WITTEN'S TOP CHERN CLASS VIA COSECTION LOCALIZATION

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ABSTRACT. For a Landau-Ginzburg space $([\mathbb{C}^n/G], W)$, we construct Witten's top Chern class as an algebraic cycle using cosection localized virtual cycles in the case where all sectors are narrow, verify all axioms of this class, and derive an explicit formula for it in the free case. We prove that this construction is equivalent to the constructions of Polishchuk-Vaintrob, Chiodo, and Fan-Jarvis-Ruan.

1. Introduction

In this paper, we construct and study Witten's top Chern class of the moduli stack of spin curves associated with a Landau-Ginzburg space using the method of cosection localization.

A Landau-Ginzburg space (LG space in short) in this paper is a pair $([\mathbb{C}^n/G], W)$ of a finite subgroup $G \le GL(n,\mathbb{C})$ and a non-degenerate quasi-homogeneous G-invariant polynomial W in n-variables (see Definition 2.2). Given such an LG space, one forms the moduli stack of smooth G-spin curves $M_{g,\ell}(G)$, which when $G = \mathrm{Aut}(W)$, takes the form

$$M_{g,\ell}(G) = \{[\mathcal{C}, \mathcal{L}_1, \cdots, \mathcal{L}_n] \mid \mathcal{C} \text{ stable}, W_a(\mathcal{L}_1, \cdots, \mathcal{L}_n) \cong \omega_{\mathcal{C}}^{log}\}. \tag{1.1}$$

Here $\mathcal{C}$ is a smooth ℓ-pointed twisted (orbifold) curve, $\mathcal{L}_j$'s are invertible sheaves on $\mathcal{C}$, and W_a's are the monomials of W (see details in §2). In ([Wi]), Witten demonstrated how to construct a "topological gravity coupled with matter" using solutions to his equation (i.e. the Witten equation), which takes the form

$$\overline{\partial} s + (k+1)\bar{s}^k = 0, \quad s \in C^\infty(\mathcal{C}, \mathcal{L}),$$

in the case where $([\mathbb{C}/\mathbb{Z}_{k+1}], W = x^{k+1})$ and for $[\mathcal{C}, \mathcal{L}] \in M_{g,n}(G)$. He conjectured that the partition functions of such A_k singularities, and also other singularities of DE type, satisfy ADE integrable hierarchies.

The mathematical theory of Witten's "topological gravity coupled with matter" has satisfactorily been derived. The proper moduli stacks of nodal spin curves have been worked out by Abramovich and Jarvis ([Ja1, Ja2, AJ]). "Witten's top Chern class" has been constructed by Polishchuk-Vaintrob ([PV1]), alternatively by Chiodo ([Ch2, Ch3]) via K-theory, and by Mochizuki ([Mo]) following Witten's approach.

The case for a general LG space was solved later by Fan-Jarvis-Ruan ([FJR1, FJR2]). Their construction is analytic in nature, and uses the Witten equations for

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W,

$$\overline{\partial} s_j + \overline{\partial_j W}(s_1, \cdots, s_n) = 0, \quad s_j \in C^\infty(\mathcal{C}, \mathcal{L}_j), \tag{1.2}$$

to construct Witten's top Chern class of $M_{g,\ell}(G)$. They also proved all expected properties of the class. In line with the fact that the GW theory of a smooth variety is a virtual counting of maps, this theory, now commonly referred to as the FJRW-theory, is an (enumeration) theory for the singularity $(W = 0)/G$. We add that since the domain curves $\mathcal{C}$ in $[\mathcal{C}, \mathcal{L}_j] \in M_{g,\ell}(G)$ are pointed twisted curves, each $\mathcal{L}_j$ yields a representation (sector) of the automorphism group of a marked point of $\mathcal{C}$. We use γ to denote this collection of representations. Those γ's such that all factors of the representation at all marked points are non-trivial are called narrow ("Neveu-Schwarz"). The FJRW theory treats all situations, including but not restricted to the narrow case. Witten's ADE integrable conjecture was solved by Faber-Shadrin-Zvonkine for the A_n case ([FSZ]), and by Fan-Jarvis-Ruan for the DE case ([FJR2]).

In this paper, using cosection localized virtual cycles, we construct Witten's top Chern class for an LG space $([\mathbb{C}^n/G], W)$ in the case where all sectors are narrow, and also verify all expected properties of the virtual class using this construction. Our construction is an algebraic analogue of Witten's argument using his equation. This allows us to prove that our construction yields the same class as those of Polishchuk-Vaintrob, Chiodo, and Fan-Jarvis-Ruan. Note that the equivalence of Polishchuk-Vaintrob's definition and that of Chiodo is known, but the equivalence of Polishchuk-Vaintrob's definition and that of Fan-Jarvis-Ruan is new.

We define Witten's top Chern class as the cosection localized virtual class of the moduli of G-spin curves with fields. Let $([\mathbb{C}^n/G], W)$ be an LG space. Given integers g, ℓ, and a collection of representations γ, denote by $\overline{M}_{g,\gamma}(G)$ the moduli stack of G-spin ℓ-pointed genus g twisted nodal curves banded by γ, which parameterizes $[\mathcal{C}, \mathcal{L}_1, \cdots, \mathcal{L}_n]$ such that $\mathcal{C}$ is a stable ℓ-pointed genus g twisted nodal curve and $\mathcal{L}_j$'s are invertible sheaves on $\mathcal{C}$ such that, in addition to the constraint in (1.1) (when $G = \mathrm{Aut}(W)$), the representations of $\mathcal{L}_j$ restricted to the marked points of $\mathcal{C}$ are given by the collection γ. Following the work of Chang and Li ([CL]), we form the moduli of G-spin curves with fields:

$$\overline{M}_{g,\gamma}(G)^p = \{[\mathcal{C}, \mathcal{L}_j, \rho_j]_{j=1}^n \mid [\mathcal{C}, \mathcal{L}_j] \in \overline{M}_{g,\gamma}(G),\ \rho_j \in \Gamma(\mathcal{L}_j)\}.$$

It is a DM stack, and has a perfect obstruction theory relative to $\overline{M}_{g,\gamma}(G)$. The forgetful morphism

$$\overline{M}_{g,\gamma}(G)^p \longrightarrow \overline{M}_{g,\gamma}(G) \tag{1.3}$$

has linear fibers, and has a zero section if we set all $\rho_j = 0$.

Given that $G \subset \mathrm{Aut}(W)$, and that γ is narrow, we use the polynomial W to construct a cosection (homomorphism) of the obstruction sheaf of $\overline{M}_{g,\gamma}(G)^p$:

$$\sigma : \mathcal{O}b_{\overline{M}_{g,\gamma}(G)^p} \longrightarrow \mathcal{O}_{\overline{M}_{g,\gamma}(G)^p}. \tag{1.4}$$

We prove that the non-surjective locus of σ is contained in the zero section of (1.3), which is $\overline{M}_{g,\gamma}(G)$ and is proper. Applying the cosection localized virtual class of Kiem and Li ([KL]), we obtain a cosection localized virtual class of $\overline{M}_{g,\gamma}(G)^p$, which we define as

$$[\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}} \in A_*(\overline{M}_{g,\gamma}(G)). \tag{1.5}$$

Definition-Theorem 1.1. *Let* $([\mathbb{C}^n/G], W)$ *be an LG space, let* g, ℓ *be non-negative integers, and let* γ *be a collection of representations. Suppose* γ *is narrow, define (cosection localized) Witten's top Chern class* $\overline{M}_{g,\gamma}(G)$ *of* $([\mathbb{C}^n/G], W)$ *as the cosection localized virtual class* $[\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}}$.

We will verify the expected properties (axioms) of Witten's top Chern classes in §4, and prove the following comparison theorem in §5.

Theorem 1.2. *Witten's top Chern class* $[\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}}$ *constructed via cosection localization coincides with Witten's top Chern class constructed by Polishchuk-Vaintrob when their construction applies. Its associated homology class coincides with the analytic construction of Witten's top Chern class by Fan-Jarvis-Ruan (in the narrow case).*

Here are some comments on the proof of the comparison theorem. Looking more closely at Witten's equation (1.2), and realizing that the term $\overline{\partial}s_j$ gives the obstruction class for extending a holomorphic section of $\mathcal{L}_j$, the equation (1.2) in effect gives a (differentiable) section of the obstruction sheaf of the moduli of spin curves with fields. Witten's top Chern class could be viewed as obtained via the homology class generated by the solution space of transverse perturbation of equation (1.2).

Working algebraically, we substitute the complex conjugation used in (1.2) by Serre duality, and thus transform equation (1.2) into the cosection (1.4). As the cosection has a proper non-surjective locus, the cosection localized Gysin map gives us a virtual cycle of $\overline{M}_{g,\gamma}(G)^p$ supported in $\overline{M}_{g,\gamma}(G)$. Using the topological nature of the cosection localized virtual class, we can show that our construction yields the same class as that constructed by the FJRW theory when pushed to the ordinary homology group.

The algebraic-geometric construction of Witten's top Chern class for $(\mathbb{C}, x^{k+1})$ by Polishchuk-Vaintrob relies on resolving the universal family of the moduli of spin curves, and they define Witten's top Chern class as a certain combination of Chern classes of complexes derived from the resolution. We show in §5 that because the relative obstruction theory of $\overline{M}_{g,\gamma}(G)^p$ to $\overline{M}_{g,\gamma}(G)$ is linear, the cosection localized virtual cycle can be expressed in terms of localized Chern classes of certain complexes. This leads to a proof of the comparison theorem.

This paper is organised as follows. In §2, we recall the notion of G-spin curves with fields. Witten's top Chern class is constructed in §3. In §4, we verify the axioms of this class, and derive a closed formula for it in the free case. The comparison theorem is proved in §5.

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2. GENERALIZED SPIN CURVES WITH FIELDS

We work with complex numbers and let $\mathbb{G}_{\mathrm{m}} = GL(1, \mathbb{C})$. In this section we recall the moduli of G-spin curves with fields for a finite subgroup $G \le \mathbb{G}_{\mathrm{m}}^n$. Our treatment follows that of [FJR2, PV2], except that we use the term "G-spin curve" instead of "W-curve" as in [FJR2] or "Γ-spin curve" as in [PV2], as the moduli only depends on the group G.

2.1. **LG spaces.** We fix an integer $d > 1$ and a primitive n-tuple

$$\delta = (\delta_1, \cdots, \delta_n) \in \mathbb{Z}_+^n.$$

Let $\zeta_d = \exp(\frac{2\pi\sqrt{-1}}{d})$ be the standard generator of the group $\mu_d \le \mathbb{G}_{\rm m}$ of the d-th root of unity. We define

$$\jmath_\delta = (\zeta_d^{\delta_1}, \cdots, \zeta_d^{\delta_n}) \in (\mu_d)^n \quad \text{and} \quad \text{the subgroup } \langle \jmath_\delta \rangle \le (\mu_d)^n.$$

Definition 2.1. *A finite subgroup $G \le \mathbb{G}_{\rm m}^n$ containing $\langle \jmath_\delta \rangle$ is called a (d, δ)-group.*

Definition 2.2. *A Laurent polynomial W in variables $(x_1, \cdots, x_n)$ is quasi-homogeneous of weight (d, δ) if*

$$W(\lambda^{\delta_1} x_1, \ldots, \lambda^{\delta_n} x_n) = \lambda^d W(x_1, \ldots, x_n). \tag{2.1}$$

When W is a polynomial, it is non-degenerate if W has a single critical point at the origin, and W does not contain terms $x_i x_j$, $i \ne j$.

Let W be a quasi-homogeneous Laurent polynomial of weight (d, δ). We write $W = \sum_{a=1}^m \alpha_a W_a$, $\alpha_a \ne 0$, as a sum of monic Laurent monomials W_a. These monomials define a group homomorphism

$$\tau_W : \mathbb{G}_{\rm m}^n \longrightarrow \mathbb{G}_{\rm m}^m, \quad x = (x_1, \ldots, x_n) \mapsto (W_1(x), \ldots, W_m(x)). \tag{2.2}$$

Define

$$\mathrm{Aut}(W) := \ker(\tau_W) \le \mathbb{G}_{\rm m}^n.$$

As W is of weight (d, δ), $\jmath_\delta \in \mathrm{Aut}(W)$. When $\mathrm{Aut}(W)$ is finite, it is a (d, δ)-group. Also note that if W is a non-degenerate polynomial as in Definition 2.2, then $\mathrm{Aut}(W)$ is finite ([FJR2, Lem. 2.1.8]).

Definition 2.3. *We say $([\mathbb{C}^n/G], W)$ is an LG space if there is a pair (d, δ) such that G is a (d, δ)-group, and W is a non-degenerate quasi-homogeneous polynomial of weight (d, δ) such that $G \le \mathrm{Aut}(W)$.*

Given an LG space $([\mathbb{C}^n/G], W)$, the orbifold $[\mathbb{C}^n/G]$ with the superpotential W is called an "affine Landau-Ginzburg model" because it admits a global affine chart $\mathbb{C}^n \to [\mathbb{C}^n/G]$. One can work with the' 'Hybrid LG model", and LG spaces such as $(K_{\mathbb{P}^4}, \mathbf{W})$ given in the Guffin-Sharpe-Witten model, whose mathematical construction for all genus (after A-twist) is shown in ([CL]).

2.2. **Twisted curves.** We follow the notations and definitions in [AV, AGV].

Definition 2.4. *An ℓ-pointed twisted nodal curve over a scheme S is a datum*

$$(\Sigma_i^{\mathcal{C}} \subset \mathcal{C} \to C \to S)$$

where

- **(a)** *$\mathcal{C}$ is a proper DM stack, and is étale locally a nodal curve over S;*
- **(b)** *$\Sigma_i^{\mathcal{C}}$, for $1 \le i \le \ell$, are disjoint closed substacks of $\mathcal{C}$ in the smooth locus of $\mathcal{C} \to S$, and $\Sigma_i^{\mathcal{C}} \to S$ are étale gerbes banded by μ_{r_i} for some r_i;*
- **(c)** *the morphism $\pi \colon \mathcal{C} \to C$ makes C the coarse moduli of $\mathcal{C}$;*
- **(d)** *near stacky nodes, $\mathcal{C}$ is balanced;*

(**e**) *$\mathcal{C} \to C$ is an isomorphism over $\mathcal{C}_{\text{ord}}$, where $\mathcal{C}_{\text{ord}}$ is the complement of markings $\Sigma_i^{\mathcal{C}}$ and the stacky singular locus of the projection $\mathcal{C} \to S$.*

A stacky node is a node that is locally not a scheme. $\mathcal{C}$ is *balanced* near nodes means that, over a strictly Henselian local ring R, the strict henselization $\mathcal{C}^{sh}$ is isomorphic to the stack $[U/\mu_r]$, where $U = \mathrm{Spec}(R[x,y]/(xy-t))^{sh}$ for some $t \in R$ and $\zeta \in \mu_r$ acts via $\zeta \cdot (x,y) = (\zeta x, \zeta^{-1} y)$. Near the marking $\Sigma_i^{\mathcal{C}}$, $\mathcal{C}^{sh}$ is isomorphic to $[U/\mu_r]$, where $U = \mathrm{Spec} R[z]^{sh}$ and $\zeta \in \mu_r$ acts on U via $z \to \zeta z$ for z, a local coordinate on U defining the marking. The automorphism group of $\mathcal{C}$ at $\Sigma_i^{\mathcal{C}}$ is canonically isomorphic to μ_r. Denote by Σ_i^C the coarse moduli space of $\Sigma_i^{\mathcal{C}}$. Then the inclusion $\Sigma_i^C \subset C$ makes (C, Σ_i^C) a nodal pointed curve. Define $\Sigma^{\mathcal{C}} = \coprod_i \Sigma_i^{\mathcal{C}}$ and $\Sigma^C = \coprod_i \Sigma_i^C$. Define the log-dualizing sheaves of $\mathcal{C}/S$ and C/S as $\omega_{\mathcal{C}/S}^{log} = \omega_{\mathcal{C}/S}(\Sigma^{\mathcal{C}})$ and $\omega_{C/S}^{log} = \omega_{C/S}(\Sigma^C)$ respectively, where $\omega_{\mathcal{C}/S}$ and $\omega_{C/S}$ are dualizing sheaves of $\mathcal{C}/S$ and C/S respectively. Note $\omega_{\mathcal{C}/S}^{log} = \pi^* \omega_{C/S}^{log}$ ([AJ, Sec 1.3]).

When there is no confusion, we will use $(\mathcal{C}, \Sigma_i^{\mathcal{C}})$, or simply $\mathcal{C}$, to denote $(\Sigma_i^{\mathcal{C}} \subset \mathcal{C} \to C \to S)$. Denote by $\mathfrak{M}_{g,\ell}^{\text{tw}}$ the moduli stack of genus g ℓ-pointed twisted nodal curves.

2.3. *G*-**spin curves.** For any (d,δ)-group G, define

$$\Lambda_G \colon = \{\mathbf{m} \mid \mathbf{m} \text{ are } G\text{-invariant monic Laurent monomials in } (x_1, \cdots, x_n)\}.$$

Since $\langle \jmath_\delta \rangle \subset G$, every $\mathbf{m} \in \Lambda_G$ is of weight $(w(\mathbf{m}) \cdot d, \delta)$, where

$$w(\mathbf{m}) = d^{-1} \cdot \deg \mathbf{m}(t^{\delta_1}, \cdots, t^{\delta_n}) \in \mathbb{Z}.$$

Lemma 2.5. *Via standard multiplication, Λ_G is a free abelian group of rank n.*

Proof. Every monic Laurent monomial can be represented as $x_1^{c_1} \cdots x_n^{c_n}$ with integers $c_1, \cdots, c_n$. In this way the multiplicative group of Laurent polynomials is isomorphic to the additive group $\mathbb{Z}^n$ by identifying $x_1^{c_1} \cdots x_n^{c_n}$ with the vector $(c_1, \ldots, c_n) \in \mathbb{Z}^n$. Hence Λ_G is a subgroup of $\mathbb{Z}^n$. By definition, the $|G|$-th power of every element in $\mathbb{Z}^n/\Lambda_G$ vanishes. □

One can easily check that $G = \{x \in \mathbb{G}_{\mathrm{m}}^n \mid \mathbf{m}(x) = 1 \text{ for all } \mathbf{m} \in \Lambda_G\}$. We pick n generators $\mathbf{m}_1, \cdots, \mathbf{m}_n$ of Λ_G.

Definition 2.6. *An ℓ-pointed G-spin curve over a scheme S is $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k)$, consisting of an ℓ-pointed twisted curve $(\mathcal{C}, \Sigma_i^{\mathcal{C}})$ over S, invertible sheaves $\mathcal{L}_1, \cdots, \mathcal{L}_n$ on $\mathcal{C}$, and isomorphisms*

$$\varphi_k \colon \mathbf{m}_k(\mathcal{L}_1, \ldots, \mathcal{L}_n) \xrightarrow{\cong} (\omega_{\mathcal{C}}^{log})^{w(\mathbf{m}_k)}, \quad k = 1, \cdots, n. \tag{2.3}$$

An arrow between two G-spin curves $(\mathcal{C}, \mathcal{L}_j, \varphi_k)$ and $(\mathcal{C}', \mathcal{L}'_j, \varphi'_k)$ over S consists of (σ, η_j), where $\sigma : \mathcal{C} \to \mathcal{C}'$ is an S-isomorphism of pointed twisted curves, and $\eta_j : \sigma^* \mathcal{L}'_j \to \mathcal{L}_j$ is an isomorphism that commutes with the isomorphisms φ_k and φ'_k.

The definition of G-spin curves does not depend on the choice of the generators $\{\mathbf{m}_k\}$ (cf. [FJR2, Prop 2.1.12]), once the type (d,δ) is specified.

The notation of G-spin curves in this paper and that of W-curves or W-structures in [FJR2] are equivalent. The reason that we choose G-spin curves is to emphasize the different roles played by G and W. The moduli space essentially depends on

G, while the function W's role is to define a cosection of the obstruction sheaf of the moduli space. Note that different W's can be associated with the same group G and hence the same moduli space. Another nice feature of the concept of G-spin curves is that the moduli space is constructed uniformly no matter whether the group G is $\operatorname{Aut} W$ or a proper subgroup of $\operatorname{Aut} W$.

To be more specific about the equivalence of G-spin curves and W curves, we examine the case of $G = \operatorname{Aut} W$ for simplicity. Following the set-up and notations in [FJR2, Def. 2.1.11], we have an $s \times N$ matrix $B_W = (b_{j\ell})$ where $W = \sum_{j=1}^{s} c_j x_1^{b_{j1}} \cdots x_N^{b_{jN}}$, and the Smith normal form $B_W = VTQ$ where both V and Q are invertible matrices and T is of a special form. The link between G-spin curves and W-curves is to treat B_W as a $\mathbb{Z}$-module homomorphism from $\mathbb{Z}^s$ to $\mathbb{Z}^N$ by multiplying row vectors in $\mathbb{Z}^s$ from the right, and treating V, T, and Q similarly. From the matrix $A = TQ = V^{-1}B_W = (a_{j\ell})$, $A_\ell = (a_{\ell 1}, \cdots, a_{\ell N})$ corresponds to our monomial $\mathbf{m}_i = x_1^{a_{i1}} \cdots x_n^{a_{in}} \in \Lambda_G$. Here we swap s, ℓ, N in [FJR2] for our notations m, i, n respectively. We can easily check that u_ℓ in [FJR2, Def. 2.1.11] equals $w(\mathbf{m}_i)$, and the isomorphism in [FJR2, Def. 2.1.11] corresponds to the isomorphism (2.3). When G is a proper subgroup of $\operatorname{Aut} W$, the choice of a basis $\mathbf{m}_1, \cdots, \mathbf{m}_n$ of the lattice Λ_G explains the choice of Z in [FJR2, Def. 2.3.3].

A G-spin curve $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k)$ (over $\mathbb{C}$) has monodromy representations along marked sections and nodes. By definition, the group of automorphisms of each $\Sigma_i^{\mathcal{C}}$ is a cyclic group, say μ_{r_i}, which acts on $\oplus_{j=1}^n \mathcal{L}_j|_{\Sigma_i^{\mathcal{C}}}$, and thus defines a homomorphism $\mu_{r_i} \to \mathbb{G}_{\mathrm{m}}^n$. Because of (2.3) and that $\{\mathbf{m}_k\}_{k=1}^n$ generates Λ_G, this homomorphism factors through a homomorphism

$$\gamma_i : \mu_{r_i} \longrightarrow G \le \mathbb{G}_{\mathrm{m}}^n. \tag{2.4}$$

We call γ_i the monodromy representation along $\Sigma_i^{\mathcal{C}}$.

Similarly, for a node q of $\mathcal{C}$, we let $\hat{\mathcal{C}}_{q+}$ and $\hat{\mathcal{C}}_{q-}$ be the two branches of the formal completion of $\mathcal{C}$ along q, of the form $[\hat{U}/\mu_{r_q}]$, where $\hat{U} = \operatorname{Spec} \mathbb{C}[\![x, y]\!]/(xy)$ and μ_{r_q} acts on $\hat{U}$ via $(x, y)^\zeta = (\zeta x, \zeta^{-1} y)$, such that $\hat{\mathcal{C}}_{q+}$ (resp. $\hat{\mathcal{C}}_{q-}$) is $(y = 0) \subset \hat{U}$ (resp. $(x = 0) \subset \hat{U}$). Let

$$\gamma_{q\pm} : \mu_{r_q} \longrightarrow G \le \mathbb{G}_{\mathrm{m}}^n$$

be the monodromy representation of $\oplus_{j=1}^n \mathcal{L}_j \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}_{\hat{\mathcal{C}}_{q\pm}}$ along $[q/\mu_{r_q}] \subset \hat{\mathcal{C}}_{q\pm}$. .

Denoting by $\gamma_{q+} \cdot \gamma_{q-}$ the composition of $(\gamma_{q+}, \gamma_{q-}) : \mu_{r_q} \to G \times G$ with the multiplication $G \times G \to G$, then by the balanced condition on nodes, we have $\gamma_{q+} \cdot \gamma_{q-} = 1$, the trivial homomorphism. We call γ_{q+} the monodromy representation of the node q, after a choice of $\hat{\mathcal{C}}_{q+}$.

Definition 2.7. *A G-spin curve $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k)$ is stable if its coarse moduli space (C, Σ_i^C) is a stable pointed curve, and if the monodromy representations of marked sections and nodes are injective (representable).*

In this paper, given ℓ, we use $\gamma = (\gamma_i)_{i=1}^\ell$ to denote a collection of injective homomorphisms $\gamma_i : \mu_{r_i} \to G$ for some choices of $r_i \in \mathbb{Z}_+$. Thus every stable G-spin curve with ℓ marked sections will be associated with one such $\gamma = (\gamma_i)_{i=1}^\ell$ via monodromy representations. If $\gamma = (\gamma_i)_{i=1}^\ell$ is associated with some stable genus g G-spin curves, we call such γ g-admissible.

Lemma 2.8 ([FJR2, Prop 2.2.8]). *Given a non-negative integer, a collection of faithful representations $\gamma = (\gamma_1, \cdots, \gamma_\ell)$ is g-admissible if and only if, writing γ_i in* *the form* $\mu_{r_i} \ni e^{2\pi\sqrt{-1}/r_i} \mapsto (e^{2\pi\sqrt{-1}\Theta_1^i}, \cdots, e^{2\pi\sqrt{-1}\Theta_n^i})$, $\Theta_j^i \in [0,1)$, *the following identity holds:*

$$\delta_j(2g-2+\ell)/d - \sum_{i=1}^{\ell} \Theta_j^i \in \mathbb{Z}, \quad j=1,\cdots,n. \tag{2.5}$$

Definition 2.9. *Given g and let γ be a g-admissible collection of representations. A G-spin curve $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k)$ is said to be banded by γ if γ_i is identical to the representation $\mathrm{Aut}(\Sigma_i^{\mathcal{C}}) \to \mathrm{Aut}(\oplus_{j=1}^n \mathcal{L}_j|_{\Sigma_i^{\mathcal{C}}})$ for all i.*

Given a g-admissible γ, it is routine to define the notion of families of genus g γ-banded G-spin curves, to define arrows between two such families, and to define pullbacks. Accordingly, we define $\overline{M}_{g,\gamma}(G)$ as the groupoid of families of stable genus g, γ-banded G-curves. We define

$$\overline{M}_{g,\ell}(G) := \coprod_{\gamma} \overline{M}_{g,\gamma}(G),$$

where γ runs through all possible g-admissible γ.

The stack $\overline{M}_{g,\ell}(G)$ is a smooth proper DM stack with projective coarse moduli. The forgetful morphism from $\overline{M}_{g,\ell}(G)$ to the moduli space $\overline{M}_{g,\ell}$ of ℓ-pointed stable curves is quasi-finite (cf. [FJR2] and [PV2, Prop 3.2.6]). Thus $\overline{M}_{g,\gamma}(G)$ is a smooth proper DM stack.

When $G \le G'$, $\Lambda_{G'} \le \Lambda_G$ and the generators $\mathbf{m}_i'$ of $\Lambda_{G'}$ can be expressed as products of the generators $\mathbf{m}_i^{\pm 1}$ of Λ_G. This shows that the universal family of $\overline{M}_{g,\gamma}(G)$ induces a morphism $\overline{M}_{g,\gamma}(G) \to \overline{M}_{g,\gamma}(G')$, independent of the choices involved. The induced morphism between their coarse moduli spaces is an open as well as a closed embedding.

Now suppose G is the group in an LG space $([\mathbb{C}^n/G], W)$. Write $W = \sum_{a=1}^m \alpha_a W_a$, where $\alpha_a \neq 0$. Then $W_a \in \Lambda_G$. Let $\mathbf{m}_1, \cdots, \mathbf{m}_n$ be the chosen generators of Λ_G. Then there are Laurent monomials $\mathbf{n}_a$ such that $W_a = \mathbf{n}_a(\mathbf{m}_1, \cdots, \mathbf{m}_n)$. Consequently, the isomorphisms $\varphi_1, \cdots, \varphi_n$ in (2.3) induce isomorphisms

$$\varphi_a := \mathbf{n}_a(\varphi_1, \cdots, \varphi_n) \colon W_a(\mathcal{L}_1, \cdots, \mathcal{L}_n) \xrightarrow{\cong} \omega_{\mathcal{C}}^{log}, \quad 1 \le a \le m. \tag{2.6}$$

We next work with the universal family of $\overline{M}_{g,\gamma}(G)$. Let $\pi : \mathcal{C} \to \overline{M}_{g,\gamma}(G)$, invertible sheaves $\mathcal{L}_1, \cdots, \mathcal{L}_n$ be over $\mathcal{C}$, and n isomorphisms as in (2.3) be part of the universal family of $\overline{M}_{g,\gamma}(G)$. Following the discussion preceding (2.6), we obtain m induced isomorphisms

$$\Phi_a : W_a(\mathcal{L}_1, \cdots, \mathcal{L}_n) \xrightarrow{\cong} \omega_{\mathcal{C}/\overline{M}_{g,\gamma}(G)}^{log}, \quad 1 \le a \le m. \tag{2.7}$$

2.4. **Moduli of G-spin curves with fields.** We begin with its definition.

Definition 2.10. *A stable γ-banded G-spin curve with fields consists of a stable γ-banded G-spin curve $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k) \in \overline{M}_{g,\gamma}(G)$ together with n sections $\rho_j \in \Gamma(\mathcal{C}, \mathcal{L}_j)$, $j = 1, \cdots, n$.*

We apply the construction of direct image cones ([CL, Sect 2.1]) to form the stack of G-spin curves with fields. First for notational simplicity, we abbreviate $M = \overline{M}_{g,\gamma}(G)$ and denote by $\pi_M : \mathcal{C}_M \to M$ with $\mathcal{L}_{M,1}, \cdots, \mathcal{L}_{M,n}$ the universal invertible sheaves of M. Let $\mathcal{E}_M = \mathcal{L}_{M,1} \oplus \ldots \oplus \mathcal{L}_{M,n}$. As in [CL, Def 2.1], we denote by $C(\pi_{M*}\mathcal{E}_M)(S)$ the groupoid of (f, ρ_S), where $f : S \to M$ and $\rho_S \in \Gamma(\mathcal{C}_M \times_M$

$S, f^*\mathcal{E}_M)$. With obviously defined arrows among elements in $C(\pi_{M*}\mathcal{E}_M)(S)$ and the pullback $C(\pi_{M*}\mathcal{E}_M)(S') \to C(\pi_{M*}\mathcal{E}_M)(S)$ for $S' \to S$, we get a stack $C(\pi_{M*}\mathcal{E}_M)$.

Define

$$\overline{M}_{g,\gamma}(G)^p = C(\pi_{M*}\mathcal{E}_M)$$

as the stack of families of stable genus g, γ-banded G-spin curves with fields.

Theorem 2.11. *For the data g, G and γ given, the stack $\overline{M}_{g,\gamma}(G)^p$ is a separated DM stack of finite type.*

Proof. Using the Grothendieck duality for DM stacks (cf. [Ni]), the argument in the proof of [CL, Prop. 2.2] shows that

$$C(\pi_{M*}\mathcal{E}_M) = \operatorname{Spec}_M \operatorname{Sym} R^1\pi_{M*}(\mathcal{E}_M^\vee \otimes \omega_{\mathcal{C}_M/M})$$

is an affine cone of finite type over M. Note that M is a smooth proper DM stack with projective coarse moduli. □

We introduced the construction $C(\pi_{M*}\mathcal{E}_M)$ here and later we will quote the construction of the obstruction theory in [CL].

3. The relative obstruction theory and cosections

In the next two sections, we fix an LG space $([\mathbb{C}^n/G], W)$ of weight (d, δ). We fix g and ℓ, and also a $\gamma = (\gamma_i)_{i=1}^\ell$, where each $\gamma_i : \mu_{r_i} \to G$ is injective.

Definition 3.1. *$\gamma_i : \mu_{r_i} \to G$ is said to be narrow (Neveu-Schwarz) if the composition of $\gamma_i : \mu_{r_i} \to G \subset \mathbb{G}_m^n$ with any projection to its factor $\mathbb{G}_m^n \to \mathbb{G}_m$ is non-trivial. $\gamma = (\gamma_i)_{i=1}^\ell$ is narrow if every γ_i is narrow.*

Like before, we abbreviate

$$M = \overline{M}_{g,\gamma}(G) \quad \text{and} \quad X = \overline{M}_{g,\gamma}(G)^p.$$

3.1. The perfect obstruction theory.

Let

$$\big(\Sigma_{X,i} \subset \mathcal{C}_X \xrightarrow{\pi_X} X,\, \mathcal{L}_{X,j},\, \varphi_k,\, \rho_{X,j}\big) \tag{3.1}$$

be the universal family of X. By [CL, Prop 2.5], $X = C(\pi_{M*}\mathcal{E}_M)$ relative to M has a tautological perfect obstruction theory, (letting $\mathcal{E}_X := \oplus \mathcal{L}_{X,j}$,)

$$\phi_{X/M} : (L^\bullet_{X/M})^\vee \longrightarrow E^\bullet_{X/M} := \mathbf{R}\pi_{X*}\mathcal{E}_X. \tag{3.2}$$

A consequence of this description of the perfect obstruction theory is a formula of its virtual dimension. Let $\xi = (\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k, \rho_j)$ be a closed point of X. Following the notation of Lemma 2.8, the virtual dimension of X/M follows from the Riemann-Roch Theorem [AGV, Thm.7.2.1]:

$$\dim H^0(E^\bullet_{X/M}|_\xi) - \dim H^1(E^\bullet_{X/M}|_\xi) = n(1-g) + \sum_j \deg \mathcal{L}_j - \sum_{i,j} \Theta^i_j.$$

Combined with $\dim M = 3g - 3 + \ell$, we obtain

$$\operatorname{vir.dim} X = (n-3)(1-g) + \ell + \sum_j \deg \mathcal{L}_j - \sum_{i,j} \Theta^i_j. \tag{3.3}$$

Note that $\deg \mathcal{L}_j = \delta_j(2g - 2 + \ell)/d$.

3.2. **Construction of a cosection.** Because γ is narrow, we have the following useful isomorphism.

Lemma 3.2. *Let S be a scheme, let $\pi : \mathcal{C} \to S$ be a flat family of twisted curves, and let $\Sigma \subset \mathcal{C}$ be a closed substack such that $\Sigma \to S$ is an étale gerbe banded by μ_r. Let $\mathcal{L}$ be a line bundle on $\mathcal{C}$ such that for every $x \in \Sigma$, the homomorphism $\mathrm{Aut}(x) \to \mathrm{Aut}(\mathcal{L}|_x)$ is given by $\zeta_r \mapsto \zeta_r^k$, $1 \le k < r$. Then for each integer $1 \le c \le k$, the homomorphism*

$$\mathbf{R}\pi_*\mathcal{L}(-c\Sigma) \longrightarrow \mathbf{R}\pi_*\mathcal{L}$$

induced by $\mathcal{L}(-c\Sigma) \to \mathcal{L}$ is a quasi-isomorphism.

Proof. Let $p : \mathcal{C} \to C$ be the coarse moduli of $\mathcal{C}$. For $1 \le c \le k$, we have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{L}(-c\Sigma) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_{c\Sigma} \longrightarrow 0.$$

Since for $f = f(z) \in \mathbb{C}[z]/(z^c)$ and $1 \le k < r$, the condition $f(\zeta z) = \zeta^k f(z)$ forces $f = 0$, the conditions $1 \le k < r$ and $1 \le c \le k$ imply $\mathbf{R}\pi_*(\mathcal{L}|_{c\Sigma}) = 0$. The desired quasi-isomorphism then follows from applying $\mathbf{R}\pi_*$ to the exact sequence. □

Let $\pi_M : \mathcal{C}_M \to M$, $\Sigma_{M,i} \subset \mathcal{C}_M$ and $\mathcal{L}_{M,j}$ be the universal family of M. Define $\Sigma_M = \coprod_i \Sigma_{M,i}$. Suppose γ is narrow from now on.

Proposition 3.3. *Suppose γ is narrow. Then the morphism*

$$X' := C(\pi_{M*}\mathcal{E}_M(-\Sigma_M)) \longrightarrow X = C(\pi_{M*}\mathcal{E}_M)$$

induced by the inclusion $\mathcal{E}_M(-\Sigma_M) \to \mathcal{E}_M$ is an isomorphism. The relative perfect obstruction theory

$$\phi_{X'/M} : (L^\bullet_{X'/M})^\vee \longrightarrow \mathbf{R}\pi_{X*}\mathcal{E}_X(-\Sigma_X)$$

of $X' \to M$ constructed in [CL, Prop 2.5] *coincides with (3.2) via the isomorphism $\mathbf{R}\pi_{X*}\mathcal{E}_X(-\Sigma_X) \cong \mathbf{R}\pi_{X*}\mathcal{E}_X$.*

Proof. It follows from the construction. □

Because of Proposition 3.3, in the following, we will not distinguish X from X'. Define the relative obstruction sheaf of X/M as

$$\mathcal{O}b_{X/M} = H^1(E^\bullet_{X/M}) = R^1\pi_{X*}\mathcal{E}_X(-\Sigma_X). \tag{3.4}$$

Now construct the desired cosection

$$\sigma : \mathcal{O}b_{X/M} \longrightarrow \mathcal{O}_X. \tag{3.5}$$

Let S be a connected affine scheme. Given a morphism $S \to X$, denote by $\Sigma_{S,i} \subset \mathcal{C} \to S$, $\mathcal{L}_{S,j}$ and $\rho_S = (\rho_{S,j}) \in \oplus_j \Gamma(\mathcal{L}_{S,j}(-\Sigma_S))$ the pullback of the universal family on X, and let $\Sigma_S = \sum_i \Sigma_{S,i}$.

For each monomial $W_a(x)$ of W, define $W_a(x)_j = \frac{\partial}{\partial x_j} W_a(x)$. Substituting x_j by $\rho_{S,j}$, (2.7) gives

$$W_a(\rho_S)_j := W_a(\rho_{S,1}, \cdots, \rho_{S,n})_j \in \Gamma(\mathcal{C}, \omega^{log}_{\mathcal{C}/S} \otimes \mathcal{L}^{-1}_{S,j}).$$

For $\dot\rho_{S,j} \in \Gamma\big(R^1\pi_{S*}\mathcal{L}_{S,j}(-\Sigma_S)\big)$, define

$$\sigma(\dot\rho_{S,1}, \cdots, \dot\rho_{S,n}) = \sum_{1 \le a \le m} \sum_{1 \le j \le n} \alpha_a W_a(\rho_S)_j \cdot \dot\rho_{S,j} \in H^1(\mathcal{C}, \omega_{\mathcal{C}/S}) \cong \Gamma(\mathcal{O}_S). \tag{3.6}$$

Here we used $\omega^{log}_{\mathcal{C}/S}(-\Sigma_S) \cong \omega_{\mathcal{C}/S}$ and Serre duality for orbifolds. Because of Lemma 3.2, $\mathcal{O}b_{X/M}|_S = \oplus_{j=1}^n R^1\pi_{S*}\mathcal{L}_{S,j}(-\Sigma_S)$. Thus the above construction gives us the desired cosection (homomorphism) (3.5).

Now we can define the absolute obstruction sheaf $\mathcal{O}b_X$ of X as follows. Because M is smooth, the projection $q : X \to M$ gives a distinguished triangle

$$q^*L^\bullet_M \longrightarrow L^\bullet_X \longrightarrow L^\bullet_{X/M} \xrightarrow{\delta} q^*L^\bullet_M[1], \tag{3.7}$$

where the last term is $[q^*\Omega_M \to 0]$ of amplitude $[-1, 0]$. Taking the dual of δ and composing it with the obstruction homomorphism $\phi_{X/M}$, we obtain the homomorphism $q^*\Omega^\vee_M \to \mathcal{O}b_{X/M}$. Define $\mathcal{O}b_X$ by the exact sequence

$$0 \longrightarrow q^*\Omega^\vee_M \longrightarrow \mathcal{O}b_{X/M} \longrightarrow \mathcal{O}b_X \longrightarrow 0.$$

Proposition 3.4. *Suppose γ is narrow. Then the homomorphism $\sigma : \mathcal{O}b_{X/M} \to \mathcal{O}_X$ in (3.5) factors through a homomorphism*

$$\bar\sigma : \mathcal{O}b_X \longrightarrow \mathcal{O}_X.$$

3.3. **The Proof of factorization.** In this subsection, we will give a proof of Proposition 3.4. The proof is similar to [CL, Prop 3.5]. We first provide an equivalent construction of the cosection using evaluation maps.

Let $\mathcal{E}_M = \oplus\mathcal{L}_{M,j}$, and let $Z_M = \mathrm{Vb}(\mathcal{E}_M(-\Sigma_M))$, which is the total space of the vector bundle $\mathcal{E}_M(-\Sigma_M)$. Since all γ_i's are narrow, by Proposition 3.3, we have $X = C(\pi_{M*}(\mathcal{E}_M(-\Sigma_M)))$. Using the universal section $\rho_X = (\rho_{X,1}, \cdots \rho_{X,n})$ of X, and defining $\mathcal{C}_X = \mathcal{C}_M \times_M X$ as the universal curve over X, we obtain the evaluation (evaluating ρ_X) M-morphism

$$\mathfrak{e} : \mathcal{C}_X \longrightarrow Z_M. \tag{3.8}$$

We form the total space of the vector bundles $\mathrm{Vb}(\omega_{\mathcal{C}_M/M})$ and $Z_i = \mathrm{Vb}(\mathcal{L}_{M,i}(-\Sigma_M))$. Then the isomorphism $\Phi_a : W_a(\mathcal{L}_{M,\cdot}) \to \omega^{log}_{\mathcal{C}_M/M}$ (cf. (2.7)) and the polynomial $W = \sum \alpha_a W_a$ define an M-morphism

$$h : Z_M = Z_1 \times_M \cdots \times_M Z_n \longrightarrow \mathrm{Vb}(\omega_{\mathcal{C}_M/M}), \tag{3.9}$$

where for $\xi \in \mathcal{C}_M$ and $z = (z_j)_{j=1}^n \in Z_M|_\xi$, $h(z) = \sum \alpha_a W_a(z) \in \mathrm{Vb}(\omega_{\mathcal{C}_M/M})|_\xi$. Here we have used the tautological inclusion $\omega^{log}_{\mathcal{C}_M/M}(-\Sigma_M) \to \omega_{\mathcal{C}_M/M}$.

The morphism h induces a homomorphism of cotangent complexes

$$dh : (L^\bullet_{Z_M/\mathcal{C}_M})^\vee \longrightarrow h^*(L^\bullet_{\mathrm{Vb}(\omega_{\mathcal{C}_M/M})/\mathcal{C}_M})^\vee = h^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_M/M})/\mathcal{C}_M},$$

where dh is the relative differentiation of h relative to $\mathcal{C}_M$. To be more explicit, for $z = (z_j) \in Z_M|_\xi$ over $\xi \in \mathcal{C}_M$,

$$dh|_z(\dot z) = \sum_{1\le a\le m}\sum_{1\le j\le n} \alpha_a W_a(z)_j \cdot \dot z_j, \tag{3.10}$$

for $\dot z = (\dot z_j)_{j=1}^n \in \Omega^\vee_{Z_M/\mathcal{C}_M}\big|_z = \oplus_{j=1}^n \mathcal{L}_{M,j}(-\Sigma_M)|_\xi$.

On the other hand, pulling dh back to $\mathcal{C}_X$ via the evaluation morphism $\mathfrak{e}$ gives

$$\mathfrak{e}^*(dh) : \mathfrak{e}^*\Omega^\vee_{Z_M/\mathcal{C}_M} \longrightarrow \mathfrak{e}^*h^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_M/M})/\mathcal{C}_M}.$$

Because the right-hand side is canonically isomorphic to $\omega_{\mathcal{C}_X/X}$, applying $\mathbf{R}\pi_{X*}$ gives

$$\sigma^\bullet : \mathbf{R}\pi_{X*}\mathfrak{e}^*\Omega^\vee_{Z_M/\mathcal{C}_M} \longrightarrow \mathbf{R}\pi_{X*}(\mathfrak{e}^*h^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_M/M})/\mathcal{C}_M}) \cong \mathbf{R}\pi_{X*}\omega_{\mathcal{C}_X/X}. \tag{3.11}$$

By Proposition 3.3, we obtain the canonical isomorphism

$$E^\bullet_{X/M} \cong \mathbf{R}\pi_{X*}\mathcal{E}_X(-\Sigma_X) = \mathbf{R}\pi_{X*}\mathfrak{e}^*\Omega^\vee_{Z_M/\mathcal{C}_M}.$$

Hence (3.11) gives

$$\sigma^\bullet : E^\bullet_{X/M} \longrightarrow \mathbf{R}\pi_{X*}\omega_{\mathcal{C}_X/X}.$$

It is easy to check that $H^1(\sigma^\bullet)$ coincides with the σ constructed in (3.5):

$$\sigma = H^1(\sigma^\bullet) : \mathcal{O}b_{X/M} = H^1(E^\bullet_{X/M}) \longrightarrow R^1\pi_{X*}(\omega_{\mathcal{C}_X/X}) \cong \mathcal{O}_X. \tag{3.12}$$

We now prove the factorization using this interpretation of σ. Let $B = C(\pi_{M*}\omega_{\mathcal{C}_M/M})$, which by definition is the total space of the bundle $\pi_{M*}\omega_{\mathcal{C}_M/M}$ over M. Let $\mathcal{C}_B = \mathcal{C}_M \times_M B$ be the pullback of the universal curve, and let $\pi_B : \mathcal{C}_B \to B$ be the projection. Then the universal section of B over $\mathcal{C}_B$ induces an evaluation morphism $\mathfrak{f} : \mathcal{C}_B \to \mathrm{Vb}(\omega_{\mathcal{C}_M/M})$ as in (3.8).

Lemma 3.5. *The following composition*

$$H^1(\sigma^\bullet) \circ H^1(\phi_{X/M}) = 0 : H^1((L^\bullet_{X/M})^\vee) \longrightarrow H^1(E^\bullet_{X/M}) \longrightarrow \mathcal{O}_X.$$

is the zero map.

Proof. By Lemma 3.2, the universal section $\rho_{X,j}$ lies in $\Gamma(\mathcal{C}_X, \mathcal{L}_{X,j}(-\Sigma_X))$. Using (2.7), we have

$$W_a(\rho_X) := W_a(\rho_{X,1}, \cdots, \rho_{X,n}) \in \Gamma(\mathcal{C}_X, \omega^{log}_{\mathcal{C}_X/X}(-\Sigma_X)) = \Gamma(\mathcal{C}_X, \omega_{\mathcal{C}_X/X}).$$

Define

$$W(\rho_X) = \sum \alpha_a W_a(\rho_X) \in \Gamma(\mathcal{C}_X, \omega_{\mathcal{C}_X/X}).$$

The section $W(\rho_X)$ defines a morphism $\mathfrak{g} : X \to B$ such that $W(\rho_X)$ is the pullback of the universal section of B over $\mathcal{C}_B$. Let $\tilde{\mathfrak{g}} : \mathcal{C}_X \to \mathcal{C}_B$ be the tautological lift of $\mathfrak{g}$ using $\mathcal{C}_X \cong \mathcal{C}_M \times_M X$ and $\mathcal{C}_B \cong \mathcal{C}_M \times_M B$, which fits into the following commutative square of morphisms of stacks over $\mathcal{C}_M$:

$$\begin{array}{ccc} \mathcal{C}_X & \xrightarrow{\mathfrak{e}} & Z_M \\ \downarrow{\scriptstyle \tilde{\mathfrak{g}}} & & \downarrow{\scriptstyle h} \\ \mathcal{C}_B & \xrightarrow{\mathfrak{f}} & \mathrm{Vb}(\omega_{\mathcal{C}_M/M}), \end{array} \tag{3.13}$$

which in turn gives the following commutative diagrams of cotangent complexes:

$$\begin{array}{ccccc} \pi_X^*(L^\bullet_{X/M})^\vee & = & (L^\bullet_{\mathcal{C}_X/\mathcal{C}_M})^\vee & \longrightarrow & \mathfrak{e}^*\Omega^\vee_{Z_M/\mathcal{C}_M} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle dh} \\ \pi_X^*\mathfrak{g}^*(L^\bullet_{B/M})^\vee & = & \tilde{\mathfrak{g}}^*(L^\bullet_{\mathcal{C}_B/\mathcal{C}_M})^\vee & \longrightarrow & \tilde{\mathfrak{g}}^*\mathfrak{f}^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_M/M})/\mathcal{C}_M}, \end{array} \tag{3.14}$$

where $\pi_X^*\mathfrak{g}^*(L^\bullet_{B/M})^\vee = \tilde{\mathfrak{g}}^*\pi_B^*(L^\bullet_{B/M})^\vee = \tilde{\mathfrak{g}}^*(L^\bullet_{\mathcal{C}_B/\mathcal{C}_M})^\vee$ follows from the fiber diagrams

$$\begin{array}{ccccc} \mathcal{C}_X & \xrightarrow{\tilde{\mathfrak{g}}} & \mathcal{C}_B & \longrightarrow & \mathcal{C}_M \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_B} & & \downarrow \\ X & \xrightarrow{\mathfrak{g}} & B & \longrightarrow & M. \end{array} \tag{3.15}$$

Let $\phi_{B/M} : (L^\bullet_{B/M})^\vee \to \mathbf{R}\pi_{B*}\omega_{\mathcal{C}_B/B}$ be the standard obstruction theory of $B \to M$ (cf. [BF, CL]). Then $\mathfrak{g}^*\phi_{B/M}$ is the obstruction theory of $B \times_M X \to X$. The smoothness of $B \to M$ implies

$$0 = H^1(\mathfrak{g}^*\phi_{B/M}) : H^1(\mathfrak{g}^*(L^\bullet_{B/M})^\vee) \longrightarrow \mathfrak{g}^* R^1\pi_{B*}\omega_{\mathcal{C}_B/B}. \tag{3.16}$$

Finally, applying $R^1\pi_{X*}$ to (3.14), we see that the composition

$$H^1((L^\bullet_{X/M})^\vee) \longrightarrow R^1\pi_{X*}\mathfrak{e}^*\Omega^\vee_{Z_M/\mathcal{C}_M} \longrightarrow R^1\pi_{X*}\mathfrak{e}^*h^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_M/M})/\mathcal{C}_M} \tag{3.17}$$

coincides with the composition

$$H^1((L^\bullet_{X/M})^\vee) \longrightarrow H^1(\mathfrak{g}^*(L^\bullet_{B/M})^\vee) \stackrel{0}{\longrightarrow} \mathfrak{g}^*R^1\pi_{B*}\mathfrak{f}^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_M/M})/\mathcal{C}_M}. \tag{3.18}$$

Using (3.16), the composition in (3.18) is zero, thus the composition in (3.17) is also zero. Because $\mathfrak{e}^*h^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_M/M})/\mathcal{C}_M} = \omega_{\mathcal{C}_X/X}$, we have proved the desired vanishing. $\square$

Proof of Proposition 3.4. By (3.12), $\sigma = H^1(\sigma^\bullet)$. Hence the composition of σ with $q^*\Omega^\vee_M \to \mathcal{O}b_{X/M}$ (defined below (3.7)) is the H^1 of the composition

$$q^*(L^\bullet_M)^\vee[-1] \longrightarrow (L^\bullet_{X/M})^\vee \stackrel{\phi_{X/M}}{\longrightarrow} E^\bullet_{X/M} \stackrel{\sigma^\bullet}{\longrightarrow} \mathbf{R}\pi_{X*}\omega_{\mathcal{C}_X/X},$$

where the first arrow is the map $\delta^\vee$ in (3.7). Lemma 3.5 implies that the H^1 of the above composition is zero. $\square$

3.4. **Degeneracy locus of the cosection.** In this subsection, we investigate the locus of non-surjectivity of the cosection σ. Let $\xi = (\mathcal{C}, \Sigma^{\mathcal{C}}_i, \mathcal{L}_j, \varphi_k, \rho_j)$ be a closed point in $X = \overline{M}_{g,\gamma}(G)^p$.

Lemma 3.6. *Let the notation be as stated. Then $\sigma|_\xi = 0$ if and only if all $\rho_j = 0$. Thus the degeneracy locus of σ is $M \subset X$ with the reduced scheme structure, which is proper.*

Proof. $\sigma|_\xi = 0$ implies that for arbitrary $\dot\rho_1, \cdots, \dot\rho_n \in H^1(C, L_j)$, the term in (3.6)

$$\sigma(\dot\rho_1, \cdots, \dot\rho_n) = \sum_{1\le a\le m}\sum_{1\le j\le n} \alpha_a W_a(\rho)_j \cdot \dot\rho_j \in \Gamma(C, \omega_C) \cong \mathbb{C}$$

vanishes. By Serre duality, this forces $\sum_{1\le a\le m}\alpha_a W_a(\rho)_j = 0$ for every j. Since $0 \in \mathbb{C}^n$ is the only critical point of W, this forces $(\rho_1, \cdots, \rho_n) = 0$. $\square$

3.5. **Localized virtual cycle.** We recall the notion of the kernel stack of a cosection. Let $E = [E^0 \to E^1]$ be a two-term complex of locally free sheaves on a Deligne-Mumford stack X, and let $f : H^1(E) \to \mathcal{O}_X$ be a cosection of $H^1(E)$. Define $D(f)$ as the subset of $x \in X$ such that $f|_x = 0 : H^1(E)|_x \to \mathbb{C}$. The set $D(f)$ is closed. Let $U = X - D(f)$.

Definition 3.7. *Let the notations be as stated. Define the kernel stack as*

$$h^1/h^0(E)_f := \big(h^1/h^0(E) \times_X D(f)\big) \cup \ker\{h^1/h^0(E)|_U \to H^1(E)|_U \to \mathbb{C}_U\}.$$

Here $h^1/h^0(E)|_U \to H^1(E)|_U$ is the tautological projection and the map $H^1(E)|_U \to \mathbb{C}_U$ is $f|_U$. Since f is surjective over U, the composition in the bracket is surjective. Thus the kernel is a bundle stack over U. Clearly, the union is closed in $h^1/h^0(E)$. We endow it with the reduced structure, making it a closed substack of $h^1/h^0(E)$, denoted by $h^1/h^0(E)_f$,. We call it the kernel stack of f

We apply the theory developed in [KL] to X/M for $X = \overline{M}_{g,\gamma}(G)^p$ and $M = \overline{M}_{g,\gamma}(G)$. As σ is a cosection of $H^1(E^\bullet_{X/M})$, we form its kernel stack

$$h^1/h^0(E^\bullet_{X/M})_\sigma \subset h^1/h^0(E^\bullet_{X/M}). \tag{3.19}$$

Proposition 3.8. *The virtual normal cone cycle*

$$[\mathbf{C}_{X/M}] \in Z_*(h^1/h^0(E^\bullet_{X/M}))$$

lies inside $Z_*(h^1/h^0(E^\bullet_{X/M})_\sigma)$.

Proof. The smoothness of the morphism from $\mathbf{C}_{X/M}$ to $\mathbf{C}_X$(the intrinsic normal cone of X) and Proposition 3.4 reduce the Proposition 3.8 to the absolute case, which is proved in [KL, Prop. 4.3]. □

Following [KL], we form the localized Gysin map

$$0^!_{\sigma,\mathrm{loc}} : A_*(h^1/h^0(E^\bullet_{X/M})_\sigma) \longrightarrow A_{*-\mathrm{v}}(D(\sigma)),$$

where $\mathrm{v} = \mathrm{vir.}\dim X - \dim M$ and $\mathrm{vir.}\dim X$ is given in (3.3). Recall $D(\sigma) = M \subset X$ (cf. Lemma 3.6).

Definition-Proposition 3.9. *Let* $([\mathbb{C}^n/G], W)$ *be an LG space. For* $g \geq 0$ *and a* g*-admissible* γ*, define Witten's top Chern class of* $\overline{M}_{g,\gamma}(G)$ *as*

$$[\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}}_\sigma := 0^!_{\sigma,\mathrm{loc}}([\mathbf{C}_{\overline{M}_{g,\gamma}(G)^p/\overline{M}_{g,\gamma}(G)}]) \in A_*(\overline{M}_{g,\gamma}(G)),$$

for the cosection σ *constructed using* W*. It depends only on* (d,δ)*, not on the choice of* W*.*

Proof. We only need to prove the independence of W. Suppose $([\mathbb{C}^n/G], W_0)$ and $([\mathbb{C}^n/G], W_1)$ are two LG spaces of the same weight (d,δ). Let $W_t = tW_1 + (1-t)W_0$, $t \in \mathbb{A}^1$. Then there is a Zariski open $U \subset \mathbb{A}^1$ containing $0, 1$ such that W_t is non-degenerate for $t \in U$. Then every W_t, $t \in U$, induces a cosection σ_t of $\mathcal{O}b_{\overline{M}_{g,\gamma}(G)^p}$. Indeed, if σ_0 and σ_1 are the cosections constructed using W_0 and W_1, then $\sigma_t = t\sigma_1 + (1-t)\sigma_0$.

For $t \in U$, since W_t is non-degenerate, the degeneracy locus of σ_t is $M = \overline{M}_{g,\gamma}(G)$. The family $\sigma_t = t\sigma_1 + (1-t)\sigma_0$ forms a family of cosections of the family of moduli spaces $U \times \overline{M}_{g,\gamma}(G)^p$. As this family is a constant family, [KL, Thm 5.2] applies and hence $[\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}}_{\sigma_t} \in A_*(\overline{M}_{g,\gamma}(G))$ is independent of t. □

Because of the independence of the choice of W, the class $[\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}}_\sigma$ only depends on $G \leq \mathbb{G}_{\mathrm{m}}^n$ and the weight (d,δ). In the following, we will drop the subscript σ, and denote Witten's top Chern class of $([\mathbb{C}^n/G], W)$ as

$$[\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}} \in A_*(\overline{M}_{g,\gamma}(G)).$$

4. Witten's top Chern class of strata

In this section, we fix an LG space $([\mathbb{C}^n/G], W)$. We also fix g, ℓ, and a g-admissible $\gamma = (\gamma_i)_{i=1}^\ell$. Let $M = \overline{M}_{g,\gamma}(G)$ be the moduli of G-spin curves, and let $X = \overline{M}_{g,\gamma}(G)^p$ be the moduli of G-spin curves with fields. As before, denote by $(\pi_M : \mathcal{C}_M \to M, \mathcal{L}_{M,j})$ (part of) the universal family of M, and define $\mathcal{E}_M = \oplus_{j=1}^n \mathcal{L}_{M,j}$.

4.1. **Virtual cycles and Gysin maps.** We first recall a general fact about the cosection localized virtual cycles and Gysin maps.

Let $\mathcal{M}$ be a smooth DM stack, let $\mathcal{X}$ be a DM stack, and let $\mathcal{X} \to \mathcal{M}$ be a representable morphism. Assume $\mathcal{X} \to \mathcal{M}$ has a relative perfect obstruction theory $\phi_{\mathcal{X}/\mathcal{M}} : (L^\bullet_{\mathcal{X}/\mathcal{M}})^\vee \to F^\bullet$. Define its relative obstruction sheaf as $\mathcal{O}b_{\mathcal{X}/\mathcal{M}} := H^1(F^\bullet)$. Let $\sigma : \mathcal{O}b_{\mathcal{X}/\mathcal{M}} \to \mathcal{O}_{\mathcal{X}}$ be a cosection such that its composition with $\Omega^\vee_{\mathcal{M}} \to \mathcal{O}b_{\mathcal{X}/\mathcal{M}}$ is zero. Then by [KL, Thm 5.1], denoting by $\mathbf{C}$ ($= C_{\mathcal{X}/\mathcal{M}}$) the normal cone of $\mathcal{X}/\mathcal{M}$, and by $0^!_{\sigma,loc}$ the localized Gysin map defined in [KL], the σ-localized virtual cycle of $\mathcal{X}$ is

$$[\mathcal{X}]^{\rm vir} := 0^!_{\sigma,loc}[\mathbf{C}] \in A_*(D(\sigma)).$$

Let $\iota : \mathcal{S} \to \mathcal{M}$ be a proper representable l.c.i. (or flat) morphism between DM stacks of constant codimension. We form the Cartesian square

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{g} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow{\iota} & \mathcal{M}\,. \end{array}$$

Since $\mathcal{X} \to \mathcal{M}$ is representable, $\mathcal{Y}$ is a DM stack. The obstruction theory of $\mathcal{X} \to \mathcal{M}$ induces a perfect relative obstruction theory of $\mathcal{Y} \to \mathcal{S}$ by pullback [BF, Prop 7.2][1]:

$$\phi_{\mathcal{Y}/\mathcal{S}} : (L^\bullet_{\mathcal{Y}/\mathcal{S}})^\vee \longrightarrow E^\bullet := g^* F^\bullet.$$

The cosection σ pulls back to a cosection $\sigma' : \mathcal{O}b_{\mathcal{Y}/\mathcal{S}} \to \mathcal{O}_{\mathcal{Y}}$ whose degeneracy locus $D(\sigma') = D(\sigma) \times_{\mathcal{M}} \mathcal{S}$. Since ι is proper, $D(\sigma')$ is proper if $D(\sigma)$ is proper. Furthermore, the composition of $\Omega^\vee_{\mathcal{S}} \to \mathcal{O}b_{\mathcal{Y}/\mathcal{S}}$ with $\sigma' : \mathcal{O}b_{\mathcal{Y}/\mathcal{S}} \to \mathcal{O}_{\mathcal{M}}$ vanishes because of the vanishing assumption on the similar composition on $\mathcal{X}$. Applying [KL, Thm 5.1] and using the virtual normal cone $\mathbf{C}' = C_{\mathcal{Y}/\mathcal{S}}$ of $\mathcal{Y} \to \mathcal{S}$, we obtain the σ'-localized virtual cycle

$$[\mathcal{Y}]^{\rm vir} := 0^!_{\sigma',loc}[\mathbf{C}'] \in A_*(D(\sigma')). \tag{4.1}$$

Lemma 4.1. *There is an equality* $\iota^![\mathcal{X}]^{\rm vir} = [\mathcal{Y}]^{\rm vir}$

Proof. If ι is flat, $\iota^*\mathbf{C}_{X/\mathcal{M}} = \mathbf{C}_{\mathcal{Y}/\mathcal{M}}$ and the equality follows from the functorial property of the cosection localization. We now consider the case where ι is an l.c.i. morphism. Denote the normal sheaf $N = N_{\mathcal{S}/\mathcal{M}}$. It is a bundle stack over $\mathcal{S}$ because ι is an l.c.i morphism. A rational equivalence $R \in W_*(\mathbf{C} \times_{\mathcal{M}} N)$ was constructed in [Kr, Vi] such that $\partial R = [\mathbf{C}_{g^*\mathbf{C}/\mathbf{C}}] - [\mathbf{C}' \times_{\mathcal{S}} N]$. Let

$$\tilde\sigma : h^1/h^0(g^*F^\bullet) \times_{\mathcal{M}} N \to \mathcal{O}_{\mathcal{M}}$$

be the lift of the pair of (the induced) $\sigma_{\mathcal{S}} : h^1/h^0(g^*F^\bullet) \to \mathcal{O}_{\mathcal{M}}$ and $0 : N \to \mathcal{O}_{\mathcal{M}}$. Then the degeneracy locus of $\tilde\sigma$ is identical to the degeneracy locus of σ'. The standard property of localized Gysin maps states that

$$\iota^![\mathcal{X}]^{\rm vir} = \iota^! 0^!_{\sigma,loc}[\mathbf{C}] = 0^!_{\tilde\sigma,loc}[C_{g^*\mathbf{C}/\mathbf{C}}]. \tag{4.2}$$

Since σ annihilates the reduced part of $\mathbf{C}$, the reduced part of the stack $\mathbf{C} \times_{\mathcal{M}} N$ is also annihilated by $\tilde\sigma$ (i.e. lies in the kernel stack of $\tilde\sigma$). Thus R lies in the kernel stack of $\tilde\sigma$, and we have $0^!_{\tilde\sigma,loc}[\mathbf{C}_{g^*\mathbf{C}/\mathbf{C}}] = 0^!_{\tilde\sigma,loc}[\mathbf{C}' \times_{\mathcal{S}} N] = 0^!_{\sigma',loc}[\mathbf{C}']$, which is $[\mathcal{Y}]^{\rm vir}$. □

[1] The assumption that $\iota : \mathcal{S} \to \mathcal{M}$ is l.c.i. is sufficient for [BF, Prop 7.2] to be valid.

We will apply Lemma 4.1 to the pair $X = \overline{M}_{g,\gamma}(G)^p \to M = \overline{M}_{g,\gamma}(G)$ and some l.c.i. morphism $S \to M$ whose image is the closed substack defined by the topological (or geometrical) type of the G-spin curves in M.

4.2. **Topological strata of the moduli of G-spin curves.** In this subsection, we will associate a pointed G-spin curve with its decorated dual graph. For such a dual graph, we will construct its associated stratum in the moduli space. We will rephrase the constructions in [FJR2, Sect 2.2.2 and 2.2.3].

We choose the dual graph associated with a curve, where edges correspond to nodes, instead of the graph associated with a curve used in some papers, where edges correspond to components of the curve, for the following reasons. One is that the construction in [FJR2, Sect 2.2.2 and 2.2.3] uses the dual graph. The other is that it is easier to work with the dual graph when introducing the directions of the edges and to perform certain operations on the dual graph.

Given a pointed G-spin curve $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k)$ (not necessarily connected), following the standard procedure we associate with it a decorated (dual) graph Γ as follows[2]. Its vertices $v \in V(\Gamma)$ correspond to irreducible components $\mathcal{C}_v \subset \mathcal{C}$; its edges $e \in E(\Gamma)$ correspond to nodes $q_e \in \mathcal{C}$ so that a vertex v of e has $q_e \in \mathcal{C}_v$; its tails $t_i \in T(\Gamma)$ correspond to the markings $\Sigma_i^{\mathcal{C}}$ so that the vertex v of t_i has $\Sigma_i^{\mathcal{C}} \subset \mathcal{C}_v$; we give $e \in E(\Gamma)$ a direction that is consistent with the labeling of its two vertices v_e^- and v_e^+ (i.e. from $-$ to $+$).

We add decorations to Γ as follows. For a (directed) edge e, in case $v_e^- \neq v_e^+$, there is a unique labeling of the two branches of (the formal completion of $\mathcal{C}$ along q) $\hat{\mathcal{C}}_q$ as $\hat{\mathcal{C}}_{e-}$ and $\hat{\mathcal{C}}_{e+}$ so that $\hat{\mathcal{C}}_{e+}$ is the formal completion of $\mathcal{C}_{v_e^+}$ along q; in the case where $v_e^- = v_e^+$, we fix a labeling of the two branches of $\hat{\mathcal{C}}_q$ as $\hat{\mathcal{C}}_{e-}$ and $\hat{\mathcal{C}}_{e+}$. Under this agreement, we decorate $e \in E(\Gamma)$ with the monodromy representation $\gamma_e := \gamma_{q_e+} : \mu_{r_{q_e}} \to G$. We decorate the tail t_i with γ_i, the monodromy representation along $\Sigma_i^{\mathcal{C}}$. We decorate every vertex v with $g_v = g(\mathcal{C}_v)$, the arithmetic genus of $\mathcal{C}_v$.

We also define the coarse dual graph of $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_i, \varphi_j)$ as a no-edge graph so that its vertices $v \in V(\Gamma)$ correspond to connected components $\mathcal{C}_v \subset \mathcal{C}$, decorated with $g_v = g(\mathcal{C}_v)$, and that its tails and corresponding decorations are as before. Given a decorated graph Γ, by reversing the direction of one edge $e \in E(\Gamma)$ and replacing its decoration γ_e by γ_e^{-1}, we obtain a new decorated graph. We call this "direction reversing" operation. We say two decorated graphs are equivalent if one is derived from the other by repeated "direction reversing" operations.

In general, a G-decorated graph is the dual graph of a G-spin curve that is not necessarily connected.

Definition 4.2. *A pointed G-spin curve $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k)$ is said to be labeled by a G-decorated graph Γ if there are bijections between the sets of irreducible components (resp. nodes; resp. markings) of $(\mathcal{C}, \Sigma_i^{\mathcal{C}})$ with $V(\gamma)$ (resp. $E(\Gamma)$; resp. $T(\Gamma)$) that make Γ a decorated dual graph of $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k)$.*

In case Γ has no edges, we say the family is coarsely labeled by the Γ if we replace "irreducible components" by "connected components".

Given a G-decorated graph Γ, define its complete splitting as the graph resulting from breaking every (directed) edge $e \in E(\Gamma)$ into a pair of tails, denoted by t_{e-} and

[2]Our convention is that every edge is directed and has two vertices, and every tail has only one vertex.

t_{e+} attached to the vertices v_e^- and v_e^+ with the labelings γ_e^{-1} and γ_e respectively. Denote the complete splitting of Γ by Γ_{sp}.

Definition 4.3. *Let Γ be a G-decorated graph and Γ_{sp} its complete splitting. For a scheme S, an S-family $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k)$ of pointed G-spin curves is said to be weakly labeled by Γ if the followings hold:*

1. *there is a family of G-spin curves $(\mathcal{C}', \Sigma_i^{\mathcal{C}'}, \mathcal{L}_j', \varphi_k')$ coarsely labeled by Γ_{sp};*
2. *there is a morphism $\mathcal{C}' \to \mathcal{C}$ so that $\mathcal{C}$ is the result of identifying all pairs of marked sections of $(\mathcal{C}', \Sigma_i^{\mathcal{C}'})$ associated with t_{e-} and t_{e+} for all $e \in \Gamma$;*
3. *under $\mathcal{C}' \to \mathcal{C}$, $(\mathcal{L}_j', \varphi_k')$ is the pullback of $(\mathcal{L}_j, \varphi_k)$, and $\Sigma_i^{\mathcal{C}}$ is identified with $\Sigma_i^{\mathcal{C}'}$ for each t_i , the i-th tail of Γ.*

We review the associated operation on the dual graph by smoothing nodes of G-spin curves. Let Γ be a G-decorated dual graph and $e \in E(\Gamma)$. Define Γ/e as the graph resulting from eliminating the edge e from Γ, merging the vertices v_e^- with v_e^+ into a single vertex, denoted by $\bar v$, and decorating it with $g_{\bar v} = g_{v_e^-} + g_{v_e^+}$ when $v_e^- \neq v_e^+$, and with $g_{\bar v} = g_{v_e^-} + 1$ when $v_e^- = v_e^+$. We call Γ/e the contraction of Γ by e. This process also defines a graph map $\Gamma \to \Gamma/e$ that is the identity except that it sends e, $v_e^{\pm}$ to $\bar v \in V(\Gamma/e)$.

We call Γ' an edge-contraction of Γ if Γ' is obtained from Γ by repeatedly contracting edges. Let $\Gamma \to \Gamma'$ be the compositions of the contraction maps. Define $\mathrm{Aut}(\Gamma/\Gamma')$ as the group of automorphisms of Γ that commute with the identity of Γ' via the map $\Gamma' \to \Gamma$, up to the equivalence (defined by the edge-direction reversing operation).

We form the category of families of weakly Γ-labeled G-spin curves. It is a DM stack, denoted by $\overline{M}_\Gamma(G)$. In the case where Γ' is an edge contraction of Γ, we have a tautological morphism

$$\tau_{\Gamma\Gamma'} : \overline{M}_\Gamma(G) \longrightarrow \overline{M}_{\Gamma'}(G).$$

In the case where Γ is connected and Γ' consists of a single vertex decorated with g, this morphism takes the form $\tau_\Gamma : \overline{M}_\Gamma(G) \longrightarrow \overline{M}_{g,\gamma}(G)$, where $\gamma = (\gamma_i)$ is the collection of decorations of tails of Γ consistent with the ordering of tails.

Lemma 4.4. *Suppose Γ' is an edge contraction of Γ. Then the morphism $\tau_{\Gamma\Gamma'}$ is a finite l.c.i. morphism. Define $\overline{M}_{\Gamma'}(G)_\Gamma$ as the image stack of $\tau_{\Gamma\Gamma'}$. Then the factored map $\overline{M}_\Gamma(G) \to \overline{M}_{\Gamma'}(G)_\Gamma$ has pure degree $|\,\mathrm{Aut}(\Gamma/\Gamma')|$.*

Proof. By induction, we only need to prove the case when Γ' is obtained by contracting one edge of Γ. Let $\mathfrak{M}_{g,\ell}^{\mathrm{tw},\circ}$ be the moduli stack of genus g stable curves with ℓ-twisted markings and no twistings at nodal points. The structure of the forgetful morphism (eliminating the stacky structure along nodes) $\mathfrak{M}_{g,\ell}^{\mathrm{tw}} \to \mathfrak{M}_{g,\ell}^{\mathrm{tw},\circ}$ was studied in details by Olsson [Ol] (see [Ch1] §2.4 as well). Similar to the situation studied by Chiodo and Ruan in [CR], $\overline{M}_{g,\gamma}(G)$ is étale over $\mathfrak{M}_{g,\ell}^{\mathrm{tw}}$. Therefore we can and will use Olsson's result in [Ol] in the following.

Let $\xi = (\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k) \in \overline{M}_{g,\gamma}(G)(\mathbb{C})$. Suppose $\mathcal{C}$ has nodal points $p_1, \ldots, p_m$ with stabilizer groups μ_{r_i} at p_i. By [Ol, Remark 1.10] (see [Ch1] §2.4 and Remark 2.4.10 as well), we can find an affine étale neighborhood $S \to \mathfrak{M}_{g,\ell}^{\mathrm{tw},\circ}$ such that an open neighborhood of the product

$$\bar\xi \in U \subset \overline{M}_{g,\gamma}(G) \times_{\mathfrak{M}_{g,\ell}^{\mathrm{tw},\circ}} S,$$

where $\bar\xi$ lies over ξ, is

$$U=\bigl[\,\mathrm{Spec}\,A/\mu_{r_1}\times\ldots\times\mu_{r_m}\bigr]\text{ with } A=\mathcal{O}_S[x_1,\ldots,x_m]/(x_i^{r_i}=u_i, i=1,\cdots,m),$$

where $u_i\in\mathcal{O}_S$ defines the divisor of curves in S of which the node p_i in the coarse moduli of $\mathcal{C}$ remains nodal. μ_{r_i} acts on A via $x_j^{\zeta_{r_i}}=x_j$ when $j\neq i$ and $=\zeta_{r_i}x_i$ when $j=i$. By [Ol, Remark 1.10], $\{x_i=0\}$ is the divisor where the node p_i remains nodal.

Assume $\xi\in\overline{M}_\Gamma(G)$. Since $\Gamma\to\Gamma'$ is a contraction of one edge, by re-indexing, we can assume that it contracts the edge associated with the node p_m. By shrinking S if necessary, we have isomorphisms

$$\bar\xi\in U\cap\bigl(\overline{M}_{\Gamma'}(G)\times_{\mathfrak{M}^{\mathrm{tw},\circ}_{g,\ell}}S\bigr)=\bigl[\,\mathrm{Spec}\bigl(A/(x_1,\ldots,x_{m-1})\bigr)/\mu_{r_1}\times\ldots\times\mu_{r_{m-1}}\bigr],$$

and

$$\bar\xi\in U\cap\bigl(\overline{M}_{\Gamma}(G)\times_{\mathfrak{M}^{\mathrm{tw},\circ}_{g,\ell}}S\bigr)=\bigl[\,\mathrm{Spec}\bigl(A/(x_1,\ldots,x_{m})\bigr)/\mu_{r_1}\times\ldots\times\mu_{r_{m}}\bigr].$$

Therefore the morphism $\tau_{\Gamma\Gamma'}$, restricted to $\overline{M}_\Gamma(G)\times_{\mathfrak{M}^{\mathrm{tw},\circ}_{g,\ell}}S$, is étale covered by the morphism

$$\mathrm{Spec}\bigl(A/(x_1,\ldots,x_m)\bigr)\longrightarrow\mathrm{Spec}\bigl(A/(x_1,\ldots,x_{m-1})\bigr),$$

which is an immersion of a Cartier divisor. This proves that $\tau_{\Gamma\Gamma'}$ is a finite l.c.i. morphism.

Finally, that the degree of $\tau_{\Gamma\Gamma'}$ is $|\operatorname{Aut}(\Gamma/\Gamma')|$ can be verified over the generic point of $\overline{M}_{\Gamma'}(G)_\Gamma$. We omit the proof. □

4.3. **The forgetful morphism.** In the GW theory, by removing the last marked point and stabilizing, we obtain the forgetful morphism $\overline{M}_{g,\ell}\to\overline{M}_{g,\ell-1}$. Similar morphism exists for $\overline{M}_{g,\gamma}(G)$ if γ_ℓ in γ takes the form (cf. Subsection 2.1)

$$\gamma_\ell:\mu_d\longrightarrow G,\quad \zeta_d\mapsto \jmath_\delta. \tag{4.3}$$

In general, let Γ be a G-decorated graph whose ℓ-tails are decorated with $\gamma=(\gamma_i)_{i=1}^{\ell}$. Let Γ' be the stabilization of the graph after removing (the tail) t_ℓ from Γ. Thus the tails of Γ' are decorated with $\gamma'=(\gamma_i)_{i=1}^{\ell-1}$.

Theorem 4.5 (Forgetting Tails)**.** *Let Γ and Γ' be as stated. Suppose γ_ℓ takes the form* (4.3)*. Then there is a flat forgetful morphism*

$$\mathfrak{f}_{\Gamma,\ell}:\overline{M}_\Gamma(G)\longrightarrow\overline{M}_{\Gamma'}(G)$$

that sends $(\mathcal{C},\Sigma_i^{\mathcal{C}},\mathcal{L}_j,\varphi_k)$ to $(\mathcal{C}',\Sigma^{\mathcal{C}}_{j<\ell},\mathcal{L}'_j,\varphi'_k)$, where $\mathcal{C}'$ is obtained from $\mathcal{C}$ by removing the marked section $\Sigma^{\mathcal{C}}_\ell$, eliminating the stacky structure along $\Sigma^{\mathcal{C}}_\ell$, and stabilizing $(\mathcal{C}',\Sigma^{\mathcal{C}}_{j<\ell})$. $\mathcal{L}'_j$ is the push-forward of $\mathcal{L}_j$ under the tautological map $\mathcal{C}\to\mathcal{C}'$.

Let $\tilde\gamma=(\gamma_1,\cdots,\gamma_{\ell-1},1)$, where 1 is the trivial representation $\{1\}\to G$. Before proving Theorem 4.5, we construct an auxiliary stack $\overline{M}_{g,\tilde\gamma[\ell]}(G)$ that is the groupoid of all families $(\mathcal{C},\Sigma_i^{\mathcal{C}},\mathcal{L}_j,\varphi_k)$ obeying all requirements in Definitions 2.6 and 2.7 except that γ is replaced by $\tilde\gamma$ and (2.3) is replaced by

$$\phi_k\colon \mathbf{m}_k(\mathcal{L}_1,\ldots,\mathcal{L}_n)\xrightarrow{\cong}(\omega^{log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_\ell))^{w(\mathbf{m}_k)},\quad k=1,\cdots,n. \tag{4.4}$$

(Note that $\tilde\gamma_\ell$ is trivial implies that $\Sigma^{\mathcal{C}}_\ell\subset\mathcal{C}$ is non-stacky.)

Lemma 4.6. *Let the notation be as stated. Then $\overline{M}_{g,\tilde\gamma[\ell]}(G)$ is a smooth proper DM stack over $\mathbb{C}$.*

Proof. Following [FJR2, Thm 2.2.6] and [AJ, Sect 1.5], we will describe it as a moduli stack of twisted stable morphisms. Recall that a line bundle $\mathcal{L}$ on a twisted curve $\mathcal{C}$ induces a morphism $[\mathcal{L}]:\mathcal{C}\to B\mathbb{G}_{\mathrm{m}}$. Thus given an S-family $\xi=(\mathcal{C},\Sigma_i^{\mathcal{C}},\mathcal{L}_j,\phi_k)\in\overline{M}_{g,\tilde\gamma[\ell]}(G)(S)$, the n-tuple $(\mathcal{L}_j)_{j=1}^n$ defines a morphism $\tau(\xi)$ as shown in the following diagram. Let (C,Σ_i^C) be the coarse moduli of $(\mathcal{C},\Sigma_i^{\mathcal{C}})$. Abbreviating $\omega_{S,\ell}=\omega_{C/S}^{log}(-\Sigma_{S,\ell}^C)$ and $w_k=w(\mathbf{m}_k)$, the n-tuple of line bundles $(\omega_{S,\ell}^{w_k})_{k=1}^n$ induces a morphism $B(\omega_{S,\ell}^{w_\cdot})$ as shown in the diagram below. Furthermore, let $\mathbf{m}:\mathbb{G}_{\mathrm{m}}^n\to\mathbb{G}_{\mathrm{m}}^n$ be the homomorphism defined via $t=(t_1,\cdots,t_n)\mapsto(\mathbf{m}_1(t),\cdots,\mathbf{m}_n(t))$, and $B\mathbf{m}:B\mathbb{G}_{\mathrm{m}}^n\to B\mathbb{G}_{\mathrm{m}}^n$ be its induced morphism. Then the isomorphisms $\phi_1,\cdots,\phi_n$ in (4.4) fit into the (left-hand side) commutative square:

$$\begin{array}{ccccc}
\mathcal{C} & \longrightarrow & C & \xrightarrow{\ \rho\ } & C_{g,\ell-1}\\
\big\downarrow{\scriptstyle \tau(\xi)} & & \big\downarrow{\scriptstyle B(\omega_{S,\ell}^{w_\cdot})} & & \big\downarrow{\scriptstyle B(\omega_{g,\ell-1}^{w_\cdot})}\\
B\mathbb{G}_{\mathrm{m}}^n & \xrightarrow{\ B\mathbf{m}\ } & B\mathbb{G}_{\mathrm{m}}^n & = & B\mathbb{G}_{\mathrm{m}}^n
\end{array}$$

The square on the right is constructed as follows. Let $(C,\Sigma_{i<\ell}^C)\to(C,\Sigma_{i<\ell}^C)^{st}$ be the stabilization of the family of $(\ell-1)$-pointed curves $(C,\Sigma_{i<\ell}^C)$, and let $(C,\Sigma_{i<\ell}^C)^{st}\to(C_{g,\ell-1},\Sigma_i^{C_{g,\ell-1}})$ be the tautological morphism to the universal family of the moduli of $(\ell-1)$-pointed genus g stable curves. Let $\rho:C\to C_{g,\ell-1}$ be the composition of the stabilization and the tautological morphisms mentioned. Let $\omega_{g,\ell-1}=\omega_{C_{g,\ell-1}/\overline{M}_{g,\ell-1}}^{log}$. Because $\rho^*\omega_{g,\ell-1}=\omega_{S,\ell}$, the morphism $B(\omega_{g,\ell-1}^{w_\cdot})$ induced by the n-tuple of line bundles $(\omega_{g,\ell-1}^{w_k})_{k=1}^n$ makes the right hand side square commutative.

In conclusion, the two squares define a morphism

$$F(\xi):\mathcal{C}\longrightarrow\mathcal{C}_{g,\ell-1,\mathbf{m}}:=C_{g,\ell-1}\times_{B\mathbb{G}_{\mathrm{m}}^n}B\mathbb{G}_{\mathrm{m}}^n \tag{4.5}$$

such that

(1) $F(\xi)$ is an S-family of balanced ℓ-pointed genus g twisted stable morphisms with the fiber class of $C_{g,\ell-1}/\overline{M}_{g,\ell-1}$ as the fundamental class;

(2) $F(\xi)(\Sigma_i^{\mathcal{C}})$ lies in the sector $\Sigma_i^{\mathcal{C}_{g,\ell-1}}:=\Sigma_i^{C_{g,\ell-1}}\times_{B\mathbb{G}_{\mathrm{m}}^n}B\mathbb{G}_{\mathrm{m}}^n$ given by the representation γ_i for $1\le i\le\ell-1$;

(3) the last marked section $\Sigma_\ell^{\mathcal{C}}\subset\mathcal{C}$ is non-stacky.

We let $\mathcal{K}_{g,\tilde\gamma}^{bal}(\mathcal{C}_{g,\ell-1,\mathbf{m}},F)$ be the groupoid of balanced ℓ-pointed genus g twisted stable morphisms to $\mathcal{C}_{g,\ell-1,\mathbf{m}}$ of fundamental class F satisfying conditions (2) and (3) above. Following the study in [FJR2] and [AJ], the groupoid $\mathcal{K}_{g,\tilde\gamma}^{bal}(\mathcal{C}_{g,\ell-1,\mathbf{m}},F)$ forms a smooth proper DM stack over $\mathbb{C}$. The above correspondence gives a transformation of groupoids

$$\overline{M}_{g,\tilde\gamma[\ell]}(G)\longrightarrow\mathcal{K}_{g,\tilde\gamma}^{bal}(\mathcal{C}_{g,\ell-1,\mathbf{m}},F). \tag{4.6}$$

Conversely, following [FJR2], an S-family in $\mathcal{K}_{g,\tilde\gamma}^{bal}(\mathcal{C}_{g,\ell-1,\mathbf{m}},F)$ produces an S-family ξ in $\overline{M}_{g,\tilde\gamma[\ell]}(G)$ whose induced $F(\xi)$ is the given S-family in $\mathcal{K}_{g,\tilde\gamma}^{bal}(\mathcal{C}_{g,\ell-1,\mathbf{m}},F)$. This shows that the transformation is an equivalence. This proves that $\overline{M}_{g,\tilde\gamma[\ell]}(G)$ is a smooth proper DM stack over $\mathbb{C}$. □

Proof of Theorem 4.5. Let $\mathcal{K}_{g,\gamma'}^{bal}(\mathcal{C}_{g,\ell-1,\mathbf{m}},F)$ be the moduli of $(\ell-1)$-pointed genus g balanced twisted stable maps to $\mathcal{C}_{g,\ell-1,\mathbf{m}}$ of fundamental class F that satisfy the requirement (2) above. It is a smooth proper DM stack over $\mathbb{C}$ ([FJR2, Thm 2.2.6]). Let

$$\mathfrak{f}:\mathcal{K}^{bal}_{g,\tilde\gamma}(\mathcal{C}_{g,\ell-1,\mathbf{m}},F)\longrightarrow\mathcal{K}^{bal}_{g,\gamma'}(\mathcal{C}_{g,\ell-1,\mathbf{m}},F)$$

be the forgetful morphism (forgetting the ℓ-th marked point followed by stabilization) in [AV, Coro 9.1.3]. By the construction in [FJR2], canonically $\mathcal{K}^{bal}_{g,\gamma'}(\mathcal{C}_{g,\ell-1,\mathbf{m}},F)\cong\overline{M}_{g,\gamma'}(G)$. Combining with the isomorphism (4.6) gives the morphism

$$\mathfrak{v}_1:\overline{M}_{g,\tilde\gamma[\ell]}(G)\to\overline{M}_{g,\gamma'}(G).$$

Next, we construct a morphism $\mathfrak{v}_2:\overline{M}_{g,\gamma}(G)\to\overline{M}_{g,\tilde\gamma[\ell]}(G)$. Given any family $\xi=(\mathcal{C},\Sigma_i^{\mathcal{C}},\mathcal{L}_j,\varphi_k)$ in $\overline{M}_{g,\gamma}(G)$, we construct a new family $\xi'=(\mathcal{C}',\Sigma_i^{\mathcal{C}'},\mathcal{L}'_j,\varphi'_k)$ as follows. We let $\mathcal{C}'$ be $\mathcal{C}$ with the (possible) stacky structure along $\Sigma_\ell^{\mathcal{C}}$ eliminated. Let $\mathfrak{t}:\mathcal{C}\to\mathcal{C}'$ be the tautological morphism. We define $\mathcal{L}'_j=\mathfrak{t}_*\mathcal{L}_j$. Because γ_ℓ is of the form $\zeta_d\mapsto\jmath_\delta$, one verifies that the isomorphism φ_k induces isomorphism (4.4). The forgetful morphism we aimed at is

$$\mathfrak{f}_{\gamma,\ell}=\mathfrak{v}_1\circ\mathfrak{v}_2:\overline{M}_{g,\gamma}(G)\longrightarrow\overline{M}_{g,\gamma'}(G).$$

Finally, one can easily check that the restriction of $\mathfrak{f}_{\gamma,\ell}$ on $\overline{M}_\Gamma(G)$ factors through $\overline{M}_{\Gamma'}(G)$. Thus it lifts to the morphism $\mathfrak{f}_{\Gamma,\ell}$. Also, it follows directly from the flatness of $\overline{M}_{g,\ell}\to\overline{M}_{g,\ell-1}$ that both $\mathfrak{f}_{\gamma,\ell}$ and $\mathfrak{f}_{\Gamma,\ell}$ are flat. □

4.4. **Property of Witten's top Chern classes.** Let $\overline{M}_\Gamma(G)^p$ be the moduli of weakly Γ-labeled G-spin curves with fields. Suppose Γ is connected and has ℓ-tails and total genus g. Then

$$\overline{M}_\Gamma(G)^p=\overline{M}_\Gamma(G)\times_{\overline{M}_{g,\gamma}(G)}\overline{M}_{g,\gamma}(G)^p.$$

Following the discussion in the previous subsection, we have an induced perfect relative obstruction theory of $\overline{M}_\Gamma(G)^p\to\overline{M}_\Gamma(G)$, a W-induced cosection of the obstruction sheaf of $\overline{M}_\Gamma(G)^p$ assuming all γ_i's are narrow, and its Witten top Chern class $[\overline{M}_\Gamma(G)^p]^{\mathrm{vir}}$.

We state and prove the algebraic analogue (i.e. cycle version) of the "topological axioms" stated and proved in [FJR2, Thm 4.1.8] (also cf. [JKV] and [Po]).

Theorem 4.7 (Degeneration)**.** *Let Γ be a G-decorated graph so that every tail t_i is decorated with a narrow $\gamma_i\in G$. Suppose $\Gamma\to\Gamma'$ by contracting edges of Γ. Then*

$$\tau^!_{\Gamma\Gamma'}[\overline{M}_\Gamma(G)^p]^{\mathrm{vir}}=[\overline{M}_{\Gamma'}(G)^p]^{\mathrm{vir}}.$$

Proof. The identity follows from applying Lemma 4.1 to $\tau_{\Gamma\Gamma'}$. □

Theorem 4.8 (Disjoint Union of Graphs)**.** *Let notations be as in Theorem 4.7. Suppose Γ is a disjoint union of Γ_i, $i=1,\cdots,k$. Then*

$$[\overline{M}_\Gamma(G)^p]^{\mathrm{vir}}=[\overline{M}_{\Gamma_1}(G)^p]^{\mathrm{vir}}\times\cdots\times[\overline{M}_{\Gamma_d}(G)^p]^{\mathrm{vir}}.$$

Proof. The canonical isomorphism $\overline{M}_\Gamma(G)\cong\prod_{i=1}^d\overline{M}_{\Gamma_i}(G)$ is induced by the disjoint union of spin curves and sections, thus compatible with the construction of virtual cycles. □

Let notations be as in Theorem 4.7. Let $e\in E(\Gamma)$ be such that its decoration $\gamma_e:\mu_e\to G$ is narrow, and let Γ_e be the graph obtained by breaking e up into two tails $t_{e,-}$ and $t_{e,+}$, decorated with γ_e^{-1} and γ_e (see the convention after (2.4)). In the case where e has two distinct vertices, we define $\overline{M}_\Gamma(G)_e:=\overline{M}_\Gamma(G)$; otherwise, we form the double cover $\overline{M}_\Gamma(G)_e \to \overline{M}_\Gamma(G)$ whose closed points are data $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \hat{\mathcal{C}}_{q+})$ such that $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j)$ is a closed point in $\overline{M}_\Gamma(G)$, $q \in \mathcal{C}$ is the node associated with $e \in E(\Gamma)$, and $\hat{\mathcal{C}}_{q+}$ is a branch of the formal completion $\hat{\mathcal{C}}_q$. We form

$$\overline{M}_\Gamma(G)_e^p = \overline{M}_\Gamma(G)^p \times_{\overline{M}_\Gamma(G)} \overline{M}_\Gamma(G)_e.$$

We now construct the gluing morphisms η and $\mathfrak{g}$ fitting into the following two commutative squares:

$$\begin{array}{ccccc} \overline{M}_\Gamma(G)_e^p & \xleftarrow{\ \mathfrak{g}\ } & \overline{M}_\Gamma(G)_e \times_{\overline{M}_{\Gamma_e}(G)} \overline{M}_{\Gamma_e}(G)^p & \longrightarrow & \overline{M}_{\Gamma_e}(G)^p \\ \downarrow & & \downarrow & & \downarrow \\ \overline{M}_\Gamma(G)_e & = & \overline{M}_\Gamma(G)_e & \xrightarrow{\ \eta\ } & \overline{M}_{\Gamma_e}(G) \end{array} \tag{4.7}$$

For η, by Definition 4.3, any S-family $\xi = (\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k)$ in $\overline{M}_\Gamma(G)(S)$ comes from gluing an S-family of G-spin curves $\tilde{\xi}$ coarsely labeled by Γ_{sp}. As $(\Gamma_e)_{sp} = \Gamma_{sp}$, the family $\tilde{\xi}$ glues to the family $\xi' = (\mathcal{C}', \Sigma_i^{\mathcal{C}'}, \mathcal{L}_j', \varphi_k')$ in $\overline{M}_{\Gamma_e}(G)(S)$. This transformation $\xi \mapsto \xi'$ defines the morphism η.

To define $\mathfrak{g}$, we note that any S-family in the fiber product consists of a triple $(\xi, \xi', (\rho_j'))$, where $\xi \in \overline{M}_\Gamma(G)_e(S)$, ξ' and $\rho_j' \in \oplus_j \Gamma(\mathcal{L}_j')$, etc., are as mentioned. Because γ_e is narrow, each section ρ_j' vanishes along the marked sections labeled by t_{e-} and t_{e+}. In particular, ρ_j' glues to a global section $\rho_j \in \Gamma(\mathcal{L}_j)$. The transformation $(\xi, \xi', (\rho_j')) \mapsto (\xi, (\rho_j))$ defines the morphism $\mathfrak{g}$.

Theorem 4.9 (Composition Law)**.** *Let notations be as in Theorem 4.7, let $e \in E(\Gamma)$ be such that its decoration $\gamma_e : \mu_e \to G$ is narrow, and let Γ_e be as constructed. Then the morphism $\mathfrak{g}$ is an isomorphism and η is a G-torsor (cf. (4.7)). Furthermore,*

$$[\overline{M}_\Gamma(G)_e^p]^{\mathrm{vir}} = \eta^! [\overline{M}_{\Gamma_e}(G)^p]^{\mathrm{vir}}. \tag{4.8}$$

Proof. We first prove that η is a G-torsor. By the construction of the transformation $\xi \mapsto \xi'$, we see that the morphism η is étale and finite. We next construct a G-action on $\overline{M}_\Gamma(G)_e$. For simplicity, we assume that e has two distinct vertices. Let $\xi = (\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}_j, \varphi_k) \in \overline{M}_\Gamma(G)(\mathbb{C})$ and let $\eta(\xi) = (\mathcal{C}', \Sigma_i^{\mathcal{C}'}, \mathcal{L}_j', \varphi_k') \in \overline{M}_{\Gamma_e}(G)(\mathbb{C})$. Let $f : \mathcal{C}' \to \mathcal{C}$ be the gluing morphism along the node $q \in \mathcal{C}$, namely, $f^{-1}(q) = \{q_+, q_-\}$ with $\hat{\mathcal{C}}_{q+} \cong \hat{\mathcal{C}}'_{q+}$ under f. By definition, the sheaves $\mathcal{L}_j$ and $\mathcal{L}_j'$ fit into the (gluing) exact sequence

$$0 \longrightarrow \mathcal{L}_j \longrightarrow f_* \mathcal{L}_j' \xrightarrow{(a_j, b_j)} \mathcal{O}_q \longrightarrow 0, \tag{4.9}$$

where a_j (reps. b_j) factors through $f_* \mathcal{L}_j' \xrightarrow{a_j'} \mathcal{L}_j' \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}_{\hat{\mathcal{C}}_{q+}} \xrightarrow{a_j''} \mathcal{O}_q$ (resp. write out the whole thing but with $+$ replaced by $-$). Here we used the tautological isomorphism $\hat{\mathcal{C}}_{q\pm} \cong \hat{\mathcal{C}}'_{q\pm}$.

For $c \in \mathbb{G}_{\mathrm{m}}$, we define $a_j^c := (c \cdot a_j'') \circ a_j'$, and define

$$\mathcal{L}_j^c = \ker\{(a_j^c, b_j) : f_* \mathcal{L}_j' \longrightarrow \mathcal{O}_q\}. \tag{4.10}$$

By lifting to an étale covering of $\mathcal{C}$, we see that $\mathcal{L}_j^c$ is an invertible sheaf on $\mathcal{C}$.

We now investigate the choices of $c = (c_1, \cdots, c_n) \in \mathbb{G}_{\mathrm{m}}^n$ so that

$$\varphi_k|_{\mathcal{C}-q} : \mathbf{m}_k(\mathcal{L}_1^{c_1}, \ldots, \mathcal{L}_n^{c_n})|_{\mathcal{C}-q} \xrightarrow{\cong} (\omega_{\mathcal{C}}^{log})^{w(\mathbf{m}_k)}|_{\mathcal{C}-q}, \quad k = 1, \cdots, n \tag{4.11}$$

extend to isomorphisms over $\mathcal{C}$. Because $\varphi_k : \mathbf{m}_k(\mathcal{L}_1, \ldots, \mathcal{L}_n) \to (\omega_{\mathcal{C}}^{log})^{w(\mathbf{m}_k)}$ is an isomorphism, by the construction of $\mathcal{L}_j^{c_j}$, we see that (4.11) extend if and only if $\mathbf{m}_k(\mathcal{L}_1|_q, \cdots, \mathcal{L}_n|_q) \equiv \mathbf{m}_k(c_1\mathcal{L}_1|_q, \cdots, c_n\mathcal{L}_n|_q)$, which holds if and only if $\mathbf{m}_k(c_1, \cdots, c_n) = 1$ for all k. By the definition of G, we conclude that (4.11) extend to isomorphisms if and only if $c \in G$.

Conversely, we claim that if $\bar{\xi} = (\mathcal{C}, \Sigma_i^{\mathcal{C}}, \bar{\mathcal{L}}_j, \bar{\varphi}_k) \in \eta^{-1}(\eta(\xi))$, then $\bar{\mathcal{L}}_j = \mathcal{L}_j^{c_j}$ for a $c \in G$ and $\bar{\varphi}_k$ are extensions of (4.11). Indeed, because $\eta(\xi) = \eta(\bar{\xi})$, there is an isomorphism $\beta_j : \mathcal{L}_j|_{\mathcal{C}-q} \cong \bar{\mathcal{L}}_j|_{\mathcal{C}-q}$, extending to $\hat{\mathcal{C}}_{q\pm}$, so that it commutes with φ_k and $\bar{\varphi}_k$ for all k. Therefore, there is a $c = (c_1, \cdots, c_n) \in \mathbb{G}_{\mathrm{m}}^n$ so that $\bar{\mathcal{L}}_j = \mathcal{L}_j^{c_j}$, and β_j is the identity map: $\bar{\mathcal{L}}_j|_{\mathcal{C}-q} = \mathcal{L}_j^{c_j}|_{\mathcal{C}-q} = \mathcal{L}_j|_{\mathcal{C}-q}$. Then as $\bar{\varphi}_k$ is an extension of (4.11), the previous discussion shows that $c \in G$. This proves that η is a G-torsor.

Next, for any $(\xi, (\rho_j))$ in $\overline{M}_\Gamma(G)_e^p$, taking $\xi' = \eta(\xi)$ and $\rho_j' = f^*\rho_j$ defines $(\xi, \xi', (\rho_j'))$. This is the inverse of $\mathfrak{g}$ and thus $\mathfrak{g}$ is an isomorphism. This proves the first part of the Theorem.

We now prove the identity of virtual cycles. Let $(\mathcal{C}_\Gamma, \mathcal{L}_{\Gamma,j})$ and $(\mathcal{C}_{\Gamma_e}, \mathcal{L}_{\Gamma_e,j})$ be the universal families of $\overline{M}_\Gamma(G)_e$ and $\overline{M}_{\Gamma_e}(G)$, respectively. We form the gluing morphism $\mathfrak{f}$ as shown in the following commutative diagrams

$$\begin{array}{ccccc} \mathcal{C}_\Gamma & \xleftarrow{\ \mathfrak{f}\ } & \mathcal{C}'_\Gamma := \mathcal{C}_{\Gamma_e} \times_{\overline{M}_{\Gamma_e}(G)} \overline{M}_\Gamma(G)_e & \xrightarrow{\ \eta'\ } & \mathcal{C}_{\Gamma_e} \\ \downarrow{\scriptstyle \pi_\Gamma} & & \downarrow{\scriptstyle \pi'_\Gamma} & & \downarrow{\scriptstyle \pi_{\Gamma_e}} \\ \overline{M}_\Gamma(G)_e & \xleftarrow{\ =\ } & \overline{M}_\Gamma(G)_e & \xrightarrow{\ \eta\ } & \overline{M}_{\Gamma_e}(G). \end{array} \tag{4.12}$$

As usual, we let $\mathcal{E}_{\Gamma_e} = \oplus_j \mathcal{L}_{\Gamma_e,j}$ and $\mathcal{E}_\Gamma = \oplus_j \mathcal{L}_{\Gamma,j}$. Let $\mathcal{E}'_\Gamma = \eta'^*\mathcal{E}_{\Gamma_e}$. The square on the right above induces a morphism

$$C(\pi'_{\Gamma*}\mathcal{E}'_\Gamma) \longrightarrow C(\pi_{\Gamma_e*}\mathcal{E}_{\Gamma_e}) = \overline{M}_{\Gamma_e}(G)^p$$

compatible with relative obstruction theories. By étaleness of η and Lemma 4.1

$$\eta^![\overline{M}_{\Gamma_e}(G)^p]^{\mathrm{vir}} = [C(\pi'_{\Gamma*}\mathcal{E}'_\Gamma)]^{\mathrm{vir}}. \tag{4.13}$$

On the other hand, the gluing morphism $\mathfrak{f}$ induces a canonical isomorphism $\mathfrak{f}^*\mathcal{E}_\Gamma \cong \mathcal{E}'_\Gamma$, and

$$\overline{M}_\Gamma(G)_e^p = C(\pi_{\Gamma*}\mathcal{E}_\Gamma) \xrightarrow{\cong} C(\pi'_{\Gamma*}\mathcal{E}'_\Gamma), \tag{4.14}$$

which is the inverse of $\mathfrak{g}$. Taking cohomology of (4.9) and applying Lemma 3.2's proof, one can verify that (4.14) identifies deformation theories and cosections. Thus it induces

$$[\overline{M}_\Gamma(G)_e^p]^{\mathrm{vir}} = [C(\pi'_{\Gamma*}\mathcal{E}'_\Gamma)]^{\mathrm{vir}} \in A_*(\overline{M}_\Gamma(G)_e) \tag{4.15}$$

□

We state and prove the forgetting tails theorem.

Theorem 4.10 (Forgetting Tails). *Suppose Γ has its last tail t_ℓ decorated by $\gamma_\ell : \mu_d \to G$ via $\zeta_d \mapsto \jmath_\delta$ (cf. subsection 2.1). Let Γ' be the stabilization of the graph after removing t_ℓ from Γ, and let*

$$\mathfrak{f}_{\Gamma,\ell} : \overline{M}_\Gamma(G) \longrightarrow \overline{M}_{\Gamma'}(G)$$

be the forgetful morphism constructed in Theorem 4.5. Then $\mathfrak{f}_{\Gamma,\ell}$ is flat and satisfies $[\overline{M}_\Gamma(G)^p]^{\mathrm{vir}} = (\mathfrak{f}_{\Gamma,\ell})^*[\overline{M}_{\Gamma'}(G)]^{\mathrm{vir}}$.

Proof. Let $\pi_\Gamma : \mathcal{C}_\Gamma \to \overline{M}_\Gamma(G)$ and $\pi_{\Gamma'} : \mathcal{C}_{\Gamma'} \to \overline{M}_{\Gamma'}(G)$ be the universal curves, and let

$$\mathfrak{u}_1 : \mathcal{C}_\Gamma \longrightarrow \mathcal{C}_{\Gamma'/\Gamma} := \mathcal{C}_{\Gamma'} \times_{\overline{M}_{\Gamma'}(G)} \overline{M}_\Gamma(G)$$

be the tautological morphism induced by the construction of the morphisms $\mathfrak{v}_1$ and $\mathfrak{v}_2$ in the proof of Theorem 4.5. We let $\mathfrak{u}_2 : \mathcal{C}_{\Gamma'/\Gamma} \to \mathcal{C}_{\Gamma'}$ be the projection.

For $\{\mathcal{L}_j\}_{j=1}^n$ and $\{\mathcal{L}'_j\}_{j=1}^n$ being the universal invertible sheaves on $\mathcal{C}_\Gamma$ and $\mathcal{C}_{\Gamma'}$ respectively, by the constructions of $\mathfrak{v}_1$ and $\mathfrak{v}_2$, we have $\mathfrak{u}_2^*\mathcal{L}'_j = \mathfrak{u}_{1*}\mathcal{L}_j$. Because of the assumption on γ_ℓ, we have

$$(\mathfrak{u}_2 \circ \mathfrak{u}_1)^*\mathcal{L}'_j = \mathfrak{u}_1^*\mathfrak{u}_{1*}\mathcal{L}_j \cong \mathcal{L}_j(-\delta_j \Sigma_\ell^{\mathcal{C}_\Gamma}) \xrightarrow{\subset} \mathcal{L}_j. \tag{4.16}$$

Defining $\mathcal{E}' = \oplus_j \mathcal{L}'_j$ and $\mathcal{E} = \oplus_j \mathcal{L}_j$, and applying Lemma 3.2 to $(\mathfrak{u}_2 \circ \mathfrak{u}_1)^*\mathcal{E}' \subset \mathcal{E}$, we obtain $R^\bullet\pi_{\Gamma*}(\mathfrak{u}_2 \circ \mathfrak{u}_1)^*\mathcal{E}' \cong R^\bullet\pi_{\Gamma*}\mathcal{E}$ and the induced isomorphism

$$C(\pi_{\Gamma*}\mathcal{E}) \xrightarrow{\cong} C(\pi_{\Gamma'*}\mathcal{E}') \times_{\overline{M}_{\Gamma'}(G)} \overline{M}_\Gamma(G).$$

Since the isomorphism is compatible with deformation theories and cosections,

$$[\overline{M}_\Gamma(G)^p]^{\mathrm{vir}} = [C(\pi_{\Gamma*}\mathcal{E})]^{\mathrm{vir}} = \mathfrak{f}_{\Gamma,\ell}^*[C(\pi_{\Gamma'*}\mathcal{E}')]^{\mathrm{vir}} = \mathfrak{f}_{\Gamma,\ell}^*[\overline{M}_{\Gamma'}(G)]^{\mathrm{vir}}.$$

□

Let $([\mathbb{C}^n/G], W) = ([\mathbb{C}^{n_1} \times \mathbb{C}^{n_2}/G_1 \times G_2], W_1 + W_2)$ be the product of two LG spaces $([\mathbb{C}^{n_i}/G_i], W_i)$. Let $\gamma^i \in (G_i)^\ell$ be g-admissible and narrow, and let $\gamma := \gamma^1 \times \gamma^2$ be the direct product, i.e., $\gamma_i = \gamma_i^1 \times \gamma_i^2$ for $1 \le i \le \ell$. Let Γ be a G-decorated graph with ℓ ordered tails decorated with $\gamma_1, \cdots, \gamma_\ell$. Also let Γ_j be Γ with the decorations of its i-th tails replaced by γ_i^j for all i.

Theorem 4.11 (Product of LG Spaces)**.** *Let notations be as stated. Then there is a partial forgetting morphism* $f_j : \overline{M}_\Gamma(G_1 \times G_2) \longrightarrow \overline{M}_{\Gamma_j}(G_j)$*. Furthermore,*

$$(f_1 \times f_2)^*([\overline{M}_{\Gamma_1}(G_1)^p]^{\mathrm{vir}} \times [\overline{M}_{\Gamma_2}(G_2)^p]^{\mathrm{vir}}) = [\overline{M}_\Gamma(G_1 \times G_2)^p]^{\mathrm{vir}}.$$

Proof. We first show that the map

$$f_1 \times f_2 : \overline{M}_\Gamma(G_1 \times G_2) \to \overline{M}_{\Gamma_1}(G_1) \times \overline{M}_{\Gamma_2}(G_2)$$

is an l.c.i. morphism. Using the étale forgetful morphisms from $\overline{M}_{\Gamma_1}(G_1)$, $\overline{M}_{\Gamma_2}(G_2)$ and $\overline{M}_\Gamma(G_1 \times G_2)$ to $\mathfrak{M}_{g,\ell}^{\mathrm{tw}}$, $f_1 \times f_2$ is the composition

$$\overline{M}_\Gamma(G_1 \times G_2) \longrightarrow \overline{M}_{\Gamma_1}(G_1) \times_{\mathfrak{M}_{g,\ell}^{\mathrm{tw}}} \overline{M}_{\Gamma_2}(G_2) \longrightarrow \overline{M}_{\Gamma_1}(G_1) \times \overline{M}_{\Gamma_2}(G_2).$$

By construction, the first arrow is an open embedding, and the second arrow is a lift of the diagonal morphism $\Delta : \mathfrak{M}_{g,\ell}^{\mathrm{tw}} \to \mathfrak{M}_{g,\ell}^{\mathrm{tw}} \times \mathfrak{M}_{g,\ell}^{\mathrm{tw}}$. Since $\mathfrak{M}_{g,\ell}^{\mathrm{tw}}$ is smooth, $f_1 \times f_2$ is an l.c.i. morphism.

The remainder of the proof is analogous to the previous proofs, and is therefore omitted. □

4.5. **Free cases.** Let S be a pure dimensional proper DM stack and $\iota : S \to M$ be a closed embedding. We form $Y = X \times_M S$, with $\lambda : Y \to S$ being the projection. Since the relative obstruction theory of $X \to M$ pulls back to that of $Y \to S$, $\mathcal{O}b_{Y/S} = \iota^*\mathcal{O}b_{X/M}$, and the cosection $\sigma : \mathcal{O}b_{X/M} \to \mathcal{O}_X$ pulls back to a cosection $\sigma_S : \mathcal{O}b_{Y/S} \to \mathcal{O}_S$, which has all the properties required to define the cosection localized virtual cycle

$$[Y]^{\mathrm{vir}} = 0^!_{\sigma_S,loc}[\mathbf{C}_{Y/S}] \in A_*(D(\sigma_S)).$$

Definition 4.12. *ι is called free if all $H^i(\iota^*\mathbf{R}\pi_{M*}\mathcal{L}_{M,j})$ are locally free sheaves of $\mathcal{O}_S$-modules.*

In this subsection, we derive an explicit formula of $[Y]^{\mathrm{vir}}$ in the case where ι is free.

We first generalize a lemma on the Segre class to the case of a weighted projective bundle. Let S be a proper integral DM stack over $\mathbb{C}$. Let $F_1, \cdots, F_n$ be vector bundles on S of rank $r_1, \cdots, r_n$. We define $F = \oplus_{i=1}^n F_i$, $r = \operatorname{rank} F$, and let 1_S be the trivial line bundle on S. For relatively prime positive integers $(e_1, \cdots, e_n)$, we introduce a $\mathbb{G}_{\mathrm{m}}$-action on $F \oplus 1_S$ via

$$(v_1, \cdots, v_n, c)^t = (t^{e_1} v_1, \cdots, t^{e_n} v_n, t \cdot c). \tag{4.17}$$

Let $0_S \subset F \oplus 1_S$ be the zero section. We define

$$Z = (F \oplus 1_S - 0_S)/\mathbb{G}_{\mathrm{m}}. \tag{4.18}$$

The total space of F is embeded in Z via $F \to F \oplus 1_S$: $v \mapsto [v, 1]$. Let $D = Z - F$ be its complement, which is a divisor in Z. Let $\pi : Z \to S$ be the projection. For a vector bundle E, we define $c(E)(t) = \sum c_i(E) t^i$, the total Chern polynomial of E.

Lemma 4.13. *There is an identity of cycles*

$$\sum_i \pi_*\big(c_1(\mathcal{O}_Z(D))^i \cap [Z]\big) \cdot t^i = t^r \big(\prod_{j=1}^n (e_j)^{r_j} c(F_j)(t/e_j)\big)^{-1}.$$

Proof. First, using projection formula, one can see that if the lemma holds for vector bundles over S' and there is a generic finite proper morphism $S' \to S$, then the lemma holds for vector bundles over S as stated. By Chow's Lemma, we can find a generic finite morphism from an integral projective scheme to S. Thus, to prove the lemma, we only need to treat the case where S is projective and integral. Next, by splitting principle, we can assume that all F_j's are direct sums of line bundles. Thus without loss of generality, we can assume that all F_j's are line bundles on S. Applying the covering trick ([BG, Lemma 2.1]), after passing to an integral projective scheme generically finite over S, we can assume that there is a line bundle L_j on S so that $L_j^{\otimes e_j} \cong F_j$ for every j.

Let $L = \oplus_j L_j$, $Z' = \mathbb{P}(L \oplus 1_S)$ with projection $\pi' : Z' \to S$, and $\rho : Z' \to Z$ be induced by the map

$$L \oplus 1_S \ni ((u_1, \cdots, u_n), c) \mapsto ((u_1^{e_1}, \cdots, u_n^{e_n}), c) \in F \oplus 1_S.$$

The map is a flat, finite S-morphism of degree $\bar{e} = e_1^{r_1} \cdots e_n^{r_n}$. Let $D' \subset Z'$ be the divisor at infinity (defined by $c = 0$). Then $\rho^*\mathcal{O}(D) = \mathcal{O}(D')$ and $\pi \circ \rho = \pi'$. Applying the projection formula gives

$$\sum_i \pi'_*\big(c_1(\mathcal{O}(D'))^i\big) \cdot t^i = \sum_i \pi_*\rho_*\rho^*\big(c_1(\mathcal{O}(D))^i\big) \cdot t^i = \bar{e} \sum_i \pi_*\big(c_1(\mathcal{O}(D))^i\big) t^i.$$

On the other hand, since Z' is a projective bundle over S, we have $\big(c(L)(t)\big)^{-1} = \sum_i \pi'_*\big(c_1(\mathcal{O}(D'))^i\big) \cdot t^{i-r}$. Therefore,

$$\sum_i \pi_*\big(c_1(\mathcal{O}(D))^i\big) t^{i-r} = (\bar{e})^{-1}\big(c(L)(t)\big)^{-1} = (\bar{e})^{-1}\Big(\prod_{j=1}^n c(F_j)(t/e_j)\Big)^{-1}.$$

Here the second identity is from $F = \oplus_j F_j = \oplus_j L_j^{\otimes e_j}$. $\square$

Let us go back to the situation introduced at the beginning of this subsection. Suppose ι is free. As usual, let $\pi_S : \mathcal{C}_S \to S$ be the map $\mathcal{C}_M \times_M S \to S$, and $\mathcal{L}_{S,j}$ be the pull-back of $\mathcal{L}_{M,j}$. Let $F_j = R^0\pi_{S*}\mathcal{L}_{S,j}$, $F = \oplus_j F_j$, $G_j = R^1\pi_{S*}\mathcal{L}_{S,j}$, and $G = \oplus_j G_j$. Since ι is free, they are locally free sheaves (vector bundles) on S. Because the pull-back of the relative obstruction sheaf of X/M to Y is the relative obstruction sheaf of Y/S, we have $\mathcal{O}b_{Y/S} = \lambda^* G$.

As G is locally free, by the base change property, we know that Y is the total space of F, as a stack over S. In particular, Y is smooth over S. Thus the intrinsic normal cone is the zero section of the bundle λ^*G. Therefore, using the cosection $\sigma|_S : \mathcal{O}b_{Y/S} \to \mathcal{O}_S$, the cosection localized virtual cycle of Y is

$$[Y]^{\mathrm{vir}} = 0^!_{\sigma|_S, loc}[0_{\lambda^*G}] \in A_*(S).$$

In particular, if S_k's are the integral components of S with multiplicity a_k, then $[Y]^{\mathrm{vir}} = \sum_k a_k [T \times_S S_k]^{\mathrm{vir}}$. Therefore, it suffices to treat the case where S is integral, which we assume from now on.

We now determine the cosection $\sigma|_S : \lambda^*G \to \mathcal{O}_Y$. Since Y is integral and λ^*G is locally free, we only need to know $\sigma(y) = \sigma|_y$ for any closed point $y \in Y$. Recall that $W_a(x) = x_1^{m_{a1}} \cdots x_n^{m_{an}}$ is a monomial of $W(x)$, $W_a(x)_j = \partial_{x_j} W_a(x)$, and $\tilde W_a(x)_j = (m_{aj})^{-1} W_a(x)_j$. Following the construction of (3.12), let $\tilde y = (\tilde y_j) \in \oplus_j \lambda^* F_j|_y = \lambda^*F|_y$ be the point $y \in Y$ under the identification $Y = \lambda^*F$, the total space of λ^*F. For any $\dot\rho_j \in \lambda^*G_j|_y$, define

$$\tilde W_a(\tilde y)_j := \tilde W_a(\tilde y_1, \cdots, \tilde y_n)_j \in \omega^{log}_{\mathcal{C}_M/M} \otimes \mathcal{L}^{-1}_{M,j}|_{\lambda(y)},$$

and for the coefficient α_a of W_a in W, we have

$$\sigma(y) = (\sigma_j(y)) : \oplus_{j=1}^n \lambda^* G_j|_y \longrightarrow \mathbb{C}, \quad \sigma_j(y)(\dot\rho_j) = \big(\sum_a \alpha_a \cdot m_{aj} \cdot \tilde W_a(\tilde y)_j\big) \cdot \dot\rho_j.$$

Here we used the condition that γ is narrow.

Proposition 4.14. *Let $r_j = \operatorname{rank} F_j$, $r = \sum_j r_j$, $s_j = \operatorname{rank} G_j$, and $s = \sum_j s_j$. Set $\varepsilon_j = \delta_j - d$. Then*

$$[Y]^{\mathrm{vir}} = \mathrm{Coeff}_{t^{s-r}} \left(\frac{\prod_{j=1}^n \varepsilon_j^{s_j} c(G_j)(t/\varepsilon_j)}{\prod_{j=1}^n \delta_j^{r_j} c(F_j)(t/\delta_j)} \right). \tag{4.19}$$

Proof. For the vector bundle $F_j (= R\pi_{S*}\mathcal{L}_{S,j})$ introduced, we form the quotient Z as in (4.18) with $e_j = \delta_j$. Let $\pi : Z \to S$ be the projection. As Y is the total space of $F = \oplus_j F_j$, $Y \subset Z$ is open whose complement is the divisor $D = (c = 0)$, and $\pi|_Y = \lambda$. Let $V_j = \pi^* G_j$, and let $V = \oplus_j V_j$. Thus $\sigma_j : V_j \to \mathcal{O}_Y$ and $\sigma : V \to \mathcal{O}_Y$.

Consider the duals $\sigma_j^\vee \in \Gamma(Y, V_j^\vee)$ and $\sigma^\vee \in \Gamma(Y, V^\vee)$. The zero locus $(\sigma^\vee = 0)$ with reduced scheme structure equals $D(\sigma)$. Hence by Lemma 3.6, it is supported on $S \subset Y$ (as the zero section of F).

Since $\tilde W_a(x)_j$ is quasi-homogeneous of total degree $-\varepsilon_j = d - \delta_j$, each homomorphism σ_j extends uniquely over Z to $\bar\sigma_j : V_j \otimes \mathcal{O}_Z(\varepsilon_j D) \longrightarrow \mathcal{O}_Z$ such that the degeneracy (non-surjective) locus of $\bar\sigma := \oplus_j \bar\sigma_j$ is supported on S. Thus by denoting $\tilde V = \oplus_j V_j \otimes \mathcal{O}_Z(\varepsilon_j D)$, the section $\bar s := \bar\sigma^\vee \in \Gamma(Y, \tilde V^\vee)$ has the zero locus supported on S, and $(Z, \tilde V^\vee, \bar\sigma^\vee)$ is an extension of $(Y, V^\vee|_Y, \sigma^\vee)$.

Since Y/S is smooth, the cone $\mathbf{C}_{Y/S} \subset h^1/h^0(E^\bullet_{Y/S})$ is the zero section. By our construction of Witten's top Chern class, we have

$$0^!_{\sigma, loc}[\mathbf{C}_{Y/S}] = (-1)^s e_{\sigma^\vee, loc}(V^\vee) = e_{\bar\sigma^\vee, loc} \tilde V^\vee = \pi_* e(\tilde V^\vee) = (-1)^s \pi_* e(\tilde V) \in A_*(S).$$

Here, the second identity holds because $\bar{\sigma}^\vee$ is non-vanishing along $D = Z - Y$, and the third identity holds because Z is proper. Hence Witten's top Chern class is

$$\pi_*\Big(e(\oplus_j V_j(\varepsilon_j D))\Big) = \pi_*\Big(\prod_j e(\pi^* G_j \otimes \mathcal{O}_Z(\varepsilon_j D))\Big). \tag{4.20}$$

Defining $\mathcal{O}(1) = \mathcal{O}_Z(D)$ and letting $I = \{(i_1, \cdots, i_n) \in \mathbb{Z}^n \mid 0 \le i_j \le s_j\}$, we have

$$\begin{aligned}
& \pi_*\Big(\prod_{j=1}^{n} \sum_{i_j=0}^{s_j} c_{s_j - i_j}(\pi^* G_j) \cdot c_1(\mathcal{O}(\varepsilon_j))^{i_j}\Big) \\
= \ & \pi_*\Big(\sum_{(i_1,\cdots,i_n)\in I} \big(\prod_{j=1}^{n} c_{s_j-i_j}(\pi^* G_j)\big)\big(c_1(\mathcal{O}(1))^{i_1+\cdots+i_n}\big)\big(\varepsilon_1^{i_1}\varepsilon_2^{i_2}\cdots\varepsilon_n^{i_n}\big)\Big) \\
= \ & \sum_{(i_1,\cdots,i_n)\in I} \big(\prod_{j=1}^{n} \varepsilon_j^{s_j} c_{s_j-i_j}(G_j)\varepsilon_j^{-(s_j-i_j)}\big) \cdot \pi_*[c_1(\mathcal{O}(1))]^{i_1+\cdots+i_n} \\
= \ & \mathrm{Coeff}_{t^{s-r}}\Big(\big(\prod_{j=1}^{n} \varepsilon_j^{s_j} c(G_j)(t/\varepsilon_j)\big) \cdot \sum_i \pi_*\big(c_1(\mathcal{O}(1))^i\big)t^{i-r}\Big).
\end{aligned}$$

Applying Lemma 4.13, we obtain (4.19). □

Corollary 4.15. *In the case where $n = 1$, $\delta_1 = 1$, and letting $\varepsilon = 1 - d$,*

$$[Y]^{\mathrm{vir}} = \mathrm{Coeff}_{t^{s-r}}\big(\varepsilon^h c(G)(t/\varepsilon)/c(F)(t)\big).$$

The following propositions are the "Concavity" and "Index Zero" axioms stated and proved in [FJR2, Thm 4.1.8].

Proposition 4.16. *In the case where $r = 0$, we have $[Y]^{\mathrm{vir}} = c_{top}(G)$.*

Proposition 4.17. *If $r = s$, we have*

$$[Y]^{\mathrm{vir}} = \frac{(\varepsilon_1^{s_1}\cdots\varepsilon_n^{s_n})\prod_j s_j}{(\delta_1^{r_1}\cdots\delta_n^{r_n})\prod_j r_j} \cdot [S] \in A_{\dim S}(S).$$

Proof. The lowest-degree terms of the two power series $\prod_{j=1}^n \varepsilon_j^{s_j} c(G_j)(t/\varepsilon_j)$ and $\prod_{j=1}^n \delta_j^{r_j} c(F_j)(t/\delta_j)$ are constant terms $(\varepsilon_1^{s_1}\cdots\varepsilon_n^{s_n})\prod_j s_j$ and $(\delta_1^{r_1}\cdots\delta_n^{r_n})\prod_j r_j$, respectively. As $r - s = 0$, (4.19) is the ratio of these two constant terms. The ratio is the "degree of Witten map" in [FJR2, Thm 4.1.8 (5)(b)]. □

5. Comparison with other constructions

In this section, we will prove the equivalence between our construction and the other known constructions of Witten's top Chern class.

We fix an LG space $([\mathbb{C}^n/G], W)$, integers g, ℓ and a narrow $\gamma = (\gamma_1, \cdots, \gamma_\ell)$ of faithful cyclic representations in G. We abbreviate $M = \overline{M}_{g,\gamma}(G)$ and $X = \overline{M}_{g,\gamma}(G)^p$, and let $\pi_M : \mathcal{C}_M \to M$ and $\mathcal{L}_1, \cdots, \mathcal{L}_n$ be (part of) the universal family of M. Define $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$.

5.1. **Comparison with Fan-Javis-Ruan's construction.** In this subsection, we prove that the associated homology class of $[\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}}$ coincides with Witten's top Chern class constructed by Fan-Jarvis-Ruan in [FJR2, Thm 4.1.8].

We begin with an explicit description of $X \to M$ using complex representatives of $\mathbf{R}\pi_{M*}\mathcal{E}$. Let $F^\bullet = [\zeta : F^0 \to F^1]$ be a two-term complex of locally free sheaves of $\mathcal{O}_M$-modules so that $F^\bullet \cong R^\bullet \pi_{M*}\mathcal{E}$ as derived objects. Let Y be the total space of the associated vector bundle of F^0, let $\tilde{q} : Y \to M$ be the projection, and let $V = \tilde{q}^* F^1$. The homomorphism $\zeta : F^0 \to F^1$ induces a tautological section $\bar{\zeta} \in \Gamma(Y, V)$ whose vanishing locus $\bar{\zeta}^{-1}(0) \subset Y$, by the isomorphism $\mathbf{R}\pi_{M*}\mathcal{E} \cong F^\bullet$, is canonically isomorphic to X. In this way, we view X as a substack of Y defined by the vanishing of $\bar{\zeta}$, and write $X = (\bar{\zeta} = 0)$. Let $q = \tilde{q}|_X : X \to M$ be the projection.

By the construction of $X = (\bar{\zeta} = 0)$, we have the identity of complexes

$$[d\bar{\zeta}|_X : \Omega^\vee_{Y/M}|_X \to V|_X] = [q^*\zeta : q^*F^0 \to q^*F^1] = q^*F^\bullet.$$

Combined with (3.2), and denoting $E_\zeta = q^*F^\bullet$, we have an induced isomorphism

$$E^\bullet_{X/M} = q^*(\mathbf{R}\pi_{M*}\mathcal{E}) \cong q^*F^\bullet = E^\bullet_\zeta. \tag{5.1}$$

Let I_X be the ideal sheaf of $X \subset Y$. $X = \bar{\zeta}^{-1}(0)$ gives the (truncated) perfect obstruction theory ϕ_ζ:

$$\phi^\vee_\zeta : [V^\vee|_X \to \Omega_{Y/M}|_X] = (E^\bullet_\zeta)^\vee \longrightarrow L^{\bullet \geq -1}_{X/M} = [I_X/I_X^2 \to \Omega_{Y/M}|_X]. \tag{5.2}$$

Let $\mathbf{C}_\zeta \subset h^1/h^0(E^\bullet_\zeta)$ be the virtual normal cone of X via the obstruction theory ϕ_ζ. Using the isomorphism (5.1), $\mathbf{C}_\zeta$ is a closed substack of $h^1/h^0(E^\bullet_{X/M}) \cong h^1/h^0(E^\bullet_\zeta)$.

Lemma 5.1. *The cycle $[\mathbf{C}_\zeta] \in Z_*(h^1/h^0(E^\bullet_{X/M}))$ is independent of the choice of the complex $F^\bullet$ satisfying $F^\bullet \cong \mathbf{R}\pi_{M*}\mathcal{E}$.*

Proof. Let $F^\bullet$ and $\tilde{F}^\bullet$ be two-term complexes of locally free sheaves such that $F^\bullet \cong \tilde{F}^\bullet \cong \mathbf{R}\pi_{M*}\mathcal{E}$. Since the derived isomorphisms of complexes are compositions of quasi-isomorphic complex homomorphisms, to prove the lemma, we only need to prove the case where $F^\bullet \cong \tilde{F}^\bullet$ is induced by a complex homomorphism $f : F^\bullet \to \tilde{F}^\bullet$. In this case, the conclusion is straightforward. □

We now describe Fan-Javis-Ruan's construction of Witten's top Chern class for the narrow γ. As γ is narrow, the isomorphism Φ_a in (2.7) induces a sheaf homomorphism

$$\tau_a : W_a(\mathcal{L}_1, \cdots, \mathcal{L}_n) \longrightarrow \omega^{log}_{\mathcal{C}_M/M}. \tag{5.3}$$

We choose a complex of locally free sheaves of $\mathcal{O}_M$-modules $\mathcal{F}^\bullet_j = [\mathcal{F}^0_j \to \mathcal{F}^1_j]$ such that $\mathcal{F}^\bullet_j \cong \mathbf{R}\pi_{M*}\mathcal{L}_j$ as derived objects. Using the tensor product of complexes of locally free sheaves, we can substitute x_j in the monomial $W_a(x) = x_1^{m_{a1}} \cdots x_n^{m_{an}}$ by $\mathcal{F}^\bullet_j$ to obtain the complex

$$W_a(\mathcal{F}^\bullet) := (\mathcal{F}^\bullet_1)^{\otimes m_{a1}} \times \cdots \times (\mathcal{F}^\bullet_n)^{\otimes m_{an}}.$$

Lemma 5.2. *There exists $\mathcal{F}_j^\bullet = [\zeta_j : \mathcal{F}_j^0 \to \mathcal{F}_j^1]$, $j = 1, \cdots, n$, such that the derived morphism $W_a(\mathcal{F}^\bullet) \to \mathbf{R}\pi_*\omega_{\mathcal{C}_M/M}$ induced by τ_a's in (5.3) is realized by the homomorphism of complexes*

$$\nu_a : W_a(\mathcal{F}^\bullet) \longrightarrow \mathbf{R}\pi_{M*}\omega_{\mathcal{C}_M/M} \cong \mathcal{O}_M[-1].$$

Proof. This was proved in [PV2]. We point out that Σ_j and Σ_M defined in [PV2, (4.8)] contain marked points which are not narrow and hence are empty sets in our case. Thus the homomorphism in [PV2, Lemma 4.2.2] is the same as (5.3), and is identical to $t_M = \tau_M$ in the diagram of [PV2, (4.14)]. By the same argument as [PV2, Lemma 4.2.5], the two-term perfect resolution $\mathcal{F}_j^\bullet$ realizing τ_M (our ν_a) as a complex homomorphism exists. Since the existence of ν_a only relies on the vanishings of cohomology groups, we can choose $\mathcal{F}_j^\bullet$ so that all ν_a's are realized as complex homomorphisms. □

We fix the complex $\mathcal{F}_j^\bullet$ and the complex homomorphisms ν_a given by the Lemma 5.2. For the given $W_a(x)$, we continue to denote by $W_a(x)_j$ the partial derivative $\partial_{x_j} W_a(x)$, and define $\tilde{W}_a(x)_j = (m_{aj})^{-1} W_a(x)_j$(a monomial of coefficient 1). We abbreviate $\tilde{W}_a(\mathcal{F}^0)_j = \tilde{W}_a(\mathcal{F}_1^0, \cdots, \mathcal{F}_n^0)_j$.

Since the degree one term of the complex $W_a(\mathcal{F}^\bullet)$ is $\oplus_{j=1}^n \tilde{W}_a(\mathcal{F}^0)_j \otimes \mathcal{F}_j^1$, the degre- one part of ν_a is

$$\nu_a^1 : \oplus_{j=1}^n \tilde{W}_a(\mathcal{F}^0)_j \otimes \mathcal{F}_j^1 \longrightarrow \mathcal{O}_M. \tag{5.4}$$

Let $F^\bullet = \oplus_{j=1}^n \mathcal{F}_j^\bullet$, namely, $F^i = \mathcal{F}_1^i \oplus \cdots \oplus \mathcal{F}_n^i$ and $F^\bullet = [\zeta : F^0 \to F^1]$ given by $\zeta = \oplus_{j=1}^n \zeta_j$. For $\mathcal{E} = \oplus_{j=1}^n \mathcal{L}_j$, we have $F^\bullet \cong \mathbf{R}\pi_{M*}\mathcal{E}$ as derived objects. Following the notations developed before (5.1), let $(Y, V, \bar{\zeta})$ be constructed from $F^\bullet$, which gives us the isomorphism $\bar{\zeta}^{-1}(0) = X$, the obstruction theory ϕ_ζ, and the deformation complex $E_\zeta^\bullet = q^* F^\bullet$.

We now show that each ν_a^1 defines a homomorphism $\eta_a : V \to \mathcal{O}_Y$. Indeed, let S be an affine scheme and $\rho : S \to Y$ be a morphism, and let $\rho' = q \circ \rho : S \to M$ be the composition. As Y is the total space of F^0, ρ is given by a section $s = (s_1, \cdots, s_n) \in \oplus_{j=1}^n \Gamma(\rho'^* \mathcal{F}_j^0)$. We abbreviate $W_a(s)_j = W_a(s_1, \cdots, s_n)_j$. Define

$$\eta_{a,\rho} : \rho^* V = \oplus_{j=1}^n \rho'^* \mathcal{F}_j^1 \longrightarrow \mathcal{O}_S \tag{5.5}$$

via $(\dot{s}_1, \cdots, \dot{s}_n) \mapsto \rho'^*(\nu_a^1)(W_a(s)_1 \dot{s}_1, \cdots, W_a(s)_n \dot{s}_n)$. As this definition is stable under base change, such $\eta_{a,\rho}$ descends to a homomorphism $\eta_a : V \to \mathcal{O}_Y$. Let α_a be the coefficient of W_a in W. Define

$$\eta = \sum_a \alpha_a \eta_a : V \longrightarrow \mathcal{O}_Y.$$

Lemma 5.3. *The composition $\eta \circ \bar{\zeta} = 0$.*

Proof. It suffices to show that for the map $\rho : S \to Y$ as before, $\rho^*(\eta \circ \bar{\zeta}) = 0$. For this purpose, we give another interpretation of this composition.

First, by the definition of the symmetric product of complexes, the part of $W_a(\mathcal{F}^\bullet)$ containing the degree zero and degree one terms is

$$\delta_a^0 : W_a(\mathcal{F}^0) \longrightarrow \oplus_{j=1}^n \tilde{W}_a(\mathcal{F}^0)_j \otimes \mathcal{F}_j^1,$$

whose pullback via ρ' is

$$\rho'^*(\delta_a^0)(W_a(e)) = \sum W_a(e)_j \otimes \zeta_j(e_j), \quad e = (e_1, \cdots, e_n) \in \oplus_{j=1}^n \rho'^* \mathcal{F}_j^0. \tag{5.6}$$

Because ν_a is a homomorphism of complexes, we have $\nu_a^1 \circ \delta_a^0 = 0$. Therefore

$$\rho^*(\eta \circ \bar{\zeta}) = \rho^*\eta \circ \rho^*\bar{\zeta} = \rho'^*(\nu_a^1)(W_a(s)_j\zeta_j(s_j)) = (\rho'^*(\nu_a^1) \circ \rho'^*)(\delta_a^0)(W_a(s)) = 0.$$

□

We next endow F_1 with a Hermitian metric, thus inducing a Hermitian metric $(\cdot,\cdot)$ on V. Via the linear-antilinear pairing $(\cdot,\cdot) : V \times V \to \mathbb{C}$, we obtain an anti-$\mathbb{C}$-linear isomorphism $c : V^\vee \to V$. Viewing η as a section in $V^\vee$, $c(\eta) \in C^\infty(V)$. Define

$$d_\eta = \bar{\zeta} + c(\eta) \in C^\infty(V).$$

Lemma 5.4. *The vanishing locus of d_η is the zero section of $Y \to M$, thus is isomorphic to M and is proper.*

Proof. For a closed point $v \in Y$ lying over $w \in M$ such that $d_\eta(v) = \bar{\zeta}(v)+c(\eta)(v) = 0$, by composing with η, we have $\eta \circ c(\eta)(v) = 0$. As c is induced by the duality from a Hermitian metric, $\eta \circ c(\eta)(v) = 0$ if and only if $\eta(v) = 0$. Applying Lemma 3.6, we conclude that this is possible if and only if v lies in the zero section of $Y \to M$. □

By the compactness of the vanishing locus of the smooth section d_η, the pair (V, d_η) defines a localized Euler class

$$e_{\rm loc}(V, d_\eta) \in H_*(M, \mathbb{Q}).$$

In [FJR2, Thm 4.1.8 (5)], the authors constructed Witten's top Chern class $[W_g(\gamma)]^{\rm vir}_{FJRW}$ (analogue to the class $[\overline{M}_{g,\gamma}(G)^p]^{\rm vir}$, where W stands for the polynomial W) in a more general setting. In the case where γ is narrow, they proved that the class $[W_g(\gamma)]^{\rm vir}_{FJRW}$ is given by the class $e_{\rm loc}(V, d_\eta)$ just constructed. We state it as a proposition.

Proposition 5.5 ([FJR2, Thm 4.1.8 (5)])**.** *Let $([\mathbb{C}^n/G], W)$ be an LG space, let g, ℓ be integers and $\gamma = (\gamma_i)_{i=1}^\ell$ be narrow, and let $\epsilon = \operatorname{rank} F^0 - \operatorname{rank} F^1$. Then*

$$[W_g(\gamma)]^{\rm vir}_{FJRW} = (-1)^\epsilon \cdot e_{\rm loc}(V, d_\eta) \in H_*(M).$$

Let $\imath_* : A_*(M) \to H_*(M, \mathbb{Q})$ be the tautological homomorphism.

Theorem 5.6. *Let the notation be as in Proposition 5.5. Then*

$$\imath_*[\overline{M}_{g,\gamma}(G)^p]^{\rm vir} = (-1)^\epsilon \cdot [W_g(\gamma)]^{\rm vir}_{FJRW} \in H_*(M, \mathbb{Q}).$$

Here is our strategy for proving this theorem. Using the quasi-homogeneous polynomial W, we will construct a cosection σ_η of the obstruction sheaf of the obstruction theory ϕ_ζ with the property that its degeneracy locus is the zero section of $Y \to M$. In this way, we will obtain a cosection localized virtual cycle $[X]^{\rm vir}_{\sigma_\eta}$. Using the analytic construction of the cosection localized virtual cycle [KL, Appendix], we will conclude that $e_{\rm loc}(V, d_\eta) = \imath_*[X]^{\rm vir}_{\sigma_\eta}$. Finally, by comparing the obstruction theories, we prove $[\overline{M}_{g,\gamma}(G)^p]^{\rm vir} = [X]^{\rm vir}_{\sigma_\eta}$.

First we construct the cosection σ_η. Let $\eta_X = \eta|_X$. Since $\eta \circ \bar{\zeta} = 0$ and $\bar{\zeta}|_X = 0$, for the differential $d\bar{\zeta}|_X : T_X Y \to V|_X$, we have $\eta_X \circ d\bar{\zeta}|_X = 0$. Thus η_X lifts to a cosection

$$\sigma_\eta : \mathcal{O}b_{X/M} = H^1(E_\zeta^\bullet) \longrightarrow \mathcal{O}_X \tag{5.7}$$

whose composition with $q^*\Omega_M^\vee \to H^1(E_\zeta^\bullet)$ vanishes.

By the proof of Lemma 5.4, we know that the σ_η is surjective away from the zero section of $V \to M$. Thus the degeneracy locus $D(\sigma_\eta) = M \subset X$. This gives us the associated cosection localized virtual cycle $[X]^{\mathrm{vir}}_{\sigma_\eta} \in A_*(M)$.

Lemma 5.7. *Let notations be as stated. Then* $\imath_*[X]^{\mathrm{vir}}_{\sigma_\eta} = e_{\mathrm{loc}}(V, d_\eta)$.

Proof. First, by the construction of the smooth section d_η, the C^∞-homomorphism $\eta \circ d_\eta : V \to \mathbb{C}_Y$ takes a value of 0 along $M \subset Y$ and takes a positive value on $Y - M$. Let Γ_t be the graph of $t\bar{\zeta} \in \Gamma(V)$, and let Γ_∞ be the limit of Γ_t. Then Γ_∞ is the normal cone to $X \subset Y$, embedded in $V|_X$ via the defining equation $X = (\bar{\zeta} = 0) \subset Y$.

Since $\eta \circ \bar{\zeta} = 0$, for any $t \in [0, \infty]$, we have $\eta|_{\Gamma_t} \equiv 0$. Thus $(d_\eta \cap \Gamma_t) \cap V|_{Y-M} = \emptyset$. We then pick a perturbation d'_η of d_η so that d'_η is transversal to both the zero section $0_V \subset V$ and $\Gamma_\infty \subset V$, and that $d'_\eta = d_\eta$ away from a compact subset of Y. Therefore,

$$e_{\mathrm{loc}}(V, d_\eta) = \zeta_*[0_V \cap d'_\eta] = \zeta_*[\Gamma_\infty \cap d'_\eta] = \imath_*[X]^{\mathrm{vir}}_{\sigma_\eta} \in H_*(M, \mathbb{Q}),$$

where the first identity follows from the definition of the localized Euler classes, the second identity follows from the fact that Γ_t is a homotopy between $[0_V]$ and $[\Gamma_\infty]$ and that the union of $d'_\eta \cap \Gamma_t$ for $t \in [0, \infty]$ is contained in a compact subset of V, and the last identity follows from the analytic construction of the cosection localized virtual cycles [KL, Appendix]. □

Recall that

$$[\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}} = 0^!_{\sigma,\mathrm{loc}}\big([\mathbf{C}_{X/M}]\big) \in A_*(M)$$

is constructed via the localized Gysin map $0^!_{\sigma,\mathrm{loc}}$ (cf. Definition-Proposition 3.9) applied to the virtual normal cone cycle $[\mathbf{C}_{X/M}] \in Z_*(h^1/h^0(E^\bullet_{X/M}))$ of the relative perfect obstruction theory $\phi_{X/M}$ (cf. (3.2)). By the discussion before Lemma 5.1, we have $h^1/h^0(E^\bullet_{X/M}) \cong h^1/h^0(E^\bullet_\zeta)$ and $H^1(E^\bullet_{X/M}) \cong H^1(E^\bullet_\zeta)$.

Lemma 5.8. *Let the notation be as stated. Then*
(1). the two cosections are equal, i.e., $\sigma_\eta = \sigma : H^1(E^\bullet_\zeta) \cong H^1(E^\bullet_{X/M}) \to \mathcal{O}_X$.
(2). $[\mathbf{C}_{X/M}] = [\mathbf{C}_\zeta] \in Z_*(h^1/h^0(E^\bullet_{X/M})) \cong Z_*(h^1/h^0(E^\bullet_\zeta))$.

Proof of Theorem 5.6. By Lemma 5.7, we have $[W_g(\gamma)]^{\mathrm{vir}}_{FJRW} = \imath_*[X]^{\mathrm{vir}}_{\sigma_\eta}$. By the definitions, $[X]^{\mathrm{vir}}_{\sigma_\eta} = 0^!_{\sigma_\eta,\mathrm{loc}}[\mathbf{C}_\zeta]$ and $\overline{M}_{g,\gamma}(G)^p)^{\mathrm{vir}} = 0^!_{\sigma,\mathrm{loc}}[\mathbf{C}_{X/M}]$. Applying Lemma 5.8, we conclude $0^!_{\sigma_\eta,\mathrm{loc}}[\mathbf{C}_\zeta] = 0^!_{\sigma,\mathrm{loc}}[\mathbf{C}_{X/M}] \in A_*(M)$. □

Proof of Lemma 5.8. We prove (1) first. Let S be an affine scheme, and let $\rho : S \to X$ and $\rho' = q \circ \rho : S \to M$ be morphisms. As in the discussion leading up to (5.5), we can assume that ρ is given by $s = (s_1, \cdots, s_n) \in \oplus_{j=1}^n \Gamma(\rho'^* \mathcal{F}_j^0)$. Recall that $\rho^*(\eta) : \rho^* V \to \mathcal{O}_S$ (cf. (5.5)) sends $(\dot{s}_1, \cdots, \dot{s}_n) \in \rho^* V = \oplus_{j=1}^n \rho'^* \mathcal{F}_j^1$ to

$$\sum_a \alpha_a \rho'^*(\nu_a^1)(W_a(s)_1 \dot{s}_1, \cdots, W_a(s)_n \dot{s}_n).$$

Let $\dot{t}_j \in H^1(\rho'^* \mathcal{F}_j^\bullet)$ be the image of $\dot{s}_j$. Then

$$(\dot{t}_1, \cdots, \dot{t}_n) \in \rho^* \mathcal{O}b_{X/M} = \oplus_{j=1}^n H^1(\rho'^* \mathcal{F}_j^\bullet),$$

and because ν_a is the complex realization of the isomorphism τ_a in (5.3) and $\rho^*(\sigma_\eta)$ is the descent of $\rho^*(\eta)$, we have

$$\rho^*(\sigma_\eta)(\dot t_1,\cdots,\dot t_n)=\rho^*(\eta)(\dot s_1,\cdots,\dot s_n)=\sum_a \alpha_a \rho'^*(\nu_a^1)(W_a(s)_1\dot s_1,\cdots,W_a(s)_n\dot s_n)$$

$$=\sum_a\sum_j \alpha_a\rho'^*(\tau_a)(W_a(s)_j\dot t_j)\in\Gamma(\rho'^*R^1\pi_{M*}\omega_{\mathcal{C}_M/M}),$$

where the last identity follows from the same reasoning as that given before Lemma 3.5. The last term above is of the same form as (3.6). Thus we conclude that $\rho^*(\sigma_\eta)=\rho^*(\sigma)$. This proves (1).

We next prove (2). Applying Lemma 5.1, it suffices to prove (2) for a complex of locally free sheaves $F^\bullet=[F^0\to F^1]$ such that $F^\bullet\cong\mathbf{R}\pi_{M*}\mathcal{E}$. We choose such a complex $F^\bullet$ now.

Since $\pi_M:\mathcal{C}_M\to M$ is a family of pointed stable twisted curves, $\omega^{log}_{\mathcal{C}_M/M}$ is π_M-ample. Thus for a sufficiently large and divisible r, $\mathcal{K}=(\omega^{log}_{\mathcal{C}_M/M})^{\otimes r}$ is sufficiently π_M-ample and is the pullback of an invertible sheaf on the coarse moduli space of $\mathcal{C}_M$. Hence $\pi_M^*\pi_{M*}\mathcal{K}\to\mathcal{K}$ is surjective. By dualizing this homomorphism, taking the cokernel, and then tensoring with $\mathcal{K}$, we obtain an exact sequence of locally free sheaves of $\mathcal{O}_{\mathcal{C}_M}$-modules

$$0\longrightarrow\mathcal{O}_{\mathcal{C}_M}\longrightarrow(\pi_M^*\pi_{M*}\mathcal{K})^\vee\otimes\mathcal{K}\longrightarrow\mathcal{K}_1\longrightarrow 0.$$

Tensoring with $\mathcal{E}$, and renaming the second and third terms as $\mathcal{E}^0$ and $\mathcal{E}^1$, we obtain the following exact sequence of locally free sheaves of $\mathcal{O}_{\mathcal{C}_M}$-modules

$$0\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}^0\longrightarrow\mathcal{E}^1\longrightarrow 0. \tag{5.8}$$

Because $\mathcal{K}$ is sufficiently π_M-ample, we have $R^1\pi_{M*}\mathcal{E}^1=R^1\pi_{M*}\mathcal{E}^0=0$. Let the new complex $F^\bullet$ be

$$F^\bullet=[\zeta:F^0\to F^1]=[\pi_{M*}\mathcal{E}^0\to\pi_{M*}\mathcal{E}^1].$$

By the vanishing, we have an isomorphism of derived objects $F^\bullet\cong\mathbf{R}\pi_{M*}\mathcal{E}$.

Let $\tilde q:Y\to M$, $X=(\bar\zeta=0)\subset Y$, and the obstruction theory ϕ_ζ be constructed from this $F^\bullet$ as before. We will prove that $\phi_\zeta=\phi^{\ge -1}_{X/M}$. Note that this implies that $[\mathbf{C}_\zeta]=[\mathbf{C}_{X/M}]$ as cycles in $Z_*(h^1/h^0(E^\bullet_{X/M}))$.

Let $\mathcal{C}_Y=Y\times_M\mathcal{C}_M$ and $\mathcal{C}_X=X\times_M\mathcal{C}_M$. Denote by E, E^0 and E^1 the (total spaces of the) vector bundles associated with $\mathcal{E}$, $\mathcal{E}^0$ and $\mathcal{E}^1$, respectively. Because Y is the total space of F^0, via the projection $\tilde q:Y\to M$, the isomorphism $F^0=\pi_{M*}\mathcal{E}^0$ induces a tautological section in $\Gamma(\tilde q^*\pi_{M*}\mathcal{E}^0)$. By composing with the tautological map $\pi_M^*\pi_{M*}\mathcal{E}^0\to\mathcal{E}^0$, it induces a section $e\in\Gamma(\mathcal{C}_Y,\tilde p^*\mathcal{E}^0)$, where $\tilde p:\mathcal{C}_Y\to\mathcal{C}_M$ is the projection. Let $e_Y:\mathcal{C}_Y\to E^0$ be the evaluation morphism associated with e. Via (5.8), its restriction to $\mathcal{C}_X\subset\mathcal{C}_Y$ then lifts to a section $e_X:\mathcal{C}_X\to E$, fitting into the Cartesian square

$$\begin{array}{ccc}\mathcal{C}_X & \xrightarrow{e_X} & E\\ \downarrow & & \downarrow\\ \mathcal{C}_Y & \xrightarrow{e_Y} & E^0.\end{array}$$

Let $I_{\mathcal{C}_X}$ (resp. I_E) be the ideal sheaf of $\mathcal{C}_X \subset \mathcal{C}_Y$ (resp. $E \subset E^0$). As $\mathcal{C}_Y$ and E^0 are smooth over $\mathcal{C}_M$, we obtain the induced homomorphism

$$(5.9)\qquad \begin{array}{ccc} e_X^* L^{\bullet\geq -1}_{E/\mathcal{C}_M} & = & [e_X^*\, I_E/I_E^2 \to e_X^*\, \Omega_{E^0/\mathcal{C}_M}] \\ \downarrow & & \downarrow (h_{-1},h_0) \\ L^{\bullet\geq -1}_{\mathcal{C}_X/\mathcal{C}_M} & = & [I_{\mathcal{C}_X}/I^2_{\mathcal{C}_X} \to \Omega_{\mathcal{C}_Y/\mathcal{C}_M}|_{\mathcal{C}_X}] \end{array}$$

Let $\tau : E \to \mathcal{C}_M$ be the projection. Then $I_E/I_E^2 = N^\vee_{E/E^0} = \tau^*\mathcal{E}^{1\vee}$ and $\Omega_{E^0/\mathcal{C}_M}|_E = \tau^*\mathcal{E}^{0\vee}$. Let $f = \tau \circ e_X : \mathcal{C}_X \to \mathcal{C}_M$ be the projection. Thus $e_X^*\tau^* = f^*$, and

$$e_X^*\, I_E/I_E^2 = f^*\mathcal{E}^{1\vee} \quad \text{and} \quad e_X^*\, \Omega_{E^0/\mathcal{C}_M} = f^*\mathcal{E}^{0\vee}.$$

Let I_X be the ideal sheaf of $X \subset Y$. Because $\pi_X : \mathcal{C}_X \to X$ is flat, $I_{\mathcal{C}_X}/I^2_{\mathcal{C}_X} = \pi_X^*(I_X/I_X^2)$ and $L^{\bullet\geq -1}_{\mathcal{C}_X/\mathcal{C}_M} \cong \pi_X^* L^{\bullet\geq -1}_{X/M}$. Thus (5.9) is identical to

$$(5.10)\qquad \begin{array}{ccc} e_X^* L^{\bullet\geq -1}_{E/\mathcal{C}_M} & = & [f^*\mathcal{E}^{1\vee} \to f^*\mathcal{E}^{0\vee}] \\ \downarrow h & & \downarrow (h_{-1},h_0) \\ \pi_X^* L^{\bullet\geq -1}_{X/M} & = & [\pi_X^*(I_X/I_X^2) \to \pi_X^*\Omega_{Y/M}]. \end{array}$$

Applying the Grothendieck duality

$$\varphi : \mathrm{Hom}_{\mathcal{C}_X}(G_1^\vee, \pi_X^* G_2) \xrightarrow{\cong} \mathrm{Hom}_X((\mathbf{R}\pi_{X*}G_1)^\vee, G_2),$$

for $G_1 \in D^b(\mathcal{C}_X)$ and $G_2 \in D^b(X)$, to the columns of (5.10), and using $R^1\pi_{X*}f^*\mathcal{E}^i = 0$ because $R^1\pi_{M*}\mathcal{E}^i = 0$, $i = 0, 1$, we obtain the arrow $\varphi(h) = (\varphi(h_{-1}), \varphi(h_0))$ shown below:

$$(5.11)\qquad \begin{array}{ccc} (F^\bullet)^\vee & = & [(\pi_{X*}f^*\mathcal{E}^1)^\vee \to (\pi_{X^*}f^*\mathcal{E}^0)^\vee] \\ \downarrow \varphi(h) & & \downarrow (\varphi(h_{-1}),\varphi(h_0)) \\ L^{\bullet\geq -1}_{X/M} & = & [I_X/I_X^2 \to \Omega_{Y/M}|_X]. \end{array}$$

Following [CL, Prop 2.5], we have $\phi_{X/M} = \varphi(h)^\vee$. Also $\varphi(h_0)$ is given by $(\pi_{X^*}f^*\mathcal{E}^0)^\vee = (q^*\mathcal{E}^0)^\vee = \Omega_{Y/M}|_X$. On the other hand, via $\pi_{X*}f^*\mathcal{E}^1 = q^*\pi_{M*}\mathcal{E}^1 = q^*F^1 = V|_X$, one can easily check that $\varphi(h_{-1}) = \bar\zeta$. Thus (5.11) is identical to (5.2). This proves that $(\phi^{\geq -1}_{X/M})^\vee = \phi_\zeta^\vee$, thus $\phi^{\geq -1}_{X/M} = \phi_\zeta$. □

5.2. **Comparison with Polishchuk-Vaintrob's construction.** In this subsection, we prove the equivalence of our construction and Polishchuk-Vaintrob's construction [PV1] of Witten's top Chern class.

We first introduce the initial data (triplet) (V, ζ, η) over which Polishchuk-Vaintrob's construction applies. The triplet consists of a pure rank vector bundle V over a pure dimensional DM stack Y over $\mathbb{C}$, and sections $\zeta \in \Gamma(Y, V)$ and $\eta \in \Gamma(Y, V^\vee)$ such that $\eta \circ \zeta = 0$. Note that the (Y, ζ, η) constructed in the previous subsection satisfies this requirement.

Let $X = (\zeta = 0) \subset Y$, and let $D = (\eta = 0) \cap X \subset X$. We first recall the previous construction that gives the cosection localized virtual cycle $[X]^{\mathrm{vir}}_{\mathrm{loc}}$. Let $\mathbf{C}_{X/Y}$ be the normal cone of X relative to Y, which is a subcone of $V^\vee|_X$. We claim that $\eta|_{\mathbf{C}_{X/Y}} = 0$. Indeed, since $\eta \circ \zeta = 0$, and since $\eta : V \to \mathcal{O}_Y$ is a homomorphism, $\eta \circ (t\zeta) = 0$. As $\mathbf{C}_{X/Y}$ is the specialization of the graph of $t\zeta$ for $t \to \infty$, we have $\eta|_{\mathbf{C}_{X/Y}} = 0$.

Define $\sigma = \eta|_X : V|_X \to \mathcal{O}_X$. Its degeneracy locus (non-surjective locus) is $X \times_Y (\eta = 0) = D$. Let $V|_X(\sigma) = V|_D \cup \ker\{V|_{X-D} \to \mathcal{O}_{X-D}\}$ be the kernel stack of σ. Then $[\mathbf{C}_{X/Y}] \in Z_*(V|_X(\sigma))$. Applying the cosection localized Gysin map $0^!_{\sigma,loc} : A_*(V|_X(\sigma)) \to A_*(D)$, we obtain the class

$$[X]^{\mathrm{vir}}_{\sigma,loc} := 0^!_{\sigma,loc}[\mathbf{C}_{X/Y}] \in A_*(D).$$

Note that when (V, ζ, η) equals (E_0, V, ζ, η) in the previous subsection, we have $X = \overline{M}_{g,\gamma}(G)^p$ and $0^!_{\sigma,loc}[\mathbf{C}_{X/Y}] = [\overline{M}_{g,\gamma}(G)^p]^{\mathrm{vir}}$, which is Witten's top Chern class constructed using cosection localization.

In [PV1], Polishchuk-Vaintrob applied MacPherson's graph construction to a double-periodic complex to construct a cycle. By applying it to the case $(Y, V, \zeta, \eta) = (E_0, Y, \zeta, \eta)$, they obtained their construction of Witten's top Chern class of the LG space $([\mathbb{C}^n/G], W)$. We briefly describe their construction.

For (Y, V, ζ, η), one can form the vector bundles over Y:

$$S^+ = \bigoplus_{k \text{ even}} \wedge^k V^\vee, \quad S^- = \bigoplus_{k \text{ odd}} \wedge^k V^\vee, \quad \text{and} \quad S = S^+ \oplus S^-.$$

The section $s = (\zeta, \eta) \in \Gamma(Y, V \oplus V^\vee)$ defines a two-periodic complex

$$S^\bullet = [\cdots \xrightarrow{\wedge s} S^+ \xrightarrow{\wedge s} S^- \xrightarrow{\wedge s} S^+ \xrightarrow{\wedge s} S^- \xrightarrow{\wedge s} \cdots]. \tag{5.12}$$

Clearly, $S^\bullet$ is exact outside $D = (s = 0)$.

By adopting MacPherson's graph construction in this case, they constructed a localized Chern character $ch^Y_D(S^\bullet) \in A^*(D \to Y)$ of the two-periodic complex $S^\bullet$, and proved that ([PV1, Thm 3.2])

$$td(V|_D) \cdot ch^Y_D(S^\bullet) \in A^r(D \to Y), \quad r = \operatorname{rank} V. \tag{5.13}$$

Definition 5.9 (Polishchuk-Vaintrob [PV1])**.** *Let (V, ζ, η) be as stated. Then Witten's top Chern class of (V, ζ, η) is*

$$c_{top}(V, \zeta, \eta) := td(V|_D) \cdot ch^Y_D(S^\bullet) \cdot [Y] \in A_*(D). \tag{5.14}$$

Witten's top Chern class constructed by Polishchuk-Vaintrob is obtained by applying this definition to the (E_0, V, ζ, η) constructed in the previous subsection.

Proposition 5.10. *Let the notation be as in Definition 5.9. Suppose Y is smooth. Then*

$$[X]^{\mathrm{vir}}_{\sigma,loc} = c_{top}(V, \zeta, \eta) \in A_*(D).$$

Corollary 5.11. *Let (V, ζ, η) be (E_0, V, ζ, η) constructed in the previous subsection. Then $D = \overline{M}_{g,\gamma}(G)$, $X = \overline{M}_{g,\gamma}(G)^p$, and*

$$[\overline{M}_{g,\gamma}(G)^p)]^{\mathrm{vir}} = c_{top}(V, \zeta, \eta) \in A_*(\overline{M}_{g,\gamma}(G)).$$

We divide the proof of the proposition into several lemmas.

Lemma 5.12. *Let the situation be as in Definition 5.9. Suppose $\eta = 0$, then*

$$td(V|_D) \cdot ch^Y_D(S^\bullet) = \zeta^*[\iota_Y] \in A^r(X \to Y), \quad r = \operatorname{rank} V,$$

where $\iota_Y : Y \to V$ is the zero section, which defines the class $[\iota_Y] \in A^r(Y \to V)$, and $X \to Y$ is the inclusion. Consequently, $c_{top}(V, \zeta, 0) = [X]^{\mathrm{vir}}_{\sigma,loc}$.

Proof. Let $p : V \to Y$ be the projection and $e \in \Gamma(V, p^*V)$ be the tautological section. Let $\wedge^{-\bullet} p^*V^\vee$ be the Koszul complex (concentrated in degrees $[-r, 0]$) induced by e. Applying [PV1, Prop 2.3 (vi)] gives

$$ch_Y^V(\wedge^{-\bullet} p^*V^\vee) = td(V)^{-1}[\iota_Y]. \tag{5.15}$$

Applying ζ^* to (5.15) and using the compatibility between the pullback and the product of invariant classes, we have

$$ch_D^Y(\zeta^* \wedge^{-\bullet} p^*V^\vee) = \zeta^* ch_Y^V(\wedge^{-\bullet} p^*V^\vee) = td(V|_D)^{-1}\zeta^*[\iota_Y], \tag{5.16}$$

where the first identity is by [PV1, Prop 2.3(iii)]. Observe that the complex $\zeta^* \wedge^{-\bullet} p^*V^\vee$ is identical to $[\cdots \to \wedge^k V^\vee \xrightarrow{\zeta} \wedge^{k-1} V^\vee \to \cdots]$, and its associated two-periodic complex following [PV1, Prop 2.2] is identical to (5.12) because $\eta = 0$. Hence [PV1, Prop 2.2] implies the first term in (5.16) is equal to $ch_D^Y(S^\bullet)$. □

Corollary 5.13. *Let $Z \subset Y$ be a Cartier divisor, and let $U^\bullet = [\mathcal{O}_Y \xrightarrow{1} \mathcal{O}_Y(Z)]$ of amplitude $[0, 1]$. Denote by $ch_Z^Y(U^{\bullet\vee}) \in A^*(Z \to Y)$ the localized Chern character defined in* [Fu, Sect 18.1]*. Then*

$$td[\mathcal{O}_Y(Z)] \cdot ch_Z^Y(U^{\bullet\vee}) = [\iota_Z] \in A^1(Z \xrightarrow{\iota_Z} Y)$$

and

$$td[\mathcal{O}_Y(-Z)] \cdot ch_Z^Y(U^\bullet) = -[\iota_Z] \in A^1(Z \xrightarrow{\iota_Z} Y).$$

Proof. Applying Lemma 5.12 to $V = \mathcal{O}(Z)$, $\zeta = 1$ and $\eta = 0$ gives the first identity. The second identity follows from [Fu, Example 18.1.2 (b)] and the first identity. □

Lemma 5.14. *Let notations be as stated, and suppose Y is smooth. Then*

$$c_{top}(V, \zeta, \eta) = [X]^{\mathrm{vir}}_{\sigma,loc}.$$

Proof. Let $\lambda : \tilde{Y} \to Y$ be the blowing up of Y along $(\eta = 0)$, $\tilde{Y}$. Let $(\tilde{V}, \tilde{\zeta}, \tilde{\eta}) := (\lambda^*V, \lambda^*\zeta, \lambda^*\eta)$ be the pullback of (V, ζ, η) by λ. Let $\tilde{X} := (\tilde{\zeta} = 0) = \lambda^{-1}(X)$, $\tilde{D} := (\tilde{\zeta} = 0) \cap (\tilde{\eta} = 0) = \lambda^{-1}(D)$, and $\tilde{\sigma} = \tilde{\eta}|_{\tilde{X}}$.

Define $\lambda' = \lambda|_{\tilde{D}} : \tilde{D} \to D$. Since Y is smooth, using the deformation to normal cone construction, we conclude that the cycle $\mathbf{C}_{X/Y} \in Z_*(V)$ is the push-forward of $\mathbf{C}_{\tilde{X}/\tilde{Y}} \in Z_*(\tilde{V})$ under the proper morphism $\tilde{V} \to V$. Because the Gysin map commutes with proper push-forwards, we conclude

$$\lambda'_*(0^!_{\tilde{\sigma},loc}[\mathbf{C}_{\tilde{X}/\tilde{Y}}]) = 0^!_{\sigma,loc}[\mathbf{C}_{X/Y}] \in A_*(V).$$

On the other hand, [PV1, Prop 2.3(iii)] implies $\lambda'_*(c_{top}(\tilde{V}, \tilde{\zeta}, \tilde{\eta})) = c_{top}(V, \zeta, \eta)$. Thus to prove the lemma, it suffices to prove the case when (the triplet (V, ζ, η)) $\eta : V \to \mathcal{O}_Y$ factors through a surjection $\eta' : V \to \mathcal{O}_Y(-Z) \subset \mathcal{O}_Y$ for a Cartier divisor $Z \subset Y$ where Y is not necessarily smooth. Let $K \subset V$ be the kernel of η', which is a (locally free) vector bundle since η' is surjective. Since $\eta(\zeta) = 0$, $\zeta \in \Gamma(V)$ lifts to $\zeta' \in \Gamma(K)$.

The exact sequence

$$0 \longrightarrow K \longrightarrow V \longrightarrow \mathcal{O}_Y(-Z) \longrightarrow 0 \tag{5.17}$$

is usually non-split. We will use a standard argument to reduce the general case to the split case. Let $\bar{Y} := Y \times \mathbb{A}^1$ with the projection $\pi_Y : \bar{Y} \to Y$. We form the vector bundle $\bar{V}$ on $\bar{Y}$ that fits into the exact sequence

$$0 \longrightarrow \pi_Y^* K \longrightarrow \bar{V} \longrightarrow \pi_Y^* \mathcal{O}_Y(-Z) \longrightarrow 0$$

such that its restriction to $t \neq 0$ (resp. 0) is the exact sequence (5.17) (resp. splits).

We let $\bar\zeta = \pi_Y^*\zeta' \in \Gamma(\pi_Y^* K)$. We also view it as a section in $\bar V$ via the inclusion $\pi_Y^* K \subset \bar V$. We let $\bar\eta : \bar V \to \mathcal{O}_{\bar Y}$ be the composition of $\bar V \to \pi_Y^*\mathcal{O}_Y(-Z)$ with the inclusion $\pi_Y^*\mathcal{O}_Y(-Z) \subset \mathcal{O}_{\bar Y}$.

The triplet $(\bar V, \bar\zeta, \bar\eta)$ over $\bar Y$ has the property that $\bar X = (\bar\zeta = 0) = X \times \mathbb{A}^1$ and $\bar D := \bar X \cap (\bar\eta = 0) = D \times \mathbb{A}^1$. Let $\iota_t : D = \bar D \times_{\mathbb{A}^1} t \subset \bar D$ be the inclusion, and define $\bar Y_y = \bar Y \times_{\mathbb{A}^1} t$. Then [PV1, Prop 2.3(iii)] and [Fu, Def 17.1 (C3)] imply that

$$\iota_t^! c_{top}(\bar V, \bar\zeta, \bar\eta) = c_{top}(\bar V|_{\bar Y_t}, \bar\zeta|_{\bar Y_t}, \bar\eta|_{\bar Y_t}) \in A_*(\bar D_t) = A_*(D)$$

is constant in $t \in \mathbb{A}^1$.

On the other hand, let $\bar X_t = \bar X \times_{\mathbb{A}^1} t$, and let $[\bar X_t]^{\rm vir}_{loc}$ and $[\bar X]^{\rm vir}_{loc}$ be the cosection localized virtual classes of the data $(\bar V_t, \bar\zeta_t, \bar\eta_t)$ and $(\bar V, \bar\zeta, \bar\eta)$ respectively. Since $\bar X$ is a constant family over $\mathbb{A}^1$, Theorem 5.2 in [KL] implies that $\iota_t^![\bar X]^{\rm vir}_{loc} = [\bar X_t]^{\rm vir}_{loc} \in A_*(D)$ is constant in t. Thus to prove the lemma, we only need to show that $\iota_0^! c_{top}(\bar V, \bar\zeta, \bar\eta) = [\bar X]^{\rm vir}_{loc}$. Equivalently, we only need to prove that in the case where (5.17) splits, $c_{top}(V, \zeta, \eta) = [X]^{\rm vir}_{\sigma, loc}$.

Because $X = (\zeta = 0)$ and ζ lifts to a section in $K \subset V$, we have the normal cone $\mathbf{C}_{X/Y} \subset K|_X$, and $0_K^![\mathbf{C}_{X/Y}] \in A_*(X)$. Following the definition of the cosection localization [KL], we conclude

$$[X]^{\rm vir}_{\sigma,loc} = -\iota_Z^! 0^! [\mathbf{C}_{X/Y}] \in A_*(D) \tag{5.18}$$

where $\iota_Z^! : A_*(X) \to A_{*-1}(D)$ is the Gysin map associated with the Cartesian square

$$\begin{array}{ccc} D & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\iota_Z} & Y. \end{array}$$

Next we analyze $c_{top}(V, \zeta, \eta)$. The Koszul complex

$$C^\bullet := [\cdots \to \wedge^k K^\vee \xrightarrow{\zeta} \wedge^{k-1} K^\vee \to \cdots]$$

(with $-k$ as the amplitude of $\wedge^k K^\vee$) is of finite length. Also let

$$U^\bullet := [\mathcal{O}_Y \xrightarrow{\subset} \mathcal{O}_Y(Z)]$$

be of amplitude $[0,1]$. As $C^\bullet$ is exact outside X and $U^\bullet$ is exact outside Z, their tensor product

$$\mathfrak{V}^\bullet := C^\bullet \otimes U^\bullet = [\cdots \to \mathfrak{V}^k \xrightarrow{d_k} \mathfrak{V}^{k-1} \to \cdots]$$

is exact outside $X \times_Y Z = D$ and of finite length. This complex leads to another complex

$$S^\bullet_{\mathfrak{V}} = [\cdots \to S^+_{\mathfrak{V}} \xrightarrow{d^+} S^-_{\mathfrak{V}} \xrightarrow{d^-} S^+_{\mathfrak{V}} \to \cdots],$$

where $S^+_{\mathfrak{V}} = \oplus_k \mathfrak{V}^{2k}$ and $S^-_{\mathfrak{V}} = \oplus_k \wedge \mathfrak{V}^{2k+1}$, with differentials $d^+ = \oplus_k d_{2k}$ and $d^- = \oplus_k d_{2k+1}$.

We next introduce

$$J_+ = \oplus_k \wedge^{2k} V^\vee \quad \text{and} \quad J_- = \oplus_k \wedge^{2k-1} V^\vee,$$

and let $\zeta_\pm : J_\pm \to J_\mp$ be induced by contracting with ζ. We have

$$S^+_{\mathfrak{V}} = J_+ \oplus J_-(Z) \quad \text{and} \quad S^-_{\mathfrak{V}} = J_- \oplus J_+(Z).$$

For $a \in J_+, a' \in J_-(Z)$ and $b \in J_-, b' \in J_+(Z)$, the boundary maps take the form

$$d_+(a,a') = (\zeta_+(a), a\otimes 1_Z + \zeta_-(a')) \quad \text{and} \quad d_-(b,b') = (\zeta_-(b), b\otimes 1_Z + \zeta'(b')),$$

where 1_Z is the section of $\mathcal{O}_Y(Z)$ as the image of 1 under $\mathcal{O}_Y \to \mathcal{O}_Y(Z)$. Using the splitting $V = K \oplus \mathcal{O}_Y(-Z)$ and its compatibility with ζ, η, one can easily check that the two-periodic complex $S^\bullet$ is identical to $S^\bullet_{\mathfrak{V}}$. Thus

$$ch^Y_D(S^\bullet) = ch^Y_D(S^\bullet_{\mathfrak{V}}) = ch^Y_D(\mathfrak{V}^\bullet),$$

where the last term is the localized Chern character defined in [PV1, Sect. 18] (for the complex $\mathfrak{V}^\bullet$) and the last equality holds by [PV1, Prop 2.2]. Now as $\mathfrak{V}^\bullet = C^\bullet \otimes U^\bullet$, Example 18.1.5 in [Fu] implies

$$ch^Y_D(\mathfrak{V}^\bullet) = ch^Y_Z(U^\bullet) \cup ch^Y_X(C^\bullet).$$

Using the splitting $V = K \oplus \mathcal{O}_Y(-Z)$, and letting $\iota_X : X \to Y$ be the inclusion, (5.14) becomes

$$\begin{aligned} c_{top}(Y,V,\zeta,\eta) &:= td(V|_D)\cdot ch^Y_D(S^\bullet)\cdot[Y] \\ &= [td(\mathcal{O}_Y(-Z)|_Z)\cdot ch^Y_Z(U^\bullet)] \cup [td(K|_X)\cdot ch^Y_X(C^\bullet)]\cdot[Y] \\ &= \iota_X^*([td(\mathcal{O}_Y(-Z)|_Z)\cdot ch^Y_Z(U^\bullet)])(0^![\mathbf{C}_{X/Y}]) \\ &= -\iota_Z^! 0^![\mathbf{C}_{X/Y}] = [X]^{\mathrm{vir}}_{\sigma,loc}, \end{aligned}$$

where the second equality follows from [Fu, Prop 17.8.2], the third equality follows from Lemma 5.12 for $(Y,K,\zeta,0)$, and the last equality follows from Corollary 5.13. □

Combining Lemmas 5.12 and 5.14 gives the proof of Proposition 5.10.

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