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Incompressible limit of solutions of multidimensional steady compressible Euler equations

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Abstract. A compactness framework is formulated for the incompressible limit of approximate solutions with weak uniform bounds with respect to the adiabatic exponent for the steady Euler equations for compressible fluids in any dimension. One of our main observations is that the compactness can be achieved by using only natural weak estimates for the mass conservation and the vorticity. Another observation is that the incompressibility of the limit for the homentropic Euler flow is directly from the continuity equation, while the incompressibility of the limit for the full Euler flow is from a combination of all the Euler equations. As direct applications of the compactness framework, we establish two incompressible limit theorems for multidimensional steady Euler flows through infinitely long nozzles, which lead to two new existence theorems for the corresponding problems for multidimensional steady incompressible Euler equations.

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1. Introduction

We are concerned with the incompressible limit of solutions of multidimensional steady compressible Euler equations. The steady compressible full Euler equations take the form:

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \\ \operatorname{div}(\rho u E + u p) = 0, \end{cases} \tag{1.1}$$

while the steady homentropic Euler equations have the form:

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \end{cases} \tag{1.2}$$

where $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $n \geq 2$, $u := (u_1, \ldots, u_n) \in \mathbb{R}^n$ is the flow velocity,

$$|u| = \left(\sum_{i=1}^{n} u_i^2 \right)^{1/2} \tag{1.3}$$

is the flow speed, ρ, p, and E represent the density, pressure, and total energy, respectively, and $u \otimes u := (u_i u_j)_{n \times n}$ is an $n \times n$ matrix (cf. [8]).

For the full Euler case, the total energy is

$$E = \frac{|u|^2}{2} + \frac{p}{(\gamma - 1)\rho}, \tag{1.4}$$

with adiabatic exponent $\gamma > 1$, the local sonic speed is

$$c = \sqrt{\frac{\gamma p}{\rho}}, \tag{1.5}$$

and the Mach number is

$$M = \frac{|u|}{c} = \frac{1}{\sqrt{\gamma}}|u|\sqrt{\frac{\rho}{p}}. \tag{1.6}$$

For the homentropic case, the pressure-density relation is

$$p = \rho^{\gamma}, \qquad \gamma > 1. \tag{1.7}$$

The local sonic speed is

$$c = \sqrt{\gamma \rho^{\gamma-1}} = \sqrt{\gamma}\, p^{\frac{\gamma-1}{2\gamma}}, \tag{1.8}$$

and the Mach number is defined as

$$M = \frac{|u|}{c} = \frac{1}{\sqrt{\gamma}}|u|p^{\frac{1-\gamma}{2\gamma}}. \tag{1.9}$$

The incompressible limit is one of the fundamental fluid dynamic limits in fluid mechanics. Formally, the steady compressible full Euler equations (1.1) converge to the steady inhomogeneous incompressible Euler equations:

$$\begin{cases} \operatorname{div} u = 0, \\ \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \end{cases} \tag{1.10}$$

while the homentropic Euler equations (1.2) converge to the steady homogeneous incompressible Euler equations:

$$\begin{cases} \operatorname{div} u = 0, \\ \operatorname{div}(u \otimes u) + \nabla p = 0. \end{cases} \tag{1.11}$$

However, the rigorous justification of this limit for weak solutions has been a challenging mathematical problem, since it is a singular limit for which singular phenomena usually occur in the limit process. In particular, both the uniform estimates and the convergence of the nonlinear terms in the incompressible models are usually difficult to obtain. Moreover, tracing the boundary conditions of the solutions in the limit process is a tricky problem.

Generally speaking, there are two processes for the incompressible limit: the adiabatic exponent γ tending to infinity and the Mach number M tending to zero [19,20]. The latter is also called the low Mach number limit. A general framework for the low Mach number limit for local smooth solutions for compressible flow was established in Klainerman–Majda [14,15]. In particular, the incompressible limit of local smooth solutions of the Euler equations for compressible fluids was established with well-prepared initial data; i.e., the limiting velocity satisfies the incompressible condition initially, in the whole space or torus. Indeed, by analyzing the rescaled linear group generated by the penalty operator (cf. [23]), the low Mach number limit can also be verified for the case of general data, for which the velocity in the incompressible fluid is the limit of the Leray projection of the velocity in the compressible fluids. This method also applies to global weak solutions of the isentropic Navier–Stokes equations with general initial data and various boundary conditions [9,10,17]. In particular, in [17], the incompressible limit on the stationary Navier–Stokes equations with the Dirichlet boundary condition was also shown, in which the gradient estimate on the velocity played the major role. For the one-dimensional Euler equations, the low Mach number limit has been proved by using the BV space in [3]. For the limit $\gamma \to \infty$, it was shown in [18] that the compressible isentropic Navier–Stokes flow would converge to the homogeneous

incompressible Navier–Stokes flow. Later, the similar limit from the Korteweg barotropic Navier–Stokes model to the homogeneous incompressible Navier–Stokes model was also considered in [16].

For the steady flow, the uniqueness of weak solutions of the steady incompressible Euler equations is still an open issue. Thus, the incompressible limit of the steady Euler equations becomes more fundamental mathematically; it may serve as a selection principle of physical relevant solutions for the steady incompressible Euler equations since a weak solution should not be regarded as the compressible perturbation of the steady incompressible Euler flow in general. Furthermore, for the general domain, it is quite challenging to obtain directly a uniform estimate for the Leray projection of the velocity in the compressible fluids.

In this paper, we formulate a suitable compactness framework for weak solutions with weak uniform bounds with respect to the adiabatic exponent γ by employing the weak convergence argument. One of our main observations is that the compactness can be achieved by using only natural weak estimates for the mass conservation and the vorticity, which was introduced in [4,7,13]. Another observation is that the incompressibility of the limit for the homentropic Euler flow follows directly from the continuity equation, while the incompressibility of the limit for the full Euler flow is from a combination of all the Euler equations. Finally, we find a suitable framework to satisfy the boundary condition without the strong gradient estimates on the velocity. As direct applications of the compactness framework, we establish two incompressible limit theorems for multidimensional steady Euler flows through infinitely long nozzles. As a consequence, we can establish the new existence theorems for the corresponding problems for multidimensional steady incompressible Euler equations.

The rest of this paper is organized as follows. In Sect. 2, we establish the compactness framework for the incompressible limit of approximate solutions of the steady full Euler equations and the homentropic Euler equations in $\mathbb{R}^n$ with $n \geq 2$. In Sect. 3, we give a direct application of the compactness framework to the full Euler flow through infinitely long nozzles in $\mathbb{R}^2$. In Sect. 4, the incompressible limit of homentropic Euler flows in the three-dimensional infinitely long axisymmetric nozzle is established.

2. Compactness framework for approximate steady Euler flows

In this section, we establish the compensated compactness framework for approximate solutions of the steady Euler equations in $\mathbb{R}^n$ with $n \geq 2$. We first consider the homentropic case, that is, the approximate solutions $(u^{(\gamma)}, p^{(\gamma)})$ satisfy

$$\begin{cases} \mathrm{div}\big(\rho^{(\gamma)} u^{(\gamma)}\big) = e_1(\gamma), \\ \mathrm{div}\big(\rho^{(\gamma)} u^{(\gamma)} \otimes u^{(\gamma)}\big) + \nabla p^{(\gamma)} = e_2(\gamma), \end{cases} \tag{2.1}$$

where $e_1(\gamma)$ and $e_2(\gamma) := (e_{21}(\gamma), \ldots, e_{2n}(\gamma))^\top$ are sequences of distributional functions depending on the parameter γ.

Remark 2.1. The distributional functions $e_i(\gamma)$, $i = 1, 2$, here present possible error terms from different types of approximation. If $(u^{(\gamma)}, p^{(\gamma)})$ with $\rho^{(\gamma)} := \big(p^{(\gamma)}\big)^{\frac{1}{\gamma}}$ are the exact solutions of the steady Euler flows, $e_i(\gamma)$, $i = 1, 2$, are both equal to zero. Moreover, the same remark is true for the full Euler case, where $e_i(\gamma)$, $i = 1, 2, 3$, are the distributional functions as introduced in (2.17).

Let the sequences of functions $u^{(\gamma)}(x) := (u_1^{(\gamma)}, \ldots, u_n^{(\gamma)})(x)$ and $p^{(\gamma)}(x)$ be defined on an open bounded subset $\Omega \subset \mathbb{R}^n$ such that the following qualities:

$$\rho^{(\gamma)} := (p^{(\gamma)})^{\frac{1}{\gamma}}, \quad |u^{(\gamma)}| := \sqrt{\sum\nolimits_{i=1}^{n} (u_i^{(\gamma)})^2}, \quad c^{(\gamma)} := \sqrt{\gamma}\big(p^{(\gamma)}\big)^{\frac{\gamma-1}{2\gamma}}, \quad M^{(\gamma)} := \frac{|u^{(\gamma)}|}{c^{(\gamma)}}, \tag{2.2}$$

$$E^{(\gamma)} := \frac{|u^{(\gamma)}|^2}{2} + \frac{\left(p^{(\gamma)}\right)^{\frac{\gamma-1}{\gamma}}}{\gamma - 1}. \tag{2.3}$$

can be well defined. Moreover, the following conditions hold:

(A.1). $M^{(\gamma)}$ are uniformly bounded by $\bar{M}$;

(A.2). $|u^{(\gamma)}|^2$ and $p^{(\gamma)} \geq 0$ are uniformly bounded in $L^1_{\text{loc}}(\Omega)$;

(A.3). $e_1(\gamma)$ and $\operatorname{curl} u^{(\gamma)}$ are in a compact set in $H^{-1}_{\text{loc}}(\Omega)$;

(H). As $\gamma \to \infty$,

$$\int_\Omega \ln\left(E^{(\gamma)}\right) \mathrm{d}x = o(\gamma).$$

Remark 2.2. In the limit $\gamma \to \infty$, the energy sequence $E^{(\gamma)}$ may tend to zero. Condition (H) is designed to exclude the case that $E^{(\gamma)}$ exponentially decays to zero as $\gamma \to \infty$. In fact, in the two applications in Sects. 3 and 4 below, both of the energy sequences $E^{(\gamma)}$ go to zero with polynomial rate so that condition (H) is satisfied automatically. It is noted that condition (H) could be replaced equivalently by a pressure condition:

$$\int_\Omega \ln\left(p^{(\gamma)}\right) \mathrm{d}x = o(\gamma) \qquad \text{as } \gamma \to \infty.$$

Indeed, from (A.1) and (2.2), we have

$$\frac{1}{\gamma - 1}(p^{(\gamma)})^{1 - \frac{1}{\gamma}} \leq E^{(\gamma)} = \frac{|u^{(\gamma)}|^2}{2} + \frac{\left(p^{(\gamma)}\right)^{\frac{\gamma-1}{\gamma}}}{\gamma - 1} \leq \frac{(\gamma - 1)\gamma \bar{M}^2 + 2}{2(\gamma - 1)}(p^{(\gamma)})^{1 - \frac{1}{\gamma}}, \tag{2.4}$$

which directly implies the equivalence of the two conditions.

Remark 2.3. Conditions (A.1)–(A.3) are naturally satisfied in the applications in Sects. 3 and 4 below.

Then, we have

Theorem 2.1. (Compensated compactness framework for the homentropic Euler case). *Let a sequence of functions* $u^{(\gamma)}(x) = (u_1^{(\gamma)}, \ldots, u_n^{(\gamma)})(x)$ *and* $p^{(\gamma)}(x)$ *satisfy conditions* (A.1)–(A.3) *and* (H). *Then, there exists a subsequence (still denoted by)* $(u^{(\gamma)}, p^{(\gamma)})(x)$ *such that, when* $\gamma \to \infty$,

$$\begin{aligned} u^{(\gamma)}(x) &\to (\bar{u}_1, \ldots, \bar{u}_n)(x) & \text{a.e. in } x \in \Omega, \\ p^{(\gamma)}(x) &\rightharpoonup \bar{p} & \text{as bounded measures.} \end{aligned} \tag{2.5}$$

Proof. We divide the proof into four steps.

1. From condition (A.2), we can see that $p^{(\gamma)}$ weakly converges to $\bar{p}$ as bounded measures when $\gamma \to \infty$.

2. Now we show that $\rho^{(\gamma)} = (p^{(\gamma)})^{\frac{1}{\gamma}}(x) \to 1$ *a.e. in* $x \in \Omega$ as $\gamma \to \infty$.

Since $\gamma \to \infty$, for given $q \geq 1$, we may assume $\gamma > q$. Then, we find by Jensen's inequality that

$$\left(\frac{\int_K (p^{(\gamma)})^{\frac{q}{\gamma}} \mathrm{d}x}{|\Omega|}\right)^{\frac{\gamma}{q}} \leq \frac{\int_\Omega p^{(\gamma)} \mathrm{d}x}{|\Omega|}, \tag{2.6}$$

where $|\Omega|$ is the Lebesgue measure of Ω. Then, for $(p^{(\gamma)})^{\frac{1}{\gamma}}$, we have

$$\left(\int_\Omega (p^{(\gamma)})^{\frac{q}{\gamma}} \mathrm{d}x\right)^{\frac{1}{q}} \leq \left(\int_\Omega p^{(\gamma)} \mathrm{d}x\right)^{\frac{1}{\gamma}} |\Omega|^{\frac{1}{q} - \frac{1}{\gamma}}. \tag{2.7}$$

On the other hand, since $\ln y$ is concave with respect to y, we have

$$\frac{1}{|\Omega|} \int_\Omega \ln((p^{(\gamma)})^{\frac{1}{\gamma}}) \, \mathrm{d}x \leq \ln \left(\frac{1}{|\Omega|} \int_\Omega (p^{(\gamma)})^{\frac{1}{\gamma}} \, \mathrm{d}x \right), \tag{2.8}$$

which implies from the Hölder inequality that

$$|\Omega|^{\frac{1}{q}} \exp\left\{ \frac{1}{\gamma|\Omega|} \int_\Omega \ln(p^{(\gamma)}) \, \mathrm{d}x \right\} \leq |\Omega|^{\frac{1}{q}} \exp\left\{ \ln \left(\frac{1}{|\Omega|} \int_\Omega (p^{(\gamma)})^{\frac{1}{\gamma}} \, \mathrm{d}x \right) \right\}$$

$$= |\Omega|^{\frac{1}{q}-1} \int_\Omega (p^{(\gamma)})^{\frac{1}{\gamma}} \, \mathrm{d}x$$

$$\leq \left(\int_\Omega (p^{(\gamma)})^{\frac{q}{\gamma}} \, \mathrm{d}x \right)^{\frac{1}{q}}. \tag{2.9}$$

Moreover, from (2.4), we have

$$\ln \left(E^{(\gamma)} \right) \leq \frac{\gamma-1}{\gamma} \ln p^{(\gamma)} - \ln \left(\frac{2(\gamma-1)}{(\gamma-1)\gamma\bar{M}^2+2} \right), \tag{2.10}$$

which, together with (2.7) and (2.9), gives

$$|\Omega|^{\frac{1}{q}} \exp\left\{ \frac{\int_\Omega \left(\ln(E^{(\gamma)}) + \ln(\frac{2(\gamma-1)}{(\gamma-1)\gamma\bar{M}^2+2}) \right) \mathrm{d}x}{(\gamma-1)|\Omega|} \right\} \leq \|(p^{(\gamma)})^{\frac{1}{\gamma}}\|_{L^q(\Omega)} \leq \left(\int_\Omega p^{(\gamma)} \, \mathrm{d}x \right)^{\frac{1}{\gamma}} |\Omega|^{\frac{1}{q}-\frac{1}{\gamma}}. \tag{2.11}$$

Note that both the left and right sides of the above inequality tend to $|\Omega|^{\frac{1}{q}}$ as $\gamma \to \infty$, owing to condition (H). Then, we have

$$\lim_{\gamma\to\infty} \|\rho^{(\gamma)}\|_{L^q(\Omega)} = |\Omega|^{\frac{1}{q}}, \tag{2.12}$$

where $\rho^{(\gamma)} := (p^{(\gamma)})^{\frac{1}{\gamma}}$. In particular, taking $q=1$ and $q=2$, respectively, we have

$$\lim_{\gamma\to\infty} \|\rho^{(\gamma)}\|_{L^2(\Omega)} = |\Omega|^{\frac{1}{2}}, \quad \lim_{\gamma\to\infty} \|\rho^{(\gamma)}\|_{L^1(\Omega)} = |\Omega|. \tag{2.13}$$

This implies that $\rho^{(\gamma)}$ are uniformly bounded in $L^2(\Omega)$. Then, there exists a subsequence of $\rho^{(\gamma)}$ (still denoted by $\rho^{(\gamma)}$) such that $\rho^{(\gamma)}$ weakly converges to $\bar{\rho}$ in $L^2(\Omega)$. By a simple computation, we obtain from (2.13) that

$$\int_\Omega (\bar{\rho}-1)^2 dx = \int_\Omega (\bar{\rho}^2 - 2\bar{\rho} + 1) \, \mathrm{d}x \leq \lim_{\gamma\to\infty} \int_\Omega \left((\rho^{(\gamma)})^2 - 2\rho^{(\gamma)} + 1 \right) \mathrm{d}x = 0.$$

That is, $\rho^{(\gamma)}$ converges to 1 *a.e.* in $x \in \Omega$, as $\gamma \to \infty$.

3. By the div-curl lemma of Murat [21] and Tartar [22], the Young measure representation theorem for a uniformly bounded sequence of functions in L^p (cf. Tartar [22]; also see Ball [1]), we use $(2.1)_1$ and (A.3) to obtain the following commutation identity:

$$\sum_{i=1}^n \langle \nu(\rho, u), u_i \rangle \langle \nu(\rho, u), \rho u_i \rangle = \left\langle \nu(\rho, u), \sum_{i=1}^n \rho u_i^2 \right\rangle, \tag{2.14}$$

where we have used that $\nu(\rho, u)$ is the associated Young measure (a probability measure) for the sequence $(\rho^{(\gamma)}, u^{(\gamma)})(x)$.

Then, the main point in the compensated compactness framework is to prove that $\nu(\rho, u)$ is in fact a Dirac measure, which in turn implies the compactness of the sequence $(\rho^{(\gamma)}, u^{(\gamma)})(x)$. On the other hand, from

$$\lim_{\gamma \to \infty} \rho^{(\gamma)}(x) = 1 \qquad a.e.$$

we see that

$$\nu(\rho, u) = \delta_1(\rho) \otimes \nu(u),$$

where $\delta_1(\rho)$ is the Delta mass concentrated at $\rho = 1$.

4. We now show $\nu(u)$ is a Dirac measure.

Combining both sides of (2.14) together, we have

$$\left\langle \nu(u^{(1)}) \otimes \nu(u^{(2)}), \sum_{i=1}^{n} u_i^{(1)} \left(u_i^{(1)} - u_i^{(2)} \right) \right\rangle = 0. \tag{2.15}$$

Exchanging indices (1) and (2), we obtain the following symmetric commutation identity:

$$\left\langle \nu(u^{(1)}) \otimes \nu(u^{(2)}), \sum_{i=1}^{n} \left(u_i^{(1)} - u_i^{(2)} \right)^2 \right\rangle = 0, \tag{2.16}$$

which immediately implies that,

$$u^{(1)} = u^{(2)},$$

i.e., $\nu(u)$ concentrates on a single point.

In fact, if this would not be true, we could suppose that there are two different points $\acute{u}$ and $\grave{u}$ in the support of ν. Then, $(\acute{u}, \acute{u})$, $(\acute{u}, \grave{u})$, $(\grave{u}, \acute{u})$, and $(\grave{u}, \grave{u})$ would be in the support of $\nu \otimes \nu$, which contradicts with $u^{(1)} = u^{(2)}$.

Therefore, the Young measure ν is a Dirac measure, which implies the strong convergence of $u^{(\gamma)}$. This completes the proof. $\qquad \square$

For the full Euler case, we assume that the approximate solutions $(\rho^{(\gamma)}, u^{(\gamma)}, p^{(\gamma)})$ satisfy

$$\begin{cases} \operatorname{div}\big(\rho^{(\gamma)} u^{(\gamma)}\big) = e_1(\gamma), \\ \operatorname{div}\big(\rho^{(\gamma)} u^{(\gamma)} \otimes u^{(\gamma)}\big) + \nabla p^{(\gamma)} = e_2(\gamma), \\ \operatorname{div}\big(\rho^{(\gamma)} u^{(\gamma)} E^{(\gamma)} + u^{(\gamma)} p^{(\gamma)}\big) = e_3(\gamma), \end{cases} \tag{2.17}$$

where $e_1(\gamma)$, $e_2(\gamma) = (e_{21}(\gamma), \dots, e_{2n}(\gamma))^\top$, and $e_3(\gamma)$ are sequences of distributional functions depending on the parameter γ. In this case, the energy function is

$$E^{(\gamma)} := \frac{|u^{(\gamma)}|^2}{2} + \frac{p^{(\gamma)}}{(\gamma - 1)\rho^{(\gamma)}},$$

and the entropy function is

$$S^{(\gamma)} := \frac{\rho^{(\gamma)}}{(p^{(\gamma)})^{\frac{1}{\gamma}}} \geq 0,$$

so that condition (H) for the homentropic case is replaced by

(F.1). As $\gamma \to \infty$,

$$\int_{\Omega} \ln(p^{(\gamma)}) \, \mathrm{d}x = o(\gamma);$$

(F.2). $S^{(\gamma)}$ converges to a bounded function $\overline{S}$ $a.e.$ in Ω as $\gamma \to \infty$.

Remark 2.4. Conditions (A.1)–(A.3) and (F.1)–(F.2) in the framework are naturally satisfied in the applications for the full Euler case in Sect. 3 below.

Similar to Theorems 2.1, we have

Theorem 2.2. (Compensated compactness framework for the full Euler case). *Let a sequence of functions* $\rho^{(\gamma)}(x)$, $u^{(\gamma)}(x) = (u_1^{(\gamma)}, \ldots, u_n^{(\gamma)})(x)$, *and* $p^{(\gamma)}(x)$ *satisfy conditions* (A.1)–(A.3) *and* (F.1)–(F.2). *Then, there exists a subsequence (still denoted by)* $(\rho^{(\gamma)}, u^{(\gamma)}, p^{(\gamma)})(x)$ *such that, as* $\gamma \to \infty$,

$$p^{(\gamma)}(x) \rightharpoonup \bar{p} \qquad\qquad \text{as bounded measures,}$$
$$\rho^{(\gamma)}(x) \to \bar{\rho}(x) \qquad\qquad \text{a.e. in } x \in \Omega,$$
$$u^{(\gamma)}(x) \to (\bar{u}_1, \ldots, \bar{u}_n)(x) \qquad \text{a.e. in } x \in \{x \,:\, \bar{\rho}(x) > 0, x \in \Omega\}.$$

Proof. We follow the same arguments as in the homentropic case.

First, the weak convergence of $p^{(\gamma)}$ is obvious. On the other hand, we observe that (2.7) and (2.9) still hold for the full Euler case. Then, for any $\gamma > q \geq 1$,

$$|\Omega|^{\frac{1}{q}} \exp\left\{ \frac{1}{\gamma|\Omega|} \int_\Omega \ln(p^{(\gamma)}) \, \mathrm{d}x \right\} \leq \left(\int_\Omega (p^{(\gamma)})^{\frac{q}{\gamma}} \, \mathrm{d}x \right)^{\frac{1}{q}} \leq \left(\int_\Omega p^{(\gamma)} \, \mathrm{d}x \right)^{\frac{1}{\gamma}} |\Omega|^{\frac{1}{q} - \frac{1}{\gamma}}. \tag{2.18}$$

Thanks to condition (F.1), we obtain

$$\lim_{\gamma \to \infty} \|(p^{(\gamma)})^{\frac{1}{\gamma}}\|_{L^q(\Omega)} = |\Omega|^{\frac{1}{q}}. \tag{2.19}$$

Taking $q = 1$ and $q = 2$, respectively, and following the same line of argument as in the homentropic case, we conclude that $(p^{(\gamma)})^{\frac{1}{\gamma}}$ converges to 1 *a.e. in* $x \in \Omega$ as $\gamma \to \infty$. Then, from condition (F.2), $\rho^{(\gamma)} = S^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}$ converges to $\bar{\rho} := \bar{S} \geq 0$ *a.e. in* $x \in \Omega$.

The remaining proof is the same as that for the homentropic case, except the strong convergence of $u^{(\gamma)}$ only stands on $\{x \,:\, \bar{\rho}(x) > 0, x \in \Omega\}$ since the vacuum can not be excluded. This completes the proof. $\qquad\square$

Remark 2.5. Consider any function $Q(\rho, u, p) := (Q_1, \ldots, Q_n)(\rho, u, p)$ satisfying

$$\mathrm{div}\,(Q(\rho^{(\gamma)}, u^{(\gamma)}, p^{(\gamma)})) = e_Q(\gamma), \tag{2.20}$$

where $e_Q(\gamma) \to 0$ in the distributional sense as $\gamma \to \infty$. The similar statement is also valid for Theorem 2.2, via replacing (2.1) by (2.17).

Then, as direct corollaries, we conclude the following propositions.

Proposition 2.3. (Convergence of approximate solutions of the homentropic Euler equations). *Let* $u^{(\gamma)}(x) = (u_1^{(\gamma)}, \ldots, u_n^{(\gamma)})(x)$ *and* $p^{(\gamma)}(x)$ *be a sequence of approximate solutions satisfying conditions* (A.1)–(A.3) *and* (H), *and*

$$e_i(\gamma) \to 0 \qquad \text{as } \gamma \to \infty$$

in the distributional sense for $i = 1, 2$. *Then, there exists a subsequence (still denoted by)* $(u^{(\gamma)}, p^{(\gamma)})(x)$ *that converges a.e. to a weak solution* $(\bar{u}, \bar{p})$ *of the homogeneous incompressible Euler equations as* $\gamma \to \infty$:

$$\begin{cases} \mathrm{div}\,\bar{u} = 0, \\ \mathrm{div}(\bar{u} \otimes \bar{u}) + \nabla\bar{p} = 0. \end{cases} \tag{2.21}$$

Proof. From Theorem 2.1, we know that $(u^{(\gamma)}, p^{(\gamma)})$ converges to $(\bar{u}, \bar{p})$ as $\gamma \to \infty$. For the approximate continuity equation, we see that, for any test function $\phi \in C_c^\infty$,

$$\int e_1(\gamma)\,\phi\,\mathrm{d}x = \int \phi\,\mathrm{div}(\rho^{(\gamma)}u^{(\gamma)})\,\mathrm{d}x$$
$$= -\int \nabla\phi \cdot u^{(\gamma)}\rho^{(\gamma)}\,\mathrm{d}x$$
$$= -\int \nabla\phi \cdot u^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}\,\mathrm{d}x. \tag{2.22}$$

Letting $\gamma \to \infty$, we conclude

$$\int \nabla\phi \cdot \bar{u}\,\mathrm{d}x = 0, \tag{2.23}$$

which implies $(2.21)_1$ in the distributional sense. With a similar argument, we can show that $(2.21)_2$ holds in the distributional sense. $\qquad\square$

Proposition 2.4. (Convergence of approximate solutions for the full Euler flow) *Let $\rho^{(\gamma)}(x)$, $u^{(\gamma)}(x) = (u_1^{(\gamma)}, \ldots, u_n^{(\gamma)})(x)$, and $p^{(\gamma)}(x)$ be a sequence of approximate solutions satisfying conditions* (A.1)–(A.3) *and* (F.1)–(F.2), *and*

$$e_i(\gamma) \to 0 \qquad \text{for } i = 1, 2,$$
$$(p^{(\gamma)})^{-1}\Big(e_3(\gamma) - u^{(\gamma)} \cdot e_2(\gamma) + \frac{|u^{(\gamma)}|^2}{2}e_1(\gamma)\Big) \to 0$$

in the distributional sense as $\gamma \to \infty$. Then, there exists a subsequence (still denoted by) $(\rho^{(\gamma)}, u^{(\gamma)}, p^{(\gamma)})(x)$ that converges a.e. to a weak solution $(\bar{\rho}, \bar{u}, \bar{p})$ of the inhomogeneous incompressible Euler equations as $\gamma \to \infty$:

$$\begin{cases} \mathrm{div}\,\bar{u} = 0, \\ \mathrm{div}(\bar{\rho}\bar{u}) = 0, \\ \mathrm{div}(\bar{\rho}\bar{u} \otimes \bar{u}) + \nabla\bar{p} = 0. \end{cases} \tag{2.24}$$

Proof. From a direct calculation, we have

$$\mathrm{div}\big((p^{(\gamma)})^{\frac{1}{\gamma}}u^{(\gamma)}\big) = \frac{\gamma - 1}{\gamma}(p^{(\gamma)})^{\frac{1}{\gamma}-1}\Big(e_3(\gamma) - \sum_{i=1}^n u^{(\gamma)} \cdot e_2(\gamma) + \frac{|u^{(\gamma)}|^2}{2}e_1(\gamma)\Big). \tag{2.25}$$

Then, for any test function $\phi \in C_c^\infty$, we find

$$\int \frac{\gamma - 1}{\gamma}(p^{(\gamma)})^{\frac{1}{\gamma}-1}\Big(e_3(\gamma) - \sum_{i=1}^n u^{(\gamma)} \cdot e_2(\gamma) + \frac{|u^{(\gamma)}|^2}{2}e_1(\gamma)\Big)\phi\,\mathrm{d}x$$
$$= -\int \nabla\phi \cdot u^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}\,\mathrm{d}x. \tag{2.26}$$

Taking $\gamma \to \infty$, we have

$$\int \nabla\phi \cdot \bar{u}\,\mathrm{d}x = 0, \tag{2.27}$$

which implies $(2.24)_1$ in the distributional sense.

The fact that $(2.24)_2$ and $(2.24)_3$ hold in the distributional sense can be shown similarly from $e_j(\gamma) \to 0$ as $\gamma \to \infty$, $j = 1, 2$, respectively. $\qquad\square$

Remark 2.6. The main difference between Propositions 2.3 and 2.4 is that, when $\gamma \to \infty$, the compressible homentropic Euler equations converge to the homogeneous incompressible Euler equations with the unknown variables (u, p), while the full Euler equations converge to the inhomogeneous incompressible Euler equations with the unknown variables (ρ, u, p). Furthermore, the incompressibility of the limit for the homentropic case follows directly from the approximate continuity equation $(2.1)_1$, while the incompressibility for the full Euler case is from a combination of all the equations in (2.17).

There are various ways to construct approximate solutions by either numerical methods or analytical methods such as numerical/analytical vanishing viscosity methods. As direct applications of the compactness framework, we now present two examples in Sects. 3 and 4 for establishing the incompressible limit for the multidimensional steady compressible Euler flows through infinitely long nozzles.

3. Incompressible limit for two-dimensional steady full Euler flows in an infinitely long nozzle

In this section, as a direct application of the compactness framework established in Theorem 2.2, we establish the incompressible limit of steady subsonic full Euler flows in a two-dimensional, infinitely long nozzle.

The infinitely long nozzle is defined as

$$\Omega = \{(x_1, x_2) : f_1(x_1) < x_2 < f_2(x_1), \ -\infty < x_1 < \infty\},$$

with the nozzle walls $\partial\Omega := W_1 \cup W_2$, where

$$W_i = \{(x_1, x_2) : x_2 = f_i(x_1) \in C^{2,\alpha}, \ -\infty < x_1 < \infty\}, \qquad i = 1, 2.$$

Suppose that W_1 and W_2 satisfy

$$
\begin{aligned}
& f_2(x_1) > f_1(x_1) \qquad \text{for } x_1 \in (-\infty, \infty), \\
& f_1(x_1) \to 0, \quad f_2(x_1) \to 1 \qquad \text{as } x_1 \to -\infty, \\
& f_1(x_1) \to a, \quad f_2(x_1) \to b > a \qquad \text{as } x_1 \to \infty,
\end{aligned}
\tag{3.1}
$$

and there exists $\alpha > 0$ such that

$$\|f_i\|_{C^{2,\alpha}(\mathbb{R})} \leq C, \qquad i = 1, 2, \tag{3.2}$$

for some positive constant C. It follows that Ω satisfies the uniform exterior sphere condition with some uniform radius $r > 0$. See Fig. 1.

Suppose that the nozzle has impermeable solid walls so that the flow satisfies the slip boundary condition:

$$u \cdot \nu = 0 \qquad \text{on } \partial\Omega, \tag{3.3}$$
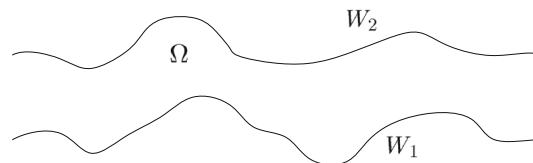
where ν is the unit outward normal to the nozzle wall.

FIG. 1. Two-dimensional infinitely long nozzle

It follows from $(1.1)_1$ and (3.3) that

$$\int_{\Sigma} (\rho u) \cdot l \, \mathrm{d}s \equiv m \tag{3.4}$$

holds for some constant m, which is the mass flux, where Σ is any curve transversal to the x_1-direction, and l is the normal of Σ in the positive x_1-axis direction.

We assume that the upstream entropy function is given, i.e.,

$$\frac{\rho}{p^{1/\gamma}} \longrightarrow S_-(x_2) \qquad \text{as } x_1 \to -\infty, \tag{3.5}$$

and the upstream Bernoulli function is given, i.e.,

$$\frac{|u|^2}{2} + \frac{\gamma p}{(\gamma - 1)\rho} \longrightarrow B_-(x_2) \qquad \text{as } x_1 \to -\infty, \tag{3.6}$$

where $S_-(x_2)$ and $B_-(x_2)$ are the functions defined on $[0,1]$.

Problem 1 (m, γ). *Solve the full Euler system* (1.1) *with the boundary condition* (3.3), *the mass flux condition* (3.4), *and the asymptotic conditions* (3.5)–(3.6).

Set

$$\underline{S} = \inf_{x_2 \in [0,1]} S_-(x_2), \qquad \underline{B} = \inf_{x_2 \in [0,1]} B_-(x_2).$$

For this problem, the following theorem has been established in Chen–Deng–Xiang [5].

Theorem 3.1. *Let the nozzle walls $\partial\Omega$ satisfy (3.1)–(3.2), and let $\underline{S} > 0$ and $\underline{B} > 0$. Then, there exists $\delta_0 > 0$ such that, if $\|(S_- - \underline{S}, B_- - \underline{B})\|_{C^{1,1}([0,1])} \le \delta$ for $0 < \delta \le \delta_0$, $(S_- B_-)'(0) \le 0$, and $(S_- B_-)'(1) \ge 0$, there exists $\hat{m} \ge 2\delta_0^{\frac{1}{8}}$ such that, for any $m \in (\delta^{\frac{1}{4}}, \hat{m})$, there is a global solution (i.e., a full Euler flow) $(\rho, u, p) \in C^{1,\alpha}(\overline{\Omega})$ of **Problem** $1(m, \gamma)$ such that the following hold:*

(i) *Subsonic state and horizontal direction of the velocity: The flow is uniformly subsonic with positive horizontal velocity in the whole nozzle, i.e.,*

$$\sup_{\overline{\Omega}}(|u|^2 - c^2) < 0, \quad u_1 > 0 \qquad \text{in } \overline{\Omega}; \tag{3.7}$$

(ii) *The flow satisfies the following asymptotic behavior in the far field: As $x_1 \to -\infty$,*

$$p \to p_- > 0, \qquad u_1 \to u_-(x_2) > 0, \qquad (u_2, \rho) \to (0, \rho_-(x_2; p_-)), \tag{3.8}$$

$$\nabla p \to 0, \qquad \nabla u_1 \to (0, u_-'(x_2)), \qquad \nabla u_2 \to 0, \qquad \nabla \rho \to (0, \rho_-'(x_2; p_-)) \tag{3.9}$$

uniformly for $x_2 \in K_1 \Subset (0,1)$, where $\rho_-(x_2; p_-) = p_-^{\frac{1}{\gamma}} S_-(x_2)$, the constant p_- and function $u_-(x_2)$ can be determined by m, $S_-(x_2)$, and $B_-(x_2)$ uniquely.

Next, we take the incompressible limit of the full Euler flows.

Theorem 3.2 (Incompressible limit of two-dimensional full Euler flows). *Let $(\rho^{(\gamma)}, u^{(\gamma)}, p^{(\gamma)})(x)$ be the corresponding sequence of solutions to **Problem** 1 $(m^{(\gamma)}, \gamma)$. Then, as $\gamma \to \infty$, the solution sequence possesses a subsequence (still denoted by) $(\rho^{(\gamma)}, u^{(\gamma)}, p^{(\gamma)})(x)$ that converges strongly a.e. in Ω to a vector function $(\bar{\rho}, \bar{u}, \bar{p})(x)$ which is a weak solution of (1.10). Furthermore, the limit solution $(\bar{\rho}, \bar{u}, \bar{p})(x)$ also satisfies the boundary condition (3.3) as the normal trace of the divergence-measure field u on the boundary in the sense of Chen–Frid [6].*

Proof. We divide the proof into four steps.

1. From (1.1), we can obtain the following linear transport parts:

$$\begin{cases} \partial_{x_1}\big((p^{(\gamma)})^{\frac{1}{\gamma}}u_1^{(\gamma)}\big) + \partial_{x_2}\big((p^{(\gamma)})^{\frac{1}{\gamma}}u_2^{(\gamma)}\big) = 0, \\ \partial_{x_1}\big((p^{(\gamma)})^{\frac{1}{\gamma}}B^{(\gamma)}u_1^{(\gamma)}\big) + \partial_{x_2}\big((p^{(\gamma)})^{\frac{1}{\gamma}}B^{(\gamma)}u_2^{(\gamma)}\big) = 0, \\ \partial_{x_1}\big((p^{(\gamma)})^{\frac{1}{\gamma}}S^{(\gamma)}u_1^{(\gamma)}\big) + \partial_{x_2}\big((p^{(\gamma)})^{\frac{1}{\gamma}}S^{(\gamma)}u_2^{(\gamma)}\big) = 0. \end{cases} \tag{3.10}$$

From $(3.10)_1$, we can introduce the potential function $\psi^{(\gamma)}$:

$$\begin{cases} \partial_{x_1}\psi^{(\gamma)} = -(p^{(\gamma)})^{\frac{1}{\gamma}}u_2^{(\gamma)}, \\ \partial_{x_2}\psi^{(\gamma)} = (p^{(\gamma)})^{\frac{1}{\gamma}}u_1^{(\gamma)}. \end{cases} \tag{3.11}$$

From the far-field behavior of the Euler flows, we can define

$$\psi_-^{(\gamma)}(x_2) := \lim_{x_1 \to -\infty} \psi^{(\gamma)}(x_1, x_2).$$

Since both the upstream Bernoulli and entropy functions are given, $B^{(\gamma)}$ and $S^{(\gamma)}$ have the following expressions:

$$B^{(\gamma)}(x) = B_-((\psi_-^{(\gamma)})^{-1}(\psi^{(\gamma)}(x))), \qquad S^{(\gamma)}(x) = S_-((\psi_-^{(\gamma)})^{-1}(\psi^{(\gamma)}(x))),$$

where $(\psi_-^{(\gamma)})^{-1}\psi^{(\gamma)}(x)$ is a function from Ω to $[0,1]$, and

$$\begin{cases} B_- = \frac{u_-^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_-}{\rho_-}, \\ S_- = \rho_- p_-^{-\frac{1}{\gamma}}, \end{cases}$$

with uniformly upper and lower bounds with respect to γ.

Since the flow is subsonic so that the Mach number $M^{(\gamma)} \leq 1$, then we have

$$|u^{(\gamma)}| < \sqrt{\frac{2(\gamma-1)}{\gamma+1}\max B_-}, \tag{3.12}$$

and

$$\left(\frac{2(\gamma-1)}{\gamma(\gamma+1)}\min(B_-S_-)\right)^{\frac{\gamma}{\gamma-1}} < p^{(\gamma)} \leq \left(\frac{\gamma-1}{\gamma}\max(B_-S_-)\right)^{\frac{\gamma}{\gamma-1}}. \tag{3.13}$$

Since $|u^{(\gamma)}|^2$ and $p^{(\gamma)}$ are uniformly bounded, we conclude that $|u^{(\gamma)}|^2$ and $p^{(\gamma)}$ are uniformly bounded in $L^1_{\text{loc}}(\Omega)$. Thus, conditions (A.1)–(A.2) are satisfied. It is observed that, even though the lower bound of pressure $p^{(\gamma)}$ may tend to zero as $\gamma \to \infty$ with polynomial rate, so that (F.1) holds for any bounded domain.

2. For fixed x_1, $(\psi_-^\gamma)^{-1}(\psi^\gamma(\cdot))$ can be regarded as a backward characteristic map with

$$\frac{\partial((\psi_-^{(\gamma)})^{-1}(\psi^{(\gamma)}))}{\partial x_2} = \frac{(p^{(\gamma)})^{\frac{1}{\gamma}}u_1^{(\gamma)}}{p_-^{\frac{1}{\gamma}}u_-} > 0.$$

The uniform boundedness and positivity of $p_-^{\frac{1}{\gamma}} u_-$ and $(p^{(\gamma)})^{\frac{1}{\gamma}} u_1^{(\gamma)}$ implies that the map is not degenerate. Then, we have

$$
\begin{cases}
\partial_{x_1} S^{(\gamma)}(x) = -S'_-((\psi_-^{(\gamma)})^{-1}(\psi^{(\gamma)}(x))) \dfrac{(p^{(\gamma)})^{\frac{1}{\gamma}} u_2^{(\gamma)}}{p_-^{\frac{1}{\gamma}} u_-}, \\[2mm]
\partial_{x_2} S^{(\gamma)}(x) = S'_-((\psi_-^{(\gamma)})^{-1}(\psi^{(\gamma)}(x))) \dfrac{(p^{(\gamma)})^{\frac{1}{\gamma}} u_1^{(\gamma)}}{p_-^{\frac{1}{\gamma}} u_-}.
\end{cases}
\tag{3.14}
$$

Thus, $S^{(\gamma)}$ is uniformly bounded in BV, which implies its strong convergence. Then, condition (F.2) follows.

3. Similar to [7], the vorticity sequence $\omega^{(\gamma)} := \partial_{x_1} u_2^{(\gamma)} - \partial_{x_2} u_1^{(\gamma)}$ can be written as

$$
\begin{cases}
\partial_{x_1} B^{(\gamma)} = u_2^{(\gamma)} \omega^{(\gamma)} - \frac{\gamma}{\gamma-1} (\rho^{(\gamma)})^{-2} (p^{(\gamma)})^{\frac{\gamma+1}{\gamma}} \partial_{x_1} S^{(\gamma)}, \\[2mm]
\partial_{x_2} B^{(\gamma)} = -u_1^{(\gamma)} \omega^{(\gamma)} - \frac{\gamma}{\gamma-1} (\rho^{(\gamma)})^{-2} (p^{(\gamma)})^{\frac{\gamma+1}{\gamma}} \partial_{x_2} S^{(\gamma)}.
\end{cases}
\tag{3.15}
$$

By direct calculation, we have

$$
\begin{aligned}
\omega^{(\gamma)} &= \frac{1}{|u^{(\gamma)}|^2} \Big(u_2^{(\gamma)} \big(\partial_{x_1} B^{(\gamma)} + \frac{\gamma}{\gamma-1} (\rho^{(\gamma)})^{-2} (p^{(\gamma)})^{\frac{\gamma+1}{\gamma}} \partial_{x_1} S^{(\gamma)} \big) \\
&\quad - u_1^{(\gamma)} \big(\partial_{x_2} B^{(\gamma)} + \frac{\gamma}{\gamma-1} (\rho^{(\gamma)})^{-2} (p^{(\gamma)})^{\frac{\gamma+1}{\gamma}} \partial_{x_2} S^{(\gamma)} \big) \Big) \\
&= -\frac{1}{p_-^{\frac{1}{\gamma}} u_-} \Big((p^{(\gamma)})^{\frac{1}{\gamma}} B'_- + \frac{\gamma}{\gamma-1} (p^{(\gamma)})^{\frac{\gamma+2}{\gamma}} (\rho^{(\gamma)})^{-2} S'_- \Big),
\end{aligned}
\tag{3.16}
$$

which implies that ω^ε as a measure sequence is uniformly bounded so that it is compact in H_{loc}^{-1}. Therefore, the flows satisfy condition (A.3).

Then, Proposition 2.4 immediately implies that the solution sequence has a subsequence (still denoted by) $(\rho^{(\gamma)}, u^{(\gamma)}, p^{(\gamma)})(x)$ that converges *a.e.* in Ω to a vector function $(\bar{\rho}, \bar{u}, \bar{p})(x)$ as $\gamma \to \infty$.

4. Since $\bar{u}$ is uniformly bounded, the normal trace $\bar{u} \cdot \nu$ on $\partial\Omega$ exists and is in $L^\infty(\partial\Omega)$ in the sense of Chen–Frid [6]. On the other hand, for any $\phi \in C^\infty(\mathbb{R}^2)$, we have

$$
\langle (\bar{u} \cdot \nu)|_{\partial\Omega}, \phi \rangle = \int_\Omega \bar{u}(x) \cdot \nabla\phi(x)\,\mathrm{d}x + \int_\Omega \phi \operatorname{div} \bar{u}\,\mathrm{d}x.
\tag{3.17}
$$

Since $\int_\Omega \phi \operatorname{div} \bar{u}\,\mathrm{d}x = 0$, and

$$
\int_\Omega \bar{u}(x) \cdot \nabla\phi(x)\,\mathrm{d}x = \lim_{\gamma \to \infty} \int_\Omega ((p^{(\gamma)})^{\frac{1}{\gamma}} u^{(\gamma)})(x) \cdot \nabla\phi(x)\,\mathrm{d}x = 0,
\tag{3.18}
$$

then we have

$$
\langle (\bar{u} \cdot \nu)|_{\partial\Omega}, \phi \rangle = 0,
\tag{3.19}
$$

for any $\phi \in C^\infty(\mathbb{R}^2)$. By approximation, we conclude that the normal trace $(\bar{u} \cdot \nu)|_{\partial\Omega} = 0$ in $L^\infty(\partial\Omega)$. This completes the proof. $\qquad\square$

Remark 3.1. In the two-dimensional homentropic case, the subsonic results in [2,24] can also be extended to the incompressible limit by using Proposition 2.3.

4. Incompressible limit for the three-dimensional homentropic Euler flows in an infinitely long axisymmetric nozzle

We consider Euler flows through an infinitely long axisymmetric nozzle in $\mathbb{R}^3$ given by

$$\Omega = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq \sqrt{x_2^2 + x_3^2} < f(x_1), \ -\infty < x_1 < \infty \right\},$$

where $f(x_1)$ satisfies

$$f(x_1) \to 1 \qquad \text{as } x_1 \to -\infty,$$
$$f(x_1) \to r_0 \qquad \text{as } x_1 \to \infty,$$
$$\|f\|_{C^{2,\alpha}(\mathbb{R})} \leq C \qquad \text{for some } \alpha > 0 \text{ and } C > 0, \tag{4.1}$$
$$\inf_{x_1 \in \mathbb{R}} f(x_1) = b > 0. \tag{4.2}$$

See Fig. 2.

The boundary condition is set as follows: Since the nozzle wall is solid, the flow satisfies the slip boundary condition:

$$u \cdot \nu = 0 \qquad \text{on } \partial\Omega, \tag{4.3}$$

where ν is the unit outward normal to the nozzle wall. The continuity equation in $(1.1)_1$ and the boundary condition (4.3) imply that the mass flux

$$\int_{\Sigma} (\rho u) \cdot l \, \mathrm{d}s \equiv m_0 \tag{4.4}$$

remains for some positive constant m_0, where Σ is any surface transversal to the x_1-axis direction and l is the normal of Σ in the positive x_1-axis direction.

In Du–Duan [11], axisymmetric flows without swirl are considered for the fluid density $\rho = \rho(x_1, r)$ and the velocity

$$u = (u_1, u_2, u_3) = \left(U(x_1, r), V(x_1, r)\frac{x_2}{r}, V(x_1, r)\frac{x_3}{r} \right)$$

in the cylindrical coordinates, where U and V are the axial velocity and radial velocity respectively, and $r = \sqrt{x_2^2 + x_3^2}$. Then, instead of (1.2), we have
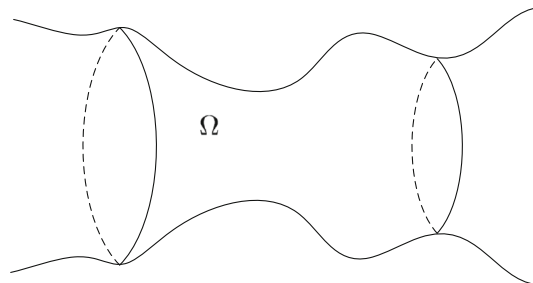

FIG. 2. Infinitely long axisymmetric nozzle

$$\begin{cases} \partial_{x_1}(r\rho U) + \partial_r(r\rho V) = 0, \\ \partial_{x_1}(r\rho U^2) + \partial_r(r\rho UV)_r + r\partial_{x_1}p = 0, \\ \partial_{x_1}(r\rho UV) + \partial_r(r\rho V^2)_r + r\partial_r p = 0. \end{cases} \tag{4.5}$$

Rewrite the axisymmetric nozzle as

$$\Omega = \{(x_1, r) : 0 \le r < f(x_1), \ -\infty < x_1 < \infty\}$$

with the boundary of the nozzle:

$$\partial\Omega = \{(x_1, r) : r = f(x), \ -\infty < x_1 < \infty\}.$$

The boundary condition (4.3) becomes

$$(U, V, 0) \cdot \tilde{\nu} = 0 \qquad \text{on } \partial\Omega, \tag{4.6}$$

where $\tilde{\nu}$ is the unit outer normal of the nozzle wall in the cylindrical coordinates. The mass flux condition (4.4) can be rewritten in the cylindrical coordinates as

$$\int_{\Sigma} (r\rho U, r\rho V, 0) \cdot \tilde{l} \, \mathrm{d}s \equiv m := \frac{m_0}{2\pi}, \tag{4.7}$$

where Σ is any curve transversal to the x_1-axis direction and $\tilde{l}$ is the unit normal of Σ.

Notice that the quantity

$$B = \frac{\gamma}{\gamma - 1}\rho^{\gamma-1} + \frac{U^2 + V^2}{2}$$

is constant along each streamline. For the homentropic Euler flows in the axisymmetric nozzle, we assume that the upstream Bernoulli is given, that is,

$$\frac{\gamma}{\gamma - 1}\rho^{\gamma-1} + \frac{U^2 + V^2}{2} \longrightarrow B_-(r) \qquad \text{as } x_1 \to -\infty, \tag{4.8}$$

where $B_-(r)$ is a function defined on $[0, 1]$.

Set

$$\underline{B} = \inf_{r\in[0,1]} B_-(r), \quad \sigma = ||B'_-||_{C^{0,1}([0,1])}, \tag{4.9}$$

We denote the above problem as **Problem 2**(m, γ). It is shown in [11] that

Theorem 4.1. *Suppose that the nozzle satisfies* (4.1). *Let the upstream Bernoulli function $B(r)$ satisfy $\underline{B} > 0$, $B'(r) \in C^{1,1}([0,1])$, $B'(0) = 0$, and $B'(r) \ge 0$ on $r \in [0, 1]$. Then, we have*

(i) *There exists $\delta_0 > 0$ such that, if $\sigma \le \delta_0$, then there is $\hat{m} \le 2\delta_0^{\frac{1}{8}}$. For any $m \in (\delta_0^{\frac{1}{4}}, \hat{m})$, there exists a global C^1–solution (i.e., a homentropic Euler flow) $(\rho, U, V) \in C^1(\overline{\Omega})$ through the nozzle with mass flux condition* (4.7) *and the upstream asymptotic condition* (4.8). *Moreover, the flow is uniformly subsonic, and the axial velocity is always positive, i.e.,*

$$\sup_{\overline{\Omega}}(U^2 + V^2 - c^2) < 0 \quad and \quad U > 0 \qquad in \ \overline{\Omega}. \tag{4.10}$$

(ii) *The subsonic flow satisfies the following properties: As $x_1 \to -\infty$,*

$$\rho \to \rho_- > 0, \quad \nabla\rho \to 0, \quad p \to \rho_-^{\gamma}, \quad (U, V) \to (U_-(r), 0), \quad \nabla U \to (0, U'_-(r)), \tag{4.11}$$

uniformly for $r \in K_1 \Subset (0, 1)$, where ρ_- is a positive constant and ρ_- and $U_-(r)$ can be determined by m and $B(r)$ uniquely.

As above, we have the following incompressible limit theorem for this case.

Theorem 4.2. (Incompressible limit of three-dimensional Euler flows through an axisymmetric nozzle) *Let $u^{(\gamma)} = (u_1^{(\gamma)}, u_2^{(\gamma)}, u_3^{(\gamma)})$, and $p^{(\gamma)} = (\rho^{(\gamma)})^\gamma$ be the corresponding solutions to* **Problem 2** $(m^{(\gamma)}, \gamma)$. *Then, as $\gamma \to \infty$, the solution sequence possesses a subsequence (still denoted by) $(u^{(\gamma)}, p^{(\gamma)})$ that converges strongly a.e. in Ω to a vector function $(\bar{u}, \bar{p})$ with $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ which is a weak solution of* (1.11). *Furthermore, the limit solution $(\bar{u}, \bar{p})$ also satisfies the boundary conditions* (4.3) *as the normal trace of the divergence-measure field $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ on the boundary in the sense of Chen–Frid* [6].

Proof. For the approximate solutions, $B^{(\gamma)}$ satisfy

$$\partial_{x_1}(rU^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}} B^{(\gamma)}) + \partial_r(rV^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}} B^{(\gamma)}) = 0. \tag{4.12}$$

Based on the equation:

$$\partial_{x_1}(rU^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}) + \partial_r(rV^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}) = 0,$$

we introduce $\psi^{(\gamma)}$ as

$$\begin{cases} \partial_{x_1}\psi^{(\gamma)} = -rV^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}, \\ \partial_r\psi^{(\gamma)} = rU^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}. \end{cases} \tag{4.13}$$

From the far-field behavior of the Euler flows, we define

$$\psi_-^{(\gamma)}(r) := \lim_{x_1 \to -\infty} \psi^{(\gamma)}(x_1, r).$$

Similar to the argument in Theorem 3.2, $(\psi_-^{(\gamma)})^{-1}(\psi^{(\gamma)})$ are nondegenerate maps. A direct calculation yields

$$B^{(\gamma)}(x_1, x_2, x_3) = B_-\big((\psi_-^{(\gamma)})^{-1}\big(\psi^{(\gamma)}\big(x_1, \sqrt{x_2^2 + x_3^2}\big)\big)\big),$$

with

$$B_-^{(\gamma)} = \frac{U_-^2}{2} + \frac{\gamma}{\gamma - 1}\rho_-^{\gamma-1}, \tag{4.14}$$

Similar to the previous case, the flow is subsonic so that the Mach number $M^{(\gamma)} \le 1$,

$$|(U^{(\gamma)}, V^{(\gamma)})| < \sqrt{2\max B_-}, \tag{4.15}$$

and

$$\left(\frac{2(\gamma-1)}{\gamma(\gamma+1)}\min B_-\right)^{\frac{\gamma}{\gamma-1}} < p^{(\gamma)} \le \left(\frac{\gamma-1}{\gamma}\max B_-\right)^{\frac{\gamma}{\gamma-1}}. \tag{4.16}$$

From (1.4), we have

$$\frac{2}{\gamma(\gamma+1)}\min B_- < E^{(\gamma)} \le \frac{(\gamma-1)\gamma+2}{2\gamma}\max B_-. \tag{4.17}$$

Therefore, conditions (A.1)–(A.2) and (H) are satisfied for any bounded domain.

On the other hand, the vorticity $\omega^{(\gamma)}$ has the following expressions:

$$\begin{cases} \omega_{1,2}^{(\gamma)} = \partial_{x_1}u_2^{(\gamma)} - \partial_{x_2}u_1^{(\gamma)} = \frac{x_2}{r}(\partial_{x_1}V^{(\gamma)} - \partial_r U^{(\gamma)}), \\ \omega_{2,3}^{(\gamma)} = \partial_{x_2}u_3^{(\gamma)} - \partial_{x_3}u_2^{(\gamma)} = 0, \\ \omega_{3,1}^{(\gamma)} = \partial_{x_3}u_1^{(\gamma)} - \partial_{x_1}u_3^{(\gamma)} = -\frac{x_3}{r}(\partial_{x_1}V^{(\gamma)} - \partial_r U^{(\gamma)}). \end{cases} \tag{4.18}$$

A direct calculation yields

$$\partial_{x_1}V^{(\gamma)} - \partial_r U^{(\gamma)} = -\frac{r}{(\psi_-^{(\gamma)})'}(p^{(\gamma)})^{\frac{1}{\gamma}} B_-', \tag{4.19}$$

which implies that $\omega^{(\gamma)}$ is uniformly bounded in the bounded measure space and (A.3) is satisfied.

Then, the sequence $(u^{(\gamma)}, p^{(\gamma)})(x)$ satisfies conditions (A.1)–(A.3) and (H). Moreover, (1.2) holds for $(u^{(\gamma)}, p^{(\gamma)})(x)$.

Similar to Theorem 3.2, we conclude that there exists a subsequence (still denoted by) $(u^{(\gamma)}, p^{(\gamma)})$ that converges to a vector function $(\bar{u}, \bar{p})$ *a.e.* in Ω satisfying (1.11) in the distributional sense.

Since $\bar{u}$ is uniformly bounded, the normal trace $\bar{u} \cdot \nu$ on $\partial\Omega$ exists and is in $L^\infty(\partial\Omega)$ in the sense of Chen–Frid [6]. On the other hand, for any $\phi \in C^\infty(\mathbb{R}^2)$, we have

$$\langle (\bar{u} \cdot \nu)|_{\partial\Omega}, \phi \rangle = \int_\Omega \bar{u}(x) \cdot \nabla\phi(x) \, \mathrm{d}x + \int_\Omega \phi \, \mathrm{div}\, \bar{u} \, \mathrm{d}x. \tag{4.20}$$

Since $\int_\Omega \phi \, \mathrm{div}\, \bar{u} \, \mathrm{d}x = 0$, and

$$\int_\Omega \bar{u}(x) \cdot \nabla\phi(x) \, \mathrm{d}x = 0, \tag{4.21}$$

then we have

$$\langle (\bar{u} \cdot \nu)|_{\partial\Omega}, \phi \rangle = 0, \tag{4.22}$$

for any $\phi \in C^\infty(\mathbb{R}^2)$. By approximation, we conclude that the normal trace $(\bar{u} \cdot \nu)|_{\partial\Omega} = 0$ in $L^\infty(\partial\Omega)$. This completes the proof. $\square$

Remark 4.1. For the full Euler flow case, the subsonic results of [12] can be also extended to the incompressible limit by Proposition 2.4.

Acknowledgments

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