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AN OPTIMAL MEAN-REVERSION TRADING RULE UNDER A MARKOV CHAIN MODEL

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Abstract. This paper is concerned with a mean-reversion trading rule. In contrast to most market models treated in the literature, the underlying market is solely determined by a two-state Markov chain. The major advantage of such Markov chain model is its striking simplicity and yet its capability of capturing various market movements. The purpose of this paper is to study an optimal trading rule under such a model. The objective of the problem under consideration is to find a sequence stopping (buying and selling) times so as to maximize an expected return. Under some suitable conditions, explicit solutions to the associated HJ equations (variational inequalities) are obtained. The optimal stopping times are given in terms of a set of threshold levels. A verification theorem is provided to justify their optimality. Finally, a numerical example is provided to illustrate the results.

1. **Introduction.**

Major market models treated in mathematical finance in recent years can be classified into two broad categories: Cox-Ross-Rubinstein's binomial tree model (BTM) and Brownian motion based models. See related books by Hull [15], Elliott and Kopp [8], Fouque et al. [10], Karatzas and Shreve [17], and Musiela and Rutkowski [20] among others.

The BTM is widely used in option pricing due to its simplicity and yet clear advantage when pricing American type options. However, a main drawback of the BTM is its non-Markovian nature. The lack of Markovian property makes it difficult to work with mathematically, not to mention closed-form solutions. To preserve the BTM's simplicity and enhance its mathematical tractability, a two-state Markov chain model was considered in Zhang [29]. The Markov chain model in [29] appears to be natural for not frequently traded securities such as illiquid stocks. Its capability of capturing this type of markets makes it preferable in related applications. In addition, it can be seen from Example 1 to follow that the Markov chain model is closely related to the traditional mean-reversion diffusion when the jump rates of the Markov chain are large. Related Markov chain based models can be found in Van der Hoek and Elliott [23] and Norberg [21]. Van der Hoek and Elliott introduced a stock price model based on stock dividend rates and a Markov chain noise. Norberg used a Markov chain to represent interest rate and considered a market model driven

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by a Markov chain. In particular, the market model in [21] resembles a GBM in which the 'drift' is approximated by the duration between jumps and the 'diffusion' is given in terms of jump times. An additional advantage of a Markov chain driven model is that its price is almost everywhere differentiable. Such differentiability is desirable in the optimal control type market analysis proposed by Barmish and Primbs [1]. In connection with dynamic programming problems, the corresponding Hamilton-Jacobi (HJ) equations are of the first order, which are easier to analyze than those under traditional Brownian motion based models.

Mean-reversion models are often used in financial markets to capture price movements that have the tendency to move towards an 'equilibrium.' There are many studies in connection with mean-reversion stock returns; see e.g., Cowles and Jones [5], Fama and French [9], and Gallagher and Taylor [11] among others. In addition to equity markets, mean-reversion models are used for stochastic interest rates (Vasicek [24] and Hull [15]); stochastic volatility (Hafner and Herwartz [14]); and energy markets (Blanco and Soronow [2]. See also related results in option pricing with a mean-reversion asset by Bos, Ware and Pavlov [3].

Mathematical trading rules have been studied for many years. For example, Zhang [28] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. Such optimal threshold levels are obtained by solving a set of two-point boundary value problems. Guo and Zhang [13] studied the optimal selling rule under a model with regime switching. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. These papers are concerned with the selling side of trading in which the underlying price models are of GBM type. Recently, Dai et al. [7] developed a trend following rule based on a conditional probability indicator. They showed that the optimal trading rule can be determined by two threshold curves which can be obtained by solving the associated HJ equations. Similar idea was developed following a confidence interval approach by Iwarere and Barmish [16]. In addition, Merhi and Zervos [18] studied an investment capacity expansion/reduction problem following a dynamic programming approach under a geometric Brownian motion market model. In connection with mean-reversion trading, Zhang and Zhang [27] obtained a buy-low and sell-high policy by charactering the 'low' and 'high' levels in terms of two threshold levels. Further development along this line can be found in pairs trading in Song and Zhang [22]. Under a mean-reversion model, a pair of stocks can be traded simultaneously, when their price divergence exceeds a predetermined level, to bet on their eventual convergence. More details in connection with pairs trading and extensive empirical market tests can be found in Gatev et al. [12] and references therein.

To take advantage of many new features of Markov chain models and explore related mean-reversion markets, it is the purpose of this paper to study the corresponding trading strategies. The objective is to buy and sell the mean-reversion security to maximize a reward function. A fixed (commission or slippage) cost will be imposed to each transaction. We study the problem following a dynamic programming approach and establish the associated HJ equations for the value functions. We show that the corresponding optimal stopping (buying and selling) times can be determined by four threshold levels depending on market states. These key levels can be obtained by solving a set of algebraic like equations. In solving the associated HJ equations, hypergeometric series arise naturally in the analysis. Various boundary conditions and convexity conditions of hypergeometric functions are involved extensively. We provide a set of sufficient conditions that guarantee the optimality of the trading rules. Finally, to illustrate the results, we present a market backtesting example with the daily closing prices of Dow Jones Industrial Average from 1961 to 1980.

The main contributions of this paper include: (1) Introduction of a new mean-reversion model driven by a two-state Markov chain. Such model is very simple in structure and yet powerful enough to capture many important market features. (2) Development of mean-reversion trading strategies that are simple to implement. The corresponding optimal stopping problem is solved using a smooth-fit technique and hypergeometric analysis is involved in a substantial way.

This paper is organized as follows. In §2, we formulate the mean-reversion trading problem under consideration. In §3, we study preliminary properties of the value functions. In §4, we consider the associate HJ equations and their solutions. Then in §5, we discuss additional inequalities including the variational inequalities arising from the HJ equations and related convexity conditions. In §6, we give a verification theorem under suitable conditions. A numerical example is considered in §7. Several technical lemmas are postponed and provided in Appendix.

2. **Problem formulation.**

Let $\{\alpha_t : t \geq 0\}$ denote a two-state Markov chain with state space $\mathcal{M} = \{1,2\}$ and the generator $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$. Here λ_1 and λ_2 are positive.

Let S_t denote the stock price (or a function of the price) at time t given by the equation

$$\frac{dS_t}{dt} = a(b - S_t) + f(\alpha_t),\ t \geq 0, \tag{1}$$

where $a > 0$ is the rate of reversion and b the equilibrium. Here $f(1) = f_1 > 0$ represents the uptick return rate and $f(2) = f_2 < 0$ the downtick return rate.

Example 2.1. (Convergence to a Mean-Reversion Diffusion). In this example, we present a simple Markov chain model and demonstrate how it approaches the corresponding mean-reversion diffusion. Given $\varepsilon > 0$, we consider $f_1 = \sigma/\sqrt{\varepsilon}$, $f_2 = -\sigma/\sqrt{\varepsilon}$, $\lambda_1 = \lambda_2 = 1/\varepsilon$. Using the asymptotic normality given in Yin and Zhang [26, Theorem 5.9], we can show that the solution to (1) $S_t = S_t^\varepsilon$ converges weakly to the solution of $dS_t = a(b-S_t)dt+\sigma dW_t$, where W_t is a standard Brownian motion.

In particular, taking $a = 1$, $b = 2$, $\sigma = 0.5$, and $\varepsilon = 0.1$, we generate the Monte Carlo sample paths of $(S_t^\varepsilon, \alpha_t^\varepsilon)$. One sample is given in (a) of Figure 1. Similarly, we vary ε and plot the corresponding sample paths in (b) and (c) of Figure 1. It is clear from these pictures that as ε gets smaller and smaller, the fluctuation of α_t is more and more rapidly and the corresponding S_t^ε approaches to a mean-reversion diffusion.

Remark 2.1. The main advantage of our model is its simplicity and flexibility in capturing various market conditions. For example, in a typical sideways (trendless) market, the price range can be precisely defined. In our model, such 'price band' arises naturally. A candidate for the band can be given by the interval $[b+f_2/a, b+f_1/a]$.

We consider the state process $X_t = S_t - b$. Then X_t satisfies

$$\frac{dX_t}{dt} = -aX_t + f(\alpha_t),\ t \geq 0, \tag{2}$$

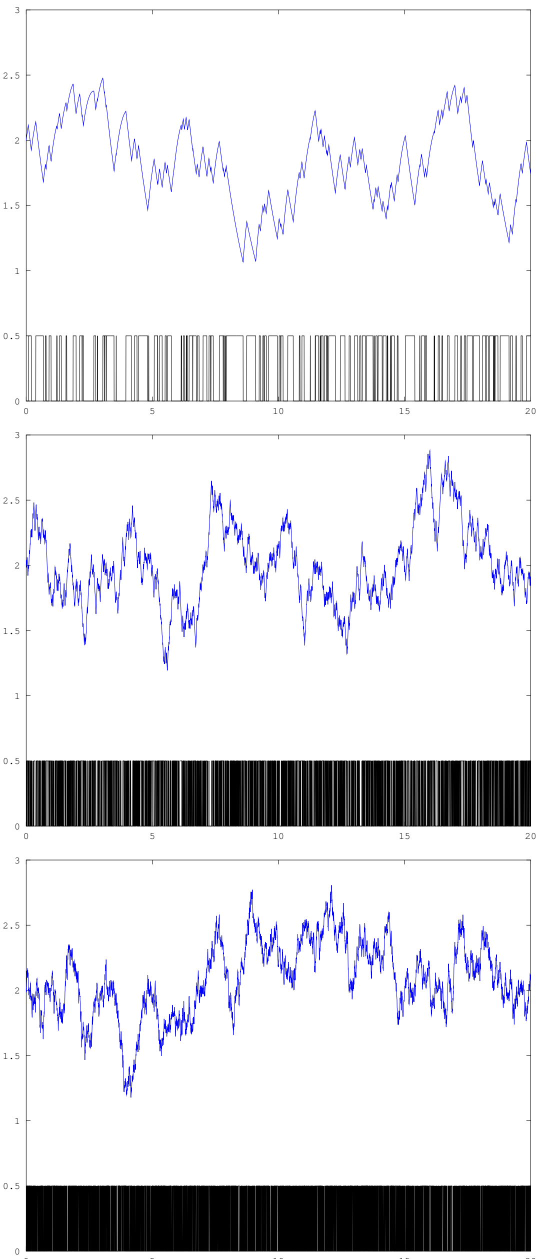

FIGURE 1. Monte Carlo Sample Paths of $(S_t^\varepsilon, \alpha_t^\varepsilon)$ with ε=0.1, 0.01, 0.001, resp.

with initial vector $(X_0, \alpha_0) = (x, \alpha)$. Let $\mathcal{I}_0 = [f_2/a, f_1/a]$. Then $X_t \in \mathcal{I}_0$ for all $t \geq 0$ if $X_0 \in \mathcal{I}_0$.

Let $\mathcal{F}_t = \{X_r : \ r \leq t\}$ denote the filtration generated by X_t. Note that α_t is observable and $\mathcal{F}_t = \{\alpha_r : \ r \leq t\}$. Let

$$0 \leq \tau_1^b \leq \tau_1^s \leq \tau_2^b \leq \tau_2^s \leq \cdots \tag{3}$$

denote a sequence of stopping times with respect to $\{\mathcal{F}_t\}$. A buying decision is made at τ_n^b and a selling decision at τ_n^s, $n = 1, 2, \ldots$.

We consider the case that the net position at any time can be either long with one share of the stock or flat with no stock position. Let $i = 0, 1$ denote the initial net position. If the initial net position is long ($i = 1$), then one should sell the stock before acquiring any future shares. The corresponding sequence of stopping times is denoted by $\Lambda_1 = (\tau_1^s, \tau_2^b, \tau_2^s, \tau_3^b, \ldots)$. Likewise, if the initial net position is flat ($i = 0$), then one should start to buy a share of the stock. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_2^s, \ldots)$.

Let $K > 0$ denote the fixed transaction cost (e.g., slippage and/or commission). Given the initial state vector $(X_0, \alpha_0) = (x, \alpha)$, the initial net position $i = 0, 1$, and the decision sequences (Λ_0 and Λ_1), we consider the corresponding reward functions

$$J_i(x,\alpha,\Lambda_i) = \begin{cases} E\left\{ \displaystyle\sum_{n=1}^{\infty} \left[e^{-\rho\tau_n^s}(X_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(X_{\tau_n^b} + K) \right] \right\}, & \text{if } i = 0, \\ E\left\{ e^{-\rho\tau_1^s}(X_{\tau_1^s} - K) \right. \\ \quad \left. + \displaystyle\sum_{n=2}^{\infty} \left[e^{-\rho\tau_n^s}(X_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(X_{\tau_n^b} + K) \right] \right\}, & \text{if } i = 1, \end{cases} \tag{4}$$

where $\rho > 0$ is a given discount factor.

In this paper, given random variables x_n, the term $E\{\sum_{n=1}^{\infty} x_n\}$ is interpreted as

$$\limsup_{N\to\infty} E \sum_{n=1}^{N} x_n.$$

For $i = 0, 1$, let $V_i(x, \alpha)$ denote the value functions with the initial vector $(X_0, \alpha_0) = (x, \alpha)$ and initial net positions $i = 0, 1$. That is,

$$V_i(x,\alpha) = \sup_{\Lambda_i} J_i(x,\alpha,\Lambda_i). \tag{5}$$

It follows from (2) that $X_t = xe^{-at} + e^{-at}\int_0^t e^{ar} f(\alpha_r)dr$. Therefore, $J_i(x, \alpha, \Lambda_i)$ is linear in x and $V_i(x, \alpha)$ is convex in x for fixed $i = 0, 1$ and $\alpha = 1, 2$.

In practice, the transaction cost $K > 0$ is typically small when compared to all other parameters. Here we assume it is smaller than both (f_1/a) and $(-f_2/a)$. In addition, both λ_1 and λ_2 are large numbers relative to a. So we also assume $\lambda_1 > a$ and $\lambda_2 > a$.

We summarize conditions to be imposed in this paper:

(A1) $f_1 > 0$ and $f_2 < 0$;
(A2) $K < \min\{f_1/a, |f_2|/a\}$.
(A3) $\lambda_1 > a$ and $\lambda_2 > a$.

Note that in this paper the stock price X_t is differentiable and the value of α_t can be given in terms of the derivative of X_t with respect to t.

3. Properties of the value functions.

Lemma 3.1. *The following inequalities hold.*

(a) $0 \le V_0(x, \alpha) \le \left(\dfrac{\rho + 2a}{\rho a}\right) \max\{f_1, |f_2|\}$, *for all* $(x, \alpha) \in \mathcal{I}_0 \times \mathcal{M}$.

(b) $-K \leq V_0(x,\alpha) - V_1(x,\alpha) + x \leq K$, *for all* $(x,\alpha) \in \mathcal{I}_0 \times \mathcal{M}$. *Therefore,* $V_1(x,\alpha)$ *is also bounded on* $\mathcal{I}_0 \times \mathcal{M}$.

Proof. It is clear that $V_0(x,\alpha) \geq 0$. To show it is bounded from above, we first note that

$$\frac{d(e^{-\rho t}X_t)}{dt} = e^{-\rho t}(-(\rho+a)X_t + f(\alpha_t)).$$

Given Λ_0, integrate both sides over $[\tau_n^b, \tau_n^s]$ to obtain

$$e^{-\rho\tau_n^s}X_{\tau_n^s} - e^{-\rho\tau_n^b}X_{\tau_n^b} = \int_{\tau_n^b}^{\tau_n^s} e^{-at}(-(\rho+a)X_t + f(\alpha_t))dt.$$

It follows that, for $(x,\alpha) \in \mathcal{I}_0 \times \mathcal{M}$,

$$\begin{aligned}
J_0(x,\alpha,\Lambda_0) \quad &= E\sum_{n=1}^{\infty}\left[e^{-\rho\tau_n^s}(X_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(X_{\tau_n^b} + K)\right] \\
&\leq E\sum_{n=1}^{\infty}\left[e^{-\rho\tau_n^s}X_{\tau_n^s} - e^{-\rho\tau_n^b}X_{\tau_n^b}\right] \\
&\leq E\sum_{n=1}^{\infty}\int_{\tau_n^b}^{\tau_n^s} e^{-at}(-(\rho+a)X_t + f(\alpha_t))dt \\
&\leq E\int_0^{\infty} e^{-at}((\rho+a)|X_t| + |f(\alpha_t)|)dt \\
&\leq \left(\frac{\rho+2a}{\rho a}\right)\max\{f_1, |f_2|\}.
\end{aligned}$$

This implies (a).

To show that $V_0(x,\alpha) - V_1(x,\alpha) + x \leq K$, or $V_1(x,\alpha) \geq V_0(x,\alpha) + x - K$, we note that, for a given Λ_1 with $\tau_1^s = 0$,

$$V_1(x,\alpha) \geq J_1(x,\alpha,\Lambda_1) + x - K.$$

Since the rest stopping times in Λ_1 are arbitrary, it follows that $V_1(x,\alpha) \geq V_0(x,\alpha) + x - K$. Similarly, we can show the second inequality in (b). $\square$

4. **HJ equations.**

Let $\mathcal{A}$ denote the generator of (X_t, α_t), i.e., for any differentiable functions $h(x,i)$, $i = 1,2$,

$$\begin{cases} \mathcal{A}h(x,1) = (-ax + f_1)h'(x,1) + \lambda_1(h(x,2) - h(x,1)), \\ \mathcal{A}h(x,2) = (-ax + f_2)f_2h'(x,2) + \lambda_2(h(x,1) - h(x,2)), \end{cases}$$

where h' denotes the derivative of h with respect to x. The associated HJ equations should have the form:

$$\begin{cases} \min\{\rho v_0(x,\alpha) - \mathcal{A}v_0(x,\alpha),\ v_0(x,\alpha) - v_1(x,\alpha) + x + K\} = 0, \\ \min\{\rho v_1(x,\alpha) - \mathcal{A}v_1(x,\alpha),\ v_1(x,\alpha) - v_0(x,\alpha) - x + K\} = 0, \end{cases} \tag{6}$$

for $\alpha = 1,2$.

In this paper, our goal is to find the value functions by solving these HJ equations with all value function properties satisfied. Then we proceed to show that such functions $v_i(x,\alpha)$ are equal to the value functions.

4.1. **Solving** $(\rho - \mathcal{A})v_i(x,\alpha) = 0$**.** To solve these HJ equations, for each fixed $i = 0, 1$, we consider equations when $\rho v_i(x,\alpha) - \mathcal{A}v_i(x,\alpha) = 0$, $\alpha = 1, 2$. Using the generator $\mathcal{A}$, we can write

$$\left\{ \begin{array}{l} (\rho + \lambda_1)v_i(x,1) = (-ax + f_1)v_i'(x,1) + \lambda_1 v_i(x,2), \\ (\rho + \lambda_2)v_i(x,2) = (-ax + f_2)v_i'(x,2) + \lambda_2 v_i(x,1). \end{array} \right. \tag{7}$$

For each fixed $i = 0, 1$ and $\alpha = 1, 2$, let $y_\alpha(x) = v_i(x,\alpha)$. Then, using the first equation, we can write y_2 in terms of y_1:

$$y_2(x) = \frac{1}{\lambda_1}[(\rho + \lambda_1)y_1(x) + (ax - f_1)y_1'(x)].$$

Its derivative with respect to x is given by

$$y_2'(x) = \frac{1}{\lambda_1}[(\rho + \lambda_1 + a)y_1'(x) + (ax - f_1)y_1''(x)].$$

Substitute these into the second equation in (7) to obtain

$$\begin{aligned} &(ax - f_1)(ax - f_2)y_1''(x) + [(\rho + \lambda_2)(ax - f_1) \\ &\quad +(\rho + \lambda_1 + a)(ax - f_2)]y_1'(x) + \rho(\rho + \lambda_1 + \lambda_2)y_1(x) = 0. \end{aligned}$$

To simplify this equation, let

$$\xi = \frac{ax - f_2}{f_1 - f_2}$$

and $u(\xi) = y_1(x)$. Then, it follows that

$$ax - f_2 = (f_1 - f_2)\xi \quad \text{and} \quad ax - f_1 = (f_1 - f_2)(\xi - 1).$$

Moreover, applying the chain rule to obtain

$$y_1'(x) = \left(\frac{a}{f_1 - f_2}\right)u'(\xi) \quad \text{and} \quad y_1''(x) = \left(\frac{a}{f_1 - f_2}\right)^2 u''(\xi).$$

This reduces the equation for $y_1(x)$ into an equation for $u(\xi)$ as

$$a^2\xi(\xi - 1)u''(\xi) + a[(\rho + \lambda_2)(t - 1) + (\rho + \lambda_1 + a)t]u'(\xi) + \rho(\rho + \lambda_1 + \lambda_2)]u(\xi) = 0.$$

Next, we introduce the additional parameters

$$\gamma_1 = \frac{\rho}{a}, \ \gamma_2 = \frac{\rho + \lambda_1 + \lambda_2}{a}, \ \gamma_3 = \frac{\rho + \lambda_2}{a}.$$

Then, we can write the above differential equation in terms of $(\gamma_1, \gamma_2, \gamma_3)$ as

$$\xi(1 - \xi)u''(\xi) + [\gamma_3 - (\gamma_1 + \gamma_2 + 1)t]u'(\xi) - \gamma_1\gamma_2 u(\xi) = 0. \tag{8}$$

This is exactly the classical hypergeometric equation. Related properties of hypergeometric functions can be found in [19] and [25].

For any real numbers η_i, $i = 1, 2, 3$, let

$$F(\eta_1, \eta_2; \eta_3, \xi) = \sum_{n=0}^{\infty} \frac{(\eta_1)_n(\eta_2)_n}{n!(\eta_3)_n}\xi^n,$$

where

$$(\eta)_0 = 1 \text{ and } (\eta)_n = \eta(\eta + 1)\cdots(\eta + n - 1) = \frac{\Gamma(\eta + n)}{\Gamma(\eta)},$$

for any $\eta = \eta_i$. Then, for $0 < \xi < 1$, two independent solutions of (8) can be given by

$$F\left(\frac{\rho}{a}, \frac{\rho + \lambda_1 + \lambda_2}{a}; \frac{\rho + \lambda_2}{a}, \xi\right) \text{ and } F\left(\frac{\rho}{a}, \frac{\rho + \lambda_1 + \lambda_2}{a}; \frac{\rho + \lambda_2 + a}{a}, 1 - \xi\right).$$

Let

$$\begin{aligned} F_1(x) &= F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2}{a}, \frac{ax-f_2}{f_1-f_2}\right), \\ \widetilde{F}_1(x) &= F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2+a}{a}, 1-\frac{ax-f_2}{f_1-f_2}\right). \end{aligned} \tag{9}$$

We consider these solutions over a neighborhood in $\mathcal{I}_0$ containing the lower end f_2/a. Then any solution over this interval can be written as a linear combination of F_1 and $\widetilde{F}_1$, i.e., for some constants C_1 and $\widetilde{C}_1$, $v_i(x,1) = C_1F_1(x) + \widetilde{C}_1\widetilde{F}_1(x)$, $i = 0, 1$. In particular, we need this solution to be bounded near f_2/a. This requires $\widetilde{C}_1 = 0$ because $\widetilde{F}_1(x) \to \infty$ as $x \to f_2/a$ (see Lemma 8.1 in Appendix).

Therefore, on the neighborhood of $\mathcal{I}_0$ containing f_2/a we can assume that $v_1(x,1)$ $= C_1F_1(x)$. Using this solution, we can write $v_1(x,2)$ in terms of $F_1(x)$:

$$v_1(x,2) = \frac{C_1}{\lambda_1}[(\rho+\lambda_1)F_1(x) + (ax-f_1)F_1'(x)] := C_1G_1(x). \tag{10}$$

Alternatively, we can use the second equation in (7) and write y_1 in terms of y_2:

$$y_1(x) = \frac{1}{\lambda_2}[(\rho+\lambda_2)y_2(x) + (ax-f_2)y_2'(x)]$$

and its derivative in terms of y_2' and y_2''

$$y_1'(x) = \frac{1}{\lambda_2}[(\rho+\lambda_2+a)y_2'(x) + (ax-f_2)y_2''(x)].$$

Substituting these into the first equation in (7) to obtain

$$\begin{aligned} &(ax-f_1)(ax-f_2)y_2''(x) + [(\rho+\lambda_1)(ax-f_2) \\ &\quad +(\rho+\lambda_2+a)(ax-f_1)]y_1'(x) + \rho(\rho+\lambda_1+\lambda_2)y_2(x) = 0. \end{aligned}$$

Similarly, let $w(\xi) = y_2(x)$ with

$$\xi = \frac{f_1-ax}{f_1-f_2},$$

Then, we can write the above differential equation in terms of $w(\xi)$:

$$\xi(1-\xi)w''(\xi) + [\gamma_3' - (\gamma_1+\gamma_2+1)t]w'(\xi) - \gamma_1\gamma_2 w(\xi) = 0,$$

with $\gamma_3' = (\rho+\lambda_1)/a$. Its two independent solutions can be given by

$$F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_1}{a}, \xi\right) \text{ and } F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_1+a}{a}, 1-\xi\right).$$

Let

$$\begin{aligned} F_2(x) &= F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_1}{a}, \frac{f_1-ax}{f_1-f_2}\right), \\ \widetilde{F}_2(x) &= F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2+a}{a}, 1-\frac{f_1-ax}{f_1-f_2}\right). \end{aligned} \tag{11}$$

Similarly, we consider these solutions over a neighborhood in $\mathcal{I}_0$ containing the upper end f_1/a. Again, any solution can be written as a linear combination of F_2 and $\widetilde{F}_2$, i.e., for some constants C_2 and $\widetilde{C}_2$, $v_i(x,2) = C_2F_2(x) + \widetilde{C}_2\widetilde{F}_2(x)$, $i = 0, 1$. We now need this solution to be bounded near f_1/a, which implies $\widetilde{C}_2 = 0$ because $\widetilde{F}_2(x) \to \infty$ as $x \to f_1/a$.

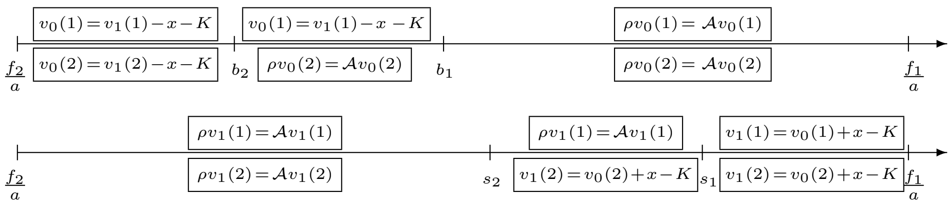

FIGURE 2. Buying Thresholds: b_1 and b_2; Selling Thresholds: s_1 and s_2.

Therefore, on the neighborhood of $\mathcal{I}_0$ containing f_1/a we can assume that $v_0(x,2) = C_2F_2(x)$. Using this solution, we have

$$v_0(x,1) = \frac{C_2}{\lambda_2}[(\rho+\lambda_2)F_2(x) + (ax - f_2)F_2'(x)] := C_2G_2(x). \tag{12}$$

Lemma 4.1. *The solutions of $(\rho - \mathcal{A})v_i(x,\alpha) = 0$ can be given as*

$$v_0(x,1) = C_2G_1(x),\ v_0(x,2) = C_2F_2(x),\ v_1(x,1) = C_1F_1(x),\ v_1(x,2) = C_1G_1(x),$$

where F_1, F_2, G_1, and G_2 are defined in (9), (11), (10), and (12), respectively.

4.2. **Order of threshold levels.** If $\alpha_t = 1$, let b_1 denote the buying threshold (to buy when $X_t < b_1$) and s_1 the selling threshold (to sell when $X_t > s_1$), respectively. Likewise, if $\alpha_t = 2$, let b_2 and s_2 denote the corresponding buying and selling thresholds, respectively.

In the purely selling case, it is shown in [29] that $s_1 > s_2$. Intuitively, b_1 should be greater than b_2 as well.

In addition, in this paper, we consider the case when $b_1 < s_2$.

The opposite inequality $b_1 \geq s_2$ means to buy when $\alpha_t = 1$ and $X_t < b_1$ and sell as soon as α_t jumps to 2 and $X_t > s_2$. This case rarely arises in practice because the jump rates λ_1 and λ_2 are typically large so that α_t switches between 1 and 2 frequently.

In view of these, we consider the threshold levels having the following order

$$\frac{f_2}{a} < b_2 < b_1 < s_2 < s_1 < \frac{f_1}{a}.$$

The corresponding equalities in the HJ equations in (6) are marked in Figure 2.

Using convexity of the value functions and variational inequalities in (6), it is shown in Lemma 8.2 (Appendix) that

$$b_2 = \frac{f_2 - \rho K}{\rho + a} \text{ and } s_1 = \frac{f_1 + \rho K}{\rho + a}.$$

It is easy to check, under Assumption (A2), that b_2 and s_1 given above indeed satisfy the inequalities $b_2 > f_2/a$ and $s_1 < f_1/a$.

4.3. **Solving $(\rho - \mathcal{A})v_1(x,1) = 0$ with $v_1(x,2) = v_0(x,2) + x - K$ and $(\rho - \mathcal{A})v_0(x,2) = 0$ with $v_0(x,1) = v_1(x,1) - x - K$.** For $x \in \mathcal{I}_0$, let

$$g_1(x) = \left(\frac{as_1 - f_1}{ax - f_1}\right)^{\frac{\rho+\lambda_1}{a}} \text{ and } g_2(x) = \left(\frac{ab_2 - f_2}{ax - f_2}\right)^{\frac{\rho+\lambda_2}{a}}.$$

Next, we solve the HJ equations on $[s_2, s_1]$. Note that on this interval, $v_0(x,1) = C_2G_2(x)$, $v_0(x,2) = C_2F_2(x)$, and $v_1(x,2) = v_0(x,2) + x - K = C_2F_2(x) + x - K$. To solve $\rho v_1(x,1) = \mathcal{A}v_1(x,1)$ on this interval, we write it as

$$(ax - f_1)v_1'(x,1) + (\rho + \lambda_1)v_1(x,1) = \lambda_1 v_1(x,2).$$

Differentiate $(1/g_1(x))v_1(x,1)$ and then integrate from s_1 to x to obtain

$$v_1(x,1) = v_1(s_1,1)g_1(x) - \lambda_1(ax - f_1)^{-\frac{\rho+\lambda_1}{a}} \int_{s_1}^{x} (f_1 - as)^{\frac{\rho+\lambda_1}{a}-1} v_1(s,2)ds.$$

Recall that $v_1(x,2) = C_2F_2(x) + x - K$. We write

$$v_1(x,1) = g_1(x)A + k_1(x)C_2 + k_2(x),$$

where $A = v_1(s_1,1)$ and

$$\begin{cases} k_1(x) = -\lambda_1(f_1 - ax)^{-\frac{\rho+\lambda_1}{a}} \displaystyle\int_{s_1}^{x} (f_1 - as)^{\frac{\rho+\lambda_1}{a}-1} F_2(s)ds, \\ k_2(x) = -\lambda_1(f_1 - ax)^{-\frac{\rho+\lambda_1}{a}} \displaystyle\int_{s_1}^{x} (f_1 - as)^{\frac{\rho+\lambda_1}{a}-1} (s - K)ds. \end{cases}$$

Similarly, we solve the HJ equations on $[b_2, b_1]$. Note that on this interval, $v_1(x,1) = C_1F_1(x)$, $v_1(x,2) = C_1G_1(x)$, and $v_0(x,1) = v_1(x,1) - x - K = C_1F_1(x) - x - K$. To solve $\rho v_0(x,2) = \mathcal{A}v_0(x,2)$ on this interval, we first write it as

$$(ax - f_2)v_0'(x,2) + (\rho + \lambda_2)v_0(x,2) = \lambda_2 v_0(x,1).$$

Differentiate $(1/g_2(x))v_0(x,2)$ and then integrate over $[b_2, x]$ to obtain

$$v_0(x,2) = v_0(b_2,2)g_2(x) + \lambda_2(ax - f_2)^{-\frac{\rho+\lambda_2}{a}} \int_{b_2}^{x} (as - f_2)^{\frac{\rho+\lambda_2}{a}-1} v_0(s,1)ds.$$

Recall that $v_0(x,1) = C_1F_1(x) - x - K$. We write

$$v_0(x,2) = g_2(x)B + h_1(x)C_1 + h_2(x),$$

where $B = v_0(b_2,2)$ and

$$\begin{cases} h_1(x) = \lambda_2(ax - f_2)^{-\frac{\rho+\lambda_2}{a}} \displaystyle\int_{b_2}^{x} (as - f_2)^{\frac{\rho+\lambda_2}{a}-1} F_1(s)ds, \\ h_2(x) = -\lambda_2(ax - f_2)^{-\frac{\rho+\lambda_2}{a}} \displaystyle\int_{b_2}^{x} (as - f_2)^{\frac{\rho+\lambda_2}{a}-1} (s + K)ds. \end{cases}$$

Using the power series form of $F_1(x) = \displaystyle\sum_{n=0}^{\infty} p_n \left(\frac{ax - f_2}{f_1 - f_2}\right)^n$, we can write

$$h_1(x) = \lambda_2 \sum_{n=0}^{\infty} \frac{p_n}{\rho + \lambda_2 + an} \left[\left(\frac{ax - f_2}{f_1 - f_2}\right)^n - g_2(x)\left(\frac{ab_2 - f_2}{f_1 - f_2}\right)^n \right].$$

Similarly, we have

$$h_2(x) = -\frac{\lambda_2}{a}\left[\frac{1}{\rho + \lambda_2 + a}\left((ax - f_2) - (ab_2 - f_2)g_2(x)\right) + \frac{f_2 + aK}{\rho + \lambda_2}(1 - g_2(x))\right].$$

Likewise, write $F_2(x) = \displaystyle\sum_{n=0}^{\infty} q_n \left(\frac{f_1 - ax}{f_1 - f_2}\right)^n$ to obtain

$$k_1(x) = -\lambda_1 \sum_{n=0}^{\infty} \frac{q_n}{\rho + \lambda_1 + an} \left[g_1(x)\left(\frac{f_1 - as_1}{f_1 - f_2}\right)^n - \left(\frac{f_1 - ax}{f_1 - f_2}\right)^n \right]$$

and

$$k_2(x) = \frac{\lambda_1}{a}\left[-\frac{1}{\rho+\lambda_1+a}\left((f_1-ax)-(f_1-as_1)g_1(x)\right)+\frac{f_1-aK}{\rho+\lambda_1}(1-g_1(x))\right].$$

4.4. **Smooth-fit conditions.** Note that the convexity of $v_i(x,\alpha)$ in x on $\mathcal{I}_0$ implies that they are continuous in the interior of $\mathcal{I}_0$. The continuity of these functions at the threshold levels leads to the following conditions:

$$\begin{cases} \boxed{x=b_2}: & C_1G_1(b_2)-b_2-K=g_2(b_2)B+h_1(b_2)C_1+h_2(b_2);\\ \boxed{x=b_1}: & \begin{cases} C_1F_1(b_1)-b_1-K=C_2G_2(b_1),\\ g_2(b_1)B+h_1(b_1)C_1+h_2(b_1)=C_2F_2(b_1),\end{cases}\\ \boxed{x=s_2}: & \begin{cases} C_1F_1(s_2)=g_1(s_2)A+k_1(s_2)C_2+k_2(s_2),\\ C_1G_1(s_2)=C_2F_2(s_2)+s_2-K,\end{cases}\\ \boxed{x=s_1}: & C_2G_2(s_1)+s_1-K=g_1(s_1)A+k_1(s_1)C_2+k_2(s_1).\end{cases} \tag{13}$$

Recall that $b_2=(f_2-\rho K)/(\rho+a)$ and $s_1=(f_1+\rho K)/(\rho+a)$. Using the first and last equations in (13), we can write

$$\begin{cases} B=\left(\dfrac{G_1(b_2)-h_1(b_2)}{g_2(b_2)}\right)C_1-\dfrac{b_2+K+h_2(b_2)}{g_2(b_2)}=:B_1C_1+B_2\\ A=\left(\dfrac{G_2(s_1)-k_1(s_1)}{g_1(s_1)}\right)C_2+\dfrac{s_1-K-k_2(s_1)}{g_1(s_1)}=:A_1C_2+A_2.\end{cases}$$

Then, we can write the second and the third equations in terms of C_1 and C_2:

$$\begin{pmatrix} F_1(b_1) & -G_2(b_1)\\ g_2(b_1)B_1+h_1(b_1) & -F_2(b_1)\end{pmatrix}\begin{pmatrix} C_1\\ C_2\end{pmatrix}=\begin{pmatrix} b_1+K\\ -g_2(b_1)B_2-h_2(b_1)\end{pmatrix}.$$

Similarly, we can write the fourth and the fifth equations as follows:

$$\begin{pmatrix} G_1(s_2) & -F_2(s_2)\\ F_1(s_2) & -[g_1(s_2)A_1+k_1(s_2)]\end{pmatrix}\begin{pmatrix} C_1\\ C_2\end{pmatrix}=\begin{pmatrix} s_2-K\\ g_1(s_2)A_2+k_2(s_2)\end{pmatrix}.$$

If the above 2×2 matrices are invertible, then we can eliminate C_1 and C_2 and obtain

$$\begin{aligned} &\begin{pmatrix} F_1(b_1) & -G_2(b_1)\\ g_2(b_1)B_1+h_1(b_1) & -F_2(b_1)\end{pmatrix}^{-1}\begin{pmatrix} b_1+K\\ -g_2(b_1)B_2-h_2(b_1)\end{pmatrix}\\ &\quad=\begin{pmatrix} G_1(s_2) & -F_2(s_2)\\ F_1(s_2) & -[g_1(s_2)A_1+k_1(s_2)]\end{pmatrix}^{-1}\begin{pmatrix} s_2-K\\ g_1(s_2)A_2+k_2(s_2)\end{pmatrix}.\end{aligned} \tag{14}$$

The solutions to the HJ equations (6) should have the form:

$$\begin{cases} v_0(x,1)=\begin{cases} C_1F_1(x)-x-K & \text{if } f_2/a\le x\le b_1,\\ C_2G_2(x) & \text{if } b_1<x\le f_1/a,\end{cases}\\ v_0(x,2)=\begin{cases} C_1G_1(x)-x-K & \text{if } f_2/a\le x\le b_2,\\ g_2(x)B+h_1(x)C_1+h_2(x) & \text{if } b_2<x\le b_1,\\ C_2F_2(x) & \text{if } b_1<x\le f_1/a,\end{cases}\\ v_1(x,1)=\begin{cases} C_1F_1(x) & \text{if } f_2/a\le x\le s_2,\\ g_1(x)A+k_1(x)C_2+k_2(x) & \text{if } s_2<x\le s_1,\\ C_2G_2(x)+x-K & \text{if } s_1<x\le f_1/a,\end{cases}\\ v_1(x,2)=\begin{cases} C_1G_1(x) & \text{if } f_2/a\le x\le s_2,\\ C_2F_2(x)+x-K & \text{if } s_2<x\le f_1/a,\end{cases}\end{cases} \tag{15}$$

In Figure 3, these functions are so labelled in their corresponding intervals in Figure 2.

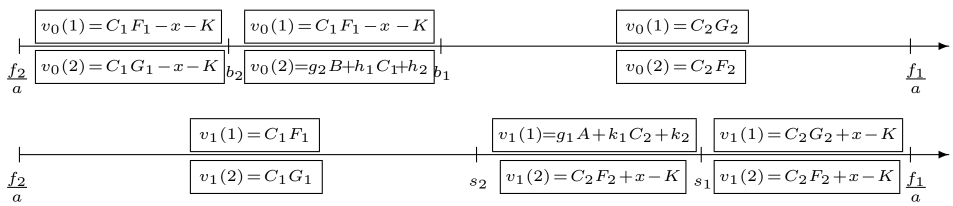

FIGURE 3. HJ equation solutions.

5. **Variational inequalities and convexity.** In this section, we consider the variational inequalities in (6) and related convexity inequalities.

5.1. **Variational inequalities.** Note that with the functions given in (15), all variational inequalities in (6) have to be satisfied, i.e., for $(x,\alpha) \in \mathcal{I}_0 \times \mathcal{M}$,

$$\begin{cases} \rho v_0(x,\alpha) - \mathcal{A}v_0(x,\alpha) \geq 0, \\ v_0(x,\alpha) - v_1(x,\alpha) + x + K \geq 0, \\ \rho v_1(x,\alpha) - \mathcal{A}v_1(x,\alpha) \geq 0, \\ v_1(x,\alpha) - v_0(x,\alpha) - x + K \geq 0. \end{cases} \tag{16}$$

In what follows, we consider these inequalities on each intervals: $[f_2/a, b_2]$, $[b_2, b_1]$, $[b_1, s_2]$, $[s_2, s_1]$, and $[s_1, f_1/a]$.

First, we show that these inequalities on $[f_2/a, b_2]$ are automatically satisfied. Note that on $[f_2/a, b_2]$, we have

$$\begin{cases} v_0(x,1) = v_1(x,1) - x - K, \\ v_0(x,2) = v_1(x,2) - x - K, \\ \rho v_1(x,1) = \mathcal{A}v_1(x,1), \\ \rho v_1(x,2) = \mathcal{A}v_1(x,2). \end{cases} \tag{17}$$

We need to show

$$\begin{cases} \rho v_0(x,1) \geq \mathcal{A}v_0(x,1), \\ \rho v_0(x,2) \geq \mathcal{A}v_0(x,2), \\ v_1(x,1) \geq v_0(x,1) + x - K, \\ v_1(x,2) \geq v_0(x,2) + x - K. \end{cases} \tag{18}$$

It is easy to see that the last two inequalities follow directly from the first two equalities in (17). To see the first inequality $\rho v_0(x,1) \geq \mathcal{A}v_0(x,1)$, note that it can be written in terms of $v_1(x,1)$and $v_1(x,2)$:

$$\rho(v_1(x,1) - x - K) \geq (f_1 - ax)(v_1'(x,1) - 1) + \lambda_1(v_1(x,2) - v_1(x,1)),$$

which is equivalent to

$$-\rho(x+K) \geq -(f_1 - ax), \text{ for } x < b_2,$$

because $\rho v_1(x,1) = \mathcal{A}v_1(x,1)$. The above inequality is clearly satisfied because $b_2 = (f_2 - \rho K)/(\rho + a)$ and $f_2 < f_1$.

Similarly, on $[s_1, f_1/a]$, we can use $s_1 = (f_1 + \rho K)/(\rho + a)$ and show that all VIs in (16) hold.

Next, we consider the VIs on $[b_2, b_1]$. We have

$$\begin{cases} v_0(x,1) = v_1(x,1) - x - K, \\ \rho v_0(x,2) = \mathcal{A} v_0(x,2), \\ \rho v_1(x,1) = \mathcal{A} v_1(x,1), \\ \rho v_1(x,2) = \mathcal{A} v_1(x,2). \end{cases} \tag{19}$$

We need the following inequalities to hold:

$$\begin{cases} \rho v_0(x,1) \geq \mathcal{A} v_0(x,1), \\ v_0(x,2) \geq v_1(x,2) - x - K, \\ v_1(x,1) \geq v_0(x,1) + x - K, \\ v_1(x,2) \geq v_0(x,2) + x - K. \end{cases} \tag{20}$$

The third inequality is automatically satisfied. The second and forth ones can be written as:

$$-K \leq v_0(x,2) - v_1(x,2) + x \leq K.$$

Finally, the first inequality in (20) can be given as

$$v_0(x,2) - v_1(x,2) + x \leq \frac{1}{\lambda_1}[f_1 - \rho K - (\rho + a)x] - K.$$

We can work out all needed inequalities this way on all other intervals and summarize these inequalities as follows:

$$\begin{cases} \text{On } [b_2, b_1]: \quad -K \leq W_2(x) \leq \min\left\{K, \dfrac{1}{\lambda_1}[f_1 - \rho K - (\rho + a)x] - K\right\}, \\ \text{On } [b_1, s_2]: \quad -K \leq W_1(x) \leq K \text{ and } -K \leq W_2(x) \leq K, \\ \text{On } [s_2, s_1]: \quad \max\left\{-K, \dfrac{1}{\lambda_2}[f_2 + \rho K - (\rho + a)x] + K\right\} \leq W_1(x) \leq K, \end{cases} \tag{21}$$

where $W_1(x) = v_0(x,1) - v_1(x,1) + x$ and $W_2(x) = v_0(x,2) - v_1(x,2) + x$.

5.2. **Convexity.** Recall that the value functions are convex on $\mathcal{I}_0$. We expect the solutions to the HJ equations in (6) to be also convex.

Recall also that

$$G_i(x) = \frac{1}{\lambda_i}\left((\rho + \lambda_i)F_i(x) + (ax - f_i)F_1'(x)\right),$$

for $i = 1, 2$.

Lemma 5.1. *F_1, F_2, G_1, G_2 are convex on $[f_2/a, f_1/a]$.*

Proof. Clearly, both F_1 and F_2 are convex because their second derivatives are positive.

To show the convexities of G_1 and G_2, we first consider

$$g(\xi) = (\gamma_1 + \gamma_2 - \gamma_3)u(\xi) - (1 - \xi)\frac{du}{d\xi}(\xi),$$

with $u(\xi) = F(\gamma_1, \gamma_2; \gamma_3, \xi)$. Multiply its both sides by $(-(1-\xi)^{\gamma_1+\gamma_2-\gamma_3-1})$ to obtain

$$-(1-\xi)^{\gamma_1+\gamma_2-\gamma_3-1}g(\xi) = -(\gamma_1+\gamma_2-\gamma_3)(1-\xi)^{\gamma_1+\gamma_2-\gamma_3-1}u(\xi) + (1-\xi)^{\gamma_1+\gamma_2-\gamma_3}\frac{du}{d\xi}(\xi).$$

The right hand side of the above equation is equal to

$$\frac{d}{d\xi}\left[(1-\xi)^{\gamma_1+\gamma_2-\gamma_3}F(\gamma_1, \gamma_2; \gamma_3, \xi)\right].$$

We next apply the formula

$$\frac{d^n}{d\xi^n}\left[(1-\xi)^{\gamma_1+\gamma_2-\gamma_3}F(\gamma_1,\gamma_2;\gamma_3,\xi)\right] = \frac{(\gamma_3-\gamma_1)_n(\gamma_3-\gamma_2)_n}{(\gamma_3)_n}(1-\xi)^{\gamma_1+\gamma_2-\gamma_3-n}F(\gamma_1,\gamma_2;\gamma_3+n,\xi),$$

with $n=1$ and get

$$-(1-\xi)^{\gamma_1+\gamma_2-\gamma_3-1}g(\xi) = \frac{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)}{\gamma_3}(1-\xi)^{\gamma_1+\gamma_2-\gamma_3-1}F(\gamma_1,\gamma_2;\gamma_3+1,\xi).$$

Hence we conclude that

$$g(\xi) = -\frac{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)}{\gamma_3}F(\gamma_1,\gamma_2;\gamma_3+1,\xi).$$

For $u(\xi) = F\left(\frac{\rho}{a},\frac{\rho+\lambda_1+\lambda_2}{a};\frac{\rho+\lambda_2}{a},\xi\right)$, the corresponding $g(\xi)$ is given by

$$g(\xi) = \frac{\lambda_2\lambda_1}{\rho+\lambda_2}F\left(\frac{\rho}{a},\frac{\rho+\lambda_1+\lambda_2}{a};\frac{\rho+\lambda_2}{a}+1,\xi\right).$$

Then the convexity of $F\left(\frac{\rho}{a},\frac{\rho+\lambda_1+\lambda_2}{a};\frac{\rho+\lambda_2}{a}+1,\xi\right)$ implies that $g(\xi)$ is convex. It is easy to check with change of variable $t=(ax-f_2)/(f_1-f_2)$ that $G_1(x)=(\lambda_1/a)g(\xi)$. So G_1 is convex. Similarly, we can show the convexity of G_2. □

Lemma 5.2. *$h_1(x)$ and $k_1(x)$ are convex.*

Proof. Here, we only show the convexity of $h_1(x)$. The convexity for $k_1(x)$ can be given in a similar way. Recall that

$$h_1(x) = \lambda_2(ax-f_2)^{-\frac{\rho+\lambda_2}{a}}\int_{b_2}^{x}(as-f_2)^{\frac{\rho+\lambda_2}{a}-1}F_1(s)ds,$$

where

$$F_1(x) = F\left(\gamma_1,\gamma_2;\gamma_3,\frac{ax-f_2}{f_1-f_2}\right) \text{ with } \gamma_1=\frac{\rho}{a},\ \gamma_2=\frac{\rho+\lambda_1+\lambda_2}{a} \text{ and } \gamma_3=\frac{\rho+\lambda_2}{a}.$$

Let

$$w_1(\xi) = h_1(x) = h_1\left(\frac{(f_1-f_2)t+f_2}{a}\right), \text{ with } t=\frac{ax-f_2}{f_1-f_2}.$$

It follows that

$$w_1(\xi) = \frac{\lambda_2}{a}\xi^{-\gamma_3}\int_{\widetilde{b}_2}^{\xi}r^{\gamma_3-1}F(\gamma_1,\gamma_2;\gamma_3,r)dr,$$

where $\widetilde{b}_2=(ab_2-f_2)/(f_1-f_2)$. Note that

$$\int F(\alpha,\beta;\gamma,\xi)\xi^{\gamma-1}dt = \frac{\xi^\gamma}{\gamma}F(\alpha,\beta;\gamma+1,\xi)+k.$$

Apply the above formula, we obtain

$$w_1(\xi) = \frac{\lambda_2}{a\gamma_3}\left[F(\gamma_1,\gamma_2;\gamma_3+1,\xi)-F(\gamma_1,\gamma_2;\gamma_3+1,\widetilde{b}_2)\right].$$

Recall the derivative formula for hypergeometric function:

$$\frac{d^m}{d\xi^m}F(\alpha,\beta;\gamma,\xi) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}F(\alpha+m,\beta+m;\gamma+m,\xi).$$

It follows that

$$w_1''(\xi) = \frac{\lambda_2\gamma_1(\gamma_1+1)\gamma_2(\gamma_2+1)}{a\gamma_3(\gamma_3+1)(\gamma_3+2)} F(\gamma_1+2,\gamma_2+2;\gamma_3+3,\xi).$$

In view of this and the convexity of the hypergeometric functions with positive parameters, the convexity of $w_1(\xi)$ follows on $[0,1]$. So is the convexity of $h_1(x)$. $\square$

Lemma 5.3. *$g_2B + h_1C_1 + h_2$ and $g_1A + k_1C_2 + k_2$ are convex provided that*

$$C_1 \geq 0,\ C_2 \geq 0,\ A + A_0 \geq 0,\ B + B_0 \geq 0, \tag{22}$$

where

$$A_0 = \frac{\lambda_1}{a}\left(\frac{f_1 - as_1}{\rho+\lambda_1+a} - \frac{f_1 - aK}{\rho+\lambda_1}\right) \text{ and } B_0 = \frac{\lambda_2}{a}\left(\frac{ab_2 - f_2}{\rho+\lambda_2+a} + \frac{f_2 + aK}{\rho+\lambda_2}\right).$$

Proof. It is easy to see that both $g_1(x)$ and $g_2(x)$ are convex. In addition, note that $g_1A + k_2$ can be written as $g_1(A + A_0)+$ a linear function. So $A + A_0 > 0$ implies that $g_1A + k_2$ is convex. Then in view of the convexity of $k_1(x)$ in Lemma 5.2, it follows that $g_1A + k_1C_2 + k_2$ is convex. Similarly, we can show the convexity of $g_2B + h_1C_1 + h_2$. $\square$

To guarantee the convexity of the functions $v_i(x,\alpha)$, it is sufficient to require the convexity of $v_0(x,1)$ at $x = b_1$ and $v_1(x,2)$ at $x = s_2$.

The convexity of $v_0(x,1)$ at $x = b_1$ is equivalent to

$$(v_0(b_1,1))'_- \leq (v_0(b_1,1))'_+.$$

Therefore, we need

$$C_1F_1'(b_1) - 1 \leq C_2G_2'(b_1). \tag{23}$$

Similarly, The convexity of $v_1(x,2)$ at $x = s_2$ requires

$$C_1G_1'(s_2) \leq C_2F_2'(s_2) + 1. \tag{24}$$

Theorem 5.1. *Assume that there exist b_1 and s_2 such that* (a) *the equalities in* (14) *hold with invertible matrices;* (b) *the additional VI inequalities in* (21) *hold;* (c) *the inequalities in* (22) *hold;* (d) *the convexity inequalities in* (23) *and* (24) *hold. Then, $v_i(x,\alpha)$ defined in* (15) *satisfy the HJ equations* (6).

6. **A verification theorem.**

Theorem 6.1. *Assume the conditions of Theorem 5.1. Then,*

(a) *$v(x,i) = V(x,i)$, $i = 1,2$.*

(b) *Moreover, let*

$$D_b = (b_1, f_1/a) \times \{1\} \cup (b_2, f_1/a) \times \{2\},$$
$$D_s = (f_2/a, s_1) \times \{1\} \cup (f_2/a, s_2) \times \{2\}$$

denote the continuation regions. The set of stopping times in Λ_0 and Λ_1 can be defined as the first exit times from these continuation regions in sequence. Let $\Lambda_0^* = (\tau_1^{b*}, \tau_1^{s*}, \tau_2^{b*}, \tau_2^{s*}, \ldots)$, where $\tau_1^{b*} = \inf\{t : (X_t,\alpha_t) \notin D_b\}$, $\tau_1^{s*} = \inf\{t \geq \tau_1^{b*} : (X_t,\alpha_t) \notin D_s\}$,.., $\tau_{n+1}^{b*} = \inf\{t \geq \tau_n^{s*} : (X_t,\alpha_t) \notin D_b\}$, $\tau_{n+1}^{s*} = \inf\{t \geq \tau_{n+1}^{b*} : (X_t,\alpha_t) \notin D_s\}$,..., and let $\Lambda_1^* = (\tau_1^{s*}, \tau_2^{b*}, \tau_2^{s*}, \ldots)$, where $\tau_1^{s*} = \inf\{t \geq 0 : (X_t,\alpha_t) \notin D_s\}$,.., $\tau_{n+1}^{b*} = \inf\{t \geq \tau_n^{s*} : (X_t,\alpha_t) \notin D_b\}$, $\tau_{n+1}^{s*} = \inf\{t \geq \tau_{n+1}^{b*} : (X_t,\alpha_t) \notin D_s\}$,... If $v_0(x) \geq 0$ and $\tau_n^{s*} \to \infty$, a.s., then, Λ_0^* and Λ_1^* are optimal.

Proof. We only sketch the proof because it is similar to that of Zhang and Zhang [27, Theorem 5]. First, for any stopping times $0 \leq \theta_1 \leq \theta_2$, we can show as in [27] that

$$Ee^{-\rho\theta_1} v_i(X_{\theta_1}, \alpha_{\theta_1}) \geq Ee^{-\rho\theta_2} v_i(X_{\theta_2}, \alpha_{\theta_2}),$$

for $i = 0, 1$. For any given Λ_0, it follows from this inequality (with $\theta_1 = 0$ and $\theta_2 = \tau_1^b$) and the VIs in (6) that

$$\begin{aligned}
v_0(x,\alpha) \geq\ & E\left(e^{-\rho\tau_1^b} v_0(X_{\tau_1^b}, \alpha_{\tau_1^b})\right) \\
\geq\ & E\left(e^{-\rho\tau_1^b} [v_1(X_{\tau_1^b}, \alpha_{\tau_1^b}) - X_{\tau_1^b} - K]\right) \\
\geq\ & E\left(e^{-\rho\tau_1^s} v_1(X_{\tau_1^s}, \alpha_{\tau_1^s})\right) - E\left(e^{-\rho\tau_1^b}(X_{\tau_1^b} - K)\right) \\
\geq\ & E\left(e^{-\rho\tau_1^s} [v_0(X_{\tau_1^s}, \alpha_{\tau_1^s}) + X_{\tau_1^s} - K]\right) - E\left(e^{-\rho\tau_1^b}(X_{\tau_1^b} - K)\right) \\
=\ & E\left(e^{-\rho\tau_1^s} v_0(X_{\tau_1^s}, \alpha_{\tau_1^s})\right) + E\left(e^{-\rho\tau_1^s}(X_{\tau_1^s} - K) - e^{-\rho\tau_1^b}(X_{\tau_1^b} - K)\right).
\end{aligned} \tag{25}$$

Continue this way and recall $v_0 \geq 0$ to obtain

$$v_0(x,\alpha) \geq E\sum_{n=1}^{\infty}\left[e^{-\rho\tau_n^s}(X_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(X_{\tau_n^b} + K)\right].$$

It follows that $v_0(x,\alpha) \geq V_0(x,\alpha)$. Similarly, we can show $v_1(x,\alpha) \geq V_1(x,\alpha)$.

To establish the equalities with $\Lambda_i = \Lambda_i^*$, first note that, in view of Lemma 8.3 (Appendix), τ_n^{b*} and τ_n^{s*} are finite a.s. for each n. Using this condition, the inequalities in (25) become equalities when τ_i^s is replaced by τ_i^{s*} and τ_i^b is replaced by τ_i^{b*}, respectively. Finally, the boundedness of v_0 and $\tau_n^{s*} \to \infty$ imply, for $i = 0$,

$$v_0(x,\alpha) = E\sum_{n=1}^{\infty}\left[e^{-\rho\tau_n^{s*}}(X_{\tau_n^{s*}} - K) - e^{-\rho\tau_n^{b*}}(X_{\tau_n^{b*}} + K)\right] = V_0(x,\alpha).$$

Similarly, we can show the case when $i = 1$. □

Remark 6.1. We point out that a sufficient condition for $\tau_n^{s*} \to \infty$ can be given by $s_2 - b_1 > 2K$ following the proof in [27]. (This inequality is satisfied in all cases in our numerical example in the next section.)

7. A numerical example.

7.1. Model calibration and state estimation.

First we give a model calibration method. Let $Z_k = X_{k\delta}$ with $\delta > 0$ being a discretization step size. Then, Z_k can be approximated by the solution of the follow equation:

$$Z_{k+1} = (1 - a\delta)Z_k + ab\delta + \varepsilon_k, \tag{26}$$

where $\varepsilon_k = \int_{k\delta}^{(k+1)\delta} f(\alpha_t)dt$ represents system noise.

In view of this, $Z_{n\delta}$ is an autoregressive process of order 1. Following standard AR(1) estimation (least square) method, we can obtain the estimates for a, b, and its variance Σ^2. Plugging back these values to (26) to obtain the values for ε_k. An estimate for the state process α_t can be given in terms of ε_k: for $t \in (k\delta, (k+1)\delta]$,

$$\hat{\alpha}_t = \begin{cases} 1 & \text{if } \varepsilon_k \geq 0, \\ 2 & \text{if } \varepsilon_k < 0, \end{cases}$$

To estimate f_1, f_2, λ_1, and λ_2, let $f_1 = \mu + \sigma_1$ and $f_2 = \mu - \sigma_2$, for some μ, σ_1, and σ_2, with $\nu_1\sigma_1 - \nu_2\sigma_2 = 0$, where $(\nu_1, \nu_2) = (\lambda_2/(\lambda_1 + \lambda_2), \lambda_1/(\lambda_1 + \lambda_2))$.

Let $\overline{\varepsilon} = \left(\sum_{k=0}^{n-1} \varepsilon_k\right)/n$. Then, $\mu \approx \overline{\varepsilon}/\delta$. In addition, using Yin an Zhang [26, Theorem 5.9], we can show

$$E\left(\varepsilon_k - \overline{\varepsilon}\right)^2 \approx \delta \left(\frac{2\lambda_1\lambda_2(\sigma_1+\sigma_2)^2}{(\lambda_1+\lambda_2)^3}\right).$$

Let

$$\sigma_0^2 = \frac{\sum_{k=0}^{n-1}(\varepsilon_k - \overline{\varepsilon})^2}{n-1}.$$

Then, by the Law of Large Numbers, we have

$$\sigma_0^2 \approx \delta \left(\frac{2\lambda_1\lambda_2(\sigma_1+\sigma_2)^2}{(\lambda_1+\lambda_2)^3}\right).$$

Using $\nu_1\sigma_1 = \nu_2\sigma_2$, we have

$$\sigma_1 = \frac{\sigma_0}{\sqrt{\delta}}\sqrt{\frac{\lambda_1(\lambda_1+\lambda_2)}{2\lambda_2}} \text{ and } \sigma_2 = \frac{\sigma_0}{\sqrt{\delta}}\sqrt{\frac{\lambda_2(\lambda_1+\lambda_2)}{2\lambda_1}},$$

Finally, we estimate λ_1 and λ_2. Let $R = \nu_1/\nu_2$. Then, $\lambda_2 = R\lambda_1$.

$$\begin{aligned} R_1 &= \#\{k : \varepsilon_k < 0 \text{ and } \varepsilon_{k+1} \geq 0\}, \\ R_2 &= \#\{k : \varepsilon_k > 0 \text{ and } \varepsilon_{k+1} \leq 0\}. \end{aligned}$$

Then, it follows that

$$\frac{R_1}{\lambda_1} + \frac{R_2}{\lambda_2} = T.$$

Therefore, the jump rates are given by

$$\begin{cases} \lambda_1 = \dfrac{1}{T}\left(R_1 + \dfrac{R_2}{R}\right), \\ \lambda_2 = \dfrac{1}{T}\left(RR_1 + R_2\right). \end{cases}$$

7.2. **Backtesting with DJIA 1961-1980.** We use Dow Jones Industrial Average daily closing prices from 1961 to 1980 to backtest our trading rules. In particular, let $c_0, c_1, c_2, \dots$ denote the daily closing prices of the index. We take $S_{k\delta} = \log(c_k)$ with $\delta = 1/252$ and use the daily prices from the beginning of 1961 to the end of 1970 to calibrate the model. We obtain

$$a = 0.1528,\ b = 6.4813,\ f_1 = 1.09,\ f_2 = -1.20,\ \lambda_1 = 104.35,\ \lambda_2 = 114.88.$$

Then, we take $X_{k\delta} = S_{k\delta} - b$, $\rho = 1$, and $K = 0.01$ and obtain the corresponding buy and sell thresholds for trading in 1971:

$$b_2 = -1.049618,\ b_1 = -0.027672,\ s_2 = 0.032443, s_1 = 0.954198.$$

At the end of 1971, we update the system parameter values using the most recent data (1971) and dropping the 1961 prices. We keep rolling forward this way. The parameter values are given in Table 1 and the corresponding threshold levels are provided in Table 2. In addition, the conditions required in Theorem 5.1 are satisfied in all these cases.

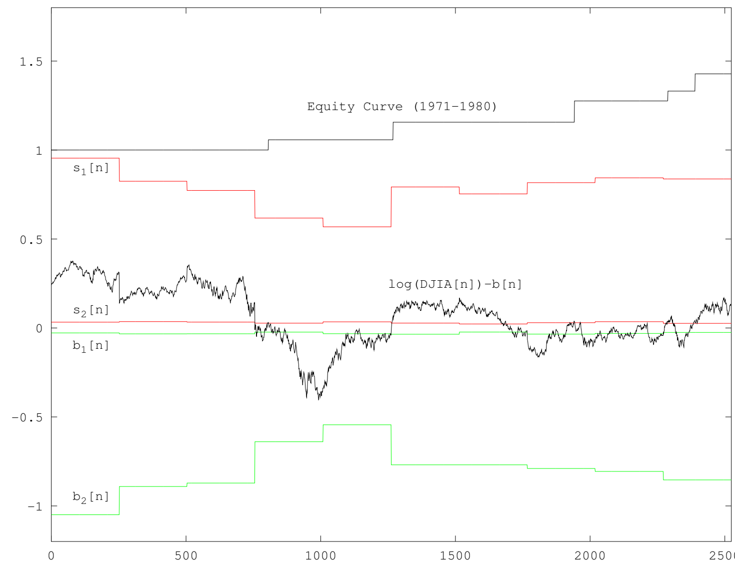

FIGURE 4. Equity curve

Calibration Data Period	a	b	f_1	f_2	λ_1	λ_2
1961-1970	0.1528	6.4813	1.09	-1.20	104.35	114.88
1962-1971	0.3582	6.6500	1.11	-1.20	103.51	112.14
1963-1972	0.3189	6.6051	1.01	-1.14	102.20	114.87
1964-1973	0.8616	6.7619	1.14	-1.18	102.97	107.18
1965-1974	1.4624	6.7885	1.39	-1.33	106.03	101.71
1966-1975	0.8552	6.7667	1.46	-1.41	105.22	101.10
1967-1976	0.9243	6.7381	1.44	-1.47	104.03	106.05
1968-1977	0.8489	6.7727	1.50	-1.45	108.77	105.18
1969-1978	0.8608	6.7738	1.56	-1.49	110.52	105.51
1970-1979	0.8507	6.7403	1.54	-1.57	110.31	112.27

Table 1. System Parameter Values.

Backtesting Year	b_2	b_1	s_2	s_1
1971	-1.049618	-0.027672	0.032443	0.954198
1972	-0.890885	-0.033132	0.035488	0.824621
1973	-0.871344	-0.032808	0.032960	0.772844
1974	-0.639235	-0.023313	0.026966	0.617748
1975	-0.544185	-0.032326	0.034438	0.568551
1976	-0.769189	-0.035258	0.027205	0.792367
1977	-0.769111	-0.023021	0.022658	0.753521
1978	-0.789659	-0.034669	0.029585	0.816702
1979	-0.806105	-0.030686	0.035307	0.843723
1980	-0.853731	-0.025018	0.025720	0.837521

Table 2. Trading Threshold Levels.

In each trading year, a buying signal is triggered when $(X_t, \hat{\alpha}_t)$ exits D_b and a selling signal is generated when $(X_t, \hat{\alpha}_t)$ exits D_s. In Figure 4, we define $b[n]$ as a piecewise constant function determined by the b-column in Table 1. For example, $b[n] = 6.4813$ for $n = 0, 1, \ldots, 252$ corresponding to the prices in 1971. Likewise, we define $s_1[n]$, $s_2[n]$, $b_1[n]$, and $b_2[n]$ piecewisely using the corresponding columns in Table 2. Also, in Figure 4, $X_{n\delta} = \log(c[n]) - b[n]$ is plotted.

Initially, we allocate the capital of \$100K. When a trading signal is triggered, buy (and sell) the index with the entire account balance less the transaction fees.

The corresponding equity curve is also provided in Figure 4. There are five trades between 1971-1980. The ending balance is \$142818.61 which amounts 42.28% gain.

8. **Conclusion.** This paper considers a mean-reversion trading rule under a Markov chain model. It is shown that the optimal trading rule can be given in terms of four threshold levels that can be obtained by solving algebraic like equations. Such feature is attractive from practical viewpoints because they are easy to implement.

It would be interesting to study related problems with regime switching. For example, the rate of mean reversion and equilibrium levels are regime dependent. In this connection, one may introduce a two-time-scale Markov chain so that the 'fast' states correspond to short term events and the 'slow' states correspond to market trends. In addition, related state estimation may arise due to incomplete observation of the 'slow' components.

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Appendix. In this section, we provide three technical lemmas used in this paper.

Lemma 8.1. *Under Assumption* (A3)*, we have*

$$\begin{aligned}
&\lim_{\xi\to 1^-} F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2}{a}, \xi\right) = \infty,\\
&\lim_{\xi\to 0^+} F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2+a}{a}, 1-\xi\right) = \infty,\\
&\lim_{\xi\to 1^-} F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_1}{a}, \xi\right) = \infty,\\
&\lim_{\xi\to 0^+} F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_1+a}{a}, 1-\xi\right) = \infty.
\end{aligned}$$

Proof. First, note that the value of $F(\gamma_1, \gamma_2; \gamma_3, 1)$ can be given in terms of the sum of the hypergeometric series

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(\gamma_1)_n(\gamma_2)_n}{n!(\gamma_3)_n}.$$

The ratio of the consecutive terms in the hypergeometric series is given by

$$\frac{c_{n-1}}{c_n} = \frac{n(n-1+\gamma_3)}{(n-1+\gamma_1)(n-1+\gamma_2)} = 1 + \frac{\gamma_3-\gamma_1-\gamma_2+1}{n} + O(n^{-2}).$$

Then, it follows from Bromwich [4, §79, p. 240] that the series $\sum c_n$ is absolutely convergent if $\mathrm{Re}(\gamma_3-\gamma_1-\gamma_2) > 0$ and it is divergent (unless being a finite sum) if $\mathrm{Re}(\gamma_3-\gamma_1-\gamma_2) \le 0$.

For the function $F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2}{a}, \xi\right)$, clearly, the corresponding $\gamma_3-\gamma_1-\gamma_2 = -(\rho+\lambda_1)/a < 0$. This implies that $\lim_{\xi\to 1^-} F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2}{a}, \xi\right) = \infty$.

Also, for the function $F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2+a}{a}, 1-\xi\right)$, the corresponding $\gamma_3-\gamma_1-\gamma_2 = 1-(\rho+\lambda_2)/a < 0$ by (A3). Therefore, we have

$$\lim_{\xi\to 1^-} F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2+a}{a}, \xi\right) = \infty.$$

Hence,

$$\lim_{\xi\to 0^+} F\left(\frac{\rho}{a}, \frac{\rho+\lambda_1+\lambda_2}{a}; \frac{\rho+\lambda_2+a}{a}, 1-\xi\right) = \infty.$$

Similarly, we can show the divergence of the other two functions. $\square$

Lemma 8.2. *The following hold:*

(a) $b_2 = \dfrac{f_2-\rho K}{\rho+a}$ *and* $s_1 = \dfrac{f_1+\rho K}{\rho+a}$.

(b) $v_0(x,2)$ *is differentiable at* $x=b_2$ *and* $v_1(x,1)$ *is differentiable at* $x=s_1$. *Hence, each* $v_i(x,\alpha)$, $i=0,1$ *and* $\alpha=1,2$, *is differentiable at these two points.*

Proof. First, we show $b_2 = (f_2-\rho K)/(\rho+a)$. Note that, for $b_2 \le x \le b_1$, $\rho v_0(x,2) = \mathcal{A}v_0(x,2)$. It follows that

$$(v_0(b_2,2))'_+ = \frac{(\rho+\lambda_2)v_0(b_2,2)-\lambda_2 v_0(b_2,1)}{f_2-ab_2}.$$

Note also that, for x near b_2, $\rho v_1(x,2) = \mathcal{A}v_1(x,2)$. Therefore, we have

$$(v_1(b_2,2))' = \frac{(\rho+\lambda_2)v_1(b_2,2)-\lambda_2 v_1(b_2,1)}{f_2-ab_2}.$$

Recall that $v_0(x,1) = v_1(x,1)-x-K$ for x near b_2 and $v_0(x,2) = v_1(x,2)-x-K$ for $x \le b_2$. The convexity at $x=b_2$ implies that $(v_0(b_2,2))'_- \le (v_0(b_2,2))'_+$. Therefore,

$$\frac{(\rho+\lambda_2)v_1(b_2,2)-\lambda_2 v_1(b_2,1)}{f_2-ab_2} - 1 \le \frac{(\rho+\lambda_2)v_0(b_2,2)-\lambda_2 v_0(b_2,1)}{f_2-ab_2}. \tag{27}$$

Using $v_0(b_2,1) = v_1(b_2,1)-b_2-K$ and $v_0(b_2,2) = v_1(B_2,2)-b_2-K$ and the fact that $f_2-ab_2<0$, we can show

$$b_2 \ge \frac{f_2-\rho K}{\rho+a}.$$

To establish the opposite inequality, note that, for $x<b_2$, $\rho v_0(x,2) \ge \mathcal{A}v_0(x,2)$, $v_0(x,1) = v_1(x,1)-x-K$, and $v_0(x,2) = v_1(x,2)-x-K$. It follows that

$$\begin{aligned}\rho(v_1(x,2)-x-K) \ge\ & (f_2-ax)(v_1'(x,2)-1)+\lambda_2(v_0(x,1)-v_0(x,2))\\ =\ & (f_2-ax)(v_1'(x,2)-1)+\lambda_2(v_1(x,1)-v_1(x,2)).\end{aligned}$$

Recall that $\rho v_1(x,2) = \mathcal{A}v_1(x,2)$. We have

$$-\rho(x+K) \ge -(f_2-ax),\ \text{for } x<b_2.$$

This leads to

$$b_2 \le \frac{f_2-\rho K}{\rho+a}.$$

Therefore, the equality holds. This in turn implies the equality in (27) holds. So is the differentiability of $v_0(x,2)$ at $x=b_2$.

Similarly, we can show $s_1 = (f_1+\rho K)/(\rho+a)$ and $v_1(x,1)$ is differentiable at $x=s_1$. $\square$

Lemma 8.3. *For any given* $z_1,\ z_2 \in \mathcal{I}_0$, *let* $\theta_1 = \inf\{t:\ X_t \ge z_1\}$ *and* $\theta_2 = \inf\{t:\ X_t \le z_2\}$. *Then, under Assumption* (A3), $P(\theta_1<\infty) = P(\theta_2<\infty) = 1$.

Proof. Let $(z_0,z_1) \subset \mathcal{I}_0$ and $\theta_0 = \inf\{t:\ X_t \notin (z_0,z_1)\}$. Let $p(x,\alpha) = P(X_{\theta_0} = z_1 | X_0 = x, \alpha_0 = \alpha)$, for $x \in (z_0,z_1)$ and $\alpha \in \mathcal{M}$. Then,

$$\mathcal{A}p(x,\cdot)(\alpha) = 0,\ p(z_0,\alpha) = 0,\ p(z_1,\alpha) = 1,$$

for $\alpha = 1, 2$. Using the first equation in $\mathcal{A}p(x, \cdot)(\alpha) = 0$, we have

$$p(x,2) = p(x,1) + \left(\frac{ax - f_1}{\lambda_1}\right) p'(x,1),$$

and

$$p'(x,2) = \left(\frac{\lambda_1 + a}{\lambda_1}\right) p'(x,1) + \left(\frac{ax - f_1}{\lambda_1}\right) p''(x,1).$$

Substitute these into the second equation in $\mathcal{A}p(x, \cdot)(\alpha) = 0$ and simplify to obtain

$$(ax - f_1)(ax - f_2)p''(x,1) + [(\lambda_1 + a)(ax - f_2) + \lambda_2(ax - f_1)]p'(x,1) = 0.$$

Solve this equation with the boundary conditions $p(x, z_0) = 0$ and $p(x, z_1) = 1$ to obtain

$$p(x,1) = \frac{\int_{z_0}^{x} (f_1 - ar)^{-\frac{\lambda_1 + a}{a}} (ar - f_2)^{-\frac{\lambda_2}{a}} dr}{\int_{z_0}^{z_1} (f_1 - ar)^{-\frac{\lambda_1 + a}{a}} (ar - f_2)^{-\frac{\lambda_2}{a}} dr}.$$

Similarly, we have

$$p(x,2) = \frac{\int_{z_0}^{x} (f_1 - ar)^{-\frac{\lambda_1}{a}} (ar - f_2)^{-\frac{\lambda_2 + a}{a}} dr}{\int_{z_0}^{z_1} (f_1 - ar)^{-\frac{\lambda_1}{a}} (ar - f_2)^{-\frac{\lambda_2 + a}{a}} dr}.$$

Under Assumption (A3), it is easy to see, for any $\delta > 0$, that both integrals

$$\int_{z_0}^{x} (f_1 - ar)^{-\frac{\lambda_1 + a}{a}} (ar - f_2)^{-\frac{\lambda_2}{a}} dr \text{ and } \int_{z_0}^{z_0 + \delta} (f_1 - ar)^{-\frac{\lambda_1}{a}} (ar - f_2)^{-\frac{\lambda_2 + a}{a}} dr$$

tend to ∞, as $z_0 \to f_2/a$. It follows that

$$p(x,1) \to 1 \text{ and } p(x,2) \to 1, \text{ as } z_0 \to f_2/a.$$

Therefore, we have

$$P(\theta_1 < \infty) \geq p(x,1)P(\alpha_0 = 1) + p(x,2)P(\alpha_0 = 2) \to 1.$$

Hence, $P(\theta_1 < \infty) = 1$.

Similarly, we can consider the equations $\mathcal{A}p(x, \cdot)(\alpha) = 0$ on interval $(z_2, z_0) \subset \mathcal{I}_0$ with boundary conditions $p(z_0, \alpha) = 0$ and $p(z_2, \alpha) = 1$ and show that

$$p(x,1) \to 1 \text{ and } p(x,2) \to 1, \text{ as } z_0 \to f_1/a.$$

These lead to $P(\theta_2 < \infty) = 1$. □

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