \mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_k = \mathbf{v}]$ (see Section 2). Then $|\mathbf{u}| \ge (\xi|\eta) \ge n_1$, and hence

$$\begin{aligned} |\mathrm{Tr} f(\xi) - \mathrm{Tr} f(\eta)| &\le |\mathrm{Tr} f(\xi) - f(\mathbf{u})| + |f(\mathbf{u}) - f(\mathbf{v})| + |f(\mathbf{v}) - \mathrm{Tr} f(\eta)| \\ &< \frac{\varepsilon}{3} + \sum_{i=0}^{k-1} |f(\mathbf{u}_i) - f(\mathbf{u}_{i+1})| + \frac{\varepsilon}{3} \\ &< \frac{2\varepsilon}{3} + M\sqrt{C\mathcal{E}_X[f]\tau^{n_1}} < \varepsilon. \qquad \text{(by (3.2))} \end{aligned}$$

This concludes that $\mathrm{Tr} f \in C(K)$. $\square$

Theorem 3.5. *Suppose $\{S_i\}_{i=1}^N$ satisfies the OSC, and let $\{Z_n\}_{n=0}^\infty$ be a λ-NRW with ratio $\lambda \in (0, r^\alpha)$ on the augmented tree $(X, \mathfrak{E})$. Then $\mathrm{Tr}(\mathcal{DH}_X) = \mathcal{D}_K$ where $\mathcal{DH}_X$ is the class of harmonic functions in $\mathcal{D}_X$. More precisely, $\mathrm{Tr} Hu = u$ for $u \in \mathcal{D}_K$, and $H\mathrm{Tr} f = f$ for $f \in \mathcal{DH}_X$.*

Proof. For $u \in \mathcal{D}_K$, by definition we have $Hu \in \mathcal{DH}_X$. Note that $\mathcal{D}_K \cap C(K) = \mathcal{D}_K$, as $\mathcal{D}_K = \Lambda_{2,2}^{\alpha,\beta/2}$ can be imbedded into the Hölder space $C^{(\beta-\alpha)/2}(K)$ if $\beta > \alpha$ (Proposition 2.5). By Corollary 3.2, we have $\mathrm{Tr}Hu = u$.

For $f \in \mathcal{DH}_X$, let $u = \mathrm{Tr}f \in C(K)$ (by Proposition 3.4). For any $\varepsilon > 0$, by Corollary 3.2 and Proposition 3.4, there exists a positive integer n_0 such that for $|\mathbf{x}| \geq n_0$ and $\xi \in S_{\mathbf{x}}(K)$,

$$|f(\mathbf{x}) - u(\xi)| < \frac{\varepsilon}{2} \quad \text{and} \quad |Hu(\mathbf{x}) - u(\xi)| < \frac{\varepsilon}{2}. \tag{3.5}$$

Suppose that $f \neq Hu$. Assume without loss of generality that $f(\mathbf{x}_0) > Hu(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in \mathcal{J}_m$. Let $a_n = \max_{\mathbf{x} \in \mathcal{J}_n}(f(\mathbf{x}) - Hu(\mathbf{x}))$, $n \geq 1$. Note that $f - Hu$ is harmonic. By the Maximum Principle of harmonic functions, we regard $\mathcal{J}_{n+1}$ as the boundary of $X_{n+1} = \bigcup_{k=0}^{n+1} \mathcal{J}_k$. Then $a_{n+1} \geq \max_{\mathbf{x} \in X_n}(f(\mathbf{x}) - Hu(\mathbf{x})) = a_n$, thus the sequence $\{a_n\}$ is non-decreasing. Hence $\inf_{n \geq m} a_n = a_m > 0$. This contradicts that $\lim_{n\to\infty} a_n = 0$ by (3.5). We conclude that $f = Hu = H\mathrm{Tr}f$. $\square$

In the above theorem, we can actually give a norm on $\mathcal{D}_X$ so that $H : \mathcal{D}_K \to \mathcal{DH}_X$ is a Banach space isomorphism. Indeed, by Proposition 3.4 and the continuity of functions in $\mathcal{D}_K$, we know that functions in $\mathcal{D}_X$ are bounded. Fix $w \in (0, r^\alpha)$, let $\|f\|^2_{\ell^2(X,w)} = \sum_{\mathbf{x} \in X} |f(\mathbf{x})|^2 w^{|\mathbf{x}|}$, and define $\|\cdot\|_{\mathcal{E}_X}$ on $\mathcal{D}_X$ by

$$\|f\|^2_{\mathcal{E}_X} = \mathcal{E}_X[f] + \|f\|^2_{\ell^2(X,w)}, \tag{3.6}$$

Then it is direct to check that $\|f\|^2_{\mathcal{E}_X}$ defines a complete norm on $\mathcal{D}_X$.

Corollary 3.6. *With the same assumption as in Theorem 3.5 and let $w \in (0, r^\alpha)$. Then for all $u \in L^2(K, \nu)$,*

$$\|Hu\|_{\ell^2(X,w)} \leq C\|u\|_{L^2(K,\nu)}. \tag{3.7}$$

Consequently, $H : (\mathcal{D}_K,\ \|\cdot\|_{\mathcal{E}_K}) \to (\mathcal{DH}_X,\ \|\cdot\|_{\mathcal{E}_X})$ is an isomorphism.

Proof. Let $F(\mathbf{x}, \mathbf{y})$ denote the probability that the random walk ever visits $\mathbf{y}$ from $\mathbf{x}$. For $n \geq 1$ and $|\mathbf{y}| > n$, by [KLW, Theorem 4.6],

$$F(\vartheta, \mathbf{y}) = \sum_{\mathbf{x} \in \mathcal{J}_n} F_n(\vartheta, \mathbf{x}) F(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{x} \in \mathcal{J}_n} r_{\mathbf{x}}^\alpha F(\mathbf{x}, \mathbf{y}) \geq r^{\alpha(n+1)} \sum_{\mathbf{x} \in \mathcal{J}_n} F(\mathbf{x}, \mathbf{y}).$$

Hence $\sum_{\mathbf{x} \in \mathcal{J}_n} K(\mathbf{x}, \xi) = \sum_{\mathbf{x} \in \mathcal{J}_n} \frac{F(\mathbf{x},\mathbf{y})}{F(\vartheta,\mathbf{y})} \leq r^{-\alpha(n+1)}$. It follows that for $u \in L^2(K, \nu)$,

$$\begin{aligned} \|Hu\|^2_{\ell^2(X,w)} &= \sum\nolimits_{\mathbf{x} \in X} \big(\mathbb{E}_{\mathbf{x}}(u(Z_\infty))\big)^2 w^{|\mathbf{x}|} \ \leq \ \sum\nolimits_{\mathbf{x} \in X} \big(\mathbb{E}_{\mathbf{x}}(u(Z_\infty)^2)\big) w^{|\mathbf{x}|} \\ &= \sum\nolimits_{n=0}^{\infty} w^n \sum_{\mathbf{x} \in \mathcal{J}_n} \int_K K(\mathbf{x}, \xi) |u(\xi)|^2 d\nu(\xi) \ \leq \ C\|u\|^2_{L^2(K,\nu)}. \end{aligned}$$

where $C = r^{-\alpha}\sum_{n=0}^{\infty}(w/r^{\alpha})^n$. As $w/r^{\alpha} < 1$, this yields (3.7). In view of Theorem 3.5, the norm isomorphism of the map $H : \mathcal{D}_K \to \mathcal{DH}_X$ follows from this and $\mathcal{E}_K(u,v) = \mathcal{E}_X(Hu, Hv)$, and the open mapping theorem. $\square$

4 Effective resistances of $\mathcal{E}_X$

In this section, we will set up the limiting effective resistance for the λ-NRW on the augmented tree $(X, \mathfrak{E})$ in order to prepare for the investigation of the critical exponents of $\mathcal{D}_K$ in the next section.

We will start with a general situation. Let V be a finite graph with a reversible Markov chain with conductance $c(x,y), x, y \in V$. Let $\ell(V)$ denote the class of real valued functions on V, and let $\mathcal{E}_V(f)$ be the graph energy of f. For any $V_1 \subset V$, it is well-known that each $f \in \ell(V_1)$ has an harmonic extension to V, which has a minimal energy among all $g \in \ell(V)$ with $g|_{V_1} = f$. In the following, we give an expression of the minimal energy in terms of the conductance $c(x,y)$ of the chain on V.

Proposition 4.1. *Let V be a finite set, and $V = V_1 \cup V_2$ with $\#V_1 \geq 2$. Assume that there is a reversible Markov chain on V with conductance $c(\cdot,\cdot)$. Then for $f \in V_1$,*

$$\min\left\{\mathcal{E}_V[g] : g \in \ell(V), g|_{V_1} = f\right\} = \frac{1}{2}\sum\nolimits_{x,y\in V_1, x\neq y} c_*(x,y)(f(x) - f(y))^2, \qquad (4.1)$$

where $c_*(x,y) = c(x,y) + \sum_{z,w\in V_2} c(x,z)G_{V_2}(z,w)P(w,y)$, $x, y \in V_1$, $x \neq y$ (here $G_{V_2}(\cdot,\cdot)$ is the Green function of the random walk restricted to V_2), and it defines a conductance function on V_1.

Proof. Let $F^{V_1}(x,y) = \mathbb{P}_x(Z_{t_{V_1}} = y)$ where t_{V_1} is the first hitting time of V_1. Then

$$F^{V_1}(z,y) = \sum\nolimits_{w\in V_2} G_{V_2}(z,w)P(w,y), \quad \forall\, x \in V_2,\ y \in V_1.$$

We can check directly from the definition that $c_*(x,y) = c_*(y,x)$, $x, y \in V_1$, using the reversibility of the chain (i.e., $m(x)P(x,z) = m(z)P(z,x)$ and $m(z)G_{V_2}(z,w) = m(w)G_{V_2}(w,z)$). Hence $c_*(x,y)$ defines a conductance on V_1.

To prove (4.1), we let $h(\cdot) = \sum_{y\in V_1} F^{V_1}(\cdot, y)f(y) \in \ell(V)$. Then it is easy to check that h is the unique function such that $Ph = h$ on V_2 and $h = f$ on V_1. Hence $\mathcal{E}_V[h] = \min\{\mathcal{E}_V[g] : g \in \ell(V), g|_{V_1} = f\}$. Observe that

$$\mathcal{E}_V[h] = \frac{1}{2}\sum_{x,y\in V} c(x,y)(h(x) - h(y))^2 = \sum_{x,y\in V} c(x,y)(h(x) - h(y))h(x)$$

Hence

$$
\begin{aligned}
\mathcal{E}_V[h] &= \sum_{x\in V_1} h(x) \sum_{y\in V} c(x,y)(h(x)-h(y)) \qquad (\text{by } Ph=h \text{ on } V_2) \\
&= \sum_{x\in V_1} f(x)\Big(\sum_{y\in V_1} c(x,y)(f(x)-f(y)) + \sum_{y\in V_2} c(x,y)\sum_{z\in V_1} F^{V_1}(y,z)(f(x)-f(z))\Big) \\
&= \sum_{x,y\in V_1} f(x)(f(x)-f(y))\Big(c(x,y)+\sum_{z\in V_2} c(x,z)F^{V_1}(z,y)\Big) \quad (\text{switch } y \text{ and } z) \\
&= \sum_{x,y\in V_1} c_*(x,y)f(x)(f(x)-f(y)) \\
&= \frac{1}{2}\sum_{x,y\in V_1} c_*(x,y)(f(x)-f(y))^2. \qquad (\text{use } c_*(x,y)=c_*(y,x))
\end{aligned}
$$

This yields (4.1). □

For a finite connected graph $(X,\mathfrak{E})$ with conductances, the *effective resistance* between two disjoint nonempty subsets A, $B\subseteq X$ is given by

$$
R_X(E,F) = (\min\{\mathcal{E}_X[f] : f\in \ell(X) \text{ with } f=1 \text{ on } E, \text{ and } f=0 \text{ on } F\})^{-1}. \quad (4.2)
$$

Also we set $R_X(E,F)=0$ if $E\cap F\neq\emptyset$ by convention. Clearly $R_X(\cdot,\cdot)$ is symmetric, and the energy minimizer in (4.2) is unique, bounded in between 0 and 1, and is harmonic on $X\setminus(E\cup F)$.

For the λ-NRW on $(X,\mathfrak{E})$, for convenience and the simplicity in the estimations, we will assume slightly more that the conductance on the horizontal edges satisfies

$$
c(\mathbf{x},\mathbf{y}) = r^{\alpha|\mathbf{x}|}\lambda^{-|\mathbf{x}|} \quad \text{for } \mathbf{x}\sim_h \mathbf{y}\in X\setminus\{\vartheta\}. \quad (4.3)
$$

(we use $\asymp$ in (2.2)), and there is no change of the results. Let $\{\kappa_n\}_{n=1}^\infty$ be a κ-*sequence* defined in Section 2. For any two closed subsets Φ, $\Psi\subseteq K$, we define the level-n resistance between them (depend on κ_n) by

$$
R_n^{(\lambda)}(\Phi,\Psi) := R_{X_n}(\kappa_n(\Phi),\kappa_n(\Psi)), \quad (4.4)
$$

where $X_n := \cup_{k=0}^n \mathcal{J}_k$ and has same conductance restricted from X.

Theorem 4.2. *Suppose $\{S_i\}_{i=1}^N$ satisfies the OSC, and let $\{Z_n\}_{n=0}^\infty$ be a λ-NRW on the augmented tree $(X,\mathfrak{E})$ with $\lambda\in(0,r^\alpha)$. Then for any two closed subsets Φ, $\Psi\subseteq K$, the limit $\lim_{n\to\infty} R_n^{(\lambda)}(\Phi,\Psi)$ exists, and is independent of the choice of the κ-sequence.*

We will prove a technical lemma first. For $E, F \subset \mathcal{J}_n$ such that in the graph distance, $\mathrm{dist}(E, F) > 2$, we define

$$\partial E = \{\mathbf{x} \in \mathcal{J}_n : \mathrm{dist}(\mathbf{x}, E) = 1\}, \quad \partial F = \{\mathbf{x} \in \mathcal{J}_n : \mathrm{dist}(\mathbf{x}, F) = 1\}. \tag{4.5}$$

Let $\mathcal{E}_n := \mathcal{E}_n(E, F) = \min\{\mathcal{E}_{X_n}[f] : f \in \ell(X_n),\ f = 1 \text{ on } E,\ f = 0 \text{ on } F\}$, and let f be the energy minimizing function.

Lemma 4.3. *Consider the λ-NRW on $(X, \mathfrak{E})$ with $\lambda \in (0, r^\alpha)$. Let $\{E_n\}_{n\geq 1}, \{F_n\}_{n\geq 1}$ be two sequences such that $E_n, F_n \subset \mathcal{J}_n$, and $\liminf_{n\to\infty} \mathrm{dist}(E_n, F_n) > 2$. If $\sup_{n\geq 1} \mathcal{E}_n(E_n, F_n) < \infty$, then for any $\varepsilon > 0$, there exists n_0, such that for $n \geq n_0$,*

$$\sum_{\mathbf{x}\in\partial E_n} \sum_{\mathbf{y}\in X_n\setminus E_n} c(\mathbf{x},\mathbf{y})(1 - f(\mathbf{y}))^2 < \varepsilon, \quad \sum_{\mathbf{x}\in\partial F_n} \sum_{\mathbf{y}\in X_n\setminus F_n} c(\mathbf{x},\mathbf{y}) f(\mathbf{y})^2 < \varepsilon,$$

where f is the energy minimizer of $\mathcal{E}_n := \mathcal{E}_n(E_n, F_n)$.

Proof. We observe that

$$\begin{aligned}
\mathcal{E}_n &= \frac{1}{2} \sum_{\mathbf{x},\mathbf{y}\in X_n} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y}))^2 \\
&= \sum_{\mathbf{x},\mathbf{y}\in X_n} c(\mathbf{x},\mathbf{y}) f(\mathbf{x})(f(\mathbf{x}) - f(\mathbf{y})) \\
&= \sum_{\mathbf{x}\in E_n} \sum_{\mathbf{y}\in X_n} c(\mathbf{x},\mathbf{y})(1 - f(\mathbf{y})) \ \geq\ (r^\alpha/\lambda)^n \sum_{\mathbf{y}\in\partial E_n} (1 - f(\mathbf{y}))
\end{aligned} \tag{4.6}$$

(the third equality holds because f is harmonic on $X_n \setminus (E_n \cup F_n)$). Thus $f(\mathbf{x}) \geq 1 - (\lambda/r^\alpha)^n \mathcal{E}_n$ for $\mathbf{x} \in \partial E_n$. Using a similar argument, and that f is harmonic on $X_n \setminus (E_n \cup F_n \cup \partial E_n)$, for large n, we have

$$\begin{aligned}
\mathcal{E}_n &\geq \frac{1}{2} \sum_{\mathbf{x},\mathbf{y}\in X_n} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y}))^2 - \sum_{\mathbf{x}\in\partial E_n} \sum_{\mathbf{y}\in E_n} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y}))^2 \\
&= \sum_{\mathbf{x}\in\partial E_n} f(\mathbf{x}) \sum_{\mathbf{y}\in X_n\setminus E_n} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y})) + \sum_{\mathbf{y}\in E_n} c(\mathbf{y},\mathbf{y}^-)(1 - f(\mathbf{y}^-)) \\
&\geq \left(1 - (\lambda/r^\alpha)^n \mathcal{E}_n\right) \sum_{\mathbf{x}\in\partial E_n} \sum_{\mathbf{y}\in X_n\setminus E_n} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y}))
\end{aligned} \tag{4.7}$$

(The inequality is valid because for $\mathbf{x} \in \partial E_n$, by harmonicity, $\sum_{\mathbf{y}\in X_n\setminus E_n} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y})) = -\sum_{\mathbf{y}\in E_n} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y})) = -\sum_{\mathbf{y}\in E_n} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - 1) \geq 0)$.

Now we use (4.6) and (4.7) to make the final estimate:

$$
\begin{aligned}
&\sum_{\mathbf{x}\in\partial E_n}\sum_{\mathbf{y}\in X_n\setminus E_n} c(\mathbf{x},\mathbf{y})(1-f(\mathbf{y}))^2 \\
&\le \frac{\lambda^n}{r^{\alpha(n+1)}}\Big(\sum_{\mathbf{x}\in\partial E_n}\sum_{\mathbf{y}\in X_n\setminus E_n} c(\mathbf{x},\mathbf{y})(1-f(\mathbf{y}))\Big)^2 \\
&= \frac{\lambda^n}{r^{\alpha(n+1)}}\Big(\sum_{\mathbf{x}\in\partial E_n}\sum_{\mathbf{y}\in X_n\setminus E_n} c(\mathbf{x},\mathbf{y})\big((1-f(\mathbf{x}))+(f(\mathbf{x})-f(\mathbf{y}))\big)\Big)^2 \\
&\le \frac{\lambda^n}{r^{\alpha(n+1)}}\left(k\mathcal{E}_n+\frac{\mathcal{E}_n}{1-(\lambda/r^\alpha)^n\mathcal{E}_n}\right)^2 =: \varepsilon(n) \qquad \big(\text{by (4.6), (4.7)}\big) \qquad (4.8)
\end{aligned}
$$

where $k=\sup_{\mathbf{x}\in X}\#\{\mathbf{y}:\mathbf{x}\sim_h\mathbf{y}\}$ (as the graph $(X,\mathfrak{E})$ has bounded degree, and $c(\mathbf{x},\mathbf{y})>0$ only when $\mathbf{x}\sim_h\mathbf{y}$ or $\mathbf{y}=\mathbf{x}^-$). Hence we can choose n_0 such that $\varepsilon(n)<\varepsilon$ for $n>n_0$. Analogously, using $1-f$ instead of f, we obtain the estimate for F as well. $\square$

Proof of Theorem 4.2. We fix a $\lambda\in(0,r^\alpha)$ and omit the superscript (λ) in this proof. First we fix a κ-sequence $\{\kappa_n\}_{n=0}^\infty$, and prove that $\lim_{n\to\infty}R_n(\Phi,\Psi)$ exists. For brevity, we write $\Phi_n:=\kappa_n(\Phi)$ and $\Psi_n:=\kappa_n(\Psi)$. If $\Phi\cap\Psi\neq\emptyset$, then by the property of geodesic rays in $(X,\mathfrak{E})$, for any n, either $\Phi_n\cap\Psi_n\neq\emptyset$ or $\min\{d(\mathbf{x},\mathbf{y}):\mathbf{x}\in\Phi_n,\ \mathbf{y}\in\Psi_n\}=1$ (by Lemma 2.1). In both situations, we have $\lim_{n\to\infty}R_n(\Phi,\Psi)=0$ (for the second case, by (4.2), $R_n(\Phi,\Psi)\le(r^{\alpha n}\lambda^{-n})^{-1}=(\lambda/r^\alpha)^n$).

Hence we assume that $\Phi\cap\Psi=\emptyset$. Then there exists $\ell>0$ such that for $n\ge\ell$, $\mathrm{dist}(\Phi_n,\Psi_n)>3$. By (4.2) and (4.4), for $n\ge\ell$,

$$
R_n(\Phi,\Psi)=(\min\{\mathcal{E}_{X_n}[f]:f=1 \text{ on } \Phi_n,\ \text{and } f=0 \text{ on } \Psi_n\})^{-1}. \qquad (4.9)
$$

Let $\mathcal{E}_n$ denote the minimal energy, and let $f_n\in\ell(X_n)$ be the energy minimizer in (4.9). Let $\{n_k\}_{k\ge1}$ with $n_k\ge\ell$ be the subsequence such that $\lim_{k\to\infty}R_{n_k}(\Phi,\Psi)=\limsup_{n\to\infty}R_n(\Phi,\Psi)>0$. (otherwise $\lim_{n\to\infty}R_n(\Phi,\Psi)=0$). Then $\sup_k\mathcal{E}_{n_k}<\infty$. For $n<n_k$ and $\xi\in K$, by Lemma 3.3(i), we have

$$
\begin{aligned}
|f_{n_k}(\kappa_n(\xi))-f_{n_k}(\kappa_{n_k}(\xi))| \;&\le\; \sum_{m=1}^{n_k-n}|f_{n_k}(\kappa_{n+m-1}(\xi))-f_{n_k}(\kappa_{n+m}(\xi))| \\
&\le\; C\Big(\frac{\lambda}{r^\alpha}\Big)^{n/2}:=\varepsilon(n). \qquad (4.10)
\end{aligned}
$$

As $\lambda\in(0,r^\alpha)$, $\lim_{n\to\infty}\varepsilon(n)=0$. Let $V_1=\Phi_n\cup\Psi_n$. Then for sufficiently large n

and $n_k > n$, we have $\varepsilon(n) < \frac{1}{2}$, and

$$
\begin{aligned}
\mathcal{E}_{n_k} &\geq \mathcal{E}_{X_n}[f_{n_k}] \geq \min\{\mathcal{E}_{X_n}[f] : f \in \ell(X_n), f = f_{n_k} \text{ on } V_1\} \\
&= \frac{1}{2} \sum_{\mathbf{x},\mathbf{y} \in V_1} c_*(\mathbf{x},\mathbf{y})(f_{n_k}(\mathbf{x}) - f_{n_k}(\mathbf{y}))^2 \qquad \text{(by Proposition 4.1)} \\
&\geq \sum_{\mathbf{x} \in \Phi_n} \sum_{\mathbf{y} \in \Psi_n} c_*(\mathbf{x},\mathbf{y})(f_{n_k}(\mathbf{x}) - f_{n_k}(\mathbf{y}))^2 \\
&\geq \sum_{\mathbf{x} \in \Phi_n} \sum_{\mathbf{y} \in \Psi_n} c_*(\mathbf{x},\mathbf{y})((1 - 2\varepsilon(n))^2 \qquad \text{(by (4.10))} \\
&= \min\{\mathcal{E}_{X_n}[f] : f \in \ell(X_n), f|_{\Phi_n} = 1 - 2\varepsilon(n), f|_{\Psi_n} = 0\} \\
&\qquad \text{(by Proposition 4.1)} \\
&= (1 - 2\varepsilon(n))^2 \mathcal{E}_n.
\end{aligned}
$$

Therefore, $R_n(\Phi, \Psi) \geq (1 - 2\varepsilon(n))^2 R_{n_k}(\Phi, \Psi)$ for any large n and $n_k > n$. Taking limit, we have

$$\liminf_{n\to\infty} R_n(\Phi, \Psi) \geq \lim_{k\to\infty} R_{n_k}(\Phi, \Psi) = \limsup_{n\to\infty} R_n(\Phi, \Psi).$$

Hence $\lim_{n\to\infty} R_n(\Phi, \Psi)$ exists.

Next we show that the above limit is independent of the choice of the κ-sequence. For this, we define

$$\partial\Phi_n = \{\mathbf{x} \in \mathcal{J}_n : d(\mathbf{x}, \Phi_n) = 1\}, \quad \partial\Psi_n = \{\mathbf{x} \in \mathcal{J}_n : d(\mathbf{x}, \Psi_n) = 1\}$$

as in (4.5). For any other κ-sequences $\{\kappa'_n\}_n$, it follows from Lemma 2.1 that $\kappa'_n(\Phi) \subset \Phi_n \cup \partial\Phi_n$ and $\kappa'_n(\Psi) \subset \Psi_n \cup \partial\Phi_n$. Hence it suffices to show that

$$\lim_{n\to\infty} R_{X_n}(\Phi_n \cup \partial\Phi_n, \Psi_n \cup \partial\Phi_n) = \lim_{n\to\infty} R_n(\Phi, \Psi). \tag{4.11}$$

Without loss of generality, we assume $\lim_{n\to\infty} R_n(\Phi, \Psi) > 0$. Then $\sup_n \mathcal{E}_n < \infty$. Let $h_n \in \ell(X_n)$ with $h_n = 1$ on $\partial\Phi_n$, $h_n = 0$ on $\partial\Psi_n$, and $h_n = f_n$ on $X_n \setminus (\partial\Phi_n \cup \partial\Psi_n)$.

$$0 \leq R_{X_n}(\Phi_n \cup \partial\Phi_n, \Psi_n \cup \partial\Phi_n)^{-1} - R_n(\Phi, \Psi)^{-1} \leq \mathcal{E}_{X_n}[h_n] - \mathcal{E}_{X_n}[f_n].$$

Then by Lemma 4.3, for given ε, and for large n, $\mathcal{E}_{X_n}[h_n] - \mathcal{E}_{X_n}[f_n] \leq 2\varepsilon$. This implies (4.11) and proves the theorem. $\square$

Theorem 4.2 implies the following definition is well defined.

Definition 4.4. *With the same assumption as in Theorem 4.2, we define the* (limiting) effective resistance *between two closed subsets* Φ *and* Ψ *in* K *by*

$$R^{(\lambda)}(\Phi, \Psi) := \lim_{n\to\infty} R_n^{(\lambda)}(\Phi, \Psi). \tag{4.12}$$

(We omit the superscript (λ) *if there is no confusion.)*

5 The critical exponents of $\mathcal{D}_X$

We use the trace function to establish a basic result on the existence of nonconstant functions in $\mathcal{D}_K$.

Theorem 5.1. *With the same assumption as in Theorem 4.2, suppose Φ, Ψ are two closed subsets of K satisfying $R(\Phi,\Psi) > 0$. Then there exists $u := u_{\Phi,\Psi} \in \mathcal{D}_K$ such that $u = 1$ on Φ, and $u = 0$ on Ψ. Moreover, $u_{\Phi,\Psi}$ is the unique energy minimizer in $\mathcal{D}_K$ in the following sense*

$$R(\Phi,\Psi)^{-1} = \mathcal{E}_K[u_{\Phi,\Psi}] = \inf\{\mathcal{E}_K[u'] : u' \in \mathcal{D}_K \text{ with } u' = 1 \text{ on } \Phi,\ u' = 0 \text{ on } \Psi\}. \tag{5.1}$$

Proof. First we show that the set on the right in (5.1) is non-empty. Clearly $\Phi \cap \Psi = \emptyset$ (otherwise $R(\Phi,\Psi) = 0$). Fix a κ-sequence $\{\kappa_n\}_n$. As in the proof of Theorem 4.2, there exists a positive integer ℓ such that $\kappa_n(\Phi) \cap \kappa_n(\Psi) = \emptyset$ for all $n \geq \ell$, let $f_n \in \ell(X_n)$ be the energy minimizer for $\kappa_n(\Phi)$ and $\kappa_n(\Psi)$ as in (4.9). We extend f_n to X by setting $f_n(\mathbf{x}) = 0$ for $\mathbf{x} \in X \setminus X_n$, then f_n is harmonic on X_{n-1}. Note that $0 \leq f_n \leq 1$ for all $n \geq \ell$. Hence for each $\mathbf{x} \in X$, there exists a convergent subsequence of $\{f_n(\mathbf{x})\}_{n\geq \ell}$. By the diagonal argument, we can find a subsequence $\{f_{n_k}\}_{k\geq 1}$ with $n_1 \geq \ell$ such that f_{n_k} converges to a function $f \in \ell(X)$ pointwise. We claim that

(a) $f \in \mathcal{DH}_X$ and $0 \leq f \leq 1$ on X;

(b) For any $\xi \in \Phi$, $\lim_{n\to\infty} f(\kappa_n(\xi)) = 1$;

(c) For any $\eta \in \Psi$, $\lim_{n\to\infty} f(\kappa_n(\eta)) = 0$.

In fact, as f_{n_k} is harmonic on X_{n_k-1}, the pointwise limit f is harmonic on X. For $k \geq 1$, let g_k be the function on the edge set $\mathfrak{E}$ defined by: for $(\mathbf{x},\mathbf{y}) \in \mathfrak{E}$,

$$g_k(\mathbf{x},\mathbf{y}) = \begin{cases} c(\mathbf{x},\mathbf{y})(f_{n_k}(\mathbf{x}) - f_{n_k}(\mathbf{y}))^2, & \text{if } \mathbf{x},\mathbf{y} \in X_{n_k}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{E}_{n_k} := \mathcal{E}_{X_{n_k}}[f_{n_k}] = \frac{1}{2}\sum_{(\mathbf{x},\mathbf{y})\in\mathfrak{E}} g_k(\mathbf{x},\mathbf{y})$, and $\lim_{k\to\infty} g_k(\mathbf{x},\mathbf{y}) = c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y}))^2$. By Fatou's Lemma, we have

$$\begin{aligned} \mathcal{E}_X[f] &= \frac{1}{2}\sum_{(\mathbf{x},\mathbf{y})\in\mathfrak{E}} c(\mathbf{x},\mathbf{y})(f(\mathbf{x}) - f(\mathbf{y}))^2 = \frac{1}{2}\sum_{(\mathbf{x},\mathbf{y})\in\mathfrak{E}} \left(\lim_{k\to\infty} g_k(\mathbf{x},\mathbf{y})\right) \\ &\leq \frac{1}{2}\liminf_{k\to\infty} \sum_{(\mathbf{x},\mathbf{y})\in\mathfrak{E}} g_k(\mathbf{x},\mathbf{y}) = \lim_{k\to\infty} \mathcal{E}_{n_k} = R(\Phi,\Psi)^{-1} < \infty. \end{aligned} \tag{5.2}$$

Hence (a) follows.

To prove (b), observe that $R(\Phi,\Psi)>0$ implies that $\sup_{k\ge 1}\mathcal{E}_{n_k}<\infty$. Hence for any $k\ge 1$, $n<n_k$ and $\xi\in\Phi$, by Lemma 3.3(i)

$$|f_{n_k}(\kappa_n(\xi))-1|\le\sum_{m=n}^{n_k-1}|f_{n_k}(\kappa_m(\xi))-f_{n_k}(\kappa_{m+1}(\xi))|\le C_1(\lambda/r^\alpha)^{n/2}.$$

Letting $k\to\infty$, we have $|f(\kappa_n(\xi))-1|\le C_2(\lambda/r^\alpha)^{n/2}$, hence (b) follows by letting $n\to\infty$. With a similar argument, we can also conclude (c).

By the claim and Theorem 3.5, let $u=\mathrm{Tr}f\in\mathcal{D}_K$. Then $0\le u\le 1$ on K, $u(\xi)=\lim_{n\to\infty}f(\kappa_n(\xi))=1$ for all $\xi\in\Phi$, and $u(\eta)=\lim_{n\to\infty}f(\kappa_n(\eta))=0$ for all $\eta\in\Psi$.

Now we complete the proof of the theorem. By (5.2), $\mathcal{E}_K[u_{\Phi,\Psi}]=\mathcal{E}_X[f_{\Phi,\Psi}]\le R(\Phi,\Psi)^{-1}$. For the reverse inequality, it suffices to show that $R(\Phi,\Psi)^{-1}\le\mathcal{E}_K[u]$ for all $u\in\mathcal{D}_K$ with $u=1$ on Φ and $u=0$ on Ψ. Fix a κ-sequence $\{\kappa_n\}_n$. For any $\varepsilon\in(0,\frac{1}{2})$, by Proposition 3.2, there exists a positive integer $n_0=n_0(\varepsilon)$ such that $|Hu(\kappa_n(\xi))-u(\xi)|\le\varepsilon$ whenever $n\ge n_0$ and $\xi\in K$. Taking $V_1=\kappa_n(\Phi)\cup\kappa_n(\Psi)$ and $g=Hu|_{V_1}$ as in Proposition 4.1, then we have, for $n\ge n_0$,

$$\min\{\mathcal{E}_{X_n}[f]:f\in\ell(X_n),f=Hu\text{ on }V_1\}=\frac{1}{2}\sum_{\mathbf{x},\mathbf{y}\in V_1}c_*(\mathbf{x},\mathbf{y})(Hu(\mathbf{x})-Hu(\mathbf{y}))^2. \tag{5.3}$$

Hence

$$\begin{aligned}\mathcal{E}_{X_n}[Hu]&\ge\min\{\mathcal{E}_{X_n}[f]:f\in\ell(X_n),f=Hu\text{ on }V_1\}\\&\ge\sum_{\mathbf{x}\in\kappa_n(\Phi)}\sum_{\mathbf{y}\in\kappa_n(\Psi)}c_*(\mathbf{x},\mathbf{y})(Hu(\mathbf{x})-Hu(\mathbf{y}))^2\qquad\text{(by (5.3))}\\&\ge\sum_{\mathbf{x}\in\kappa_n(\Phi)}\sum_{\mathbf{y}\in\kappa_n(\Psi)}c_*(\mathbf{x},\mathbf{y})\big((1-\varepsilon)-\varepsilon\big)^2\\&=(1-2\varepsilon)^2R_n(\Phi,\Psi)^{-1}.\end{aligned}$$

As ε can be arbitrarily small, we have $R(\Phi,\Psi)^{-1}\le\lim_{n\to\infty}\mathcal{E}_{X_n}[Hu]=\mathcal{E}_K[u]$. Hence (5.1) follows.

The uniqueness of $u_{\Phi,\Psi}$ as an energy minimizer follows from the fact that $\mathcal{E}_K$ is strictly convex in $\mathcal{D}_K$. $\square$

The function $f\in\mathcal{DH}_X$ thus constructed is called a *harmonic function induced by* Φ *and* Ψ. The function $u=\mathrm{Tr}f\in\mathcal{D}_K$ is referred as the *energy minimizer of* Φ *and* Ψ. We denote them by $f_{\Phi,\Psi}$ and $u_{\Phi,\Psi}$ respectively.

Corollary 5.2. *With the same assumption as in Theorem 4.2, the following conditions are equivalent: for two distinct points $\xi, \eta \in K$,*

(i) there exists $u \in \mathcal{D}_K$ with range $[0,1]$ such that $u(\xi) = 1$ and $u(\eta) = 0$;

(ii) there exists $u \in \mathcal{D}_K$ such that $u(\xi) \neq u(\eta)$;

(iii) $R(\xi, \eta) > 0$.

In this case, $R(\xi,\eta) = \sup\big\{\dfrac{|u(\xi)-u(\eta)|^2}{\mathcal{E}_K(u,u)} : \ u \in \mathcal{D}_K, \ \mathcal{E}_K(u,u) > 0\big\}$.

Proof. Note that (i) $\Rightarrow$ (ii) is trivial, and (iii) $\Rightarrow$ (i) follows from Theorem 5.1. We need only prove (ii) $\Rightarrow$ (iii). We observe that the given $u \in \mathcal{D}_K$ is continuous (Proposition 2.5). Fix any κ-sequence, by Corollary 3.2, there exists $n_0 > 0$ such that for $n \geq n_0$, $|Hu(\kappa_n(\xi)) - u(\xi)| \leq \frac{1}{3}|u(\xi) - u(\eta)|$, and the same for η. Hence $|Hu(\kappa_n(\xi)) - Hu(\kappa_n(\eta))| \geq \frac{1}{3}|u(\xi) - u(\eta)|$. Then by (4.2), for $n > n_0$

$$\begin{aligned} \frac{|u(\xi)-u(\eta)|^2}{9R_n(\xi,\eta)} &\leq \frac{|Hu(\kappa_n(\xi)) - Hu(\kappa_n(\eta))|^2}{R_n(\xi,\eta)} \\ &\leq c(\kappa_n(\xi), \kappa_n(\eta)) \ |Hu(\kappa_n(\xi)) - Hu(\kappa_n(\eta))|^2 \leq \mathcal{E}_{X_n}[Hu]. \end{aligned}$$

Taking the limit on n, we have $\frac{|u(\xi)-u(\eta)|^2}{9R(\xi,\eta)} \leq \mathcal{E}_K[u] < \infty$. Hence $R(\xi,\eta) > 0$. $\square$

Corollary 5.3. *With the same assumption as in Theorem 4.2, if $R^{(\lambda)}(\xi,\eta) = 0$, then $\beta_1^* \leq \log\lambda/\log r$ where $\beta_1^* := \sup\{\beta > 0 : \mathcal{D}_K^{(\beta)} \cap C(K)$ is dense in $C(K)\}$.*

Proof. If $R^{(\lambda)}(\xi,\eta) = 0$, then every $u \in \mathcal{D}_K$ must satisfy $u(\xi) = u(\eta)$, so $\mathcal{D}_K$ is not dense in $C(K)$, which implies $\beta_1^* \leq \log\lambda/\log r$. $\square$

Remark. For the implication of (ii) $\Rightarrow$ (iii) in Corollary 5.2, we can omit $\lambda \in (0, r^\alpha)$ (i.e., $\beta > \alpha$), but consider $u \in \mathcal{D}_K \cap C(K)$, and replace $R(\xi,\eta)$ by $\underline{R}(\xi,\eta) := \liminf_{n\to\infty} R_n(\xi,\eta)$, then the implication still holds. Consequently, Corollary 5.3 is still valid.

In the following, we will apply Corollary 5.2 to give some criteria to determine the critical exponents for $\beta_2^* := \sup\{\beta > 0 : \dim \mathcal{D}_K^{(\beta)} = \infty\}$ and $\beta_3^* := \sup\{\beta > 0 : \mathcal{D}_K^{(\beta)}$ contains nonconstant functions$\}$.

Let $\mathbf{i}_n = ii\cdots i \in \mathcal{J}_n$ denote the unique word in level n consisting of symbol $i \in \Sigma$, and let $i^\infty = ii\cdots \in \Sigma^\infty$ (identified with the point $\xi = \lim_n S_{\mathbf{i}_n}(K)$ in K). Then for two distinct symbols i, $j \in \Sigma$, we use $R(i^\infty, j^\infty)$ to denote the effective resistance for the corresponding two points in K, and $R(i^\infty, j^\infty) = \lim_{n\to\infty} R_n(i^\infty, j^\infty)$.

Theorem 5.4. *With the same assumption as in Theorem 4.2, then $\mathcal{D}_K$ consists of only constant functions if and only if*

$$R^{(\lambda)}(i^\infty, j^\infty) = 0, \qquad \forall\, i, j \in \Sigma. \tag{5.4}$$

Consequently, $\beta_3^* = \log \lambda_3^* / \log r$ if

$$\lambda_3^* := \sup\{\lambda > 0 : R^{(\lambda)}(i^\infty, j^\infty) = 0,\ \forall\, i, j \in \Sigma\} \in (0, r^\alpha), \tag{5.5}$$

and $\beta_3^* = \infty$ if the above set of λ is empty.

Proof. If for some $i, j \in \Sigma$, $R(i^\infty, j^\infty) > 0$, then there exists $u \in \mathcal{D}_K$ with $u(i^\infty) \neq u(j^\infty)$ by Proposition 5.1 (or by Corollary 5.2 (iii) $\Rightarrow$ (ii)). Thus it suffices to show that (5.4) implies $\mathcal{D}_K = \{\text{constant functions}\}$.

First we claim that for $u \in C(K)$, if $u(\mathbf{x}i^\infty) = u(\mathbf{x}j^\infty)$ for any $\mathbf{x} \in \Sigma^*$ and $i, j \in \Sigma$, then u is a constant function. Indeed, let $c = u(1^\infty)$, then choosing $\mathbf{x} = \vartheta$ we have $u(i^\infty) = c$ for any $i \in \Sigma$. Choosing $\mathbf{x} = i$ we have $u(ij^\infty) = u(i^\infty) = c$ hence $u(\mathbf{x}j^\infty) = c$ for any $\mathbf{x} \in \Sigma^1$ and $j \in \Sigma$. Following the similar arguments, inductively we have $u(\mathbf{x}j^\infty) = c$ for any $\mathbf{x} \in \Sigma^*$ and $j \in \Sigma$. By continuity, $u \equiv c$ is constant.

For nonconstant $u \in C(K)$, by the claim we can pick $\mathbf{x} \in \Sigma^*$ and $i, j \in \Sigma$ such that $u(\mathbf{x}i^\infty) \neq u(\mathbf{x}j^\infty)$. We telescope u on the cell $S_\mathbf{x}(K)$ to get $\widetilde{u} = u \circ S_\mathbf{x}$. Then $\widetilde{u}(i^\infty) \neq \widetilde{u}(j^\infty)$. By Proposition 5.3 (or by Corollary 5.2 (ii) $\Rightarrow$ (iii)) and assumption (5.4), we must have $\widetilde{u} \notin \mathcal{D}_K$. Note that

$$\begin{aligned}\mathcal{E}_K[u] &\geq c_1 \int_{S_\mathbf{x}(K)} \int_{S_\mathbf{x}(K)} \frac{|u(\xi) - u(\eta)|^2}{|\xi - \eta|^{\alpha+\beta}} d\nu(\xi) d\nu(\eta) \\ &\geq c_2 \int_K \int_K \frac{|\widetilde{u}(\xi) - \widetilde{u}(\eta)|^2}{|\xi - \eta|^{\alpha+\beta}} d\nu(\xi) d\nu(\eta) \geq c_3 \mathcal{E}_K[\widetilde{u}],\end{aligned} \tag{5.6}$$

hence $u \notin \mathcal{D}_K$. Finally as $\mathcal{D}_K \cap C(K) = \mathcal{D}_K$ by Theorem 2.2, $\mathcal{D}_K$ contains constant functions only. $\square$

Next we will show that $\beta_2^* = \beta_3^*$ under the connectedness of the self-similar set. The following lemma is a key step to include more non-trivial functions in $\mathcal{D}_K$.

Lemma 5.5. *With the same assumption as in Theorem 4.2, suppose $\xi \in K$ and Ψ is a closed subset in K satisfying $R(\xi, \Psi) > 0$. Let $u = u_{\xi,\Psi} \in \mathcal{D}_K$ be the limiting harmonic function. Then for $\eta \in K$ such that $0 < u(\eta) < 1$, we have $R(\eta, \Psi) > 0$ and $R(\xi, \Psi \cup \{\eta\}) > 0$.*

Proof. Let $f = f_{\xi,\Psi} = Hu$ and $\varepsilon = \min\{u(\eta), 1-u(\eta)\} > 0$. Fix a κ-sequence $\{\kappa_n\}_n$. By Proposition 3.2, there exists a positive integer m_0 such that

$$|f(\kappa_n(\eta)) - u(\eta)| < \varepsilon/4, \qquad \forall\, n \geq m_0. \tag{5.7}$$

Following the same argument as in the proof of Proposition 5.1, let $f_n \in \ell(X_n)$ be the energy minimizer in (4.9) with $\Phi = \{\xi\}$. By passing to subsequence, we assume, without loss of generality, that $f_n \in \ell(X)$ converges to f pointwise.

Note that for $n \geq 1$ and $k < n$, by Lemma 3.3(i),

$$|f_n(\kappa_k(\eta)) - f_n(\kappa_n(\eta))| \leq \sum_{m=k}^{n-1} |f_n(\kappa_m(\eta)) - f_n(\kappa_{m+1}(\eta))| \leq C_1(\lambda/r^\alpha)^{k/2}.$$

Thus we can pick a positive integer $m_1 \geq m_0$ such that

$$|f_n(\kappa_{m_1}(\eta)) - f_n(\kappa_n(\eta))| < \varepsilon/4, \qquad \forall\, n \geq m_1. \tag{5.8}$$

Since $f_n(\kappa_{m_1}(\eta)) \to f(\kappa_{m_1}(\eta))$ as $n \to \infty$, there exists a positive integer n_0 such that $n_0 \geq m_1$ and

$$|f_n(\kappa_{m_1}(\eta)) - f(\kappa_{m_1}(\eta))| < \varepsilon/4, \qquad \forall\, n \geq n_0. \tag{5.9}$$

Combining (5.7)–(5.9), we have $f_n(\kappa_n(\eta)) \in (\varepsilon/4, 1-\varepsilon/4)$ for all $n \geq n_0$. Using (4.2) and (4.4), for $n \geq n_0$, we have

$$R_n(\eta, \Psi) \geq \frac{f_n(\kappa_n(\eta))^2}{\mathcal{E}_{X_n}[f_n]} > \frac{\varepsilon^2}{16} R_n(\xi, \Psi). \tag{5.10}$$

Hence $R(\eta, \Psi) > 0$ by passing limit.

To prove $R(\xi, \Psi \cup \{\eta\}) > 0$, let $g_n \in \ell(X_n)$ be the energy minimizer in (4.9) with $\Phi = \{\eta\}$. By passing to subsequence if necessary, we let $\gamma_{1,n} = f_n(\kappa_n(\eta))$ and $\gamma_{2,n} = g_n(\kappa_n(\xi))$. Then $\gamma_{1,n}\gamma_{2,n} \in [0, 1-\varepsilon/4)$ as $\gamma_{1,n} \in (\varepsilon/4, 1-\varepsilon/4)$ (by last part) and $\gamma_{2,n} \in [0,1]$. For $n \geq 1$, we can check that the function

$$h_n := \frac{1}{1-\gamma_{1,n}\gamma_{2,n}} f_n - \frac{\gamma_{1,n}}{1-\gamma_{1,n}\gamma_{2,n}} g_n \in \ell(X_n)$$

satisfies $h_n(\kappa_n(\xi)) = 1$, and $h_n = 0$ on $\kappa_n(\Psi \cup \{\eta\})$. Moreover, h_n is harmonic on $X_n \setminus \kappa_n(\Psi \cup \{\xi, \eta\})$, thus $\mathcal{E}_{X_n}[h_n] = (R_n(\xi, \Psi \cup \{\eta\}))^{-1}$ by (4.4). Hence

$$\begin{aligned} R(\xi, \Psi \cup \{\eta\}) &= \lim_{n\to\infty} (\mathcal{E}_{X_n}[h_n])^{-1} \geq \lim_{n\to\infty} \left(\frac{2\mathcal{E}_{X_n}[f_n]}{(1-\gamma_{1,n}\gamma_{2,n})^2} + \frac{2\gamma_{1,n}^2 \mathcal{E}_{X_n}[g_n]}{(1-\gamma_{1,n}\gamma_{2,n})^2} \right)^{-1} \\ &\geq \lim_{n\to\infty} \left(\frac{2}{(\varepsilon/4)^2 R_n(\xi, \Psi)} + \frac{2(1-\varepsilon/4)^2}{(\varepsilon/4)^2 R_n(\eta, \Psi)} \right)^{-1} \\ &= \left(\frac{2}{(\varepsilon/4)^2 R(\xi, \Psi)} + \frac{2(1-\varepsilon/4)^2}{(\varepsilon/4)^2 R(\eta, \Psi)} \right)^{-1} > 0. \end{aligned}$$

$\square$

Theorem 5.6. *With the assumptions in Theorem 4.2, assume further K is connected, and there exists $\beta > \alpha$ such that $\mathcal{D}_K(= \mathcal{D}_K^{(\beta)})$ is non-trivial. Then $\beta_2^* = \beta_3^*$.*

Proof. It suffices to verify that for $\lambda \in (0, r^\alpha)$, $\dim \mathcal{D}_K > 1$ ($\Leftrightarrow \mathcal{D}_K$ contains nonconstant functions) implies that $\dim \mathcal{D}_K = \infty$. We have $R(\xi, \eta) > 0$ for some $\xi, \eta \in K$ by Corollary 5.2 (ii) $\Rightarrow$ (iii). The energy minimizer $u_1 = u_{\xi,\eta} \in \mathcal{D}_K$ is continuous with $u_1(\xi) = 1$ and $u_1(\eta) = 0$, hence there exists $\eta_1 \in K$ such that $u_1(\eta_1) = 1/2$. By Lemma 5.5, we have $R(\xi, \{\eta, \eta_1\}) > 0$ and this induces another energy minimizer $u_2 = u_{\xi,\{\eta,\eta_1\}} \in \mathcal{D}_K$ with $u_2(\xi) = 1$ and $u_2(\eta) = u_2(\eta_1) = 0$. By the continuity, we can pick $\eta_2 \in K$ such that $u_2(\eta_2) = 1/2$. Setting $\eta_0 = \eta$ and repeating the above argument, we get a sequence of energy minimizers $\{u_n\}_{n=1}^\infty$ together with a sequence of points $\{\eta_k\}_{k=0}^\infty$ in K such that $u_n(\xi) = 1$, $u_n(\eta_n) = 1/2$, and $u_n(\eta_k) = 0$ for any $0 \le k < n$. Thus $[u_i(\eta_j)]_{i,j\ge1}$ is an infinite upper triangular matrix with constant diagonal entries $1/2$. Hence $\{u_n\}_{n=1}^\infty$ is a sequence of linearly independent functions in $\mathcal{D}_K$, so that $\dim \mathcal{D}_K = \infty$. $\square$

Remark. The connectivity of K is necessary in Theorem 5.6. For example, if we let $\{S_i\}_{i=1}^4$ be an IFS on $\mathbb{R}$ as follows:

$$S_1(x) = \frac{x}{4}, \quad S_2(x) = \frac{x}{4} + \frac{1}{12}, \quad S_3(x) = \frac{x}{4} + \frac{2}{3}, \quad S_4(x) = \frac{x}{4} + \frac{3}{4}.$$

Then $K = [0, 1/3] \cup [2/3, 1]$, and it is easy to check that the IFS satisfies the OSC (let $O = (0, 1/3) \cup (2/3, 1)$ as the open set). As K consists of two intervals as connected components, we have $\beta_2^* = 2, \beta_3^* = \infty$.

In the rest of this section, we focus on the *post critically finite* (=p.c.f.) self-similar sets [Ki1], and provide a criterion to determine β_1^*. We will need a general lemma as follow.

Lemma 5.7. *With the same assumption as in Theorem 4.2, for a finite set $E \subset K$ with $\#E \ge 2$, if $R(\xi, \eta) > 0$ for all distinct $\xi \ne \eta$ in E, then $R(\xi, E \setminus \{\xi\}) > 0$ for all $\xi \in E$.*

Proof. We prove the lemma by induction on $\#E$. It is trivial if $\#E = 2$. Suppose the lemma holds for $\#E = m$ (m ≥ 2). Now let $\#E = m+1$. We choose arbitrarily three distinct points $\xi_1, \xi_2, \xi_3 \in E$. Then it suffices to show that $R(\xi_1, E \setminus \{\xi_1\}) > 0$. By induction hypothesis, we have three positive effective resistances $R_1 := R(\xi_1, E \setminus \{\xi_1, \xi_2\})$, $R_2 := R(\xi_2, E \setminus \{\xi_2, \xi_3\})$ and $R_3 := R(\xi_3, E \setminus \{\xi_3, \xi_1\})$.

For sufficiently large n, let $f_{1,n},\ f_{2,n},\ f_{3,n} \in \ell(X_n)$ be the energy minimizer in (4.9) with $(\Phi, \Psi) = (\{\xi_1\}, E \setminus \{\xi_1, \xi_2\})$, $(\{\xi_2\}, E \setminus \{\xi_2, \xi_3\})$, $(\{\xi_3\}, E \setminus \{\xi_3, \xi_1\})$ respectively. Fix a κ-sequence $\{\kappa_n\}_n$. Let $\gamma_{1,n} = f_{1,n}(\kappa_n(\xi_2))$, $\gamma_{2,n} = f_{2,n}(\kappa_n(\xi_3))$, and $\gamma_{3,n} = f_{3,n}(\kappa(\xi_1))$. Then $\gamma_{i,n} \in [0,1]$ for $i = 1,2,3$. For sufficiently large n, we can check that the function

$$h_n := \frac{1}{1+\gamma_{1,n}\gamma_{2,n}\gamma_{3,n}}\big(f_{1,n} - (\gamma_{1,n} f_{2,n} + \gamma_{1,n}\gamma_{2,n} f_{3,n})\big) \tag{5.11}$$

satisfies $h_n(\kappa_n(\xi_1)) = 1$, and $h_n = 0$ on $\kappa_n(E \setminus \xi_1)$. Moreover, h_n is harmonic on $X_n \setminus \kappa_n(E)$, thus $\mathcal{E}_{X_n}[h_n] = (R_n(\xi_1, E \setminus \{\xi_1\}))^{-1}$ by (4.4). Hence

$$\begin{aligned} R(\xi_1, E \setminus \{\xi_1\}) &= \lim_{n\to\infty} (\mathcal{E}_{X_n}[h_n])^{-1} \\ &\geq \lim_{n\to\infty} \left(\frac{3(\mathcal{E}_{X_n}[f_{1,n}] + \gamma_{1,n}^2 \mathcal{E}_{X_n}[f_{2,n}] + \gamma_{1,n}^2\gamma_{2,n}^2 \mathcal{E}_{X_n}[f_{3,n}])}{(1+\gamma_{1,n}\gamma_{2,n}\gamma_{3,n})^2} \right)^{-1} \\ &= \left(\frac{3(R_1^{-1} + \gamma_{1,n}^2 R_2^{-1} + \gamma_{1,n}^2\gamma_{2,n}^2 R_3^{-1})}{(1+\gamma_{1,n}\gamma_{2,n}\gamma_{3,n})^2} \right)^{-1} > 0. \end{aligned}$$

This completes the proof of the induction. □

Following Kigami [Ki1], for an IFS $\{S_j\}_{j=1}^N$ with a self-similar set K, we let $C_K = \bigcup_{i,j\in\Sigma, i\neq j}(S_i(K) \cap S_j(K))$, and define a *critical set* by $\mathcal{C} = \pi^{-1}(C_K)$, a *post critical set* by $\mathcal{P} = \bigcup_{n\geq 1} \sigma^n(\mathcal{C})$. We call K *post critically finite* (p.c.f.) if $\mathcal{P}$ is a finite set.

It is known that for the similitudes $S_j = r_j(R_j x + b_j), j = 1, \cdots, N$, if the $\{R_j\}_{j=1}^N$ are commensurable, then the p.c.f. property implies the OSC [DL], and the statement is not true without the commensurable assumption [TKV]. We introduce two geometric conditions on the p.c.f. sets:

(C) *for any family of distinct subcells $S_{i_1}(K), \cdots, S_{i_k}(K)$ that intersects at a point p, there exists $0 < \delta < 1$ and closed cones $\mathcal{C}_j, 1 \leq j \leq k$ with vertex at p such that*

$$S_{i_j}(K) \cap B(p,\delta) \subset \mathcal{C}_j, \quad \textit{and} \quad \mathcal{C}_j \cap \mathcal{C}_\ell = \{p\} \quad \forall\ 1 \leq j,\ \ell \leq k,\ j \neq \ell;$$

(H) *there exists constant $\gamma > 0$ such that for any $\mathbf{x}, \mathbf{y} \in X$ with $|\mathbf{x}| = |\mathbf{y}|$, if $S_{\mathbf{x}}(K) \cap S_{\mathbf{y}}(K) = \emptyset$, then*

$$\operatorname{dist}(S_{\mathbf{x}}(K), S_{\mathbf{y}}(K)) > \gamma \cdot r^{|\mathbf{x}|}.$$

Condition (C) says the intersecting cells are separated by closed cones (except at the vertices), and the geometric meaning is clear. Condition (H) says that if two cells are disjoint, then they are "strongly" separate; it has been used in [Jo], [LW1] and [GuL]. Note that the familiar self-similar sets satisfies this condition, and it is proved in [GuL] the if the IFS is of the form $S_j(x) = r(x + b_j)$ and is p.c.f., then K satisfies condition (H).

Lemma 5.8. *Let K be a p.c.f. self-similar set that satisfies either (C) or (H). Suppose for $\alpha < \beta < \beta'$, u satisfies $u \circ S_i \in \Lambda_{2,2}^{\alpha,\beta'/2}$ for each $i \in \Sigma$, then $u \in \Lambda_{2,2}^{\alpha,\beta/2}$.*

Proof. First suppose that K satisfies (C). By the separation of the cones, and the cosine law of a triangle, we can show that there exists $c > 0$ such that if $S_i(K)$ intersects $S_j(K)$ at p, and for $\xi \in S_i(K) \cap B(p,\delta),\ \eta \in S_j(K) \cap B(p,\delta)$,

$$|\xi - \eta| \geq c(|\xi - p| + |\eta - p|) \geq 2c|\xi - p|^{1/2} \cdot |\eta - p|^{1/2}. \tag{5.12}$$

Since $u \circ S_i \in \Lambda_{2,2}^{\alpha,\beta'/2}$, it follows from Theorem 2.2 that $u \circ S_i \in C^{(\beta'-\alpha)/2}(K)$. As $u(\xi) = \sum_{i=1}^N u(\xi)\chi_{S_i(K)}(\xi)$, we show that u is also Hölder continuous of order $(\beta' - \alpha)/2$ at any $p \in S_i(K) \cap S_j(K)$. Indeed we observe that for $\xi \in S_i(K) \cap B(p,\delta), \eta \in S_j(K) \cap B(p,\delta)$,

$$\begin{aligned}
|u(\xi) - u(\eta)| &\leq |u(\xi) - u(p)| + |u(\eta) - u(p)| \\
&\leq C(|\xi - p|^{(\beta'-\alpha)/2} + |\eta - p|^{(\beta'-\alpha)/2}) \\
&\leq 2C(|\xi - p| + |\eta - p|)^{(\beta'-\alpha)/2} \\
&\leq C_1|\xi - \eta|^{(\beta'-\alpha)/2} \qquad \text{(by (5.12))}.
\end{aligned}$$

This together with (5.12) imply

$$\begin{aligned}
&\int_{S_i(K)\cap B(p,\delta)} \int_{S_j(K)\cap B(p,\delta)} \frac{|u(\xi) - u(\eta)|^2}{|\xi - \eta|^{\alpha+\beta}} d\nu(\xi) d\nu(\eta) \\
\leq{}& C_2 \int_{S_i(K)\cap B(p,\delta)} \frac{d\nu(\xi)}{|\xi - p|^{\alpha+\beta-\beta'}} \cdot \int_{S_j(K)\cap B(p,\delta)} \frac{d\nu(\eta)}{|\eta - p|^{\alpha+\beta-\beta'}} < \infty.
\end{aligned} \tag{5.13}$$

Now as $u(\xi) = \sum_{i=1}^N u(\xi)\chi_{S_i(K)}(\xi)$, we have

$$\begin{aligned}
\mathcal{E}_K[u] &= \sum_{i,j=1}^N \int_{S_i(K)} \int_{S_j(K)} \frac{|u(\xi) - u(\eta)|^2}{|\xi - \eta|^{\alpha+\beta}} d\nu(\xi) d\nu(\eta) \\
&= \Big(\sum_{i=j} + \sum_{i\neq j}\Big) \int_{S_i(K)} \int_{S_j(K)} \frac{|u(\xi) - u(\eta)|^2}{|\xi - \eta|^{\alpha+\beta}} d\nu(\xi) d\nu(\eta) \\
&:= S_I + S_{II}.
\end{aligned}$$

By a change of variable,

$$S_I = \sum_{i=1}^n r_i^{\alpha-\beta} \int_K \int_K \frac{|u \circ S_i(\xi) - u \circ S_i(\eta)|^2}{|\xi - \eta|^{\alpha+\beta}} d\nu(\xi) d\nu(\eta) < \infty.$$

By (5.13), it is easy to check that $S_{II} < \infty$. This shows that $\mathcal{E}_K[u] < \infty$, so that $u \in \mathcal{D}_K = \Lambda_{2,2}^{\alpha,\beta/2}$.

Next we suppose that K satisfies (H). Assume without loss of generality that $\mathrm{diam}(K) = 1$. For $p \in S_i(K) \cap S_j(K)$, $i \neq j \in \Sigma$, let $\delta = \frac{1}{2}\min\{|p-q| : q \in S_i(K) \cap S_j(K), q \neq p\}$. Following the same argument in the last paragraph, it suffices to show that (5.12) holds for $\xi \in S_i(K) \cap B(p,\delta)$ and $\eta \in S_j(K) \cap B(p,\delta)$. Indeed, suppose that $|\eta - p| \le |\xi - p| \in (r^k, r^{k-1}]$ for some positive integer k. Let $\mathbf{x}, \mathbf{y} \in \mathcal{J}_k$ with $\xi \in S_{\mathbf{x}}(K) \subset S_i(K)$ and $\eta \in S_{\mathbf{y}}(K) \subset S_j(K)$. As $\mathrm{diam}(S_{\mathbf{x}}(K)), \mathrm{diam}(S_{\mathbf{y}}(K)) \le r^k$, $S_{\mathbf{x}}(K) \cap S_{\mathbf{y}}(K) = \emptyset$. Hence by condition (H),

$$|\xi - \eta| \ge \gamma \cdot r^k \ge \gamma r |\xi - p| \ge \frac{\gamma r}{2}(|\xi - p| + |\eta - p|).$$

This completes the proof. $\square$

We let $V_0 = \pi(\mathcal{P})$ be the "boundary" of a p.c.f. set K, and let $V_n = \cup_{\mathbf{x} \in \Sigma^n} S_{\mathbf{x}}(V_0)$, $n \ge 1$.

Theorem 5.9. *With the same assumption as in Theorem 4.2, assume further K is a p.c.f. set with boundary V_0 and satisfies* (C) *or* (H)*, then $\mathcal{D}_K$ is dense in $C(K)$ with supremum norm if*

$$R^{(\lambda-\varepsilon)}(\xi,\eta) > 0, \qquad \forall\, \xi \neq \eta \in V_0, \tag{5.14}$$

for some $\varepsilon \in (0,\lambda)$. Consequently, $\beta_1^* = \log \lambda_1^* / \log r$ if

$$\lambda_1^* := \inf\{\lambda > 0 : R^{(\lambda)}(\xi,\eta) > 0,\ \forall\, \xi \neq \eta \in V_0\} \in (0, r^\alpha), \tag{5.15}$$

and $\beta_1^* \le \alpha$ otherwise.

Proof. Let $V_0 = \{\xi_1, \xi_2, \cdots, \xi_m\}$. If $R(\xi_i, \xi_j) = 0$ for some $i \neq j$, then $\mathcal{D}_K$ is not dense in $C(K)$ by Proposition 5.3. Now suppose that (5.14) holds and let $\beta_0 = \log(\lambda - \varepsilon)/\log r$. Then $R^{(\lambda-\varepsilon)}(\xi_i, V_0 \setminus \{\xi_i\}) > 0$ for all i by Lemma 5.7. Thus we can obtain a "basis" of functions $\{u_i\}_{1 \le i \le m} \subset \Lambda_{2,2}^{\alpha,\beta_0/2}$ with $u_i(\xi_j) = \delta_{ij}$ following from Proposition 5.1. Using the linear combinations, for any $v \in \ell(V_0)$, one can check that $u = \sum_{i=1}^m v(\xi_i) u_i \in \Lambda_{2,2}^{\alpha,\beta_0/2} \subset \mathcal{D}_K$ satisfies $u|_{V_0} = v$. Let $\beta_n = \log(\lambda - \frac{\varepsilon}{2^n})/\log r$. We use induction on n to claim that for any $v \in \ell(V_n)$, there exists $u \in \Lambda_{2,2}^{\alpha,\beta_n/2} \subset \mathcal{D}_K$ such that $u|_{V_n} = v$.

Suppose the claim holds for some n. Let $v \in \ell(V_{n+1})$. Note that $V_n = S_i^{-1}(V_{n+1} \cap S_i(K))$ for all $i \in \Sigma$. By induction hypothesis, for each i, there exists $w_i \in \Lambda_{2,2}^{\alpha,\beta_n/2}$ such that $w_i|_{V_n} = v|_{V_{n+1} \cap S_i(K)} \circ S_i$. Let $u(\xi) = \sum_{i=1}^N (w_i \circ S_i^{-1})(\xi) \chi_{S_i(K)}(\xi)$. Then $u|_{V_{n+1}} = v$ and $u \circ S_i = w_i \in \Lambda_{2,2}^{\alpha,\beta_n/2}$. By Lemma 5.8, $u \in \Lambda_{2,2}^{\alpha,\beta_{n+1}/2} \subset \mathcal{D}_K$. This completes the proof of induction.

As n tends to infinity, β_n decreases to $\beta = \log \lambda / \log r$, and $\cup_{n \ge 0} V_n$ is dense in K. Hence $\mathcal{D}_K = \Lambda_{2,2}^{\alpha,\beta/2}$ is dense in $C(K)$. $\square$

6 Network reduction and examples

In this section, we will provide a device to calculate the resistances and the critical exponents of the Besov spaces on K. We first recall some formal notions and techniques on electric network theory [DS, LP].

Let $\mathcal{N} = (V, c)$ denote the *(electric) network* with vertex set V (finite or countably infinite) and conductance $c : V \times V \to [0, \infty)$ ($c(x, y) = c(y, x)$ for all $x, y \in V$). The edge set $E = \{(x, y) \in (V \times V) \setminus \Delta : c(x, y) > 0\}$. An edge $(x, y) \in E$ is referred as a *resistor* (or *conductor*) with resistance $r_{xy} = r(x, y) = c(x, y)^{-1}$. The energy of $f \in \ell(V)$ on $\mathcal{N}$ is given by

$$\mathcal{E}_{\mathcal{N}}[f] = \frac{1}{2} \sum_{x,y \in V} c(x, y)(f(x) - f(y))^2 \tag{6.1}$$

as in (2.4). Also we can define the effective resistance $R_{\mathcal{N}}(A, B)$ between two nonempty subsets $A, B \subset V$ as in (4.2).

Definition 6.1. *For two networks $\mathcal{N}_1 = (V_1, c_1)$ and $\mathcal{N}_2 = (V_2, c_2)$ with a set of common vertices $U \subset V_1 \cap V_2$, $\#U \geq 2$, we say that $\mathcal{N}_1$ and $\mathcal{N}_2$ are* equivalent *on U if for any $f \in \ell(U)$,*

$$\inf\{\mathcal{E}_{\mathcal{N}_1}[g_1] : g_1 \in \ell(V_1), g_1|_U = f\} = \inf\{\mathcal{E}_{\mathcal{N}_2}[g_2] : g_2 \in \ell(V_2), g_2|_U = f\}. \tag{6.2}$$

It is easy to show that if $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent on U, then they are also equivalent on any $U' \subset U$. As a result, $R_{\mathcal{N}_1}(A, B) = R_{\mathcal{N}_2}(A, B)$ for any nonempty $A, B \subset U$.

The two most basic transformations to reduce networks to equivalent ones are the *series law* and the *parallel law* of resistance. The third one is the Δ-Y *transform* (or *star-triangle Law*): let $\mathcal{N}_1$ be the triangle shaped network with $V_1 = \{x, y, z\}$ as on the left of Figure 1, and let $\mathcal{N}_2$ be the starlike network on the right with $V_2 = V_1 \cup \{p\}$; for the two network to be equivalent, the resistances are related by

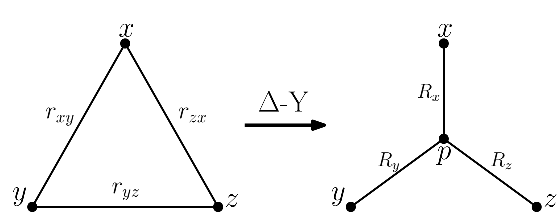

Figure 1: Δ-Y transform

$$R_x = \frac{r_{xy} r_{zx}}{r_{xy} + r_{yz} + r_{zx}}, \quad R_y = \frac{r_{xy} r_{yz}}{r_{xy} + r_{yz} + r_{zx}}, \quad R_z = \frac{r_{zx} r_{yz}}{r_{xy} + r_{yz} + r_{zx}}$$

respectively. For some network $\mathcal{N} = \{V, c\}$, $\#V > 3$ with proper symmetry, we can add one vertex and transform it to an equivalent starlike network (see the examples in the sequel and [K] for more details); we regard such transformation as a *generalized* Δ-Y transform.

More generally, we have from Proposition 4.1, that if $V = V^\circ \cup \partial V$, $\#\partial V \geq 2$ then for $f \in \ell(\partial V)$,

$$\min\{\mathcal{E}_{\mathcal{N}}[g] : g \in \ell(V), g|_{\partial V} = f\} = \frac{1}{2} \sum_{x,y \in \partial V, x \neq y} c_*(x,y)(f(x) - f(y))^2. \qquad (6.3)$$

Then the network $\mathcal{N}_* = \{\partial V, c_*\}$ is equivalent to $\mathcal{N}$ on ∂V. For proper ∂V, the graph of network $\mathcal{N}_*$ may contain a complete subgraph K_n. In this case, we say that the transform $\mathcal{N} \to \mathcal{N}_*$ is a *local completion*. For example, as in Figure 2, let $\partial V = \{x_1, x_2, \ldots, x_5\}$, then the graph of $\mathcal{N}_*$ is a complete graph K_5.

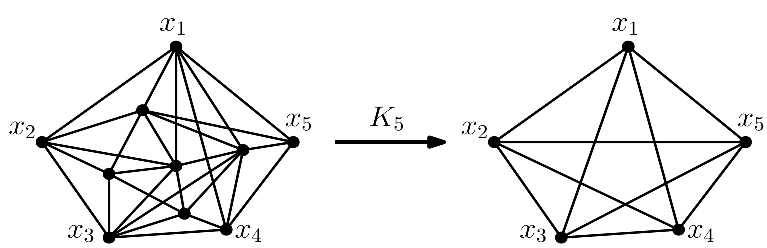

Figure 2: Local completion

Besides the above mentioned transformations, there are other basic tools in network reduction we will use: cutting and shorting, and the Rayleigh's monotonicity law, namely, *if some resistances of resistors in a network are increased (decreased), then the effective resistance between any two points in the graph can only increase (decrease).*

Example 6.2. Cantor middle third set *Let $S_1(\xi) = \frac{1}{3}\xi$ and $S_2(\xi) = \frac{1}{3}(\xi + 2)$ on $\mathbb{R}$. Then the self-similar set K is the Cantor middle-third set with ratio $r = \frac{1}{3}$. It is totally disconnected and the Hausdorff dimension is $\alpha = \frac{\log 2}{\log 3}$. The critical exponents $\beta_1^* = \beta_2^* = \beta_3^* = \infty$.*

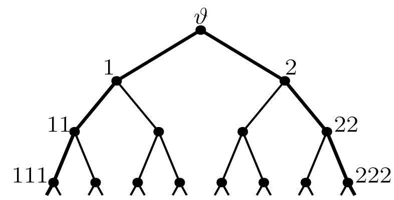

Figure 3: The limiting resistance for Cantor set

Indeed, for $\lambda \in (0, \frac{1}{2})$ ($r^\alpha = \frac{1}{2}$), the resistance between 0 and 1 (see Figure 3) is

$$\begin{aligned} R^{(\lambda)}(0,1) &= R^{(\lambda)}(1^\infty, 2^\infty) = \lim_{n\to\infty} R_n^{(\lambda)}(1^n, 2^n) \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^n c(1^k, 1^{k-1})^{-1} + \sum_{k=1}^n c(2^k, 2^{k-1})^{-1} \right) \\ &= 2 \lim_{n\to\infty} \sum_{k=1}^n (2\lambda)^{-k} = \frac{4\lambda}{1-2\lambda}, \end{aligned}$$

and Theorem 5.9 implies the result. $\square$

Example 6.3. Sierpinski gasket *It is the self-similar set K generated by the maps $S_i(\xi) = \frac{1}{2}(\xi - e_{i-1}) + e_{i-1}$ where $e_0 = 0$ and $e_i, i = 1, \dots, N-1$ are the standard basis vectors in $\mathbb{R}^{N-1}$. It is a p.c.f. set with $\mathcal{P} = \{1^\infty, 2^\infty, \dots, N^\infty\}$, and $\alpha = \dim_H K = \frac{\log N}{\log 2}$. For the λ-NRW ($r^\alpha = \frac{1}{N}$), the conductance is $c(\mathbf{x}, \mathbf{x}^-) = c(\mathbf{x}, \mathbf{y}) = (\lambda N)^{-|\mathbf{x}|}$ where $\mathbf{x} \sim_h \mathbf{y}$. The critical exponent of $\Lambda_{2,2}^{\alpha,\beta/2}$ is*

$$\beta_1^* = \beta_2^* = \beta_3^* = \frac{\log(N+2)}{\log 2} \quad at \quad \lambda = \frac{1}{N+2}.$$

(The critical exponent is known in [Jo].)

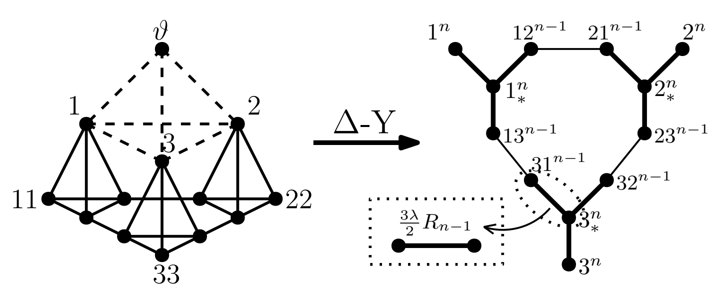

Figure 4: Cutting in Sierpinski gasket, $N = 3$

We only prove the case $N = 3$, the other case is the same (the reader is also advised to use $N = 2$ to get a clearer picture). By symmetry, it suffices to find the resistance $R^{(\lambda)}(1^\infty, 2^\infty)$. We denote $R_n^{(\lambda)} = R_n^{(\lambda)}(1^n, 2^n)$ for short.

To estimate the upper bound, we delete the edges (ϑ, i), (ij^k, ji^k), for $i \neq j \in \Sigma$, $k = 0, 1, \ldots, n-2$ in the subgraph of X_n (see Figure 4). Then we get a new subgraph consisting of 3 copies of X_{n-1} with 3 horizontal edges (ij^{n-1}, ji^{n-1}), $i \neq j \in \Sigma$ at level n connecting them; we label these copies by $1, 2, 3$ such that the copy i contains the vertex i^n. Then apply the the Δ-Y transform to the three vertices in $\mathcal{A}_i := \{ij^{n-1} : j \in \Sigma\}$ at the n-th level of each copy to get a starlike tree with center i_*^n, $i \in \Sigma$ respectively. As the resistance between any pair of vertices in $\mathcal{A}_i$ equals $3\lambda R_{n-1}$, it follows that the resistance between i_*^n and a vertex in $\mathcal{A}_i$ in the corresponding starlike tree is $\frac{3\lambda}{2} R_{n-1}$. Moreover, between any pair i_*^n, j_*^n, $i \neq j$, there is a 3-step path $[i_*^n, ij^{n-1}, ji^{n-1}, j_*^n]$. Replacing these paths with resistors, we get a triangle with vertices $\{i_*^n : i \in \Sigma\}$ and each side has resistance $3\lambda R_{n-1} + (3\lambda)^n$.

By applying the monotonicity law and the series law,

$$\begin{aligned} R_n^{(\lambda)} &\leq R(1^n, 1_*^n) + R(1_*^n, 2_*^n) + R(2_*^n, 2^n) \\ &= \frac{3\lambda}{2} R_{n-1}^{(\lambda)} + \frac{2}{3}(3\lambda R_{n-1}^{(\lambda)} + (3\lambda)^n) + \frac{3\lambda}{2} R_{n-1}^{(\lambda)} \\ &= 5\lambda R_{n-1}^{(\lambda)} + 2 \cdot 3^{n-1} \lambda^n. \end{aligned}$$

Hence $R^{(\lambda)}(1^\infty, 2^\infty) = \lim_{n\to\infty} R_n^{(\lambda)} = 0$ for $\lambda \in (0, \frac{1}{5})$. By Proposition 5.3 and Theorem 5.4, we have $\beta_1^* \leq \beta_3^* \leq \frac{\log 5}{\log 2}$.

To obtain the lower bound of the critical exponent, we need another technique. We reassign the conductance on the n-th level of the subgraph X_n ($n \geq 1$): for $\mu > 0$, let $\widetilde{c}(\mathbf{x}, \mathbf{x}^-) = (3\lambda)^{-|\mathbf{x}|}$ for $\mathbf{x} \in X_n$, and let

$$\widetilde{c}(\mathbf{x}, \mathbf{y}) = \begin{cases} (3\lambda)^{-|\mathbf{x}|}, & \text{if } |\mathbf{x}| < n, \\ \mu^{-1}(3\lambda)^{-n}, & \text{if } |\mathbf{x}| = n, \end{cases} \qquad \text{for } \mathbf{x} \sim_h \mathbf{y} \in X_n.$$

Denote the resistance between 1^n and 2^n with respect to the above $\widetilde{c}$ by $R_n^{(\lambda,\mu)}$. Then apply the generalized Δ-Y transforms to each triangle $(\mathbf{x}, \mathbf{x}1, \mathbf{x}2, \mathbf{x}3)$ for $\mathbf{x} \in \mathcal{J}_{n-1}$, and then replace each pair $\{\mathbf{x}, \mathbf{x}'\}$ by a single $\mathbf{x}$ (see Figure 5 for $N = 2$ for a clearer illustration; Figure 6 for $N = 3$ corresponds to the dotted box in Figure 5).

We have

$$R_n^{(\lambda,\mu)} \geq \frac{2\mu}{\mu + 3}(3\lambda)^n + R_{n-1}^{(\lambda,\phi(\mu))}, \tag{6.4}$$

where ϕ is given by the parallel resistance formula

$$\phi(\mu)^{-1} = \left[3\lambda\left(\frac{2\mu}{\mu+3} + \mu\right)\right]^{-1} + 1. \tag{6.5}$$

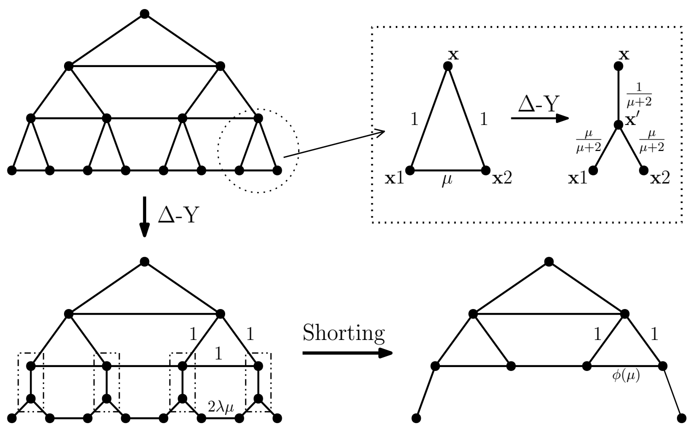

Figure 5: μ-parameter and shorting for $N = 2$

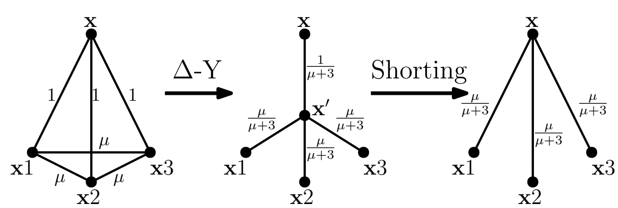

Figure 6: μ-parameter and shorting for $N = 3$

The equation $\phi(\mu) = \mu$ has a solution $\overline{\mu} \in (0, 1)$ if and only if $\lambda > \frac{1}{5}$. With such fixed point $\overline{\mu}$, by (6.4), we have $R^{(\lambda)}(1^\infty, 2^\infty) \geq \lim_{n\to\infty} R_n^{(\lambda,\overline{\mu})} \geq R_1^{(\lambda,\overline{\mu})} > 0$. By Theorem 5.7, we have $\frac{\log 5}{\log 2} \leq \beta_1^* \leq \beta_3^*$, and completes the proof. $\square$

In the next example, we adjust the above method slightly for the new situation with two different effective resistances of (i^∞, j^∞).

Example 6.4. Pentagasket *The pentagasket K is the attractor K of the five similitudes $S_i(\xi) = \frac{3-\sqrt{5}}{2}(\xi - p_i) + p_i$, here we identify $\mathbb{R}^2$ with $\mathbb{C}$, and $p_i = e^{2\pi i/5}$. It is a p.c.f. set with $\mathcal{P} = \{1^\infty, 2^\infty, \ldots, 5^\infty\}$, and $\alpha := \dim_H K = -\frac{\log 5}{\log((3-\sqrt{5})/2)}$. As $r^\alpha = \frac{1}{5}$, the λ-NRW has conductance $(5\lambda)^{-n}$ on level n. The critical exponent is*

$$\beta_1^* = \beta_2^* = \beta_3^* = \frac{\log((\sqrt{161} - 9)/40)}{\log((3 - \sqrt{5})/2)} \quad at \quad \lambda = \frac{\sqrt{161} - 9}{40}.$$

To determine the critical exponent, we need to calculate the resistances

$$R^{(\lambda)}(1^\infty, 2^\infty) \quad \text{and} \quad R^{(\lambda)}(1^\infty, 3^\infty).$$

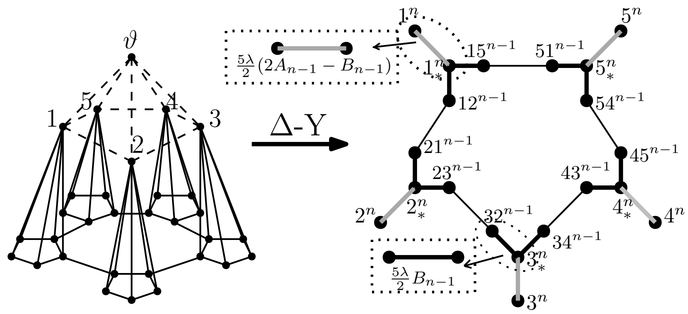

Figure 7: Cutting in pentagasket

We denote $A_n = R_n^{(\lambda)}(1^n, 3^n)$ and $B_n = R_n^{(\lambda)}(1^n, 2^n)$ for short. By referring to Figure 7, and using the same technique as before, we have

$$\begin{aligned} A_n &\le R(1^n, 1^n_*) + R(1^n_*, 3^n_*) + R(3^n_*, 3^n) \\ &= 5\lambda(2A_{n-1} - B_{n-1}) + \left[(10\lambda B_{n-1} + 2(5\lambda)^n)^{-1} + (15\lambda B_{n-1} + 3(5\lambda)^n)^{-1}\right]^{-1} \\ &= 10\lambda A_{n-1} + \lambda B_{n-1} + \frac{6}{5}(5\lambda)^n. \end{aligned}$$

Analogously, we have $B_n \le 10\lambda A_{n-1} - \lambda B_n + \frac{4}{5}(5\lambda)^n$. As the coefficient matrix $\begin{pmatrix} 10\lambda & \lambda \\ 10\lambda & -\lambda \end{pmatrix}$ has eigenvalues $\frac{9\pm\sqrt{161}}{2}\lambda$, we have $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = 0$ if $\lambda < (\frac{9+\sqrt{161}}{2})^{-1} = \frac{\sqrt{161}-9}{40}$. Hence $R^{(\lambda)}(1^\infty, 2^\infty) = R^{(\lambda)}(1^\infty, 3^\infty) = 0$ for $\lambda \in (0, \frac{\sqrt{161}-9}{40})$. By Proposition 5.3 and Theorem 5.4, we have $\beta_1^* \le \beta_3^* \le \frac{\log((\sqrt{161}-9)/40)}{\log((3-\sqrt{5})/2)}$.

To obtain the lower bound of the critical exponents, we reassign the conductance on the bottom of the subgraph X_n $(n \ge 1)$ with two parameters μ_1 and μ_2 : for $\mu_1, \mu_2 \in (0,1)$, let $\widetilde{c}(\mathbf{x}, \mathbf{x}^-) = (5\lambda)^{-|\mathbf{x}|}$ for $\mathbf{x} \in X_n$, and let

$$\widetilde{c}(\mathbf{x}, \mathbf{y}) = \begin{cases} (5\lambda)^{-|\mathbf{x}|}, & \text{if } |\mathbf{x}| < n, \\ \mu_1^{-1}(5\lambda)^{-n}, & \text{if } |\mathbf{x}| = n \text{ and } \mathbf{x}^- = \mathbf{y}^-, \\ \mu_2^{-1}(5\lambda)^{-n}, & \text{if } |\mathbf{x}| = n \text{ and } \mathbf{x}^- \ne \mathbf{y}^-, \end{cases} \qquad \text{for } \mathbf{x} \sim_h \mathbf{y} \in X_n.$$

Denote the resistance between 1^n and 3^n (or 2^n) with respect to above $\widetilde{c}$ by $A_n^{(\mu_1,\mu_2)}$ (or $B_n^{(\mu_1,\mu_2)}$). We apply the local completion to each cone $(\mathbf{x}1, \mathbf{x}11, \mathbf{x}13, \mathbf{x}14)$, $(\mathbf{x}2, \mathbf{x}22, \mathbf{x}24, \mathbf{x}25)$, $(\mathbf{x}3, \mathbf{x}33, \mathbf{x}35, \mathbf{x}31)$, $(\mathbf{x}4, \mathbf{x}44, \mathbf{x}41, \mathbf{x}42)$, $(\mathbf{x}5, \mathbf{x}55, \mathbf{x}52, \mathbf{x}53)$ for $\mathbf{x} \in \mathcal{J}_{n-2}$, and then replace each complete subgraph K_4 by a starlike network with greater energy (Figure 8). By a direct calculation, the conductance c_* in K_4 is given by

$$c_*(\mathbf{x}1, \mathbf{x}11) = \frac{\mu_1 + 4}{\mu_1 + 2}, \quad c_*(\mathbf{x}1, \mathbf{x}13) = c_*(\mathbf{x}1, \mathbf{x}14) = \frac{\mu_1 + 3}{\mu_1 + 2},$$

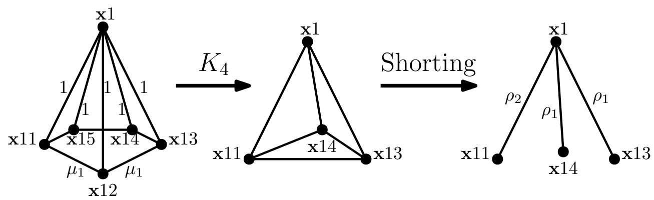

Figure 8: Shorting in pentagasket

$$c_*(\mathbf{x11},\mathbf{x13}) = c_*(\mathbf{x11},\mathbf{x14}) = \frac{1}{\mu_1(\mu_1+2)}, \quad c_*(\mathbf{x13},\mathbf{x14}) = \frac{1}{\mu_1},$$

and the resistances in the star are given by

$$\rho_1 = \frac{\mu_1(\mu_1+2)}{\mu_1^2+5\mu_1+5}, \quad \rho_2 = \frac{\mu_1(\mu_1+2)^2}{(\mu_1+1)(\mu_1^2+5\mu_1+5)}.$$

By the monotonicity law and series law,

$$A_n^{(\mu_1,\mu_2)} \geq 2\rho_2(5\lambda)^n + A_{n-1}^{(\phi_1(\mu_1,\mu_2),\phi_2(\mu_1,\mu_2))}, \tag{6.6}$$

(same inequality holds if we replace A by B) where ϕ_1 and ϕ_2 are given by the parallel resistance formulas

$$\begin{cases} \phi_1(\mu_1,\mu_2)^{-1} = [5\lambda\,(2\rho_1+\mu_2)]^{-1} + 1, \\ \phi_2(\mu_1,\mu_2)^{-1} = [5\lambda\,(2\rho_2+\mu_2)]^{-1} + 1. \end{cases} \tag{6.7}$$

The equations $\phi_i(\mu_1,\mu_2) = \mu_i$, $i=1,2$ have a solution $(\overline{\mu}_1,\overline{\mu}_2) \in (0,1)^2$ if and only if $\lambda > \frac{\sqrt{161}-9}{40}$. With such fixed point $(\overline{\mu}_1,\overline{\mu}_2)$, by (6.6), we have $R^{(\lambda)}(1^\infty,3^\infty) \geq \lim_{n\to\infty} A_n^{(\overline{\mu}_1,\overline{\mu}_2)} \geq A_1^{(\overline{\mu}_1,\overline{\mu}_2)} > 0$. Similarly we also have $R^{(\lambda)}(1^\infty,2^\infty) > 0$ if $\lambda > \frac{\sqrt{161}-9}{40}$. By Theorem 5.7, we have $\frac{\log((\sqrt{161}-9)/40)}{\log((3-\sqrt{5})/2)} \leq \beta_1^* \leq \beta_3^*$. $\square$

More computational issues on the critical exponent of nested fractals can be found in [K]. Finally, we give an example that $\beta_1^* \neq \beta_3^*$.

Example 6.5. Cantor set×interval *Let $\Sigma = \{1,2,3,4,5,6\}$ and let $p_1 = 0, p_2 = (0,\frac{1}{3}), p_3 = (0,\frac{2}{3}), p_4 = (\frac{2}{3},0), p_5 = (\frac{2}{3},\frac{1}{3}), p_6 = (\frac{2}{3},\frac{2}{3})$ in $\mathbb{R}^2$. For $i \in \Sigma$, let $S_i(\xi) = \frac{1}{3}\xi + p_i$ on $\mathbb{R}^2$. Then the self-similar set K is the product of a Cantor middle-third set and a unit interval (see the associated augmented tree in Figure 9), and $\alpha = \dim_H K = \frac{\log 2}{\log 3} + 1 = \frac{\log 6}{\log 3}$. The λ-NRW has conductance $(6\lambda)^{-n}$ on the n-th level ($r^\alpha = \frac{1}{6}$). The critical exponents are*

$$\beta_1^* = 2 \quad \textit{at} \quad \lambda = \frac{1}{9}; \qquad \beta_2^* = \beta_3^* = \infty$$

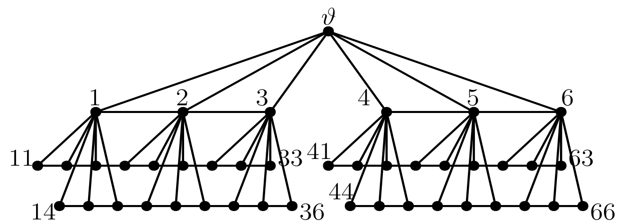

Figure 9: The graph for Cantor set×inteval

First we show that $R^{(\lambda)}(1^\infty, 4^\infty) > 0$ for any $\lambda > 0$. For $n \geq 1$, consider a function f_n on X_n defined by

$$f_n(\mathbf{x}) = \begin{cases} 1/2, & \text{if } \mathbf{x} = \vartheta, \\ 1, & \text{if } i_1 = 1, 2, 3, \\ 0, & \text{if } i_1 = 4, 5, 6, \end{cases} \qquad \text{for } \mathbf{x} = i_1 i_2 \cdots i_k \in X_n.$$

Then by (4.2), $R_n^{(\lambda)}(1^n, 4^n) \geq (\mathcal{E}_{X_n}[f_n])^{-1} = (6 \cdot (\frac{1}{2})^2 \cdot \frac{1}{6\lambda})^{-1} = 4\lambda$. Thus for any $\lambda > 0$, $R^{(\lambda)}(1^\infty, 4^\infty) = \lim_{n\to\infty} R_n^{(\lambda)}(1^n, 4^n) \geq 4\lambda > 0$. By Theorem 5.4, we have $\beta_3^* = \infty$. Also it is easy to see that $\beta_2^* = \infty$.

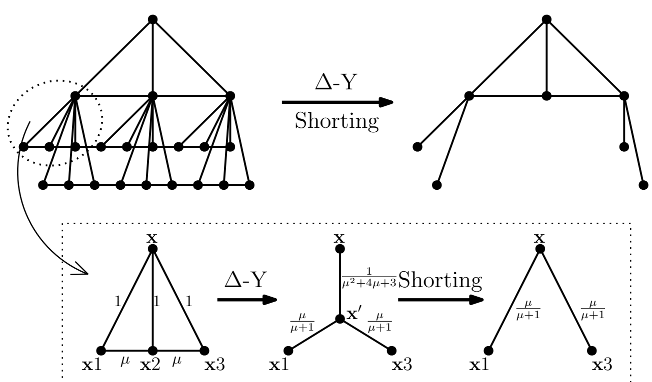

Figure 10: Shorting in Cantor set × interval

Next we consider the effective resistance $R^{(\lambda)}(1^\infty, 3^\infty)$ by using a similar shorting device as in previous examples. Denote $R_n^{(\lambda)} = R_n^{(\lambda)}(1^n, 3^n)$ for short. As in Example 6.4, we reassign the conductance on the bottom of the subgraph X_n by an additional factor μ^{-1}, and by the same method applied to triangles $(\mathbf{x}, \mathbf{x}1, \mathbf{x}3)$ (also to $(\mathbf{x}, \mathbf{x}4, \mathbf{x}6)$, see Figure 10), we have

$$R_n^{(\lambda,\mu)} \geq \frac{2\mu}{\mu+1}(6\lambda)^n + R_{n-1}^{(\lambda,\phi(\mu))}, \tag{6.8}$$

where ϕ is given by

$$\phi(\mu)^{-1} = 2\left[6\lambda\left(\frac{2\mu}{\mu+1}+\mu\right)\right]^{-1} + 1. \tag{6.9}$$

The equation $\phi(\mu) = \mu$ has a solution $\overline{\mu} \in (0,1)$ if and only if $\lambda > \frac{1}{9}$.

With such fixed point $\overline{\mu}$, by (6.8), we have $R^{(\lambda)}(1^\infty, 3^\infty) \geq \lim_{n\to\infty} R_n^{(\lambda,\overline{\mu})} \geq R_1^{(\lambda,\overline{\mu})} > 0$. On the other hand, we show that if $R^{(\lambda)}(1^\infty, 3^\infty) > 0$, then $\lambda \geq \frac{1}{9}$. Without loss of generality, we assume that $0 < \lambda < 1/6$. For $n \geq 1$, let f_n be the energy minimizer (harmonic function) on X_n with boundary conditions $f_n(1^n) = 1$ and $f_n(3^n) = 0$. Then $R_n(1^n, 3^n) = \mathcal{E}_{X_n}[f_n]^{-1}$. By Corollary 5.2 (iv) $\Rightarrow$ (iii), let $C_1 := \sup_{n\geq 1} \mathcal{E}_{X_n}[f_n] = (\inf_{n\geq 1} R_n(1^n, 3^n))^{-1} < \infty$. Pick a positive integer n_1 such that $\sum_{n=n_1+1}^{\infty} (6\lambda)^n < \frac{1}{36C_1}$. Then for $n \geq n_1$,

$$|f_n(1^n) - f_n(1^{n_1})|^2 \leq \mathcal{E}_{X_n}[f_n] R_{X_n}(1^n, 1^{n_1}) \leq C_1 \sum_{k=n_1+1}^{n} (6\lambda)^k \leq \frac{1}{36}, \tag{6.10}$$

which implies $f_n(1^{n_1}) \geq \frac{5}{6}$. Analogously we have $f_n(3^{n_1}) \leq \frac{1}{6}$. Let $m = n - n_1$. With a similar argument as in (6.10), for $\mathbf{z} \in \{1,4\}^m$,

$$|f_n(1^{n_1}\mathbf{z}) - f_n(1^{n_1})|^2 \leq \mathcal{E}_{X_n}[f_n] R_{X_n}(1^{n_1}\mathbf{z}, 1^{n_1}) \leq \frac{1}{36},$$

which implies $f_n(1^{n_1}\mathbf{z}) \geq \frac{2}{3}$. Analogously we have $f_n(3^{n_1}\mathbf{w}) \leq \frac{1}{3}$ for all $\mathbf{w} \in \{3,6\}^m$. Now, for $\mathbf{z} = i_1 i_2 \cdots i_m \in \{1,4\}^m$, denote the word $j_1 j_2 \cdots j_m \in \{3,6\}^m$ with $j_k = i_k + 2$ for all k by $\mathbf{z}'$. Note that for each $\mathbf{z} \in \{1,4\}^m$, there is a horizontal path with length $3^n - 1$ from $1^{n_1}\mathbf{z}$ to $3^{n_1}(\mathbf{z}')$. The resistance on such path is given by $R_{\mathcal{J}_n}(1^{n_1}\mathbf{z}, 3^{n_1}(\mathbf{z}')) = (3^n - 1)(6\lambda)^n$. Counting the energy on these 2^m disjoint horizontal paths, we get

$$C_1 \geq \mathcal{E}_{X_n}[f_n] \geq \sum_{\mathbf{z}\in\{1,4\}^m} \frac{[f_n(1^{n_1}\mathbf{z}) - f_n(3^{n_1}(\mathbf{z}'))]^2}{R_{\mathcal{J}_n}(1^{n_1}\mathbf{z}, 3^{n_1}(\mathbf{z}'))} \geq \frac{2^{n-n_1}}{9(3^n-1)(6\lambda)^n}$$

for arbitrary $n \geq n_1$. Hence $\lambda \geq \frac{1}{9}$ and the claim follows. By Proposition 5.3, we have $\beta_1^* = 2$. $\square$

Remark. To investigate the situation that $\beta_1^* < \beta_3^*$, it is natural to study the products of self-similar sets. But in general, if K_1 and K_2 are connected self-similar sets, then the critical exponent of the product $K_1 \times K_2$ satisfies

$$\beta_1^* \leq \max\{\dim_H K_1, \dim_H K_2\} + 1 \leq \dim_H K_1 + \dim_H K_2 = \alpha.$$

Although the criteria in the last section cannot be applied directly, it still has a similar link between the effective resistance of $\mathcal{E}_X$ and the energy on the product (see [K] for more details). For example, in the product $[0,1] \times SG$, the effective resistances $R^{(\lambda)}(i^\infty, j^\infty)$ have two critical exponents $\lambda_1^* = \frac{1}{4}$ and $\lambda_3^* = \frac{1}{5}$ for various i, j, while $2 = \beta_1^* < \frac{\log 5}{\log 2} = \beta_3^* < \alpha = \frac{\log 6}{\log 2}$. With a similar technique as in Example 6.5, it follows that $\beta_1^* = 2$ if one of K_i is a unit interval. To generalize the results above, we may leave a conjecture as

$$\beta_1^*(K_1 \times K_2) = \min\{\beta_1^*(K_1), \beta_1^*(K_2)\}, \quad \text{and} \quad \beta_3^*(K_1 \times K_2) = \max\{\beta_3^*(K_1), \beta_3^*(K_2)\}.$$

7 Remarks and open problems

The calculation of the critical exponents in Section 6 depends very much on the p.c.f. property. It is challenging to find an effective technique to estimate the non-p.c.f. sets like the Sierpinski carpet.

In our discussions, we assumed the return ratio $\lambda \in (0, r^\alpha)$ (hence $\alpha < \beta_1^*$) in order to guarantee functions in the domain of the induced bilinear form on K are continuous (Proposition 2.5). While the condition is satisfied by the well-known fractals, it also excludes the situation that $\beta_1^* \le \alpha$, which contains important examples (e.g., the classical domain, and product of fractals). We conjecture that the consideration in the paper is possible to adjust to this case. We also like to know if there is a nice sufficient condition for $\alpha < \beta_1^*$ based on the geometry of the self-similar sets.

We call a self-similar set K *mono-critical* if it has a single critical exponent $\beta^* = \beta^*(K)$, i.e., $\beta^* = \beta_1^* = \beta_2^* = \beta_3^*$. It is known that all nested fractals, Cantor-type sets, and some non-p.c.f. sets including Sierpinski carpet (see [B, BB]) are mono-critical. For these sets, the critical exponent plays an important role. It is well-known that $\Lambda_{2,2}^{\alpha,\beta^*/2}$ is trivial (see [Jo, P1]) while $\Lambda_{2,\infty}^{\alpha,\beta^*/2}$ admits a local regular Dirichlet form on $L^2(K)$. On the other hand, it is constructed in [GuL] a modified Vicsek set that is mono-critical; on this set, $\Lambda_{2,\infty}^{\alpha,\beta^*/2}$ is dense in $L^2(K,\nu)$, but is not dense in $C(K)$, and there is a local regular Dirichlet form support by K which is not $\Lambda_{2,\infty}^{\alpha,\beta^*/2}$, and does not satisfy the energy self-similar identity [Ki1].

In conclusion, the question of constructing a local Dirichlet form on a self-similar set is still unsettled. In view of the above, it will be very interesting to study this situation in general, in particular, to use the return rate λ of the random walk to study the boundary case.

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