LIOUVILLE PROPERTIES FOR p-HARMONIC MAPS WITH FINITE q-ENERGY

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ABSTRACT. We introduce and study an approximate solution of the p-Laplace equation, and a linearlization $\mathcal{L}_\epsilon$ of a perturbed p-Laplace operator. By deriving an $\mathcal{L}_\epsilon$-type Bochner's formula and Kato type inequalities, we prove a Liouville type theorem for weakly p-harmonic functions with finite p-energy on a complete noncompact manifold M which supports a weighted Poincaré inequality and satisfies a curvature assumption. This nonexistence result, when combined with an existence theorem, yields in turn some information on topology, i.e. such an M has at most one p-hyperbolic end. Moreover, we prove a Liouville type theorem for strongly p-harmonic functions with finite q-energy on Riemannian manifolds. As an application, we extend this theorem to some p-harmonic maps such as p-harmonic morphisms and conformal maps between Riemannian manifolds. In particular, we obtain a Picard-type Theorem for p-harmonic morphisms.

1. Introduction

The study of p-harmonic maps and in particular p-harmonic functions is central to p-harmonic geometry and related problems.

A real-valued C^3 function on a Riemannian m-manifold M with a Riemannian metric $\langle\ ,\ \rangle$ is said to be *strongly p-harmonic* if u is a (strong) solution of the p-Laplace equation (1.1), $p > 1$,

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0. \tag{1.1}$$

where ∇u is the gradient vector field of u on M, and $|\nabla u| = \langle \nabla u, \nabla u\rangle^{\frac{1}{2}}$.

A function $u \in W^{1,p}_{loc}(M)$ is said to be *weakly p-harmonic* if u is a (Sobolev) weak solution of the p-Laplace equation (1.1), i.e.

$$\int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \phi\rangle \, dv = 0$$

holds for every $\phi \in C_0^\infty(M)$, where dv is the volume element of M.

The p-Laplace equation (1.1) arises as the Euler-Lagrange equation of the p-energy E_p functional given by $E_p(u) = \int_M |\nabla u|^p \, dv$. Ural'tseva [45], Evans [7] and Uhlenbeck [46] proved that weak solutions of the equation (1.1) have Hölder continuous derivatives for $p \geq 2$. Tolksdorff [43], Lewis [24] and DiBenedetto [5] extended the result to $p > 1$. In fact, weak solutions of (1.1), in general do not have any regularity better than $C^{1,\alpha}_{loc}$.

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When $p = 2$, p-harmonic functions are simply harmonic functions. Liouville type properties or topological end properties have been studied by a long list of authors. We refer the reader to, for example [22], [27], [28], [29], [30], [32], [33], [34], [37], [42] for further references. In particular, P. Li and J. Wang showed Liouville type properties and splitting type properties on complete noncompact manifolds with positive spectrum λ when the Ricci curvature has a lower bound depending on λ. They also extended their work to a complete noncompact manifold with weighted Poincaré inequality (P_ρ).

For $p > 1$, We refer the works, for example [3], [6], [9], [10], [11], [12], [13], [15], [16], [21], [36], [38], [47], [49], to the reader. In particular, I. Holopainen [11] proved a sharp L^q-Liouville properties for p-harmonic functions, i.e. if $u \in L^q(M)$ is p-harmonic (or more generally, $\mathcal{A}$-harmonic) in M with $q > p-1$, then u is constant. For $q = p-1$ and $m \geq 2$, there exist a complete Riemannian m-manifold M and a nonconstant positive p-harmonic function f with $\|f\|_{L^{p-1}(M)} < \infty$. In [49][50], S.W.Wei, J.F. Li and L. Wu proved sharp Liouville Theorems for $\mathcal{A}$-harmonic function u with p-balanced growth (e.g. $u \in L^q(M)$, for $q > p-1$, cf. [47] 6.3). In [15], I. Holopainen, S. Pigola and G. Veronelli showed that if $u, v \in W^{1,p}_{loc}(M) \cap C^0(M)$ satisfy $\Delta_p u \geq \Delta_p v$ weakly and $|\nabla u|, |\nabla v| \in L^p(M)$, for $p > 1$, then $u - v$ is constant provided M is connected, possibly incomplete, p-parabolic Riemannian manifold. They also discussed L^q comparison principles in the non-parabolic setting. In [38], S. Pigola, M. Rigoli and A.G. Setti showed the constancy of p-harmonic map homotopic to a constant and with finite p-energy from p-parabolic manifolds to manifolds with non-positive sectional curvature. Moreover, if manifold M has Poincaré-Sobolev inequality, and $Ric_M \geq -k(x)$ with $k(x) \geq 0$ and the integral type of $k(x)$ has upper bound depending on Poincaré-Sobolev constant, $p \geq 2$ and $p \geq q$, then they obtained constancy properties of p-harmonic map with some finite energy types from M to manifolds with non-positive sectional curvature. In [6], by a conservation law originated from E. Noether and comparison theorems in Riemannian Geometry, Y.X. Dong and S.W. Wei obtained some vanishing theorems for vector bundle valued differential forms. In particular, they prove some Liouville type Theorems for p-harmonic maps with finite p-energy under various curvature conditions.

In [21], B. Kotschwar and L. Ni use a Bochner's formula on a neighborhood of the maximum point (i.e. the p-Laplace operator is neither degenerate nor singular elliptic on this neighborhood) to prove a gradient estimate for positive p-harmonic functions. This also implies Liouville type properties of positive p-harmonic functions on complete noncompact manifolds with nonnegative Ricci curvature, and sectional curvature bounded below.

However, the approach of Kotschwar-Ni's gradient estimate for positive p-harmonic functions, does not seem to work in this paper, since we need a Bochner's formula which is unambiguously defined at every point in the manifold.

To overcome the difficulty, in this paper, we introduce and study an approximate solution u_ϵ of the weakly p-harmonic function u. This u_ϵ is the Euler-Lagrange equation of the (p, ϵ)-energy

$$E_{p,\epsilon} = \int_\Omega \left(|\nabla u_\epsilon|^2 + \epsilon\right)^{\frac{p}{2}} dv$$

with $u - u_\epsilon \in W_0^{1,p}(\Omega)$, where Ω is a domain on M. That is, u_ϵ is the weak solution of a perturbed p-Laplace equation

$$\Delta_{p,\epsilon} u_\epsilon = \operatorname{div}\left(\left(|\nabla u_\epsilon|^2 + \epsilon \right)^{\frac{p-2}{2}} \nabla u_\epsilon \right) = 0. \tag{1.2}$$

Moreover, we consider a linearization $\mathcal{L}_\epsilon$ of the perturbed operator $\Delta_{p,\epsilon}$, given by

$$\mathcal{L}_\epsilon(\Psi) = \operatorname{div}\left(f_\epsilon^{p-2} A_\epsilon(\nabla\Psi) \right), \tag{1.3}$$

for $\Psi \in C^2(\Omega)$, where $p > 1$, $f_\epsilon = \sqrt{|\nabla u_\epsilon|^2 + \epsilon}$ and

$$A_\epsilon := \mathrm{id} + (p-2) \frac{\nabla u_\epsilon \otimes \nabla u_\epsilon}{f_\epsilon^2}.$$

We observe that since $\Delta_{p,\epsilon}$ is no longer degenerate, by the existence and ϵ-Regularization results (Proposition 6.1 and Proposition 6.2), u_ϵ exists and is infinitely differentiable. Then we can derive an $\mathcal{L}_\epsilon$-type Bochner's formula and a Kato type inequality, and apply them to u_ϵ. Hence, using the convergence of the approximate solutions u_ϵ in $W^{1,p}$ on every domain in M, as $\epsilon \to 0$, we prove a Liouville type property of weakly p-harmonic functions with finite p-energy. This nonexistence result, when combined with the result of Proposition 2.1, yields in turn the topological information that such manifold has at most one p-hyperbolic end.

We also note that, the perturbation method we employed in studying the p-Laplace equation is in contrast to the methods in [41] for harmonic maps on surfaces, in [8] for the level-set formulation of the mean curvature flow, in [17] for the inverse mean curvature flow, and in [21] for certain parabolic equations associated to the p-Laplacian.

Theorem 1.1. *Let M be a complete noncompact Riemannian m-manifold, $m \geq 2$ supporting a weighted Poincaré inequality (P_ρ), with Ricci curvature*

$$Ric_M(x) \quad \geq \quad -\tau\rho(x) \tag{1.4}$$

for all $x \in M$, where τ is a constant such that

$$\tau < \frac{4(p-1+\kappa)}{p^2},$$

in which $p > 1$, and

$$\kappa = \begin{cases} \max\left\{\frac{1}{m-1}, \min\left\{\frac{(p-1)^2}{m}, 1\right\}\right\} & \textit{if } p > 2, \\ \frac{(p-1)^2}{m-1} & \textit{if } 1 < p \leq 2. \end{cases}$$

Then every weakly p-harmonic function u with finite p-energy E_p is constant. Moreover, M has at most one p-hyperbolic end.

In Theorem 1.1, we say that M supports a weighted Poincaré inequality (P_ρ), if there exists a positive function $\rho(x)$ a.e. on M such that, for every $\Psi \in W_0^{1.2}(M)$,

$$\int_M \rho(x)\, \Psi^2(x)\, dv \quad \leq \quad \int_M |\nabla\Psi(x)|^2\, dv. \tag{1.5}$$

If $\rho(x)$ is no less than a positive constant λ, then M has positive spectrum. For example, the hyperbolic space H^m has positive spectrum, and $\rho(x) = \frac{(m-1)^2}{4}$. In $\mathbb{R}^m$, if we select $\rho(x) = \frac{(m-2)^2}{4|x|^2}(x)$, then (1.5) is Hardy's inequality. For more examples, see [4][34][48].

If u is a C^3 strongly p-harmonic function with finite q-energy, then we have a Liouville type property as follows.

Theorem 1.2. *Let M be a complete noncompact Riemannian m-manifold, $m \geq 2$, satisfying (P_ρ) , with Ricci curvature*

$$Ric_M(x) \quad \geq \quad -\tau\rho(x) \tag{1.6}$$

for all $x \in M$, where τ is a constant such that

$$\tau < \tfrac{4(q-1+\kappa+b)}{q^2}, \tag{1.7}$$

in which

$$\kappa = \min\{\tfrac{(p-1)^2}{m-1}, 1\} \text{ and } b = \min\{0, (p-2)(q-p)\}, \text{ where } p > 1.$$

Let $u \in C^3(M)$ be a strongly p-harmonic function with finite q-energy $E_q(u) < \infty$.
(I). Then u is constant under each one of the following conditions:
(1) $p = 2$ and $q > \frac{m-2}{m-1}$,
(2) $p = 4$, $q > \max\{1, 1 - \kappa - b\}$,
(3) $p > 2$, $p \neq 4$, and either

$$\max\left\{1, p - 1 - \tfrac{\kappa}{p-1}\right\} < q \leq \min\left\{2, p - \tfrac{(p-4)^2 m}{4(p-2)}\right\}$$

or

$$\max\{2, 1 - \kappa - b\} < q,$$

(II) u does not exist for $1 < p < 2$ and $q > 2$.

We remark that the curvature assumption (1.6) or the assumption (1.7) on the constant τ in (1.6) cannot be dropped, due to the nontrivial p-harmonic functions with finite q-energy that are constructed in Sect. 6.3.

As an application, we also extend Theorem 1.2 to p-harmonic morphisms and conformal maps in Sections 5.3 and 5.4 respectively. In particular, we obtain a Picard-type Theorem for p-harmonic morphisms. Some applications to such Picard-type Theorems on stable minimal hypersurfaces in Riemannian manifolds can be found in [4].

The paper is organized as follows. In section 2, we recall some facts about p-hyperbolic and p-parabolic ends from [27] and [10], and prove an existence theorem on manifolds with two p-hyperbolic ends. In section 3, we introduce the linearization $\mathcal{L}_\epsilon$ (1.3) of the perturbed operator$\Delta_{p,\epsilon}$, and derive the $\mathcal{L}_\epsilon$-type Bochner's formula (3.8) and Kato type inequalities (3.9)(3.13) for the solution u_ϵ of the perturbed equation (1.2). In section 4, by applying Bochner's formula and Kato's inequality, we show a Liouville type theorem and one p-hyperbolic end property for a weakly p-harmonic function with finite p-energy in a complete noncompact manifold which supports a weighted Poincaré inequality and satisfies a curvature assumption. In section 5, we show Liouville type theorems for strongly p-harmonic functions with finite q-energy, and we also extend our results to some p-harmonic maps such as p-harmonic morphisms and conformal maps between Riemannian manifolds. In section 6 of the Appendix, we prove the existence of the approximate solution u_ϵ, Proposition 6.2, and volume estimate of complete noncompact manifolds with p-Poincaré inequality. We also construct an example of non-trivial p-harmonic function with finite q-energy on manifolds with weighted Poincaré inequality.

2. p-Hyperbolicity

We recall some basic facts about capacities from [9], [10] and [44].

Let M be a Riemannian manifold, $G \subset M$ a connected open set in M. If D and Ω are nonempty, disjoint, and closed sets contained in the closure of G. A triple $(\Omega, D; G)$ is called a condenser. The p-capacity of $(\Omega, D; G)$ is defined by

$$Cap_p(\Omega, D; G) \quad = \quad \inf_u \int_G |\nabla u|^p \, dv,$$

for $1 \le p < \infty$, where the infimum is taken over all $u \in W^{1,p}(G) \cap C^0(G)$ with $u = 1$ in Ω and $u = 0$ in D.

Above and in what follows, $W^{1,p}(M)$ is the Sobolev space of all function $u \in L^p(M)$ and whose distributional gradient ∇u also belongs to $L^p(M)$, with respect to the Sobolev norm

$$\|u\|_{1,p} \quad = \quad \|u\|_{L_p} + \|\nabla u\|_{L_p}.$$

The space $W_0^{1,p}(M)$ is the closure of $C_0^\infty(M)$ in $W^{1,p}(M)$, with respect to the $\| \quad \|_{1,p}$ norm.

The following properties of capacities are well known (see e.g. [44]).

- $\Omega_2 \subset \Omega_1 \Longrightarrow Cap_p(\Omega_2, D; G) \le Cap_p(\Omega_1, D; G)$;
- $D_2 \subset D_1 \Longrightarrow Cap_p(\Omega, D_2; G) \le Cap_p(\Omega, D_1; G)$;
- If $\Omega_1 \supset \Omega_2 \cdots \supset \cap_i \Omega_i = \Omega$ and $D_1 \supset D_2 \cdots \supset \cap_i D_i = D$, then
$$Cap_p(\Omega, D; G) = \lim_{i\to\infty} Cap_p(\Omega_i, D_i; G).$$
- If $\overline{G \backslash (\Omega \cup D)}$ is compact, then there exists a unique weak solution $u : \overline{G \backslash (\Omega \cup D)} \to \mathbb{R}$ to the Dirichlet problem
$$\begin{cases} \Delta_p u = 0 & \text{on } G\backslash(\Omega \cup D), \\ u = 1 & \text{on } \Omega, \\ u = 0 & \text{on } D, \end{cases}$$
with $Cap_p(\Omega, D; G) = \int_G |du|^p \, dv$.

Given a compact set Ω in M, an *end* E_Ω with respect to Ω is an unbounded connected component of $M\backslash\Omega$. By a compactness argument, it is readily seen that the number of ends with respect to Ω is finite, it is also clear that if $\Omega \subset \Omega'$, then every end $E_{\Omega'}$ is contained in E_Ω, so that the number of ends increases as the compact Ω enlarges. Let $x_0 \in \Omega$. We denote $E_\Omega(R) = B_{x_0}(R) \cap E_\Omega$, $\partial E_\Omega(R) = \partial B_{x_0}(R) \cap E_\Omega$ and $\partial E_\Omega = \partial\Omega \cap E_\Omega$.

In [27] (or see e.g. [22], [28]-[30], [37]), 2-parabolic and 2-nonparabolic manifolds and ends are introduced. In [10], I. Holopainen defined the p-parabolic end as follows:

Definition 2.1. *Let E be an end of M with respect to Ω. E is p-parabolic, or, equivalently, has zero p-capacity at infinity if,*

$$\mathrm{Cap}_p(\Omega, \infty; E) \quad := \quad \lim_{i\to\infty} \mathrm{Cap}_p\left(\Omega, \overline{E}\backslash\Omega_i; E\right) \quad = \quad 0,$$

where $\{\Omega_i\}_{i=1}^{\infty}$ *is an exhaustion of* M *by relatively compact open domains with smooth boundary and* $\Omega_i \subset\subset \Omega_{i+1}$*, for every integer* i *.*

This definition also implies: if E is an end with respect to Ω, there are sequence of weakly p-harmonic functions $\{u_i\}$, $u_i \in W^{1,p}$, defined on E, satisfying

$$\Delta_p u_i \ = \ 0 \quad \text{on} \quad E\,(r_i) \tag{2.1}$$

with boundary conditions

$$u_i \ = \ \begin{cases} 1 & \text{on } \Omega, \\ 0 & \text{on } \overline{E \backslash \Omega_i}, \end{cases} \tag{2.2}$$

then $\{u_i\}$ converges (converges uniformly on each compact set of E) to the constant function $u = 1$ on E as $i \to \infty$.

Definition 2.2. *An end* E *is* p*-hyperbolic (or* p*-nonparabolic) if* E *is not* p*-parabolic.*

If h_i is a weakly p-harmonic function satisfying (2.1) and (2.2), then E is p-hyperbolic if and only if $\{h_i\}$ converges to a weakly p-harmonic function h with $h = 1$ on ∂E, $\inf_E h = 0$ and finite p-energy.

Definition 2.3. *A manifold* M *is* p*-parabolic, or, equivalently, has zero* p*-capacity at infinity if, for each compact set* $\Omega \subset M$,

$$\mathrm{Cap}_p\,(\Omega, \infty; M) \ := \ \lim_{i\to\infty} \mathrm{Cap}_p\,(\Omega, M\backslash\Omega_i; M) \ = \ 0,$$

where $\{\Omega_i\}_{i=1}^{\infty}$ *is an exhaustion of* M *by domains with smooth boundary and* $\Omega_i \subset\subset \Omega_{i+1}$*, for every integer* i *.*

Definition 2.4. *A manifold* M *is* p*-hyperbolic (or* p*-nonparabolic) if* M *is not* p*-parabolic.*

This definition also implies that a manifold M is p-parabolic if each end of M is p-parabolic, M is p-hyperbolic if M has at least one p-hyperbolic end.

Now we focus on manifold M with two p-hyperbolic ends (cf. [9]).

Proposition 2.1. *Let* M *be a complete noncompact manifold, and assume* M *has two* p*-hyperbolic ends* E_1 *and* E_2*. Then there exists a weakly* p*-harmonic function* $h : M \to \mathbb{R}$ *with finite* p*-energy such that* $0 < h < 1$, $\sup_{E_1} h = 1$ *and* $\inf_{E_2} h = 0$*. Moreover,* h *is* $C^{1,\alpha}$*.*

Proof. Given $\Omega \subset M$, we fix an exhaustion $\{\Omega_i\}$ of M by domains with smooth boundary and $\Omega_i \subset\subset \Omega_{i+1}$ for every integer i .

Denote by E_A the p-hyperbolic ends of M with respect to A . For every A, let $u_i^{E_A}$ be the p-harmonic function satisfying

$$\begin{cases} \Delta_p u_i^{E_A} = 0 & \text{in } E_A \cap \Omega_i, \\ u_i^{E_A} = 1 & \text{on } \partial E_A, \\ u_i^{E_A} = 0 & \text{on } \partial E_{A,\Omega_i} = \partial\,(E_A \cap \Omega_i)\,\backslash \partial E_A. \end{cases}$$

By the monotone property, $u_i^{E_A}$ converges uniformly to u^{E_A} on every compact subset of E_A.

For every i, let h_i be the weak solution of the boundary value problem

$$\begin{cases} \Delta_p h_i = 0 & \text{in } \Omega_i, \\ h_i = 1 & \text{on } \partial\Omega_i \cap E_1, \\ h_i = 0 & \text{on } \partial\Omega_i \cap (M \backslash E_1). \end{cases}$$

Then, $0 \le h_i \le 1$, and by gradient estimate ([21]), there are subsequence, say $\{h_i\}$, converges, locally uniformly, to a weakly p-harmonic function h on M, satisfying $0 \le h \le 1$.

On E_1, the maximum principle implies $1 - u_i^{E_1} \le h_i < 1$. Hence $1 - u^{E_1} \le h < 1$ on E_1, so that $\sup_{E_1}\left(1 - u^{E_1}\right) \le \sup_{E_A} h = 1$ gives $\sup_{E_1} h = 1$ since $\inf_{E_1} u^{E_1} = 0$.

On E_2, the maximum principle implies $0 < h_i \le u_i^{E_2}$. Hence we have $0 < h \le u^{E_2}$ on E_2, so that $0 \le \inf_{E_2} h \le \inf_{E_2} u^{E_2} = 0$.

Now we have $\sup_{E_1} h = 1$ and $\inf_{E_2} h = 0$, so h is a nonconstant p-harmonic function on M.

Finally, h has finite p-energy by

$$Cap_p\left(E_1 \backslash \Omega_i, M \backslash \left(\Omega_i \cup E_1\right); M\right) = \int_M |\nabla h_i|^p \, dv \neq 0,$$

and the monotonic properties of capacities. $\square$

3. Bochner's formula and Kato's inequality

First of all, we define $N = M \times \mathbb{R}$ with metric $g_N = g_M + dt^2$, and let

$$v_\epsilon(x,t) = u_\epsilon(x) + \sqrt{\epsilon} t \tag{3.1}$$

for $x \in \Omega \subset M$, $t \in \mathbb{R}$, and $\epsilon > 0$, where u_ϵ is the solution of the perturbed p-Laplace equation (1.2). Then $v_\epsilon \in C^\infty$ is a strongly p-harmonic function on $\Omega_N = \Omega \times \mathbb{R}$, i.e. if Δ_p^N is the p-Laplace operator on (Ω_N, g_N), we have $\Delta_p^N v_\epsilon = 0$ with $\left|\nabla^N v_\epsilon\right|^2 \ge \epsilon > 0$ and $Ric_N\left(\nabla^N v_\epsilon, \nabla^N v_\epsilon\right) = Ric\left(\nabla u_\epsilon, \nabla u_\epsilon\right)$. Moreover, if $f = |\nabla u_\epsilon|$, then $f_\epsilon = \left|\nabla^N v_\epsilon\right| = \sqrt{f^2 + \epsilon}$ which is independent of t. Hence, we have $\nabla^N f_\epsilon = \nabla f_\epsilon$ and $\Delta^N f_\epsilon = \Delta f_\epsilon$.

According to the argument of Kotschwar-Ni [21], we define the linearized operator $\mathcal{L}_0^N$ of the p-Laplace operator Δ_p^N on (Ω_N, g_N) as follows:

$$\mathcal{L}_0^N(\Psi) = \operatorname{div}^N\left(f_\epsilon^{p-2} A_0\left(\nabla^N \Psi\right)\right),$$

for $\Psi \in C^2(\Omega_N)$, where div^N is the divergence on (Ω_N, g_N) and

$$A_0 := \mathrm{id} + (p-2)\frac{\nabla^N v_\epsilon \otimes \nabla^N v_\epsilon}{f_\epsilon^2}.$$

Now we show Bochner's formula as the following:

Lemma 3.1. *Let v_ϵ be the p-harmonic function on (Ω_N, g_N), and $(\nabla d)^N v_\epsilon$ be the Hessian of v_ϵ on (Ω_N, g_N). Then for every $p > 1$,*

$$\frac{1}{2}\mathcal{L}_0^N\left(f_\epsilon^2\right) = \frac{p-2}{4} f_\epsilon^{p-4}\left|\nabla^N f_\epsilon^2\right|^2 + f_\epsilon^{p-2}\left(\left|(\nabla d)^N v_\epsilon\right|^2 + Ric_N\left(\nabla^N v_\epsilon, \nabla^N v_\epsilon\right)\right). \tag{3.2}$$

Proof. Since $f_\epsilon > 0$, for every $p > 1$, the p-harmonic equation $\Delta_p^N v_\epsilon = 0$ is equivalent to

$$\tfrac{p-2}{2}\left\langle \nabla^N f_\epsilon^2, \nabla^N v_\epsilon \right\rangle = -f_\epsilon^2 \Delta^N v_\epsilon \tag{3.3}$$

which implies

$$\tfrac{p-2}{2} f_\epsilon^{p-6}\left\langle \nabla^N v_\epsilon, \nabla^N f_\epsilon^2 \right\rangle^2 = -f_\epsilon^{p-4}\left\langle \nabla^N v_\epsilon, \nabla^N f_\epsilon^2 \right\rangle \Delta^N v_\epsilon. \tag{3.4}$$

On the other hand, taking the gradient of both sides of (3.3), and then taking the inner product with $\nabla^N v_\epsilon$, we have

$$\begin{aligned} \tfrac{p-2}{2}\left\langle \nabla^N \left\langle \nabla^N f_\epsilon^2, \nabla^N v_\epsilon \right\rangle, \nabla^N v_\epsilon \right\rangle &= -\left\langle \nabla^N f_\epsilon^2, \nabla^N v_\epsilon \right\rangle \Delta^N v_\epsilon \\ &\quad - f_\epsilon^2 \left\langle \nabla^N \left(\Delta^N v_\epsilon\right), \nabla^N v_\epsilon \right\rangle. \end{aligned} \tag{3.5}$$

Now we compute

$$\begin{aligned} \tfrac{1}{2}\mathcal{L}_0^N\left(f_\epsilon^2\right) &= \tfrac{1}{2}\mathrm{div}^N\left(f_\epsilon^{p-2}\nabla^N f_\epsilon^2 + (p-2) f_\epsilon^{p-4}\left\langle \nabla^N v_\epsilon, \nabla^N f_\epsilon^2 \right\rangle \nabla^N v_\epsilon\right) \\ &= \tfrac{p-2}{4} f_\epsilon^{p-4}\left|\nabla^N f_\epsilon^2\right|^2 + \tfrac{1}{2} f_\epsilon^{p-2}\Delta^N f_\epsilon^2 + \tfrac{(p-2)(p-4)}{4} f_\epsilon^{p-6}\left\langle \nabla^N v_\epsilon, \nabla^N f_\epsilon^2 \right\rangle^2 \\ &\quad + \tfrac{p-2}{2} f_\epsilon^{p-4}\left\langle \nabla^N\left\langle \nabla^N v_\epsilon, \nabla^N f_\epsilon^2 \right\rangle, \nabla^N v_\epsilon \right\rangle \\ &\quad + \tfrac{p-2}{2} f_\epsilon^{p-4}\left\langle \nabla^N v_\epsilon, \nabla^N f_\epsilon^2 \right\rangle \Delta^N v_\epsilon. \end{aligned} \tag{3.6}$$

Substituting (3.5) into (3.6), one gets

$$\begin{aligned} \tfrac{1}{2}\mathcal{L}_0^N\left(f_\epsilon^2\right) = &\ \tfrac{p-2}{4} f_\epsilon^{p-4}\left|\nabla^N f_\epsilon^2\right|^2 + \tfrac{p-4}{2} f_\epsilon^{p-4}\left\langle \nabla^N v_\epsilon, \nabla^N f_\epsilon^2 \right\rangle \Delta^N v_\epsilon \\ &+ \tfrac{1}{2} f_\epsilon^{p-2}\Delta^N f_\epsilon^2 - f_\epsilon^{p-2}\left\langle \nabla^N\left(\Delta^N v_\epsilon\right), \nabla^N v_\epsilon \right\rangle \\ &+ \tfrac{(p-2)(p-4)}{4} f_\epsilon^{p-6}\left\langle \nabla^N v_\epsilon, \nabla^N f_\epsilon^2 \right\rangle^2. \end{aligned} \tag{3.7}$$

Applying Bochner's formula

$$\tfrac{1}{2}\Delta^N f_\epsilon^2 = \left|(\nabla d)^N v_\epsilon\right|^2 + \left\langle \nabla^N v_\epsilon, \nabla^N\left(\Delta^N v_\epsilon\right)\right\rangle + Ric_N\left(\nabla^N v_\epsilon, \nabla^N v_\epsilon\right)$$

and the equation (3.4) to the third term and the last term of right hand side of (3.7) respectively, one obtains the desired formula (3.2). $\square$

If Ψ is independent of t, then

$$\begin{aligned} \mathcal{L}_0^N(\Psi) &= \mathrm{div}^N\left(f_\epsilon^{p-2}\nabla^N\Psi + (p-2)\left\langle \nabla^N v_\epsilon, \nabla^N \Psi \right\rangle \nabla^N v_\epsilon\right) \\ &= \mathrm{div}\left(f_\epsilon^{p-2}\nabla\Psi + (p-2) f_\epsilon^{p-4}\left\langle \nabla u_\epsilon, \nabla\Psi \right\rangle \nabla u_\epsilon\right) \\ &= \mathcal{L}_\epsilon(\Psi) \end{aligned}$$

where $\mathcal{L}_\epsilon$ is defined by (1.3). Hence Lemma 3.1 implies the following Lemma.

Lemma 3.2. *Let u_ϵ be a solution of (1.2) on $\Omega \subset M$, $f_\epsilon = \sqrt{|\nabla u_\epsilon|^2 + \epsilon}$, and ∇du_ϵ be the Hessian of u_ϵ on M. Then for every $p > 1$,*

$$\tfrac{1}{2}\mathcal{L}_\epsilon\left(f_\epsilon^2\right) = \tfrac{p-2}{4} f_\epsilon^{p-4}\left|\nabla f_\epsilon^2\right|^2 + f_\epsilon^{p-2}\left(|\nabla du_\epsilon|^2 + Ric\left(\nabla u_\epsilon, \nabla u_\epsilon\right)\right). \tag{3.8}$$

Next, we derive the following Kato type inequalities for the approximate solution u_ϵ :

Lemma 3.3. *Let u_ϵ be a solution of (1.2) on $\Omega \subset M^m$, $p > 1$. Then the Hessian of u_ϵ satisfies*

$$|du_\epsilon|^2 \, |\nabla du_\epsilon|^2 \;\geq\; \tfrac{1+\kappa_1}{4} \left|\nabla |du_\epsilon|^2\right|^2 \tag{3.9}$$

at $x \in \Omega$, where

$$\kappa_1 \;=\; \begin{cases} \frac{1}{m-1} & \text{if } p \geq 2, \\ \frac{(p-1)^2}{m-1} & \text{if } 1 < p < 2. \end{cases}$$

Proof. Fix $x \in \Omega \subset M$ with $du_\epsilon \neq 0$, we select a local orthonormal frame field $\{e_1, e_2, \dots e_m\}$ such that at x, $\nabla_{e_i} e_j = 0$, $\nabla u_\epsilon = |\nabla u_\epsilon| \, e_1$, $u_{\epsilon,1} = f$, and $u_{\epsilon,\alpha} = 0$ for all $i, j = 1, \dots, m$, $\alpha = 2, \dots, m$ where $u_{\epsilon,\alpha} = \langle \nabla u_\epsilon, e_\alpha \rangle$.

Let $f = |\nabla u_\epsilon|$, $f_\epsilon = \sqrt{|\nabla u_\epsilon|^2 + \epsilon}$ and the directional derivative $f_{\epsilon,i} = \langle \nabla f_\epsilon, e_i \rangle$. Denote the directional derivative $\langle \nabla u_{\epsilon,i}, e_j \rangle$ by $u_{\epsilon,ij}$. Then (3.3) implies

$$\begin{aligned} \Delta u_\epsilon \;=\; -\tfrac{p-2}{2f_\epsilon^2} \langle \nabla f_\epsilon^2, \nabla u_\epsilon \rangle \;&=\; -\tfrac{p-2}{2f_\epsilon^2} \textstyle\sum_{i=1}^m (f_\epsilon^2)_{,i} \, u_{\epsilon,i} \\ &=\; -\tfrac{p-2}{2f_\epsilon^2} (f_\epsilon^2)_{,1} \, u_{\epsilon,1} \\ &=\; -\tfrac{p-2}{2f_\epsilon^2} (f_\epsilon^2)_{,1} \, f. \end{aligned}$$

Moreover, by using the following property

$$\begin{aligned} (f_\epsilon^2)_{,j} \;=\; (f^2)_{,j} \;=\; \textstyle\sum_{i=1}^m \left(u_{\epsilon,i}^2\right)_{,j} \;&=\; 2\textstyle\sum_{i=1}^m u_{\epsilon,i} u_{\epsilon,ij} \\ &=\; 2u_{\epsilon,1} u_{\epsilon,1j} \\ &=\; 2f u_{\epsilon,1j}. \end{aligned}$$

We have

$$\Delta u_\epsilon = \tfrac{-(p-2)f^2}{f_\epsilon^2} u_{\epsilon,11}, \tag{3.10}$$

and

$$u_{\epsilon,1j} = f_{,j}. \tag{3.11}$$

On the other hand,

$$\begin{aligned} \textstyle\sum_{i,j=1}^m (u_{\epsilon,ij})^2 \;&\geq\; (u_{\epsilon,11})^2 + 2\textstyle\sum_{\alpha=2}^m (u_{\epsilon,1\alpha})^2 + \textstyle\sum_{\alpha=2}^m (u_{\epsilon,\alpha\alpha})^2 \\ &\geq\; (u_{\epsilon,11})^2 + 2\textstyle\sum_{\alpha=2}^m (u_{\epsilon,1\alpha})^2 + \frac{\left(\sum_{\alpha=2}^m u_{\epsilon,\alpha\alpha}\right)^2}{m-1} \\ &=\; (u_{\epsilon,11})^2 + 2\textstyle\sum_{\alpha=2}^m (u_{\epsilon,1\alpha})^2 + \frac{(\Delta u_\epsilon - u_{\epsilon,11})^2}{m-1}. \end{aligned} \tag{3.12}$$

Therefore, by using (3.10) and (3.11), the inequality (3.12) implies

$$\begin{aligned} \textstyle\sum_{i,j=1}^m (u_{\epsilon,ij})^2 \;&\geq\; (u_{\epsilon,11})^2 + 2\textstyle\sum_{\alpha=2}^m (u_{\epsilon,1\alpha})^2 + \frac{\left(\left(\frac{(p-2)f^2}{f_\epsilon^2}+1\right)u_{\epsilon,11}\right)^2}{m-1} \\ &=\; \left(1 + \frac{\left((p-1)f^2+\epsilon\right)^2}{(m-1)f_\epsilon^4}\right)(u_{\epsilon,11})^2 + 2\textstyle\sum_{\alpha=2}^m (u_{\epsilon,1\alpha})^2 \\ &\geq\; (1+\kappa)\,|\nabla f|^2 \end{aligned}$$

which can be written as

$$|du_\epsilon|^2 \, |\nabla du_\epsilon|^2 \;\geq\; \tfrac{1+\kappa_1}{4} \left|\nabla |du_\epsilon|^2\right|^2$$

for all $x \in \Omega$. This completes the proof. □

Lemma 3.4. *Let u_ϵ be a solution of (1.2) on $\Omega \subset M^m$, $p > 1$. Then the Hessian of u_ϵ satisfies*

$$\left(|du_\epsilon|^2 + \epsilon\right) |\nabla du_\epsilon|^2 \geq \tfrac{1+\kappa_2}{4} \left|\nabla |du_\epsilon|^2\right|^2 \tag{3.13}$$

at $x \in \Omega$, where $\kappa_2 = \min\left\{\frac{(p-1)^2}{m}, 1\right\}$.

Proof. Since $v_\epsilon \in C^\infty(N)$ is the strongly p-harmonic function on (Ω_N, g_N), then Kato's inequality for strongly p-harmonic function on (Ω_N, g_N) (see Lemma 5.3) implies

$$\left|(\nabla d)^N v_\epsilon\right|^2 \geq (1+\kappa_2) \left|\nabla^N \left|d^N v_\epsilon\right|\right|^2 \tag{3.14}$$

where $(\nabla d)^N$ is the Hessian on (Ω_N, g_N), and $\kappa_2 = \min\left\{\frac{(p-1)^2}{m}, 1\right\}$. Moreover, (3.14) can be rewritten as

$$\begin{aligned} \left(|du_\epsilon|^2 + \epsilon\right) |\nabla du_\epsilon|^2 &\geq (1+\kappa_2) \left(|du_\epsilon|^2 + \epsilon\right) \left|\nabla \sqrt{|du_\epsilon|^2 + \epsilon}\right|^2 \\ &= \tfrac{1+\kappa_2}{4} \left|\nabla |du_\epsilon|^2\right|^2. \end{aligned}$$

for all $x \in \Omega$. □

4. The Proof of Theorem 1.1

Now we use Lemmas 3.2 - 3.4 and weighted Poincaré inequality (1.5) to obtain the following inequality (4.1):

Lemma 4.1. *Let M be a manifold satisfying the hypothesis of Theorem 1.1. Let u_ϵ be a solution of (1.2) on $B(2R) \subset M$. Then we have*

$$C \int_{B(R)} \rho |\nabla u_\epsilon|^p \, dv \leq \tfrac{100 \cdot B}{R^2} \int_{B(2R) \setminus B(R)} \left(|\nabla u_\epsilon|^2 + \epsilon\right)^{\frac{p}{2}} dv, \tag{4.1}$$

where $C(p, m, \kappa, \tau, \epsilon_1, \epsilon_2) > 0$ and $B(p, m, \kappa, \epsilon_1, \epsilon_2) > 0$ are positive constants for sufficiently small constants $\epsilon_1, \epsilon_2 > 0$.

Proof. Let $\Omega = B(2R)$ be a geodesic ball of radius $2R$ centered at a fixed point.

Let $f = |\nabla u_\epsilon|$ and $f_\epsilon = \sqrt{f^2 + \epsilon}$. In view of Lemma 3.3 and Lemma 3.4,

$$f_\epsilon^2 |\nabla du_\epsilon|^2 \geq \tfrac{1+\kappa}{4} |\nabla f^2|^2$$

holds for all on M, where $\kappa = \max\{\kappa_1, \kappa_2\}$. Then by Lemma 3.2, we rewrite Bochner's formula as

$$\tfrac{1}{2} \mathcal{L}_\epsilon\left(f_\epsilon^2\right) \geq (p-1+\kappa) f_\epsilon^{p-2} |\nabla f_\epsilon|^2 + f_\epsilon^{p-2} Ric(\nabla u_\epsilon, \nabla u_\epsilon), \tag{4.2}$$

here we use $\nabla f_\epsilon^2 = \nabla f^2$.

We multiply both sides of (4.2) by η^2 and integrate over M,

$$\begin{aligned} \tfrac{1}{2} \int_M \eta^2 \mathcal{L}_\epsilon\left(f_\epsilon^2\right) dv \geq\; & (p-1+\kappa) \int_M \eta^2 f_\epsilon^{p-2} |\nabla f_\epsilon|^2 \, dv \\ & + \int_M \eta^2 f_\epsilon^{p-2} Ric(\nabla u_\epsilon, \nabla u_\epsilon) \, dv \end{aligned} \tag{4.3}$$

where $\eta \in C_0^\infty(M)$ is a cut-off function with $0 \le \eta(x) \le 1$ on M satisfying

$$\begin{cases} \eta(x) = 1 & \text{if } x \in \overline{B(R)}, \\ |\nabla \eta(x)| \le \frac{10}{R} & \text{if } x \in B(2R) \backslash \overline{B(R)}, \\ \eta(x) = 0 & \text{if } x \in M \backslash B(2R). \end{cases}$$

On the other hand, applying integration by parts and Cauchy-Schwarz inequality one has

$$\begin{aligned} \frac{1}{2}\int_M \eta^2 \mathcal{L}_\epsilon(f_\epsilon^2)\,dv &= \frac{-1}{2}\int_M \langle \nabla\eta^2, f_\epsilon^{p-2}\nabla f_\epsilon^2 + (p-2) f_\epsilon^{p-4}\langle \nabla u_\epsilon, \nabla f_\epsilon^2\rangle \nabla u_\epsilon\rangle\,dv \\ &\le 2\int_M \eta|\nabla\eta|\left(f_\epsilon^{p-1}|\nabla f_\epsilon| + |p-2|\, f_\epsilon^{p-3} f^2 |\nabla f_\epsilon|\right)dv \\ &\le 2(1+|p-2|)\int_M \eta|\nabla\eta|\, f_\epsilon^{p-1}|\nabla f_\epsilon|\,dv \\ &\le \epsilon_1\int_M \eta^2 f_\epsilon^{p-2}|\nabla f_\epsilon|^2\,dv + \frac{(1+|p-2|)^2}{\epsilon_1}\int_M |\nabla\eta|^2 f_\epsilon^p dv, \end{aligned}$$

where ϵ_1 is a positive constant satisfying

$$p-1-\epsilon_1 > 0.$$

Then (4.3) implies

$$\begin{aligned} \frac{(1+|p-2|)^2}{\epsilon_1}\int_M |\nabla\eta|^2 f_\epsilon^p dv \ \ge\ & \int_M (p-1+\kappa-\epsilon_1)\,\eta^2 f_\epsilon^{p-2}|\nabla f_\epsilon|^2\,dv \\ & + \int_M \eta^2 f_\epsilon^{p-2} Ric(\nabla u_\epsilon, \nabla u_\epsilon)\,dv. \end{aligned} \tag{4.4}$$

Besides, we may rewrite the first term in the right hand side of (4.4) by

$$\begin{aligned} & (p-1+\kappa-\epsilon_1)\int_M \eta^2 f_\epsilon^{p-2}|\nabla f_\epsilon|^2\,dv \\ = \ & \frac{4(p-1+\kappa-\epsilon_1)}{p^2}\int_M \eta^2\left|\nabla f_\epsilon^{\frac{p}{2}}\right|^2 dv \\ = \ & \frac{4(p-1+\kappa-\epsilon_1)}{p^2}\int_M \left|\nabla\left(\eta f_\epsilon^{\frac{p}{2}}\right) - (\nabla\eta) f_\epsilon^{\frac{p}{2}}\right|^2 dv \\ = \ & \frac{4(p-1+\kappa-\epsilon_1)}{p^2}\int_M \left\{\left|\nabla\left(\eta f_\epsilon^{\frac{p}{2}}\right)\right|^2 - 2\left\langle \nabla\left(\eta f_\epsilon^{\frac{p}{2}}\right), f_\epsilon^{\frac{p}{2}}\nabla\eta\right\rangle + |\nabla\eta|^2 f_\epsilon^p\right\} dv \\ \ge \ & \frac{4(1-\epsilon_2)(p-1+\kappa-\epsilon_1)}{p^2}\int_M \left|\nabla\left(\eta f_\epsilon^{\frac{p}{2}}\right)\right|^2 + \frac{4\left(1-\frac{1}{\epsilon_2}\right)(p-1+\kappa-\epsilon_1)}{p^2}\int_M |\nabla\eta|^2 f_\epsilon^p dv. \end{aligned}$$

where ϵ_2 is a positive constant satisfying $\epsilon_2 < 1$. Thus, we have

$$\begin{aligned} & \frac{4(1-\epsilon_2)(p-1+\kappa-\epsilon_1)}{p^2}\int_M \left|\nabla\left(\eta f_\epsilon^{\frac{p}{2}}\right)\right|^2 dv + \int_M \eta^2 f_\epsilon^{p-2} Ric(\nabla u_\epsilon, \nabla u_\epsilon)\,dv \\ \le\ & \left(\frac{(1+|p-2|)^2}{\epsilon_1} + \frac{4\left(\frac{1}{\epsilon_2}-1\right)(p-1+\kappa-\epsilon_1)}{p^2}\right)\int_M |\nabla\eta|^2 f_\epsilon^p dv. \end{aligned} \tag{4.5}$$

According to the weighted Poincaré inequality (1.5)

$$\int_M \rho\Psi^2 dv \ \le\ \int_M |\nabla\Psi|^2\,dv$$

with $\Psi = \eta f_\epsilon^{\frac{p}{2}}$, then (4.5) implies

$$\int_{B(R)} A f_\epsilon^{p-2} dv \ \le\ \frac{100\cdot B}{R^2}\int_{B(2R)\backslash B(R)} f_\epsilon^p dv, \tag{4.6}$$

for all fixed $R > 0$, where

$$A \ =\ \frac{4(1-\epsilon_2)(p-1+\kappa-\epsilon_1)}{p^2}\rho f_\epsilon^2 + Ric(\nabla u_\epsilon, \nabla u_\epsilon),$$

and

$$B = \left(\frac{(1+|p-2|)^2}{\epsilon_1} + \frac{4\left(\frac{1}{\epsilon_2}-1\right)(p-1+\kappa-\epsilon_1)}{p^2} \right).$$

Since the curvature condition (1.4) means that there exists a constant $0 < \tau < \frac{4(p-1+\kappa)}{p^2}$ such that

$$Ric_M \geq -\tau\rho,$$

Then

$$A \geq C(p, m, \kappa, \tau, \epsilon_1, \epsilon_2)\, \rho f^2$$

with $C > 0$ whenever we select ϵ_1 and ϵ_2 small enough.

Hence, (4.6) gives

$$C \int_{B(R)} \rho |\nabla u_\epsilon|^p \, dv \leq \frac{100 \cdot B}{R^2} \int_{B(2R)\setminus B(R)} \left(|\nabla u_\epsilon|^2 + \epsilon\right)^{\frac{p}{2}} dv,$$

where $C(p, m, \kappa, \tau, \epsilon_1, \epsilon_2) > 0$, and $B(p, m, \kappa, \epsilon_1, \epsilon_2) > 0$.

□

Proof of Theorem 1.1.

Proof. Given $B(R_0) \subset M$, for every $a > 0$, we let $\Omega_a = \{x \in B(R_0) : \rho(x) > 1/a\}$. It is clear the measure of Ω_a tends to zero as $a \to 0^+$. If we are able to show $\int_{B(R_0)\setminus\Omega_a} \rho |\nabla u|^p \, dv < \delta$ for any $\delta > 0$, then it implies $\nabla u = 0$ on $B(R_0)$ almost everywhere. This also infers $\nabla u = 0$ on $B(R_0)$ by the fact $u \in C^{1,\alpha}_{loc}(M)$. Moreover, since $B(R_0)$ is arbitrary, u must be constant on M.

Moreover, if we assume M has at least two p-hyperbolic ends. By Proposition 2.1, one may construct a nontrivial bounded p-harmonic function with finite p-energy on M, this gives a contradiction to our conclusion, hence M has only one p-hyperbolic end.

Now we prove the claim. By using the finite p-energy of u, we may select $0 << R < \infty$ large enough such that $B(R_0) \subset B(R)$ and

$$\frac{100B}{R^2 C} \int_{B(2R)\setminus B(R)} |\nabla u|^p \, dv < \delta$$

where B and C are defined as (4.1).

Now we construct $u_\epsilon \in C^\infty(B(2R))$ such that $u_\epsilon = u$ on $\partial B(2R)$ and u_ϵ satisfies (1.2). Then (4.1) implies

$$C \int_{B(R_0)\setminus\Omega_a} \rho |\nabla u|^p \, dv \leq \frac{100 \cdot B}{R^2} \int_{B(2R)\setminus B(R)} |\nabla u|^p \, dv,$$

as $\epsilon \to 0$, we may therefore conclude that

$$\int_{B(R_0)\setminus\Omega_a} \rho |\nabla u|^p \, dv < \delta.$$

□

If M has positive spectrum $\lambda > 0$, then M has p-Poincaré inequality

$$\lambda_p \int_M \Psi^p \leq \int_M |\nabla\Psi|^p, \ \lambda_p > 0$$

for all $\Psi \in W_0^{1,p}(M)$ and $p \geq 2$ (cf. [14] Theorem 1.8). Since p-Poincaré inequality and Caccioppoli type estimate imply decay estimate (see Lemma 6.4 which is similar to the work of [32] Lemma 1.1 and Lemma 1.2), then p-Poincaré inequality infers that M must be a p-hyperbolic manifold (see Theorem 6.1). So we have the following:

Corollary 4.1. *Let $M^m, m \geq 2$, be a complete noncompact Riemannian manifold with positive spectrum $\lambda > 0$ and*

$$Ric_M \quad \geq \quad -\tau\lambda$$

where $p \geq 2$, and constant τ is the same as in Theorem 1.1. Then every weakly p-harmonic function u with finite p-energy is constant. Moreover, M has only one p-hyperbolic end.

Remark 4.1. Similarly, if M has p-Poincaré inequality, $1 < p < 2$, then M has positive spectrum $\lambda > 0$. Hence, if M is a complete noncompact Riemannian manifold with p-Poincaré inequality, $1 < p < 2$, and $Ric_M \geq -\tau\lambda$ where $\tau < \frac{4(p-1)(p+m-2)}{p^2(m-1)}$. Then M has only one p-hyperbolic end.

5. Strongly p-harmonic functions with applications

5.1. **Bochner's formula.** Let u be a C^3 strongly p-harmonic function for $p > 1$ on M. Then $|\nabla u|^{p-2}\nabla u$ must be C^1 on M, and hence u is a solution of (5.1) as follows:

Lemma 5.1. *If $u \in C^3(M)$ is a strongly p-harmonic function for $p > 1$, then u is a solution of*

$$f^2\Delta u + \tfrac{p-2}{2}\langle \nabla f^2, \nabla u\rangle \quad = \quad 0, \tag{5.1}$$

on M, where $f = |\nabla u|$.

Proof. First, we multiply both side of (1.1) by f^4, because of $f^4 \in C^2(M)$, then

$$f^4 \mathrm{div}(f^{p-2}\nabla u) = 0$$

implies

$$\begin{aligned} 0 &= \mathrm{div}(f^{p+2}\nabla u) - 2f^p\langle \nabla f^2, \nabla u\rangle \\ &= f^{p+2}\Delta u + \langle \nabla f^{p+2}, \nabla u\rangle - 2f^p\langle \nabla f^2, \nabla u\rangle . \end{aligned}$$

Since $p > 1$ and

$$\nabla f^{p+2} = \nabla\left((f^2)^{\frac{p+2}{2}}\right) = \tfrac{p+2}{2} f^p \nabla f^2,$$

so we have

$$f^{p+2}\Delta u + \tfrac{p-2}{2} f^p\langle \nabla f^2, \nabla u\rangle = 0.$$

which implies

$$f^2\Delta u + \tfrac{p-2}{2}\langle \nabla f^2, \nabla u\rangle = 0$$

on all of M.

□

Remark 5.1. (1). If u is a solution of (5.1), u may be not a strongly p-harmonic function. For example, any constant function is a solution of (5.1), but it is not a strongly p-harmonic function for $1 < p < 2$.
(2). For $p \geq 4$, $u \in C^2(M)$ is a solution of (5.1) if and only if u is a strongly p-harmonic function.

Now we define an operator $\mathcal{L}_{s,\varepsilon}$ by

$$\mathcal{L}_{s,\varepsilon}(\Psi) = \operatorname{div}\left(f_\varepsilon^s A_\varepsilon(\nabla\Psi)\right),$$

for $\Psi \in C^2(M)$, where $s \in \mathbb{R}$, $p > 1$, $\varepsilon > 0$, $f_\varepsilon = \sqrt{f^2+\varepsilon}$ and

$$A_\varepsilon := \mathrm{id} + (p-2)\tfrac{\nabla u \otimes \nabla u}{f_\varepsilon^2}.$$

Note that $\mathcal{L}_{s,\varepsilon}$ is a linearized operator of the nonlinear equation (1.1), and $\mathcal{L}_{s,\varepsilon}(f_\varepsilon^2)(x)$ is well define for all $x \in M$ since $f_\varepsilon > 0$ and $f_\varepsilon^2 \in C^2(M)$.

Next we use the operator $\mathcal{L}_{s,\varepsilon}$ to derive the Bochner's formula for the solution of (5.1).

Lemma 5.2 (Bochner's formula)**.** *If $u \in C^3(M)$ is a strongly p-harmonic function. Let $f = |\nabla u|$ and $f_\varepsilon = \sqrt{f^2+\varepsilon}$, then for all $p > 1$ and $s \in \mathbb{R}$, the formula*

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}_{s,\varepsilon}(f_\varepsilon^2) &= \tfrac{s}{4}f_\varepsilon^{s-2}\left|\nabla f_\varepsilon^2\right|^2 + f_\varepsilon^s\textstyle\sum_{i,j=1}^m\left(u_{ij}^2 + R_{ij}u_iu_j\right)\\ &\quad+\tfrac{(p-2)(s-p+2)}{4}f_\varepsilon^{s-4}\left\langle\nabla u, \nabla f_\varepsilon^2\right\rangle^2\\ &\quad+\varepsilon\left(f_\varepsilon^{s-2}\langle\nabla u, \nabla\Delta u\rangle + \tfrac{p-4}{2}f_\varepsilon^{s-4}\left\langle\nabla u, \nabla f_\varepsilon^2\right\rangle\Delta u\right)\end{aligned}$$

holds on all of M, where $R_{ij} = \sum_{k=1}^m\langle R(e_i,e_k)e_k, e_j\rangle$ is the Ricci curvature tensor of M. In particular, if $p = 2$, then

$$\tfrac{1}{2}\mathcal{L}_{s,\varepsilon}(f_\varepsilon^2) = \tfrac{s}{4}f_\varepsilon^{s-2}\left|\nabla f_\varepsilon^2\right|^2 + f_\varepsilon^s\textstyle\sum_{i,j=1}^m\left(u_{ij}^2 + R_{ij}u_iu_j\right)$$

holds on all of M and for all $s \in \mathbb{R}$.

Proof. By Lemma 5.1, u must be a solution of (5.1). Taking the gradient of both sides of (5.1), and then taking the inner product with ∇u, we have

$$\begin{aligned}0 &= \tfrac{p-2}{2}\left\langle\nabla\left\langle\nabla f^2, \nabla u\right\rangle, \nabla u\right\rangle + \left\langle\nabla f^2, \nabla u\right\rangle\Delta u\\ &\quad+f^2\langle\nabla(\Delta u), \nabla u\rangle.\end{aligned} \tag{5.2}$$

Now we rewrite $\mathcal{L}_{s,\varepsilon}(f_\varepsilon^2)$ as the following formula,

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}_{s,\varepsilon}(f_\varepsilon^2) &= \tfrac{1}{2}\operatorname{div}\left(f_\varepsilon^s\nabla f_\varepsilon^2 + (p-2)f_\varepsilon^{s-2}\left\langle\nabla u, \nabla f_\varepsilon^2\right\rangle\nabla u\right)\\ &= \tfrac{s}{4}f_\varepsilon^{s-2}\left|\nabla f_\varepsilon^2\right|^2 + \tfrac{1}{2}f_\varepsilon^s\Delta f_\varepsilon^2\\ &\quad+\tfrac{(p-2)(s-2)}{4}f_\varepsilon^{s-4}\left\langle\nabla u, \nabla f_\varepsilon^2\right\rangle^2\\ &\quad+\tfrac{p-2}{2}f_\varepsilon^{s-2}\left\langle\nabla\left\langle\nabla u, \nabla f_\varepsilon^2\right\rangle, \nabla u\right\rangle\\ &\quad+\tfrac{p-2}{2}f_\varepsilon^{s-2}\left\langle\nabla u, \nabla f_\varepsilon^2\right\rangle\Delta u.\end{aligned} \tag{5.3}$$

Combining (5.2), one has

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}_{s,\varepsilon}(f_\varepsilon^2) &= \tfrac{s}{4}f_\varepsilon^{s-2}\left|\nabla f_\varepsilon^2\right|^2 + \tfrac{1}{2}f_\varepsilon^s\Delta f_\varepsilon^2\\ &\quad+\tfrac{(p-2)(s-2)}{4}f_\varepsilon^{s-4}\left\langle\nabla u, \nabla f_\varepsilon^2\right\rangle^2\\ &\quad-f_\varepsilon^{s-2}f^2\langle\nabla(\Delta u), \nabla u\rangle\\ &\quad+\tfrac{p-4}{2}f_\varepsilon^{s-2}\left\langle\nabla u, \nabla f_\varepsilon^2\right\rangle\Delta u,\end{aligned} \tag{5.4}$$

here we use the fact $\nabla f_\varepsilon^2 = \nabla f^2$.

According to (5.1), the last term of right hand side can be rewritten as

$$\begin{aligned} f_\varepsilon^{s-2}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle \Delta u &= f_\varepsilon^{s-4}\left(f^2+\varepsilon\right)\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle \Delta u \\ &= f_\varepsilon^{s-4} f^2\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle \Delta u + \varepsilon f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle \Delta u \\ &= -\tfrac{p-2}{2} f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle^2 + \varepsilon f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle \Delta u. \end{aligned}$$

Using Bochner's formula

$$\tfrac{1}{2}\Delta f^2 = \textstyle\sum_{i,j=1}^m u_{ij}^2 + \left\langle \nabla u, \nabla \Delta u\right\rangle + \sum_{i,j=1}^m R_{ij} u_i u_j$$

and the equality $\Delta f^2 = \Delta f_\varepsilon^2$, then (5.4) gives the desired

$$\begin{aligned} \tfrac{1}{2}\mathcal{L}_{s,\varepsilon}\left(f_\varepsilon^2\right) = \; & \tfrac{s}{4} f_\varepsilon^{s-2}\left|\nabla f_\varepsilon^2\right|^2 + f_\varepsilon^s \textstyle\sum_{i,j=1}^m \left(u_{ij}^2 + R_{ij}u_i u_j\right) \\ & + \tfrac{(p-2)(s-p+2)}{4} f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle^2 \\ & + \varepsilon\left(f_\varepsilon^{s-2}\left\langle \nabla u, \nabla \Delta u\right\rangle + \tfrac{p-4}{2} f_\varepsilon^{s-4}\left\langle \nabla u, \nabla f_\varepsilon^2\right\rangle \Delta u\right). \end{aligned}$$

$\square$

Lemma 5.3 (Refined Kato's inequality)**.** *Let $u \in C^2(M)$ be p-harmonic function on a complete manifold M^m, $p>1$ and $\kappa = \min\left\{\frac{(p-1)^2}{m-1}, 1\right\}$. Then at any $x \in M$ with $du(x) \neq 0$,*

$$|\nabla (du)|^2 \geq (1+\kappa)\left|\nabla |du|\right|^2, \tag{5.5}$$

and "$=$" holds if and only if

$$\begin{cases} u_{\alpha\beta} = 0 \text{ and } u_{11} = -\frac{m-1}{p-1} u_{\alpha\alpha}, & \text{for } (p-1)^2 = m-1, \\ u_{\alpha\beta} = 0,\ u_{1\alpha} = 0 \text{ and } u_{11} = -\frac{m-1}{p-1} u_{\alpha\alpha}, & \text{for } (p-1)^2 < m-1, \\ u_{\alpha\beta} = 0 \text{ and } u_{ii} = 0, & \text{for } (p-1)^2 > m-1, \end{cases}$$

for all $\alpha, \beta = 2, \ldots, m$, $\alpha \neq \beta$ and $i = 1, \ldots, m$.

Proof. Fix a point $x \in M$. If $du \neq 0$ at x, we are able to select a local orthonormal frame field $\{e_1, e_2, \ldots e_m\}$ such that, at x, $\nabla_{e_i} e_j = 0$, $\nabla u = |\nabla u| e_1$, and $u_\alpha = 0$ for all $i, j = 1, \ldots, m$, $\alpha = 2, \ldots, m$. Here we use the convenient notation $u_i = \langle \nabla u, e_i\rangle$.

Observing that

$$\begin{aligned} \textstyle\sum_{i,j=1}^m (u_{ij})^2 &\geq (u_{11})^2 + 2\textstyle\sum_{\alpha=2}^m (u_{1\alpha})^2 + \sum_{\alpha=2}^m (u_{\alpha\alpha})^2 \\ &\geq (u_{11})^2 + 2\textstyle\sum_{\alpha=2}^m (u_{1\alpha})^2 + \frac{\left(\sum_{\alpha=2}^m u_{\alpha\alpha}\right)^2}{m-1} \\ &= (u_{11})^2 + 2\textstyle\sum_{\alpha=2}^m (u_{1\alpha})^2 + \frac{(\Delta u - u_{11})^2}{m-1}. \end{aligned} \tag{5.6}$$

However, letting $f = |\nabla u|$ and using $f = u_1$, $f_1 = \langle \nabla f, e_1\rangle$,

$$\begin{aligned} 0 &= div\left(f^{p-2}\nabla u\right) \\ &= f^{p-2}\Delta u + (p-2) f^{p-3}\langle \nabla f, \nabla u\rangle \\ &= u_1^{p-2}\Delta u + (p-2) u_1^{p-2} f_1, \end{aligned}$$

and

$$\begin{aligned} f_j = \frac{\left(f^2\right)_{,j}}{2f} = \frac{\left(\sum_{i=1}^m u_i^2\right)_{,j}}{2f} = \frac{\sum_{i=1}^m u_i u_{ij}}{f} &= \frac{u_1 u_{1j}}{f} \\ &= u_{1j}, \end{aligned} \tag{5.7}$$

we then obtain

$$\Delta u = -(p-2)\,u_{11}. \tag{5.8}$$

Therefore the inequality (5.6) can be written as

$$\begin{array}{rcl} \sum_{i,j=1}^{m}(u_{ij})^2 & \geq & (u_{11})^2 + 2\sum_{\alpha=2}^{m}(u_{1\alpha})^2 + \frac{(p-1)^2}{m-1}(u_{11})^2 \\ & = & \left(1+\frac{(p-1)^2}{m-1}\right)(u_{11})^2 + 2\sum_{\alpha=2}^{m}(u_{1\alpha})^2 \\ & \geq & (1+\kappa)\sum_{j=1}^{m}(u_{1j})^2 \\ & = & (1+\kappa)\,|\nabla f|^2 . \end{array} \tag{5.9}$$

Then (5.5) follows.

When "=" holds in the inequality (5.5), then by (5.6), we have

$$u_{\alpha\beta} = 0 \quad \text{for all } \alpha \neq \beta, \text{ where } \alpha, \beta = 2, \ldots m$$

and

$$u_{\alpha\alpha} = u_{\beta\beta} \qquad \text{for all } \alpha, \beta = 2, \ldots m. \tag{5.10}$$

Using (5.8), (5.10) then gives

$$u_{11} = -\tfrac{m-1}{p-1}u_{\alpha\alpha} \qquad \text{for all } \alpha = 2, \ldots m.$$

Moreover, by (5.9),

- If $(p-1)^2 < m-1$, then $u_{1\alpha} = 0$ for all $\alpha = 2, \ldots m$.
- If $(p-1)^2 > m-1$, then $u_{11} = 0$, i.e. $u_{ii} = 0$ for all $i = 1, \ldots m$.

Hence we complete the proof.

□

Next, we show two examples to verify Lemma 5.3 is sharp.

Example 5.1. *If $u(x) = \log|x|$ in $\mathbb{R}^m \backslash \{0\}$, then it is easy to check that $\Delta_m u = 0$ for all $m \geq 2$. Since*

$$|\nabla du|^2 = \sum_{i,j=1}^{m}\left(\frac{\delta_{ij}}{|x|^2} - \frac{2x_ix_j}{|x|^4}\right)^2 \quad \textit{and} \quad |\nabla|\nabla u||^2 = \frac{1}{|x|^4},$$

we obtain

$$|\nabla du|^2 = m\,|\nabla|\nabla u||^2$$

for $m \geq 2$. This example implies Lemma 5.3 is sharp in the case of $p = m = 2$.

Example 5.2. *Let $u(x) = |x|^{\frac{p-m}{p-1}}$ in $\mathbb{R}^m \backslash \{0\}$, $p \neq m$, then u is a p-harmonic function. Since*

$$|\nabla du|^2 = \left(\frac{p-m}{p-1}\right)^2 |x|^{\frac{2(1-m)}{p-1}-2}\sum_{i,j=1}^{m}\left\{\delta_{ij} + \left(\frac{1-m}{p-1}-1\right)\frac{x_ix_j}{|x|^2}\right\}^2$$

and

$$|\nabla|\nabla u||^2 = \left(\frac{(p-m)(1-m)}{(p-1)^2}\right)^2 |x|^{\frac{2(1-m)}{p-1}-2},$$

we have

$$|\nabla du|^2 = \left(1+\frac{(p-1)^2}{m-1}\right)|\nabla|\nabla u||^2 .$$

This example implies Lemma 5.3 is sharp in the case of $(p-1)^2 \le m-1$.

5.2. **The Proof of Theorem 1.2.**

We need several Lemmas:

Lemma 5.4. *Suppose* M^m *is a complete noncompact Riemannian manifold satisfying* (P_ρ) *and (1.6). Let* $u \in C^3(M^m)$ *be a strongly* p*-harmonic function,* $p>1$, $p\neq 2$*. Then, for every* $0<\varepsilon<1$,

$$\int_{B(R)} A_2 f_\varepsilon^{q-2} dv + \varepsilon B_0 \le \tfrac{100\cdot B_1}{R^2}\int_{B(2R)\backslash B(R)} f_\varepsilon^q dv, \tag{5.11}$$

where $f=|\nabla u|$, $f_\varepsilon = \sqrt{f^2+\varepsilon}$, $q=s+2$, $q-1+\kappa+b>\varepsilon_1$, $b=\min\{0,(p-2)(q-p)\}$,

$$\begin{aligned} B_0 &= \int_M \eta^2 \bigg(f_\varepsilon^{s-2} \textstyle\sum_{i,j=1}^m u_{ij}^2 + f_\varepsilon^{s-2}\langle \nabla u, \nabla \Delta u\rangle \\ &\quad + \tfrac{p-4}{2} f_\varepsilon^{s-4} \langle \nabla u, \nabla f_\varepsilon^2\rangle \Delta u - b f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 \bigg) dv\,, \end{aligned} \tag{5.12}$$

$$A_2 = \tfrac{4(1-\varepsilon_2)(q-1+\kappa+b-\varepsilon_1)}{q^2}\rho f_\varepsilon^2 + \textstyle\sum_{i,j=1}^m R_{ij}u_i u_j\,,$$

and

$$B_1 = \tfrac{(1+|p-2|)^2}{\varepsilon_1} + \tfrac{4\left(\frac{1}{\varepsilon_2}-1\right)(q-1+\kappa+b-\varepsilon_1)}{q^2},$$

for some $0<\varepsilon_1,\varepsilon_2<1$.

Proof. Combining Lemma 5.3 and Lemma 5.2, and using the formula

$$f^2 |\nabla (du)|^2 \ge \tfrac{1+\kappa}{4}\left|\nabla |du|^2\right|^2$$

holds on all of M, we have the following.

$$\begin{aligned} \tfrac12 \mathcal{L}_{s,\varepsilon}(f_\varepsilon^2) \;\ge\; & (s+1+\kappa) f_\varepsilon^s |\nabla f_\varepsilon|^2 + f_\varepsilon^s \textstyle\sum_{i,j=1}^m R_{ij}u_i u_j \\ & + \tfrac{(p-2)(s-p+2)}{4} f_\varepsilon^{s-4}\langle \nabla u, \nabla f_\varepsilon^2\rangle^2 \\ & + \varepsilon\bigg(f_\varepsilon^{s-2}\textstyle\sum_{i,j=1}^m u_{ij}^2 + f_\varepsilon^{s-2}\langle \nabla u, \nabla\Delta u\rangle \\ & + \tfrac{p-4}{2} f_\varepsilon^{s-4}\langle\nabla u, \nabla f_\varepsilon^2\rangle \Delta u\bigg). \end{aligned} \tag{5.13}$$

We multiply both sides of (5.13) by a cut off function $\eta^2 \in C_0^\infty(M)$ and integrate over M,

$$\begin{aligned} & \tfrac12 \int_M \eta^2 \mathcal{L}_{s,\varepsilon}(f_\varepsilon^2)\, dv \\ \ge\; & \int_M \eta^2 f_\varepsilon^s \Big((s+1+\kappa)|\nabla f_\varepsilon|^2 + \textstyle\sum_{i,j=1}^m R_{ij}u_iu_j\Big) dv \\ & + \tfrac{(p-2)(s-p+2)}{4}\int_M \eta^2 f_\varepsilon^{s-4}\langle\nabla u, \nabla f_\varepsilon^2\rangle^2 dv \\ & + \varepsilon \int_M \eta^2 \bigg(f_\varepsilon^{s-2}\textstyle\sum_{i,j=1}^m u_{ij}^2 + f_\varepsilon^{s-2}\langle \nabla u, \nabla \Delta u\rangle \\ & + \tfrac{p-4}{2} f_\varepsilon^{s-4}\langle \nabla u, \nabla f_\varepsilon^2\rangle \Delta u\bigg) dv \end{aligned} \tag{5.14}$$

where η is a cut-off function on M satisfying

$$\begin{cases} \eta(x) = 1 & \text{if } x \in \overline{B(R)}, \\ 0 < \eta(x) < 1 & \text{if } x \in B(2R) \backslash \overline{B(R)}, \\ \eta(x) = 0 & \text{if } x \in M \backslash B(2R), \end{cases}$$

and

$$\begin{cases} |\nabla \eta(x)| = 0 & \text{if } x \in B(R) \text{ or } x \in M \backslash B(2R), \\ |\nabla \eta(x)| \leq \frac{10}{R} & \text{if } x \in B(2R) \backslash \overline{B(R)}, \end{cases}$$

Since integration by parts and Cauchy-Schwarz inequality assert that

$$\begin{array}{rcl} \frac{1}{2} \int_M \eta^2 \mathcal{L}_{s,\varepsilon}(f_\varepsilon^2)\, dv & = & \frac{-1}{2} \int_M \langle \nabla \eta^2, f_\varepsilon^s \nabla f_\varepsilon^2 + (p-2) f_\varepsilon^{s-2} \langle \nabla u, \nabla f_\varepsilon^2 \rangle \nabla u \rangle\, dv \\ & \leq & 2 \int_M \eta |\nabla \eta| \left(f_\varepsilon^{s+1} |\nabla f_\varepsilon| + (p-2) f_\varepsilon^{s-1} f^2 |\nabla f_\varepsilon| \right) dv \\ & \leq & 2(1 + |p-2|) \int_M \eta |\nabla \eta| f_\varepsilon^{s+1} |\nabla f_\varepsilon|\, dv \\ & \leq & \varepsilon_1 \int_M \eta^2 f_\varepsilon^s |\nabla f_\varepsilon|^2\, dv + \frac{(1+|p-2|)^2}{\varepsilon_1} \int_M |\nabla \eta|^2 f_\varepsilon^{s+2} dv, \end{array}$$

where $0 < \varepsilon_1 < 1$ is a positive constant such that $q - 1 + \kappa + b > \varepsilon_1$.

On the other hand,

$$\begin{array}{rl} & \frac{(p-2)(s-p+2)}{4} \int_M \eta^2 f_\varepsilon^{s-4} \langle \nabla u, \nabla f_\varepsilon^2 \rangle^2\, dv \\ \geq & \frac{b}{4} \int_M \eta^2 f_\varepsilon^{s-4} |\nabla u|^2 |\nabla f_\varepsilon^2|^2\, dv \\ = & b \int_M \eta^2 f_\varepsilon^{s-2} f^2 |\nabla f_\varepsilon|^2\, dv \\ = & b \int_M \eta^2 f_\varepsilon^s |\nabla f_\varepsilon|^2\, dv - b\varepsilon \int_M \eta^2 f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2\, dv \end{array}$$

where

$$b = \min\{0, (p-2)(s-p+2)\}.$$

Then (5.14) implies

$$\begin{array}{rl} & A_1 \int_M \eta^2 f_\varepsilon^s |\nabla f_\varepsilon|^2\, dv + \int_M \eta^2 f_\varepsilon^s \sum_{i,j=1}^m R_{ij} u_i u_j dv + \varepsilon B_0 \\ \leq & \frac{(1+|p-2|)^2}{\varepsilon_1} \int_M |\nabla \eta|^2 f_\varepsilon^{s+2} dv, \end{array} \tag{5.15}$$

where $A_1 = s + 1 + \kappa + b - \varepsilon_1 > 0$.

Now we compute the first term in the left hand side of (5.15). Since $q = s + 2$,

$$\begin{array}{rl} & \int_M \eta^2 f_\varepsilon^s |\nabla f_\varepsilon|^2\, dv \\ = & \frac{4}{q^2} \int_M \eta^2 \left| \nabla f_\varepsilon^{\frac{q}{2}} \right|^2 dv \\ = & \frac{4}{q^2} \int_M \left| \nabla \left(\eta f_\varepsilon^{\frac{q}{2}} \right) - (\nabla \eta) f_\varepsilon^{\frac{q}{2}} \right|^2 dv \\ = & \frac{4}{q^2} \int_M \left| \nabla \left(\eta f_\varepsilon^{\frac{q}{2}} \right) \right|^2 - 2 \left\langle \nabla \left(\eta f_\varepsilon^{\frac{q}{2}} \right), f_\varepsilon^{\frac{q}{2}} \nabla \eta \right\rangle + |\nabla \eta|^2 f_\varepsilon^q dv \\ \geq & \frac{4(1-\varepsilon_2)}{q^2} \int_M \left| \nabla \left(\eta f_\varepsilon^{\frac{q}{2}} \right) \right|^2 + \frac{4\left(1 - \frac{1}{\varepsilon_2}\right)}{q^2} \int_{M^m} |\nabla \eta|^2 f_\varepsilon^q dv. \end{array}$$

where ε_2 is a positive constant satisfying $0 < \varepsilon_2 < 1$. Thus, we have

$$\begin{array}{rl} & \frac{4(1-\varepsilon_2)A_1}{q^2} \int_M \left| \nabla \left(\eta f_\varepsilon^{\frac{q}{2}} \right) \right|^2 dv + \int_M \eta^2 f_\varepsilon^{q-2} \sum_{i,j=1}^m R_{ij} u_i u_j\, dv + \varepsilon B_0 \\ \leq & \left(\frac{(1+|p-2|)^2}{\varepsilon_1} + \frac{4\left(\frac{1}{\varepsilon_2} - 1\right) A_1}{q^2} \right) \int_M |\nabla \eta|^2 f_\varepsilon^q dv. \end{array} \tag{5.16}$$

According to weighted Poincaré inequality

$$\int_M \rho \Psi^2 dv \leq \int_M |\nabla \Psi|^2 dv,$$

if we select $\Psi = \eta f_\epsilon^{\frac{q}{2}}$, then (5.16) implies

$$\int_{B(R)} A_2 f_\varepsilon^{q-2} dv + \varepsilon B_0 \leq \frac{100 \cdot B_1}{R^2} \int_{B(2R)\setminus B(R)} f_\varepsilon^q dv,$$

for all fixed $R > 0$. □

Lemma 5.5. *Let B_0 be as in (5.12), $p > 1$, $p \neq 2$, $q = s + 2$ and*

$$b = \min\{0, (p-2)(q-p)\}.$$

Then
(i) if $q > 2$, then $\varepsilon B_0 \to 0$ as $\varepsilon \to 0$,
(ii) if $1 < q \leq 2$ and $b \leq -\frac{(p-4)^2 m}{4}$, then $\varepsilon B_0 \geq 0$ as $\varepsilon \to 0$.

Proof. First of all, we derive some properties.

For $s \geq 2$, it is easy to check that

$$\varepsilon f_\varepsilon^{s-2} \to 0 \text{ as } \varepsilon \to 0. \tag{5.17}$$

If $0 < s < 2$, then we also have

$$\varepsilon f_\varepsilon^{s-2} = \frac{\varepsilon}{f_\varepsilon^{2-s}} \leq \frac{\varepsilon}{\varepsilon^{1-s/2}} = \varepsilon^{s/2} \to 0 \text{ as } \varepsilon \to 0, \tag{5.18}$$

By using the estimates

$$\begin{array}{rcl} \varepsilon f_\varepsilon^{s-4} \langle \nabla u, \nabla f_\varepsilon^2 \rangle & = & 2\varepsilon f_\varepsilon^{s-4} \sum_{i,j=1}^m u_{ij} u_i u_j \\ & \leq & 2\varepsilon f_\varepsilon^{s-4} \sup_{i,j=1,\cdots,m} |u_{ij}| \sum_{i,j=1}^m |u_i u_j| \\ & \leq & 2m\varepsilon f_\varepsilon^{s-4} f^2 \sup_{i,j=1,\cdots,m} |u_{ij}| \\ & \leq & 2m\varepsilon f_\varepsilon^{s-2} \sup_{i,j=1,\cdots,m} |u_{ij}| \end{array}$$

and

$$\begin{array}{rcl} \varepsilon f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 & = & \frac{\varepsilon}{4} f_\varepsilon^{s-4} |\nabla f_\varepsilon^2|^2 \\ & = & \varepsilon f_\varepsilon^{s-4} \sum_{i,j,k=1}^m u_{ik} u_{kj} u_i u_j \\ & \leq & 2m\varepsilon f_\varepsilon^{s-4} f^2 \sup_{i,j,k=1,\cdots,m} |u_{ik}| |u_{kj}| \\ & \leq & 2m\varepsilon f_\varepsilon^{s-2} \sup_{i,j,k=1,\cdots,m} |u_{ik}| |u_{kj}| \end{array}$$

then (5.17) and (5.18) imply

$$\left\{ \begin{array}{l} \varepsilon f_\varepsilon^{s-4} |\langle \nabla u, \nabla f_\varepsilon^2 \rangle| \to 0, \\ \varepsilon f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 \to 0, \end{array} \right. \tag{5.19}$$

as $\varepsilon \to 0$, for all $s > 0$.

In the case $-1 < s \leq 0$,

$$\varepsilon f f_\varepsilon^{s-2} \leq \frac{\varepsilon}{(f^2+\varepsilon)^{1/2-s/2}} \leq \varepsilon^{1/2+s/2} \to 0 \text{ as } \varepsilon \to 0. \tag{5.20}$$

Now we prove Lemma as follows.

For any fixed $s > 0$, by (5.17), (5.18) and (5.19), then we obtain,

$$\begin{aligned} |\varepsilon B_0| &= \varepsilon \bigg| \int_M \eta^2 \bigg(f_\varepsilon^{s-2} \sum_{i,j=1}^m u_{ij}^2 + f_\varepsilon^{s-2} \langle \nabla u, \nabla \Delta u \rangle \\ &\quad + \tfrac{p-4}{2} f_\varepsilon^{s-4} \langle \nabla u, \nabla f_\varepsilon^2 \rangle \Delta u - b f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 \bigg) dv \bigg| \\ &\le \int_M \eta^2 \bigg((\varepsilon f_\varepsilon^{s-2}) \sum_{i,j=1}^m u_{ij}^2 + (\varepsilon f_\varepsilon^{s-2}) |\langle \nabla u, \nabla \Delta u \rangle| \\ &\quad + \tfrac{|p-4|}{2} (\varepsilon f_\varepsilon^{s-4} |\langle \nabla u, \nabla f_\varepsilon^2 \rangle|) |\Delta u| - b \left(\varepsilon f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 \right) \bigg) dv \\ &\to 0, \text{ as } \varepsilon \to 0. \end{aligned}$$

If $s > -1$, since $b \le -\frac{(p-4)^2 m}{4}$, then

$$\begin{aligned} & f_\varepsilon^{s-2} \sum_{i,j=1}^m u_{ij}^2 - b f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 + \tfrac{p-4}{2} f_\varepsilon^{s-4} \langle \nabla u, \nabla f_\varepsilon^2 \rangle \Delta u \\ \ge\ & f_\varepsilon^{s-2} \sum_{i,j=1}^m u_{ij}^2 - b f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 - |p-4| f_\varepsilon^{s-2} |\nabla f_\varepsilon| |\Delta u| \\ \ge\ & f_\varepsilon^{s-2} \sum_{i,j=1}^m u_{ij}^2 - b f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 - \tfrac{f_\varepsilon^{s-2} (\Delta u)^2}{m} - \tfrac{(p-4)^2 m}{4} f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 \\ \ge\ & 0, \end{aligned}$$

here we use $\sum_{i,j=1}^m u_{ij}^2 \ge \frac{(\Delta u)^2}{m}$. Hence by (5.17), (5.18) and (5.20),

$$\begin{aligned} \varepsilon B_0 &= \varepsilon \int_M \eta^2 \bigg(f_\varepsilon^{s-2} \sum_{i,j=1}^m u_{ij}^2 + f_\varepsilon^{s-2} \langle \nabla u, \nabla \Delta u \rangle \\ &\quad + \tfrac{p-4}{2} f_\varepsilon^{s-4} \langle \nabla u, \nabla f_\varepsilon^2 \rangle \Delta u - b f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 \bigg) dv \\ &\ge \varepsilon \int_M \eta^2 f_\varepsilon^{s-2} \langle \nabla u, \nabla \Delta u \rangle \\ &\ge - \int_M \eta^2 (\varepsilon f_\varepsilon^{s-2} f) |\nabla \Delta u| \\ &\to 0 \text{ whenever } s > -1 \text{ and } \varepsilon \to 0. \end{aligned}$$

In particular, if $s > -1$ and $p = 4$, by applying (5.17), (5.18), (5.20) and $b \le 0$, then

$$\begin{aligned} \varepsilon B_0 &= \varepsilon \int_M \eta^2 \bigg(f_\varepsilon^{s-2} \sum_{i,j=1}^m u_{ij}^2 + f_\varepsilon^{s-2} \langle \nabla u, \nabla \Delta u \rangle - b f_\varepsilon^{s-2} |\nabla f_\varepsilon|^2 \bigg) dv \\ &\ge - \int_M \eta^2 (\varepsilon f_\varepsilon^{s-2} f) |\nabla \Delta u| \, dv \\ &\to 0 \text{ whenever } s > -1 \text{ and } \varepsilon \to 0. \end{aligned}$$

□

Remark 5.2. In Lemma 5.5, if $p = 4$ and $q > 1$, then $\varepsilon B_0 \ge 0$ as $\varepsilon \to 0$.

Proof of Theorem 1.2. Since we assume $q-1+\kappa+b > 0$, the curvature condition (1.6) means that there exists a constant $0 < \delta < 1$ such that

$$Ric_M(\nabla u, \nabla u) = \sum_{i,j=1}^m R_{ij} u_i u_j \ \ge\ -\frac{4(q-1+\kappa+b)}{q^2} \delta \rho f^2. \tag{5.21}$$

To apply Lemmas 5.4 and 5.5, we need the following conditions:

$$(*) \begin{cases} q > 2 \text{ and } q-1+\kappa+b > 0, \\ \text{or} \quad 1 < q \le 2 \text{ and } q-1+\kappa+b > 0, \ \text{where } b \le -\frac{(p-4)^2 m}{4}. \end{cases}$$

We first assume $p \neq 2$.
For $p > 2$, the expression $1 < q \leq 2$ implies that $q < p$. Hence $b = (p-2)(q-p)$. Then

$$q - 1 + \kappa + b > 0 \quad \Longleftrightarrow \quad q > p - 1 - \frac{\kappa}{p-1}$$

(cf. Remark 5.3), and

$$b \leq -\frac{(p-4)^2 m}{4} \quad \Longleftrightarrow \quad q \leq p - \frac{(p-4)^2 m}{4(p-2)}.$$

That is, for $p > 2$, $(*)$ can be rewritten as

$$(*_1) \qquad \begin{cases} \max\{2, 1-\kappa-b\} < q \\ \text{or} \quad \max\left\{1, p-1-\frac{\kappa}{p-1}\right\} < q \leq \min\left\{2, p - \frac{(p-4)^2 m}{4(p-2)}\right\} \end{cases}$$

For $p = 4$, $b \leq -\frac{(p-4)^2 m}{4} = 0$ holds and $(*)$ can be simplified as follows:

$$(*_2) \qquad \max\{1, 1-\kappa-b\} < q.$$

For $1 < p < 2$, the expression $1 < q \leq 2$ implies that $p < q$. Or $q \leq p\,(< 2)$ would lead to $0 = b \leq -\frac{(p-4)^2 m}{4} < 0$, a contradiction. Hence $b = (p-2)(q-p)$. Then $q-1+\kappa+b > 0$ $\left(\Longleftrightarrow q > p - 1 - \frac{\kappa}{p-1}\right)$ holds . However, $b \leq -\frac{(p-4)^2 m}{4}$ $\left(\Longleftrightarrow \quad q \leq p - \frac{(p-4)^2 m}{4(p-2)}\right)$ is invalid . What remains is the following:
For $1 < p < 2$, the expression $2 < q$ implies that $b = (p-2)(q-p)$. Then $q - 1 + \kappa + b > 0$ $\left(\Longleftrightarrow \quad q > p - 1 - \frac{\kappa}{p-1}\right)$ holds .

Thus, for $1 < p < 2$, $(*)$ can be rewritten as

$$(*_3) \qquad 2 < q.$$

Similarly, for $p = 2$, we have $b = 0$ and $\kappa = \frac{1}{m-1}$. It follows that $q - 1 + \kappa + b > 0$ holds if and only if

$$(*_4) \qquad \frac{m-2}{m-1} < q.$$

In view of $(*_1), (*_2), (*_3), (*_4)$, Lemmas 5.4 and 5.5, we obtain (5.11). As $\epsilon \to 0$, (5.11), via (5.21) tends to

$$\int_{B(R)} A_3 f^q dv \quad \leq \quad \frac{100 \cdot B_1}{R^2} \int_{B(2R)\setminus B(R)} f^q dv, \tag{5.22}$$

where

$$A_3 \quad = \quad \left(\frac{4(1-\varepsilon_2)(q-1+\kappa+b-\varepsilon_1)}{q^2} - \frac{4(q-1+\kappa+b)\delta}{q^2} \right) \rho.$$

Hence one has $A_3 > 0$ whenever we select ε_1 and ε_2 small enough. Suppose $f \in L^q(M)$, then the right hand side of (5.22) tends to zero as $R \to \infty$, and then we conclude that $f(x) = 0$ for all $x \in M$ and for some $0 < \delta < 1$, i.e. $u(x)$ is a constant on M for some $0 < \delta < 1$.

In particular, if $1 < p < 2$, since constant function is not a strongly p-harmonic function, then such u does not exist. $\square$

Remark 5.3. If $p > 2$ and $p \geq q$, then

$$\begin{aligned} q - 1 + \kappa + b &= q - 1 + \kappa + (p-2)(q-p) \\ &= (p-1)q - (p-1)^2 + \kappa > 0, \end{aligned}$$

whenever $q > p - 1 - \frac{\kappa}{p-1}$.

Remark 5.4. If we replace the finite q-energy by $\int_{B(2R)\setminus B(R)} |\nabla u|^q \, dv = o(R^2)$ as $R \to \infty$, then Theorem 1.2 is still valid.

Remark 5.5. Since (P_{λ_q}) implies (P_{λ_p}) for all $p > q$ (cf. [14]). If M satisfies (P_{λ_2}), by using Lemma 6.5, then 2-hyperbolic end is equality to p-hyperbolic end since this end has infinite volume. Hence we may use the method of Theorem 2.1 of [32] to refine the conditions of Theorem 1.2 whenever M satisfies (P_{λ_2}). But we omit it in this paper.

Corollary 5.1. *Let M^m be a complete noncompact Riemannian manifold satisfying (P_ρ) and (1.6), where*

$$\tau < \frac{4\left((p-1)q-(p-1)^2+\kappa\right)}{q^2},$$

$\kappa = \min\{\frac{(p-1)^2}{m-1}, 1\}$, $p > 2$, $p \geq q$. *Let $u \in C^3(M^m)$ be a strongly p-harmonic function, with finite q-energy $E_q(u)$. Then u is a constant if p and q satisfy one of the following:*
(1) $p = 4$, $q > \frac{9-\kappa}{3}$,
(2) $p \neq 4$, and either

$$\max\left\{1, p - 1 - \frac{\kappa}{p-1}\right\} < q \leq \min\left\{2, p - \frac{(p-4)^2 m}{4(p-2)}\right\}$$

or

$$\max\{2, p - 1 - \tfrac{\kappa}{p-1}\} < q.$$

In particular, if $p = q$, then every strongly p-harmonic function u with finite p-energy is constant.

Corollary 5.2. *Let M^m be a complete noncompact Riemannian manifold satisfying (P_ρ) and (1.6), where*

$$\tau < \frac{4(p-1+\kappa)}{p^2},$$

$\kappa = \min\{\frac{(p-1)^2}{m-1}, 1\}$. *If $u \in C^3(M^m)$ is a strongly p-harmonic function for $p \geq 2$, with $E_p(u) < \infty$, then u is a constant.*

Remark 5.6. According to the following Lemma 5.6, we can replace "Let $u \in C^3(M^m)$ be a strongly p-harmonic function for $1 < p < \infty$." in Theorem 1.2 by "Let $u \in C^2(M^m)$ be a weakly p-harmonic function for $p \in \{2\} \cup [4, \infty)$, and $u \in C^3(M^m)$ be a strongly p-harmonic function for $p \in (1,2) \cup (2,4)$." Theorem 1.2 remains to be true.

Lemma 5.6. *If $u \in C^2(M)$ (resp. $u \in C^0(M)$) is a weakly p-harmonic function for $p \in [4, \infty)$ (resp. $p = 2$), then u is a strongly p-harmonic function.*

Proof. By assumption, u satisfies

$$\int_M \langle f^{p-2} \nabla u, \nabla \eta \rangle \, dv = 0$$

for every $\eta \in C_0^\infty(M)$, where $f = |\nabla u|$. Since $u \in C^2(M)$, and either $p = 2$, or $p \geq 4$, we have $f^{p-2} \in C^1(M)$. Hence $f^{p-2}\nabla u \in C^1(M)$, and the divergence theorem implies

$$0 = \int_M \langle f^{p-2}\nabla u, \nabla\eta\rangle \, dv = -\int_M \operatorname{div}(f^{p-2}\nabla u)\, \eta \, dv$$

for every $\eta \in C_0^\infty(M)$. This completes the proof. $\square$

5.3. **Application to p-harmonic morphism.** A C^2 map $u : M \to N$ is called a p-harmonic morphism if for any p-harmonic function f defined on an open set V of N, the composition $f \circ u$ is p-harmonic on $u^{-1}(V)$. Examples of p-harmonic morphisms include the Hopf fibrations. E. Loubeau and J. M. Burel ([2]) and E. Loubeau([23]) prove that a C^2 map $u : M \to N$ is a p-harmonic morphism with $p \in (1, \infty)$ if and only if u is a p-harmonic and horizontally weak conformal map. We recall a C^2 map $u : M \to N$ is *horizontally weak conformal* if for any x such that $du(x) \neq 0$, the restriction of $du(x)$ to the orthogonal complement of Ker $du(x)$ is conformal and surjective.

Theorem 5.1. *Let M^m be a complete noncompact Riemannian manifold, satisfying (P_ρ) and (1.6), where*

$$\tau < \frac{4(q-1+\kappa+b)}{q^2}, \quad \kappa = \min\{\frac{(p-1)^2}{m-1}, 1\}, \quad \text{and} \quad b = \min\{0, (p-2)(q-p)\}.$$

Let $u \in C^3(M^m, \mathbb{R}^k)$ is a p-harmonic morphism $u : M^m \to \mathbb{R}^k$, $k > 0$ of finite q-energy $E_q(u) < \infty$.
(I). Then u is constant under one of the following:
(1) $p = 2$ and $q > \frac{m-2}{m-1}$,
(2) $p = 4$, $q > 1$ and $q - 1 + \kappa + b > 0$,
(3) $p > 2$, $p \neq 4$, and either

$$\max\left\{1, p - 1 - \tfrac{\kappa}{p-1}\right\} < q \leq \min\left\{2, p - \tfrac{(p-4)^2 m}{4(p-2)}\right\}$$

or

$$\max\{2, 1 - \kappa - b\} < q.$$

(II). Then u does not exit under
(4) $1 < p < 2$, $q > 2$.

Lemma 5.7. [49]*Let M, N and K be manifolds of dimension m, n, and k respectively, and $u : M \to N$, and $w : N \to K$ be C^2. If u is horizontally weak conformal, then $|d(w \circ u)|^{p-2} = (\frac{1}{n})^{\frac{p-2}{2}} |dw|^{p-2}|du|^{p-2}$.*

Proof of Theorem 5.1. Let $u^i = \pi_i \circ u$, where $\pi_i : \mathbb{R}^k \to \mathbb{R}$ is the i-th projection. Then the linear function π_i is a p-harmonic function (cf. 2.2 in [47]). Hence u^i, a composition of a p-harmonic morphism and a p-harmonic function is p-harmonic. Since u is horizontally weak conformal, it follows from Lemma 5.7 that $E_p(u) < \infty$ implies $E_p(u^i) < \infty$. Now apply u^i to Theorem 1.2, the assertion follows. $\square$

These results are in contrast to the following:

Theorem 5.2. [49] *If $u : M^m \to \mathbb{R}^k$, $k > 0$, is a p-harmonic morphism, and if there exists i, such that $u^i = \pi_i \circ u$ is p-finite, i.e.*

$$\liminf_{r\to\infty} \tfrac{1}{r^p} \int_{B(r)} |u^i|^q \, dv < \infty$$

where $B(r)$ is a geodesic ball of radius r, for some $q > p-1$. Then u must be constant.

As further applications, one obtains

Theorem 5.3. *Let M^m be a complete noncompact Riemannian manifold, satisfying (P_ρ) and (1.6), where*

$$\tau < \frac{4(q-1+\kappa+b)}{q^2}, \quad \kappa = \min\{\frac{(p-1)^2}{m-1}, 1\} \quad \text{and} \quad b = \min\{0, (p-2)(q-p)\}.$$

Let $u \in C^3(M^m, \mathbb{R}^k)$ be a p-harmonic morphism $u : M^m \to \mathbb{R}^k$, $k > 0$, and $f : u(M^m) \subset \mathbb{R}^k \to \mathbb{R}$ be a nonconstant p-harmonic function. Assume $f \circ u$ has finite q-energy $E_q(f \circ u) < \infty$.
(I). Then u is constant under one of the following:
(1) $p = 2$ and $q > \frac{m-2}{m-1}$,
(2) $p = 4$, $q > 1$ and $q - 1 + \kappa + b > 0$,
(3) $p > 2$, $p \neq 4$, and either

$$\max\left\{1, p-1-\tfrac{\kappa}{p-1}\right\} < q \leq \min\left\{2, p - \tfrac{(p-4)^2 m}{4(p-2)}\right\}$$

or

$$\max\{2, 1-\kappa-b\} < q.$$

(II). Then u does not exit under
(4) $1 < p < 2$, $q > 2$.

Lemma 5.8. *A nonconstant p-harmonic morphism $u : M^m \to \mathbb{R}^k$ is an open map.*

Proof of Theorem 5.3. Since u is a p-harmonic morphism, then $f \circ u$ is a p-harmonic function on M^m. According to Theorem 1.2, then $f \circ u$ is a constant c. On the other hand, due to Lemma 5.8, u and f are open maps whenever they are not constant. Now we assume that u is not constant, then the image of u is an open set $u(M) \subset \mathbb{R}^k$. Hence $f \circ u(M^m)$ is an open set. This gives a contradiction to $f \circ u(M^m) = c$. Then we conclude that u is a constant. □

Theorem 5.4. (*Picard Theorem for p-harmonic morphisms*). *Let M^m be as in Theorem 5.3. Suppose that $u \in C^3(M^m, \mathbb{R}^k \backslash \{y_0\})$ is a p-harmonic morphism $u : M^m \to \mathbb{R}^k \backslash \{y_0\}$, and the function $x \mapsto |u(x) - y_0|^{\frac{p-k}{p-1}}$ has finite q-energy where $p \neq k$, for p and q satisfying one of the following:* (1), (2), *and* (3) *as in Theorem 5.3. Then u is constant. For p and q satisfying* (4) *as in Theorem 5.3, then u does not exist.*

Proof. Since $y \mapsto |y|^{\frac{p-k}{p-1}}$ is a p-harmonic function from $\mathbb{R}^k \backslash \{0\}$ to $\mathbb{R}$, the composite map $|u(x) - y_0|^{\frac{p-k}{p-1}} : M \to \mathbb{R}$ is a p-harmonic function with finite q-energy. By Theorem 5.3, in which $p \neq k$, we obtain the conclusion. □

5.4. **Application to Conformal Maps.** Our previous result can be applied to weakly conformal maps between equal dimensional manifolds based on the following:

Theorem A ([35]) $u : M \to N$ is an m-harmonic morphism, if and only if u is weakly conformal, where $m = \dim M = \dim N$.

For instance, stereographic projections $u : \mathbb{R}^m \to S^m$ are m-harmonic maps and m-harmonic morphisms, for all $m \geq 1$.

Theorem 5.5. *Let M^m be a complete noncompact m-manifold satisfying (P_ρ) and (1.6), where $\tau < \frac{4(q+b)}{q^2}$ and $b = \min\{0, (m-2)(q-m)\}$. If $u : M^m \to \mathbb{R}^m$ is a weakly conformal map of finite q-energy $E_q(u) < \infty$. Then u is a constant if m and q satisfy one of the following:*
(1) $m = 2$ and $q > 0$,
(2) $m = 4$, $q > 1$ and $q + b > 0$,
(3) $m > 2$, $m \neq 4$, and either $\frac{m(m-2)}{m-1} < q \leq \min\left\{2, m - \frac{(m-4)^2 m}{4(m-2)}\right\}$ or $q > \max\{2, -b\}$.

Proof. By Theorem A ([35]), u is an m-harmonic morphism. Now the result follows immediately from Theorem 5.1 in which $p = m$. Since $\log|x|$ is an m-harmonic function, $\log|u(x) - y_0| : M \to \mathbb{R}$ is an m-harmonic function with finite q-energy. By Theorem 5.3, in which $p = m$, we obtain the conclusion. □

6. Appendix

6.1. **The existence of the approximate solution.** In this subsection, we study an approximate solution u_ϵ of the p-Laplace equation or a solution u_ϵ of a perturbed p-Laplace equation

$$\Delta_{p,\epsilon} u_\epsilon = \operatorname{div}\left(\left(|\nabla u_\epsilon|^2 + \epsilon\right)^{\frac{p-2}{2}} \nabla u_\epsilon\right) = 0 \tag{6.1}$$

on a domain $\Omega \subset M$ with boundary condition $u_\epsilon = u$ on $\partial\Omega$. That is, u_ϵ is the Euler-Lagrange equation of the (p,ϵ)-energy $E_{p,\epsilon}$ functional given by

$$E_{p,\epsilon}(\Psi) = \int_\Omega \left(|\nabla \Psi|^2 + \epsilon\right)^{\frac{p}{2}} \, dv \tag{6.2}$$

with $\Psi \in W^{1,p}(\Omega)$, and $\Psi = u$ on $\partial\Omega$.

Proposition 6.1 (The existence of u_ϵ)**.** *Let u be a $W^{1,p}$ function on the closure $\bar{\Omega}$ of a domain $\Omega \subset M$. Then there is a solution $u_\epsilon \in W^{1,p}(\Omega)$ of the Euler-Lagrange equation of the (p,ϵ)-energy $E_{p,\epsilon}$ with $u_\epsilon = u$ on the boundary of Ω in the trace sense.*

Proof. Let H be the set of functions $v \in W^{1,p}(\Omega)$ such that $v = u$ on the boundary of Ω in the trace sense, and $I = \inf\{E_{p,\epsilon}(v) : v \in H\}$. Then by assumption, $u \in H$, H is nonempty, and I exists. Furthermore $I \leq E_{p,\epsilon}(u)$.

Take a minimizing sequence $\{v_i\}_{i=1}^\infty$ such that $E_{p,\epsilon}(v_i)$ tends to I as i tends to ∞.

Then $\{v_i\}_{i=1}^{\infty}$ is a bounded sequence in $W^{1,p}(\Omega)$. Hence there exists a subsequence, say $\{u_i\}_{i=1}^{\infty}$, converges weakly to u_ϵ in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and pointwise almost everywhere. We infer u_ϵ is in H since H is closed. Thus $I \le E_{p,\epsilon}(u_\epsilon)$.

To prove $I \ge E_{p,\epsilon}(u_\epsilon)$, it suffices to prove the lower semi-continuity of $E_{p,\epsilon}$ (two methods).

Method 1:

Since Banach-saks Theorem (see, e.g. [51] p. 120, [39] p. 80) asserts there exists some subsequence, say it again v_i for simplicity, such that the average

$$w_n = \frac{v_1+v_2+\cdots+v_n}{n}$$

converges strongly to u_ϵ in $W^{1,p}(\Omega)$. Combining this property and Lemma 6.1, we have $E_{p,\epsilon}(w_n) \to E_{p,\epsilon}(u_\epsilon)$ as $n \to \infty$.

Moreover, according to the convexity of $E_{p,\epsilon}$, one has

$$E_{p,\epsilon}(w_n) \le \frac{\sum_{i=1}^n E_{p,\epsilon}(v_i)}{n}.$$

This implies $E_{p,\epsilon}(u_\epsilon) \le I$ as $n \to \infty$.

So we obtain lower semi-continuity of $E_{p,\epsilon}$.

Method 2:

If $\dim M > 2$, we denote $T_x\Omega$ the tangent space to $\Omega \subset M$ at x. Let $\nu_i(x) \in T_x\Omega$ be a unit vector perpendicular to $\nabla u_i(x), \nabla u_\epsilon(x) \in T_x\Omega$, for a.e. $x \in \Omega$. If $\dim M = 2$, we isometrically embed M into $N = M \times \mathbb{R}$ with the standard product metric $\langle\, , \rangle_N$ and choose $\nu_i(x)$ to be a unit vector in $\mathbb{R}$.

In either case, we set $b(x) = \nabla u_i(x) + \sqrt{\epsilon}\nu_i(x)$ and $a(x) = \nabla u_\epsilon(x) + \sqrt{\epsilon}\nu_i(x)$. Then on Ω, $|b| = \sqrt{|\nabla u_i|^2 + \epsilon}$ and $|a| = \sqrt{|\nabla u_\epsilon|^2 + \epsilon}$.

If $m = 2$, applying the inequality

$$|b|^p \ge |a|^p + p\langle |a|^{p-2}a, b-a\rangle_N$$

and integrating it over Ω, we have via $\nu_i(x) \perp \nabla u_\epsilon$, and $\nu_i(x) \perp \nabla u_i$, for a.e. $x \in \Omega$,

$$\begin{aligned} E_{p,\epsilon}(u_i) &\ge E_{p,\epsilon}(u_\epsilon) + \textstyle\int_\Omega \langle (|\nabla u_\epsilon|^2+\epsilon)^{\frac{p-2}{2}}(\nabla u_\epsilon + \sqrt{\epsilon}\nu_i), \nabla u_i - \nabla u_\epsilon\rangle_N dv \\ &= E_{p,\epsilon}(u_\epsilon) + \textstyle\int_\Omega \langle (|\nabla u_\epsilon|^2+\epsilon)^{\frac{p-2}{2}}\nabla u_\epsilon, \nabla u_i - \nabla u_\epsilon\rangle_M dv \end{aligned}$$

We note that in the last term, $(|\nabla u_\epsilon|^2+\epsilon)^{\frac{p-2}{2}}\nabla u_\epsilon$ is in $L^{\frac{p}{p-1}}(\Omega)$, $\nabla u_i - \nabla u_\epsilon$ is in $L^p(\Omega)$. Thus, $\langle (|\nabla u_\epsilon|^2+\epsilon)^{\frac{p-2}{2}}\nabla u_\epsilon, \nabla u_i - \nabla u_\epsilon\rangle_M$ is in $L^1(\Omega)$. Since ∇u_i converges weakly to ∇u_ϵ in L^p, the last term tends to 0 as i tends to ∞. It follows that $E_{p,\epsilon}(u_\epsilon) \le \liminf_{i\to\infty} E_{p,\epsilon}(u_i) = I$. Similarly, if $\dim M > 2$, we obtain directly

$$E_{p,\epsilon}(u_i) \ge E_{p,\epsilon}(u_\epsilon) + \int_\Omega \langle (|\nabla u_\epsilon|^2+\epsilon)^{\frac{p-2}{2}}\nabla u_\epsilon, \nabla u_i - \nabla u_\epsilon\rangle_M dv.$$

Proceed in the same way, the assertion follows.

□

Lemma 6.1. *If v_i converges strongly to v_0 in $W^{1,p}$, then $E_{p,\epsilon}(v_i)$ converges to $E_{p,\epsilon}(v_0)$.*

Proof. Step 1: Since v_i converges strongly to v_0 in $W^{1,p}$ i.e. $\int_\Omega |\nabla v_i - \nabla v_0|^p\, dv \to 0$ as $i \to \infty$. Then

$$\int_{|\nabla v_i| \geq |\nabla v_0|} |\nabla v_i - \nabla v_0|^p\, dv \to 0 \text{ and} \int_{|\nabla v_i| < |\nabla v_0|} |\nabla v_i - \nabla v_0|^p\, dv \to 0$$

as $i \to \infty$. By using Minkowski's inequality, these also imply

$$\int_{|\nabla v_i| \geq |\nabla v_0|} \left(|\nabla v_i|^p - |\nabla v_0|^p\right) dv \to 0 \text{ and } \int_{|\nabla v_i| < |\nabla v_0|} \left(|\nabla v_0|^p - |\nabla v_i|^p\right) dv \to 0$$

as $i \to \infty$. That is, $\int_\Omega \left||\nabla v_0|^p - |\nabla v_i|^p\right| dv \to 0$ as $i \to \infty$.

Step 2: If we show that, for any positive constant $\delta > 0$,

$$\left|\left(|\nabla v_i|^2 + \epsilon\right)^{\frac{p}{2}} - \left(|\nabla v_0|^2 + \epsilon\right)^{\frac{p}{2}}\right| \leq a \left||\nabla v_i|^p - |\nabla v_0|^p\right| + \delta \tag{6.3}$$

where a is a positive constant independent of i, v_i and v_0. Then we have, by step 1,

$$\begin{array}{rcl} |E_{p,\epsilon}(v_i) - E_{p,\epsilon}(v_0)| & \leq & \int_\Omega \left|\left(|\nabla v_i|^2 + \epsilon\right)^{\frac{p}{2}} - \left(|\nabla v_0|^2 + \epsilon\right)^{\frac{p}{2}}\right| dv \\ & \leq & a \int_\Omega \left||\nabla v_i|^p - |\nabla v_0|^p\right| dv + \delta\,|\Omega| \\ & \to & \delta\,|\Omega| \ \text{ as } i \to \infty. \end{array}$$

This implies $E_{p,\epsilon}(v_i) \to E_{p,\epsilon}(v_0)$.

To show (6.3), we only claim that, $X, Y \in \mathbb{R}^n$ with $|X| \geq |Y|$,

$$\left(|X|^2 + \epsilon\right)^{\frac{p}{2}} - \left(|Y|^2 + \epsilon\right)^{\frac{p}{2}} \leq a\left(|X|^p - |Y|^p\right) + \delta. \tag{6.4}$$

Let $f(t) = \left(|X|^2 + t\right)^{\frac{p}{2}} - \left(|Y|^2 + t\right)^{\frac{p}{2}}$, $t \geq 0$. Then we have $f(0) = |X|^p - |Y|^p$ and $f(\epsilon) = \left(|X|^2 + \epsilon\right)^{\frac{p}{2}} - \left(|Y|^2 + \epsilon\right)^{\frac{p}{2}}$.

Since

$$f'(t) = \frac{p}{2}\left(\left(|X|^2 + t\right)^{\frac{p-2}{2}} - \left(|Y|^2 + t\right)^{\frac{p-2}{2}}\right),$$

then $f(t)$ is a decreasing function for $1 \leq p \leq 2$. Hence we have $f(\epsilon) \leq f(0)$ whenever $1 \leq p \leq 2$.

If $2 < p \leq 4$, then, for $s > 0$,

$$\begin{array}{rcl} f(s) - f(0) & = & \int_0^s f'(t)dt \\ & = & \frac{p}{2}\int_0^s \left(|X|^2 + t\right)^{\frac{p-2}{2}} - \left(|Y|^2 + t\right)^{\frac{p-2}{2}} dt, \\ & \leq & \frac{ps}{2}\left(|X|^{p-2} - |Y|^{p-2}\right), \end{array} \tag{6.5}$$

since $1 < p - 2 \leq 2$.

For any $\delta_1 > 0$,

$$|X|^{p-2} - |Y|^{p-2} \leq \begin{cases} |X|^{p-2} & \text{if } |X| + |Y| < \delta_1, \\ \frac{(|X|+|Y|)^2\left(|X|^{p-2} - |Y|^{p-2}\right)}{\delta_1^2} & \text{if } |X| + |Y| \geq \delta_1. \end{cases}$$

Since

$$\begin{cases} |X|^{p-2} \leq \delta_1^{p-2} & \text{if } |X| + |Y| < \delta_1, \\ \frac{(|X|+|Y|)^2\left(|X|^{p-2} - |Y|^{p-2}\right)}{\delta_1^2} \leq \frac{2}{\delta_1^2}\left(|X|^p - |Y|^p\right) & \text{if } |X| + |Y| \geq \delta_1. \end{cases}$$

So we have

$$|X|^{p-2} - |Y|^{p-2} \le \tfrac{2}{\delta_1^2}\left(|X|^p - |Y|^p + \delta_1^p\right), \tag{6.6}$$

and then (6.5) can be rewritten as

$$\left(|X|^2+s\right)^{\frac{p}{2}} - \left(|Y|^2+s\right)^{\frac{p}{2}} \le \left(1+\tfrac{ps}{\delta_1^2}\right)\left(|X|^p - |Y|^p\right) + \left(\tfrac{ps}{\delta_1^2}\right)\delta_1^p. \tag{6.7}$$

Hence we have

$$\left(|X|^2+\epsilon\right)^{\frac{p}{2}} - \left(|Y|^2+\epsilon\right)^{\frac{p}{2}} \le a\left(|X|^p - |Y|^p\right) + \delta,$$

where $a = 1 + \frac{p\epsilon}{\delta_1^2}$ and $\delta = \left(\frac{p\epsilon}{\delta_1^2}\right)\delta_1^p$.

If $4 < p \le 6$, then one has $2 < p-2 \le 4$, so (6.6) and (6.7) imply

$$\begin{aligned}
f(s) - f(0) &= \tfrac{p}{2}\int_0^s \left(|X|^2+t\right)^{\frac{p-2}{2}} - \left(|Y|^2+t\right)^{\frac{p-2}{2}}\, dt \\
&\le \tfrac{p}{2}\int_0^s \left(1+\tfrac{pt}{\delta_1^2}\right)\left(|X|^{p-2} - |Y|^{p-2}\right) + \left(\tfrac{pt}{\delta_1^2}\right)\delta_1^{p-2} dt \\
&\le \tfrac{p}{2}\left(s + \tfrac{ps^2}{2\delta_1^2}\right)\left(\tfrac{2}{\delta_1^2}\left(|X|^p - |Y|^p + \delta_1^p\right)\right) + \tfrac{p}{2}\left(\tfrac{ps^2}{2\delta_1^2}\right)\delta_1^{p-2} \\
&\le \left(\tfrac{ps}{\delta_1^2} + \tfrac{1}{2}\left(\tfrac{ps}{\delta_1^2}\right)^2\right)\left(|X|^p - |Y|^p\right) + \left(\tfrac{ps}{\delta_1^2} + \left(\tfrac{ps}{\delta_1^2}\right)^2\right)\delta_1^p.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(|X|^2+s\right)^{\frac{p}{2}} - \left(|Y|^2+s\right)^{\frac{p}{2}} \;\le\; & \left(1 + \tfrac{ps}{\delta_1^2} + \tfrac{1}{2}\left(\tfrac{ps}{\delta_1^2}\right)^2\right)\left(|X|^p - |Y|^p\right) \\
& + \left(\tfrac{ps}{\delta_1^2} + \left(\tfrac{ps}{\delta_1^2}\right)^2\right)\delta_1^p.
\end{aligned}$$

In particular, we obtain

$$\left(|X|^2+\epsilon\right)^{\frac{p}{2}} - \left(|Y|^2+\epsilon\right)^{\frac{p}{2}} \;\le\; a\left(|X|^p - |Y|^p\right) + \delta,$$

where $a = 1 + \frac{p\epsilon}{\delta_1^2} + \frac{1}{2}\left(\frac{p\epsilon}{\delta_1^2}\right)^2$ and $\delta = \left(\frac{p\epsilon}{\delta_1^2} + \left(\frac{p\epsilon}{\delta_1^2}\right)^2\right)\delta_1^p$.

By mathematical induction, we conclude that, for any $p > 2$ satisfying $2q < p \le 2q+2$, $q \in \mathbb{Z}^+$,

$$\begin{aligned}
\left(|X|^2+\epsilon\right)^{\frac{p}{2}} - \left(|Y|^2+\epsilon\right)^{\frac{p}{2}} \;\le\; & \left(1 + \textstyle\sum_{n=1}^q \tfrac{1}{n!}\left(\tfrac{p\epsilon}{\delta_1^2}\right)^n\right)\left(|X|^p - |Y|^p\right) \\
& + \left(\textstyle\sum_{n=1}^q \left(\tfrac{p\epsilon}{\delta_1^2}\right)^n\right)\delta_1^p.
\end{aligned}$$

If we select δ_1 small enough such that $\left(\sum_{n=1}^q \left(\frac{p\epsilon}{2\delta_1^2}\right)^n\right)\delta_1^p = \delta$, then we have (6.4) with $a = \left(1 + \sum_{n=1}^q \left(\frac{p\epsilon}{2\delta_1^2}\right)^n\right)$.

□

6.2. **ϵ-regularization of p-Laplacian.**

Proposition 6.2. *Let u be a weak solution of the p-Laplace equation (1.1). For every $\epsilon > 0$, let u_ϵ be a solution of the Euler-Lagrange equation (6.1) with $u - u_\epsilon \in W_0^{1,p}(\Omega)$, where Ω is a domain in M. Then $u_\epsilon \in C^\infty_{loc}(\Omega)$ is a strong solution of (6.1), and u_ϵ converges strongly to u in $W^{1,p}(\Omega)$ as $\epsilon \to 0$.*

Proof. Such solution u_ϵ exists (Proposition 6.1), and $u_\epsilon \in C^\infty_{loc}(\Omega)$ by the usual arguments of boot-strap (see, e.g. [31] Chapter 4, [40] Theorem 3.3, [20] Theorem 14.2, [19]). That is, u_ϵ is the strong solution of the partial differential equation (1.2).

Since u_ϵ and u are the minimizers of the energy functions

$$\int_\Omega \left||\nabla \phi|^2 + \epsilon\right|^{p/2} dv \text{ and } \int_\Omega |\nabla \phi|^p \, dv,$$

respectively, over all functions $\phi \in W^{1,p}(\Omega)$ and $\phi = u$ on $\partial\Omega$. Then one has

$$\int_\Omega |\nabla u|^p \, dv \le \int_\Omega |\nabla u_\epsilon|^p \, dv \tag{6.8}$$

and

$$\int_\Omega \left||\nabla u_\epsilon|^2 + \epsilon\right|^{p/2} dv \;\le \int_\Omega \left||\nabla u|^2 + \epsilon\right|^{p/2} dv. \tag{6.9}$$

Combining (6.8) and (6.9),

$$\int_\Omega |\nabla u|^p \, dv \le \int_\Omega |\nabla u_\epsilon|^p \, dv \le \int_\Omega \left||\nabla u_\epsilon|^2 + \epsilon\right|^{p/2} dv \;\le \int_\Omega \left||\nabla u|^2 + \epsilon\right|^{p/2} dv,$$

one has $\|\nabla u_\epsilon\|_p \to \|\nabla u\|_p$ as $\epsilon \to 0$. Moreover, by Lemma 6.2 $\nabla u_\epsilon \to \nabla u$ a.e. on Ω for $p > 1$, we have $\nabla u_\epsilon \to \nabla u$ in $L^p(\Omega)$, and then p-Poincaré inequality implies $u_\epsilon \to u$ in $W^{1,p}(\Omega)$. □

Lemma 6.2. *$\nabla u_\epsilon \to \nabla u$ a.e. on Ω for $p > 1$.*

First, we recall the following inequality (cf. [26] Chapter 10, or [15] Lemma 4)

Proposition 6.3. *Let X and Y be vector fields on Ω. Then*

$$\left\langle X - Y, |X|^{p-2} X - |Y|^{p-2} Y \right\rangle \ge C\Psi(X, Y), \tag{6.10}$$

where

$$\Psi(X, Y) = \begin{cases} |X - Y|^p & \text{if } p \ge 2, \\ \frac{(p-1)|X-Y|^2}{\left(1+|X|^2+|Y|^2\right)^{\frac{2-p}{2}}} & \text{if } 1 < p < 2. \end{cases} \tag{6.11}$$

Proof. Since $u - u_\epsilon \in W_0^{1,p}(\Omega)$, one has

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla(u - u_\epsilon)\rangle \, dv = 0$$

and

$$\int_\Omega \left||\nabla u_\epsilon|^2 + \epsilon\right|^{\frac{p-2}{2}} \langle \nabla u_\epsilon, \nabla(u - u_\epsilon)\rangle \, dv = 0.$$

Then

$$\begin{aligned} 0 &= \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla (u-u_\epsilon)\rangle - \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} \langle \nabla u_\epsilon, \nabla (u-u_\epsilon)\rangle \, dv \\ &= \int_\Omega |\nabla u|^{p} - |\nabla u|^{p-2} \langle \nabla u, \nabla u_\epsilon\rangle \\ &\quad - \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} \langle \nabla u_\epsilon, \nabla u\rangle + \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} |\nabla u_\epsilon|^2 \, dv. \end{aligned}$$

This equality can be rewritten as $LHS1 = RHS$, where

$$LHS1 = \int_\Omega |\nabla u|^{p} - |\nabla u|^{p-2} \langle \nabla u, \nabla u_\epsilon\rangle - |\nabla u_\epsilon|^{p-2} \langle \nabla u_\epsilon, \nabla u\rangle + \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p}{2}} dv$$

and

$$RHS = \int_\Omega \left(\left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} - |\nabla u_\epsilon|^{p-2} \right) \langle \nabla u_\epsilon, \nabla u\rangle + \epsilon \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} dv.$$

It is easy to see that $LHS1 \geq LHS2$ where

$$LHS2 = \int_\Omega |\nabla u|^{p} - |\nabla u|^{p-2} \langle \nabla u, \nabla u_\epsilon\rangle - |\nabla u_\epsilon|^{p-2} \langle \nabla u_\epsilon, \nabla u\rangle + |\nabla u_\epsilon|^{p} \, dv.$$

So, we select $X = \nabla u$ and $Y = \nabla u_\epsilon$, then Proposition 6.3 implies

$$LHS2 \geq C \int_\Omega \Psi(\nabla u, \nabla u_\epsilon) \, dv \geq 0$$

where

$$\Psi(\nabla u, \nabla u_\epsilon) = \begin{cases} |\nabla u - \nabla u_\epsilon|^p & \text{if } p \geq 2, \\ \frac{(p-1)|\nabla u - \nabla u_\epsilon|^2}{\left(1+|\nabla u|^2+|\nabla u_\epsilon|^2\right)^{\frac{2-p}{2}}} & \text{if } 1 < p < 2. \end{cases}$$

If we can show that

$$RHS \to 0 \text{ as } \epsilon \to 0,$$

Then we have

$$\int_\Omega \Psi(\nabla u, \nabla u_\epsilon) \, dv \to 0 \text{ as } \epsilon \to 0.$$

Therefore $\nabla u_\epsilon \to \nabla u$ a.e. on Ω.

Now we claim that

$$RHS = RHS1 + RHS2 \to 0$$

as $\epsilon \to 0$, where

$$RHS1 = \int_\Omega \epsilon \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} dv$$

and

$$RHS2 = \int_\Omega \left(\left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} - |\nabla u_\epsilon|^{p-2} \right) \langle \nabla u_\epsilon, \nabla u\rangle \, dv.$$

It is easy to see that, if $|\nabla u_\epsilon|^2 \geq 1$,

$$\begin{aligned} \int_\Omega \epsilon \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} dv &\leq \int_\Omega \epsilon |\nabla u_\epsilon|^2 \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} dv \\ &\leq \epsilon \int_\Omega \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p}{2}} dv, \end{aligned}$$

and if $|\nabla u_\epsilon|^2 < 1$,

$$\int_\Omega \epsilon \left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-2}{2}} dv \leq \begin{cases} \epsilon (1+\epsilon)^{\frac{p-2}{2}} \cdot vol(\Omega) & \text{if } p \geq 2 \\ \epsilon^{\frac{p}{2}} \cdot vol(\Omega) & \text{if } p < 2. \end{cases}$$

So we have $RHS1 \to 0$ as $\epsilon \to 0$.

Now we focus on the term $RHS2$,

$$\begin{aligned} RHS2 &= \int_\Omega \left(\left| |\nabla u_\epsilon|^2 + \epsilon \right|^{\frac{p-2}{2}} - |\nabla u_\epsilon|^{p-2} \right) \langle \nabla u_\epsilon, \nabla u \rangle \, dv \\ &\leq \int_\Omega \left| \left| |\nabla u_\epsilon|^2 + \epsilon \right|^{\frac{p-2}{2}} - |\nabla u_\epsilon|^{p-2} \right| |\nabla u_\epsilon| \, |\nabla u| \, dv. \end{aligned}$$

In the case $p \geq 2$, one may rewrite it as

$$\begin{aligned} RHS2 &\leq \int_\Omega \left(\left| |\nabla u_\epsilon|^2 + \epsilon \right|^{\frac{p-2}{2}} - |\nabla u_\epsilon|^{p-2} \right) |\nabla u_\epsilon| \, |\nabla u| \, dv \\ &\leq \int_\Omega \left(\left| |\nabla u_\epsilon|^2 + \epsilon \right|^{\frac{p-1}{2}} - |\nabla u_\epsilon|^{p-1} \right) |\nabla u| \, dv. \end{aligned}$$

If $p \geq 3$, using mean value theorem, we have the inequality

$$(x+\epsilon)^q - x^q = q\epsilon \left(x + \epsilon_1 \right)^{q-1} \leq q\epsilon \left(x + \epsilon \right)^{q-1}$$

here $q = \frac{p-1}{2} \geq 1$, $x \geq 0$ and $\epsilon_1 \in (0, \epsilon)$. Hence

$$\begin{aligned} RHS2 &\leq \frac{(p-1)\epsilon}{2} \int_\Omega \left| |\nabla u_\epsilon|^2 + \epsilon \right|^{\frac{p-3}{2}} |\nabla u| \, dv \\ &\leq \begin{cases} \frac{(p-1)\epsilon}{2} \int_\Omega \left| |\nabla u_\epsilon|^2 + \epsilon \right|^{\frac{p-1}{2}} |\nabla u| \, dv & \text{if } |\nabla u_\epsilon|^2 > 1 \\ \frac{(p-1)\epsilon}{2} (1+\epsilon)^{\frac{p-3}{2}} \int_\Omega |\nabla u| \, dv & \text{if } |\nabla u_\epsilon|^2 \leq 1 \end{cases} \\ &\leq \begin{cases} \frac{(p-1)\epsilon}{2} \left(\int_\Omega \left| |\nabla u_\epsilon|^2 + \epsilon \right|^{\frac{p}{2}} \right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla u|^p \right)^{\frac{1}{p}} dv & \text{if } |\nabla u_\epsilon|^2 > 1 \\ \frac{(p-1)\epsilon}{2} (1+\epsilon)^{\frac{p-3}{2}} \left(vol\left(\Omega\right) \right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla u|^p \right)^{\frac{1}{p}} dv & \text{if } |\nabla u_\epsilon|^2 \leq 1 \end{cases} \\ &\to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

If $2 \leq p \leq 3$, using the inequality

$$(x+\epsilon)^q - x^q \leq \epsilon^q$$

here $\frac{1}{2} \leq q = \frac{p-1}{2} \leq 1$, $x \geq 0$, then

$$\begin{aligned} RHS2 &\leq \int_\Omega \left(\left| |\nabla u_\epsilon|^2 + \epsilon \right|^{\frac{p-1}{2}} - |\nabla u_\epsilon|^{p-1} \right) |\nabla u| \\ &\leq \epsilon^{\frac{p-1}{2}} \int_\Omega |\nabla u| \\ &\leq \epsilon^{\frac{p-1}{2}} \left(vol\left(\Omega\right) \right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla u|^p \right)^{\frac{1}{p}} \\ &\to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

In the case $1 < p < 2$, one may rewrite $RHS2$ as

$$RHS2 \leq \int_\Omega \left(|\nabla u_\epsilon|^{p-2} - \left| |\nabla u_\epsilon|^2 + \epsilon \right|^{\frac{p-2}{2}} \right) |\nabla u_\epsilon| \, |\nabla u| \, .$$

Since $0<\frac{2-p}{2}<1$, then we have

$$\begin{aligned} RHS2 &= \int_\Omega \frac{\left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{2-p}{2}}-|\nabla u_\epsilon|^{2-p}}{\left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{2-p}{2}}}\left|\nabla u_\epsilon\right|^{p-1}|\nabla u|\,dv \\ &\leq \int_\Omega \frac{\epsilon^{\frac{2-p}{2}}}{\left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{3-2p}{2}}}\cdot\frac{|\nabla u_\epsilon|^{p-1}}{\left||\nabla u_\epsilon|^2+\epsilon\right|^{\frac{p-1}{2}}}|\nabla u| \\ &\leq \epsilon^{\frac{p-1}{2}}\int_\Omega|\nabla u|\,dv \\ &\leq \epsilon^{\frac{p-1}{2}}\left(vol\left(\Omega\right)\right)^{\frac{p-1}{p}}\left(\int_\Omega|\nabla u|^p\,dv\right)^{\frac{1}{p}} \\ &\rightarrow 0 \text{ as } \epsilon\rightarrow 0. \end{aligned}$$

Hence we conclude that

$$RHS=RHS1+RHS2\rightarrow 0 \text{ as } \epsilon\rightarrow 0.$$

□

6.3. **Non-trivial p-harmonic function with finite q-energy.** In this subsection, we construct an example of non-trivial p-harmonic function u with finite q-energy, $q>p-1$, on a complete noncompact manifold with weighted Poincaré inequality (P_ρ).

Let $M=\mathbb{R}\times N^{m-1}$, $m\geq 3$, with a metric $ds^2=dt^2+\eta^2(t)\,g_N$, where $\eta(t):\mathbb{R}\rightarrow(0,\infty)$ is a smooth function with $\eta''>0$, $(m-2)(\log\eta)''+\eta^{-2}Ric_N\geq 0$, and (N,g_N) is a compact Riemannian manifold with $vol\left(N^{m-1}\right)=1$.

According to [34] Proposition 6.1, M satisfies weighted Poincaré inequality (P_ρ) and $Ric_M\geq-\frac{m-1}{m-2}\rho$ with $\rho=(m-2)\,\eta''\eta^{-1}$.

Let $A(t)$ be the volume of $\{t\}\times N^{m-1}$, then $A(t)=\eta^{m-1}(t)$.

Now we select $\eta(t)$ such that each end of M is p-hyperbolic, and

$$A(t)\geq d_1|t|^{\frac{p-1}{q-p+1-\delta}},\ \text{if}\ |t|\geq 1,$$

where $d_1>0$ and $0<\delta<q-p+1$ are positive constants.

By using [44] Proposition 5.3,

$$Cap_p\left((-\infty,a)\times N^{m-1},(b,\infty)\times N^{m-1};M\right)=\left(\int_a^b\left(\frac{1}{A(t)}\right)^{1/(p-1)}dt\right)^{1-p},$$

for any $-\infty<a<b<\infty$. If we define u by

$$u(t)=\int_{-\infty}^t\left(\frac{1}{A(s)}\right)^{1/(p-1)}ds$$

then

$$Cap_p\left((-\infty,a)\times N^{n-1},(b,\infty)\times N^{n-1};M\right)=\left(u(b)-u(a)\right)^{1-p},$$

$u(t)\rightarrow 0$ as $t\rightarrow-\infty$, and u is uniformly bounded for all $t\in(-\infty,\infty)$.

Moreover, define a function v as follows,

$$v(t)=\begin{cases} 0, & \text{if } t\leq a, \\ \frac{u(t)-u(a)}{u(b)-u(a)}, & \text{if } a<t<b, \\ 1, & \text{if } t\geq b. \end{cases}$$

then

$$\int_M |\nabla v|^p \, dv = \int_a^b \frac{(u'(t))^p}{(u(b)-u(a))^p} A(t)\, dt = (u(b) - u(a))^{1-p}$$

which implies v is extremal of p-energy for every $-\infty < a < b < \infty$. Hence $u(t)$ is p-harmonic in M with finite q energy

$$\int_M |\nabla u|^q \, dv = \int_{-\infty}^{\infty} A^{\frac{p-1-q}{p-1}}(t)\, dt < \infty$$

for all $q > p-1$. Moreover, by [34] Proposition 6.1, we have

$$Ric_M(\nabla u, \nabla u) = -\tfrac{m-1}{m-2}\rho\, |\nabla u|^2 .$$

6.4. **Volume estimate and p-Poincaré inequality.** In this subsection, we study a complete noncompact manifold M with the p-Poincaré inequality $\left(P_{\lambda_p}\right)$, $p > 1$, that is, the inequality

$$\lambda_p \int_M |\Psi|^p \le \int_M |\nabla \Psi|^p \tag{6.12}$$

holds for every $\Psi \in W_0^{1,p}(M)$, where $\lambda_p > 0$. In particular, if $p = 2$, this formula is the general Poincaré inequality, and λ_2 is the spectrum of M. In [14], they show that a complete manifold M with positive spectrum $\lambda_2 > 0$, then it must have $\lambda_p > 0$ for all $p \ge 2$. In fact, the following inequality

$$p(\lambda_p)^{1/p} \ge 2(\lambda_2)^{1/2}$$

holds on M for all $p \ge 2$.

Lemma 6.3. *Let M be a complete noncompact manifold satisfying $\left(P_{\lambda_p}\right)$, $p > 1$. Suppose w is a positive, p-subharmonic function with a finite p-energy on M. If w satisfies*

$$\int_{B(2R)\backslash B(R)} \exp\left(-\frac{(\lambda_p)^{1/p} r(x)}{p+1}\right) |w|^p \, dv = o(R), \tag{6.13}$$

where $R \ge R_0 + 1$. Then,

$$(1-\delta)\left\|\exp\left(\frac{\delta(\lambda_p)^{1/p} r(x)}{p+1}\right) w\right\|_{L_p(M\backslash B(R_0+1))} \le C,$$

and

$$(1-\delta)\left\|\exp\left(\frac{\delta(\lambda_p)^{1/p} r(x)}{p+1}\right) \nabla w\right\|_{L_p(M\backslash B(R_0+1))} \le C,$$

for all $0 < \delta < 1$, and for some constant C depending on p and λ_p.

Proof. Let ψ be a non-negative cut-off function, then we have

$$\begin{array}{rcl} 0 & \geq & \int_M \psi^p w\left(-\Delta_p w\right) \\ & = & \int_M \left\langle \nabla(\psi^p w), |\nabla w|^{p-2} \nabla w \right\rangle \\ & = & \int_M \psi^p |\nabla w|^p + pw |\nabla w|^{p-2} \psi^{p-1} \langle \nabla\psi, \nabla w\rangle \\ & \geq & \int_M \psi^p |\nabla w|^p - p \int_M w\psi^{p-1} |\nabla w|^{p-1} |\nabla\psi| . \end{array} \tag{6.14}$$

By using Hölder inequality

$$\int_M w\psi^{p-1} |\nabla w|^{p-1} |\nabla \psi| \leq \left(\int_M |\nabla w|^p \psi^p\right)^{(p-1)/p} \left(\int_M w^p |\nabla\psi|^p\right)^{1/p},$$

then (6.14) can be rewritten as

$$\|\psi \nabla w\|_{L_p} \leq p \|\nabla\psi \cdot w\|_{L_p}, \tag{6.15}$$

and this inequality is the Caccioppoli type estimate.

Since Minkowski inequality yields

$$\|\nabla(\psi w)\|_{L_p} \leq \|\nabla\psi \cdot w\|_{L_p} + \|\psi\nabla w\|_{L_p},$$

then (6.15) implies

$$\|\nabla(\psi w)\|_{L_p} \leq (p+1) \|\nabla\psi \cdot w\|_{L_p}. \tag{6.16}$$

This inequality is not sharp enough whenever $p = 2$. In fact, if $p = 2$, one can easy to show $\|\nabla(\psi w)\|_{L_2} \leq \|\nabla\psi w\|_{L_2}$ by the similar method (cf. [32][34]).

By scaling the metric, we may assume $\lambda_p = 1$. Combining (6.16) and (6.12), then

$$\|\psi w\|_{L_p} \leq (p+1) \|\nabla\psi \cdot w\|_{L_p}, \tag{6.17}$$

where ψ is a cut off function on M.

Now we select $\psi = \phi(r(x)) \exp(a(r(x)))$, then

$$\begin{array}{rcl} \frac{1}{p+1} \|\psi w\|_{L_p} & \leq & \|(\nabla\phi + \phi\nabla a) \exp(a(x)) w\|_{L_p} \\ & \leq & \|(\nabla\phi) \exp(a(x)) w\|_{L_p} + \|(\nabla a)\, \phi \exp(a(x)) w\|_{L_p} \end{array} \tag{6.18}$$

where ϕ is a non-negative cut-off function defined by $\phi = \phi_+ + \phi_-$ where

$$\phi_+(r) = \begin{cases} r - R_0 & \text{for } R_0 \leq r \leq R_0 + 1, \\ 1 & \text{for } r > R_0 + 1, \end{cases} \qquad \phi_-(r) = \begin{cases} \frac{R-r}{R} & \text{for } R \leq r \leq 2R, \\ -1 & \text{for } r > 2R, \end{cases}$$

and we also choose $a = a_+(r(x)) + a_-(r(x))$ as

$$a_+(r) = \begin{cases} \frac{\delta r(x)}{p+1} & \text{for } r \leq \frac{K}{1+\delta}, \\ \frac{\delta K}{(1+\delta)(p+1)} & \text{for } r > \frac{K}{1+\delta}, \end{cases}$$

$$a_-(r) = \begin{cases} 0 & \text{for } r \leq \frac{K}{1+\delta}, \\ \frac{1}{p+1}\left(\frac{2K}{1+\delta} - r(x)\right) & \text{for } r > \frac{K}{1+\delta}, \end{cases}$$

for some fixed $K > (R_0+1)(1+\delta)$, $0<\delta<1$, and $R \geq \frac{K}{1+\delta}$, it's easy to check that

$$|\nabla\phi|^2(x) = \begin{cases} 1 & \text{on } B(R_0+1)\backslash B(R_0), \\ 0 & \text{on } B(R_0),\ B(R)\backslash B(R_0+1) \text{ and } M\backslash B(2R), \\ \frac{1}{R^2} & \text{on } B(2R)\backslash B(R), \end{cases}$$

and

$$|\nabla a|^2(x) = \begin{cases} \frac{\delta^2}{(p+1)^2} & \text{for } r < \frac{K}{1+\delta}, \\ \frac{1}{(p+1)^2} & \text{for } r > \frac{K}{1+\delta}. \end{cases}$$

Substituting into (6.18), we obtain

$$\begin{aligned} & \frac{1}{p+1}\|\phi\exp(a(x))w\|_{L_p(M)} \\ \leq\ & \|(\nabla\phi_+)\exp(a(x))w\|_{L_p(M)} + \|(\nabla\phi_-)\exp(a(x))w\|_{L_p(M)} \\ & + \|(\nabla a_+)\,\phi\exp(a(x))w\|_{L_p(M)} + \|(\nabla a_-)\,\phi\exp(a(x))w\|_{L_p(M)} \\ \leq\ & \|\exp(a(x))w\|_{L_p(B(R_0+1)\backslash B(R_0))} + \frac{1}{R}\|\exp(a(x))w\|_{L_p(B(2R)\backslash B(R))} \\ & + \frac{\delta}{p+1}\|\phi\exp(a(x))w\|_{L_p\left(B\left(\frac{K}{1+\delta}\right)\right)} + \frac{1}{p+1}\|\phi\exp(a(x))w\|_{L_p\left(M\backslash B\left(\frac{K}{1+\delta}\right)\right)}, \end{aligned}$$

hence

$$\begin{aligned} & \left(\frac{1-\delta}{p+1}\right)\|\phi\exp(a(x))w\|_{L_p\left(B\left(\frac{K}{1+\delta}\right)\backslash B(R_0+1)\right)} \\ \leq\ & \|\exp(a(x))w\|_{L_p(B(R_0+1)\backslash B(R_0))} + \frac{1}{R}\|\exp(a(x))w\|_{L_p(B(2R)\backslash B(R))}. \end{aligned}$$

The definition of $a(x)$ and the growth condition (6.13) imply that the last term on the right hand side tends to 0 as $R\to\infty$. Thus one has the following inequality,

$$\begin{aligned} & \left(\frac{1-\delta}{p+1}\right)\|\exp(a(x))w\|_{L_p\left(B\left(\frac{K}{1+\delta}\right)\backslash B(R_0+1)\right)} \\ \leq\ & \|\exp(a(x))w\|_{L_p(B(R_0+1)\backslash B(R_0))}. \end{aligned} \tag{6.19}$$

Since the right hand side of (6.19) is independent of K and $0<\delta<1$, by letting $K\to\infty$ we obtain that

$$(1-\delta)\|\exp(a(x))w\|_{L_p(M\backslash B(R_0+1))} \leq C_1, \tag{6.20}$$

for some constant $0 < C_1 = C_1(p) < \infty$.

Moreover, by (6.15) and similar process as above, we have

$$\begin{aligned} & \frac{1}{p}\|\psi\nabla w\|_{L_p(M)} \\ \leq\ & \|\nabla\psi\cdot w\|_{L_p(M)} \\ \leq\ & \|\exp(a(x))w\|_{L_p(B(R_0+1)\backslash B(R_0))} + \frac{1}{R}\|\exp(a(x))w\|_{L_p(B(2R)\backslash B(R))} \\ & + \frac{\delta}{p+1}\|\phi\exp(a(x))w\|_{L_p\left(B\left(\frac{K}{1+\delta}\right)\right)} + \frac{1}{p+1}\|\phi\exp(a(x))w\|_{L_p\left(B(2R)\backslash B\left(\frac{K}{1+\delta}\right)\right)} \\ \leq\ & 2\|\exp(a(x))w\|_{L_p(B(R_0+1)\backslash B(R_0))} + 3\|\phi\exp(a(x))w\|_{L_p(B(2R)\backslash B(R_0+1))} \\ \leq\ & C_2 + \frac{3C_1}{1-\delta}. \end{aligned}$$

Hence, by letting $R\to\infty$ and then letting $K\to\infty$, we conclude

$$(1-\delta)\|\exp(\delta r(x))\nabla w\|_{L_p(M\backslash B(R_0+1))} \leq C_3$$

for some constant $0 < C_3 = C_2 + \frac{3C_1}{1-\delta} < \infty$.

Then lemma now follows. $\square$

Lemma 6.4. *Let M be a complete noncompact manifold satisfying $\left(P_{\lambda_p}\right)$, $p > 1$. Suppose E is an end of M respective to a compact set, w_i is a positive, p-harmonic function with a finite p-energy on $E(R_i)$ and $w_i = 1$ on ∂E and $w_i = 0$ on $S(R_i) = \partial E(R_i) \backslash \partial E$. If $R_i \to \infty$ and $w_i \to w$ as $i \to \infty$. Then,*

$$\int_{E\backslash E(R)} |\nabla w|^p \, dv \leq C_3 R^p \exp\left(\frac{-(\lambda_p)^{1/p}(R-1)}{(p+1)}\right), \tag{6.21}$$

and

$$\int_{E(kR)\backslash E(R)} |w|^p \, dv \leq C_1 R^p \exp\left(\frac{-(\lambda_p)^{1/p}(R-1)}{p+1}\right), \tag{6.22}$$

for some constant C depending on p.

Proof. As in the proof of Lemma 6.3. If ϕ is a non-negative cut-off function defined by

$$\phi(r(x)) = \begin{cases} r(x) - R_0 & \text{on } E(R_0+1)\backslash E(R_0), \\ 1 & \text{on } E\backslash E(R_0+1), \end{cases}$$

and we choose $a = \frac{\delta r(x)}{p+1}$ for $0 < \delta < 1$. It's easy to check that

$$|\nabla\phi|^2(x) = \begin{cases} 1 & \text{on } E(R_0+1)\backslash E(R_0), \\ 0 & \text{on } E\backslash E(R_0+1), \end{cases} \quad \text{and} \quad |\nabla a|^2(x) = \frac{\delta^2}{(p+1)^2}.$$

By the formula (6.18), we obtain

$$\begin{array}{rl} & \frac{1}{p+1}\|\phi\exp(a(x))w\|_{L_p} \\ \leq & \|(\nabla\phi)\exp(a(x))w\|_{L_p} + \|(\nabla a)\,\phi\exp(a(x))w\|_{L_p} \\ \leq & \|\exp(a(x))w\|_{L_p(E(R_0+1)\backslash E(R_0))} + \frac{\delta}{p+1}\|\phi\exp(a(x))w\|_{L_p(E)} \end{array}$$

hence

$$\left(\frac{1-\delta}{p+1}\right)\|\phi\exp(a(x))w\|_{L_p(E\backslash E(R_0+1))} \leq \|\exp(a(x))w\|_{L_p(E(R_0+1)\backslash E(R_0))}.$$

Then we obtain that

$$(1-\delta)\|\exp(\delta r)w\|_{L_p(E\backslash E(R_0+1))} \leq C_1, \tag{6.23}$$

for some constant $0 < C_1 = C_1(p) < \infty$.

Moreover, since

$$\begin{array}{rcl} \frac{1}{p}\|\psi\nabla w\|_{L_p} & \leq & \|\nabla\psi w\|_{L_p} \\ & \leq & \|\exp(a(x))w\|_{L_p(E(R_0+1)\backslash E(R_0))} + \delta\|\phi\exp(a(x))w\|_{L_p(E)} \\ & \leq & C_2 + \frac{\delta C_1}{1-\delta}. \end{array}$$

Hence, we conclude

$$(1-\delta)\left\|\exp(\delta r(x))\nabla w\right\|_{L_p(E\backslash E(R_0+1))} \leq C_3 \tag{6.24}$$

for some constant $0 < C_3 = C_3(p) < \infty$.

If we select $\delta = \left(1 - \frac{1}{R}\right)$ and $R_0 > 1$, then 6.23) gives

$$\begin{aligned} C_3 &\geq \frac{1}{R^p}\int_{E\backslash E(R_0+1)} \exp\left((1-\tfrac{1}{R})\tfrac{(\lambda_p)^{1/p} r}{p+1}\right)|\nabla w|^p\,dv \\ &\geq \frac{1}{R^p}\int_{E(kR)\backslash E(R_0+1)} \exp\left((1-\tfrac{1}{R})\tfrac{(\lambda_p)^{1/p} r}{p+1}\right)|\nabla w|^p\,dv. \end{aligned}$$

Hence

$$\int_{E(kR)} \exp\left(\tfrac{(\lambda_p)^{1/p}(R-1)r}{(p+1)R}\right)|\nabla w|^p\,dv \leq C_3 R^p,$$

and then we have

$$\int_{E(kR)\backslash E(R)} |\nabla w|^p\,dv \leq C_3 R^p \exp\left(\tfrac{-(\lambda_p)^{1/p}(R-1)}{(p+1)}\right),$$

for all constant $k > 1$.

Similarly, 6.24 implies

$$\int_{E(kR)} \exp\left(\tfrac{(\lambda_p)^{1/p}(R-1)r}{(p+1)R}\right)|w|^p\,dv \leq C_1 R^p$$

and

$$\int_{E(kR)\backslash E(R)} |w|^p\,dv \leq C_1 R^p \exp\left(\tfrac{-(\lambda_p)^{1/p}(R-1)}{p+1}\right),$$

for any constant $k > 1$.

□

Lemma 6.5. *Let M be a complete noncompact manifold satisfying $\left(P_{\lambda_p}\right)$, $p > 1$. If E is a p-hyperbolic end of M^n, then*

$$V(E(R+1)) - V(E(R)) \geq CR^{-p(p-1)} \exp\left(\tfrac{(p-1)(\lambda_p)^{1/p}(R-1)}{p+1}\right).$$

for some constant $C > 0$, and for R sufficiently large. If E is p-parabolic, then

$$V(E) < \infty$$

and

$$V(E) - V(E(R)) \leq CR^p \exp\left(\tfrac{-(\lambda_p)^{1/p}(R-1)}{p+1}\right)$$

for some constant $C > 0$, for any $0 < \delta < 1$, and for R sufficiently large.

Proof. If E is p-parabolic, we select the barrier function $w = 1$ on E, then (6.22) implies

$$\int_{E\backslash E(R)} dv \leq CR^p \exp\left(\tfrac{-(\lambda_p)^{1/p}(R-1)}{p+1}\right)$$

for all R large enough and for any δ satisfying $0 < \delta < 1$. This implies $V(E) < \infty$.

If E is p-hyperbolic. Let w be the barrier function on E, and $S(R) = \partial E(R) \backslash \partial E$, then

$$\begin{aligned} C &= \int_{\partial E} |\nabla w|^{p-2} \frac{\partial w}{\partial \nu} dA \\ &\leq \int_{S(r)} |\nabla w|^{p-1} dA \\ &\leq \left(\int_{S(r)} |\nabla w|^p \, dA\right)^{(p-1)/p} \left(\int_{S(r)} dA\right)^{1/p}. \end{aligned} \tag{6.25}$$

Then (6.25) imply

$$\begin{aligned} \int_R^{R+1} \left(\int_{S(r)} dA\right)^{-1/(p-1)} dr &\leq C \int_R^{R+1} \int_{S(r)} |\nabla w|^p \, dA dr \\ &= C \int_{E(R+1)\backslash E(R)} |\nabla w|^p \, dv. \end{aligned}$$

By using Schwarz inequality,

$$\begin{aligned} 1 &= \int_R^{R+1} \left(\int_{S(r)} dA\right)^{-\frac{1}{p}} \left(\int_{S(r)} dA\right)^{\frac{1}{p}} dr \\ &\leq \left(\int_R^{R+1} \left(\int_{S(r)} dA\right)^{-\frac{1}{p-1}} dr\right)^{\frac{p-1}{p}} \cdot \left(\int_R^{R+1} \int_{S(r)} dA dr\right)^{\frac{1}{p}} \\ &\leq C \left(\int_{E(R+1)\backslash E(R)} |\nabla w|^p \, dv\right)^{\frac{p-1}{p}} \cdot \left(\int_R^{R+1} \int_{S(R)} dA dr\right)^{\frac{1}{p}}. \end{aligned}$$

Then co-area formula and (6.21) give

$$\int_{E(R+1)\backslash E(R)} dv \geq C R^{-p(p-1)} \exp\left(\frac{(p-1)(\lambda_p)^{1/p}(R-1)}{p+1}\right).$$

□

Since $\left(P_{\lambda_p}\right)$ implies the volume of M is infinity, then Lemma 6.5 implies the following property.

Theorem 6.1. *If M is a complete noncompact manifold satisfying $\left(P_{\lambda_p}\right)$, then M must be p-hyperbolic.*

Remark 6.1. One can also prove the above theorem by contradiction. That is, if M were p-parabolic, then λ_p would be zero, a contradiction by a different approach (cf. e.g. [49] proof of Theorem 6.1).

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