Calabi–Yau threefolds with infinite fundamental group

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Abstract

The present article surmmarizes our recent results [8, 9] about Calabi–Yau threefolds with infinite fundamental group. This class of Calabi–Yau manifolds is relatively simple yet rich enough to display the essential complexities of Calabi–Yau geometries, and thus it provides good testing-grounds for general theories and conjectures.

1 Introduction

Throughout the article, a Calabi–Yau threefold is a smooth complex projective threefold X with trivial canonical bundle $\Omega_X^3 \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. A fundamental gap in the classification of algebraic threefolds is the lack of understanding of Calabi–Yau threefolds, classification of which seems very challenging at this point. Therefore in this article we concentrate on a special class of Calabi–Yau threefolds, namely those with infinite fundamental group. Most Calabi–Yau threefolds we know have finite fundamental groups: for example, complete intersection Calabi–Yau threefolds in toric varieties or homogeneous spaces, and (resolutions of singularities of) finite quotients thereof. Calabi–Yau threefolds with infinite fundamental group were only partially explored before the pioneering work of Oguiso and Sakurai [13]. A Calabi–Yau threefold with infinite fundamental group always admits an étale Galois covering either by an abelian threefold (type A) or by the product of a K3 surface and an elliptic curve (type K). In [13], Calabi–Yau threefolds of type A were essentially classified[1], while the full classification of type K was unsettled. The main purpose of the present article is to give an overview of the full classification obtained in [8]. We also discuss their interesting properties with special emphasis on mirror symmetry [9].

As we will see, Calabi–Yau threefolds with infinite fundamental group are relatively simple yet rich enough to display the essential complexities of Calabi–Yau geometries, and we expect that they will provide good testing-grounds for general theories and conjectures. Indeed, the simplest example of type K, known as the Enriques Calabi–Yau threefold, has been one of the most tractable compact Calabi–Yau threefolds both in string theory and mathematics (for example [6, 1, 14, 10, 11]). We believe that many of what is known for the Enriques Calabi–Yau threefold have natural generalizations to Calabi–Yau threefolds of type K.

[1] We refine their result further in [8].

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2 Classification

In this section we will provide the full classification of Calabi–Yau threefolds with infinite fundamental group. Let X be a Calabi–Yau threefold with infinite fundamental group. Then the Bogomolov decomposition theorem implies that X admits an étale Galois covering either by an abelian threefold or by the product of a K3 surface and an elliptic curve. We call X of type A in the former case and of type K in the latter case. Among many candidates of such coverings, we can always find a unique smallest one, up to isomorphism as a covering [3]. We call the smallest covering the minimal splitting covering of X.

2.1 Calabi–Yau threefolds of type K

We begin with Calabi–Yau threefolds of type K. In this case the full classification is given in terms of the Galois group of the minimal splitting covering as follows.

Theorem 2.1 ([8]). *There exist exactly eight Calabi–Yau threefolds of type K, up to deformation equivalence. The equivalence class is uniquely determined by the Galois group G of the minimal splitting covering. Moreover, the Galois group is isomorphic to one of the following combinations of cyclic and dihedral groups*

$$C_2,\ C_2 \times C_2,\ C_2 \times C_2 \times C_2,\ D_6,\ D_8,\ D_{10},\ D_{12},\ \textit{or}\ C_2 \times D_8.$$

Moreover, an explicit presentation of the deformation class of each Calabi–Yau threefold of type K is obtained in [8]. In the following, we shall briefly summarize the construction of the deformation classes. Let X be a Calabi–Yau threefold of type K and $\pi : S \times E \to X$ its minimal splitting covering with Galois group G. There exists a canonical isomorphism $\mathrm{Aut}(S \times E) \cong \mathrm{Aut}(S) \times \mathrm{Aut}(E)$, which induces a faithful G-action on each S and E:

$$\begin{array}{ccccc} \mathrm{Aut}(S) & \xleftarrow{p_1} & \mathrm{Aut}(S \times E) & \xrightarrow{p_2} & \mathrm{Aut}(E) \\ \cup & & \cup & & \cup \\ p_1(G) & \xleftarrow[p_1|_G]{\cong} & G & \xrightarrow[p_2|_G]{\cong} & p_2(G), \end{array}$$

where p_i denotes i-th projection with respect to the isomorphism $\mathrm{Aut}(S \times E) \cong \mathrm{Aut}(S) \times \mathrm{Aut}(E)$.

Proposition 2.2 ([13, 8])**.** *Let $H := \mathrm{Ker}(G \to \mathrm{GL}(H^{2,0}(S)))$ and take any $\iota \in G \setminus H$. Then the following hold:*

1. $\mathrm{ord}(\iota) = 2$ *and* $G = H \rtimes \langle \iota \rangle$*, where the semi-direct product structure is given by* $\iota h \iota = h^{-1}$ *for all* $h \in H$*;*

2. g *acts on* S *as an Enriques involution if* $g \in G \setminus H$*;*

3. ι *acts on* E *as* -1_E *and* H *as translations of the form* $\langle t_a \rangle \times \langle t_b \rangle \cong C_n \times C_m$ *under an appropriate origin of* E*. Here* t_a *and* t_b *are translations of order* n *and* m *respectively for some* $(n, m) \in \{(1, k)(1 \le k \le 6),\ (2, 2),\ (2, 4)\}$*.*

Conversely, such a G-action on the product $S \times E$ yields a Calabi–Yau threefold $X := (S \times E)/G$ of type K. Proposition 2.2 provides us with a complete understanding of the G-action on E (see [8, Section 3] for details), and therefore a classification of Calabi–Yau threefolds of type K essentially reduces to that of K3 surfaces equipped with actions described above, which we shall call *Calabi–Yau*. In what follows, G is always one of the finite groups described in Proposition 2.2. A basic example of a K3 surface which the reader could bear in mind is the following.

Proposition 2.3 (Horikawa model [2, Section V, 23])**.** *Consider the double covering* $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ *branching along a bidegree* $(4, 4)$*-divisor* B*. Then* S *is a K3 surface if it is smooth. We denote by* θ *the covering involution on* S*. Assume that* B *is invariant under the involution* λ *of* $\mathbb{P}^1 \times \mathbb{P}^1$ *given by* $(x, y) \mapsto (-x, -y)$*, where* x*and* y *are the inhomogeneous coordinates of* $\mathbb{P}^1 \times \mathbb{P}^1$*. The involution* λ *lifts to a symplectic involution of* S*. Then* $\theta \circ \lambda$ *is an involution of* S *without fixed points unless* B *passed through one of four fixed points of* λ *on* $\mathbb{P}^1 \times \mathbb{P}^1$*. The quotient surface* $T = S/\langle \theta \circ \lambda \rangle$ *is therefore an Enriques surface.*

$$\begin{tikzcd} S \arrow[r, "id"] \arrow[d, "/\langle\theta\rangle"'] & S \arrow[d, "/\langle\theta\circ\lambda\rangle"] \\ \mathbb{P}^1 \times \mathbb{P}^1 & T \end{tikzcd}$$

The classical theory of Enriques surfaces says that any generic K3 surface with an Enriques involution is realized as a Horikawa model ([2, Propositions 18.1, 18.2]).

Example 2.4 (Enriques Calabi–Yau threefold)**.** *Let* S *be a K3 surface with an Enriques involution* ι *and* E *an elliptic curve with negation* -1_E*. The free quotient* $X := (S \times E)/\langle(\iota, -1_E)\rangle$ *is the simplest Calabi–Yau threefold of type K, known as the Enriques Calabi–Yau threefold.*

In order to obtain other Calabi–Yau threefolds of type K, we consider special classes of the Horikawa model as follows. Let $\rho_1, \rho_2 : G \to \mathrm{PGL}(2, \mathbb{C})$ be 2-dimensional complex projective representations of $G := H \rtimes C_2$, which we do not specify at this point. Let λ be the generator of the second factor C_2. We then get a G-action $\rho_1 \times \rho_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that there exists a G-stable smooth curve B of bidegree $(4, 4)$. We then obtain a Horikawa K3 surface S as the double covering $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ branching along B and the G-action on $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to S as a symplectic G-action. We further assume that the curve B does not pass through any of fixed points of $g \in G$, $g \neq 1$. With the same notation as in Proposition 2.3, it can be checked that the symplectic G-action and the covering transformation θ commute.

$$\begin{tikzcd}
 & S \arrow[r, "id"] \arrow[d, "/\langle\theta\rangle"] & S \arrow[d, "/\langle\theta\circ\lambda\rangle"] \\
G \arrow[ur, bend left, "\exists symplectic"] \arrow[r, bend left, "\rho_1\times\rho_2"] & \mathbb{P}^1 \times \mathbb{P}^1 & T
\end{tikzcd}$$

By twisting λ by θ, we obtain a new G-action on S, i.e.

$$\mathrm{Aut}(S) \supset G \times \langle\theta\rangle \supset H \rtimes \langle\theta \circ \lambda\rangle \cong G$$

The new G-action on S turns out to be a Calabi–Yau action. A fundamental result of [8, Section 3.3] confirms that a generic K3 surface equipped with a Calabi–Yau action is realized (not necessarily uniquely) in this way. To put it another way, there always exist 2-dimensional complex projective representations $\rho_1, \rho_2 : G \to \mathrm{PGL}(2, \mathbb{C})$ which satisfy all the assumptions mentioned above. Here are two examples.

Example 2.5. *Suppose that* $G \cong D_{12} := \langle a, b | a^6 = b^2 = baba = 1\rangle$. *For* $i = 1, 2$, *we define* $\rho_i : D_{12} \to \mathrm{PGL}(2, \mathbb{C})$ *by*

$$a \mapsto \begin{bmatrix} \zeta_{12}^i & 0 \\ 0 & \zeta_{12}^{12-i} \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where ζ_k *denotes a primitive* k*-th root of unity. A basis of* D_{12}*-invariant polynomials of bidegree* $(4, 4)$ *are given by* $x^4z^4 + y^4w^4, x^4zw^3 + y^4z^3w, x^2y^2z^2w^2$. *A generic linear combination of these cuts out a desired smooth curve of bidegree* $(4, 4)$.

Example 2.6. *Suppose that* $G \cong D_8 \times C_2 = \langle a, b, c | a^4 = b^2 = baba = 1, ac = ca, bc = cb\rangle$ *For* $i = 1, 2$, *we define* $\rho_i : D_8 \times C_2 \to \mathrm{PGL}(2, \mathbb{C})$ *by*

$$a \mapsto \begin{bmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^7 \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} \sqrt{-1}^{i-1} & 0 \\ 0 & \sqrt{-1}^{1-i} \end{bmatrix}.$$

A basis of $D_8 \times C_2$*-invariant polynomials of bidegree* $(4, 4)$ *are given by* $x^4z^4 + y^4w^4, x^4w^4 + y^4z^4, x^2y^2z^2w^2$. *A generic linear combination of these cuts out a desired smooth curve of bidegree* $(4, 4)$.

2.2 Calabi–Yau threefolds of type A

We now turn to Calabi–Yau threefolds of type A. Let $A := \mathbb{C}^d/\Lambda$ be a d-dimensional complex torus. There is a natural semi-direct decomposition $\mathrm{Aut}(A) = A \rtimes \mathrm{Aut}_{\mathrm{Lie}}(A)$, where the first factor is the translation group of A and $\mathrm{Aut}_{\mathrm{Lie}}(A)$ consists of elements that fix the origin of A. We call the second factor of $g \in \mathrm{Aut}(A)$ the Lie part of g and denote it by g_0.

Theorem 2.7 ([8]). *Let X be a Calabi–Yau threefold of type A. Then X is isomorphic to the étale quotient A/G of an abelian threefold A by an action of a finite group G, where A and G are given by the following.*

1. *$A = A'/T$, where A' is the direct product of three elliptic curves E_1, E_2 and E_3:*

$$A' := E_1 \times E_2 \times E_3, \quad E_i := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau_i), \quad \tau_i \in \mathbb{H}$$

and T is one of the subgroups of A' in the following table, which consists of 2-torsion points of A'.

T_1	T_2	T_3	T_4
0	$\langle (0, 1/2, 1/2)_{A'} \rangle$	$\langle (1/2, 1/2, 0)_{A'}, (1/2, 0, 1/2)_{A'} \rangle$	$\langle (1/2, 1/2, 1/2)_{A'} \rangle$

Here $(z_1, z_2, z_3)_{A'}$ denotes the image of $(z_1, z_2, z_3) \in \mathbb{C}^3$ in A'.

2. *$G \cong C_2 \times C_2$ or D_8.*

   (a) *If $G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2$, then G is generated by*

$$a\colon (z_1, z_2, z_3)_A \mapsto (z_1 + \tau_1/2, -z_2, -z_3)_A,$$
$$b\colon (z_1, z_2, z_3)_A \mapsto (-z_1, z_2 + \tau_2/2, -z_3 + \tau_3/2)_A.$$

   (b) *If $G = \langle a, b \mid a^4 = b^2 = abab = 1 \rangle \cong D_8$, then $\tau_2 = \tau_3 =: \tau$, $T = T_2$ or T_3, and G is generated by*

$$a\colon (z_1, z_2, z_3)_A \mapsto (z_1 + \tau_1/4, -z_3, z_2)_A,$$
$$b\colon (z_1, z_2, z_3)_A \mapsto (-z_1, z_2 + \tau/2, -z_3 + (1+\tau)/2)_A.$$

Moreover, each case really occurs.

In contrast to Calabi–Yau threefolds of type K, Calabi–Yau threefolds of type A are not classified by the Galois groups of the minimal splitting coverings. They are classified by the minimal totally splitting coverings, where abelian threefolds A which cover X split into the product of three elliptic curves.

2.3 Summary

Theorem 2.1 and Theorem 2.7 finally complete the full classification of Calabi–Yau threefolds with infinite fundamental group with a very explicit description of each deformation class.

Theorem 2.8 ([8])**.** *There exist precisely fourteen Calabi–Yau threefolds with infinite fundamental group, up to deformation equivalence. To be more precise, six of them are of type A and eight of them are of type K.*

3 Mirror Symmetry

Calabi–Yau threefolds of type K are, by construction, topologically self-mirror threefolds. However, mirror symmetry should involve more than the mere exchange of Hodge numbers. In this section, we focus on Calabi–Yau threefolds of type K, whose mirror symmetry bears a resemblance to mirror symmetry of Borcea–Voisin threefolds [15, 4]. Mirror symmetry of Borcea–Voisin threefolds relies on the strange duality of K3 surfaces with anti-symplectic involution discovered by Nikulin [12].

Let $M_G := (H^2(S,\mathbb{Z})^H)^\iota$ and $N_G := (H^2(S,\mathbb{Z})^H)_\iota$. We begin with the case $G \cong C_2$. A generic K3 surface S with an Enriques involution is a self-mirror K3 surface in the sense of Dolgachev [5], i.e.

$$U \oplus NS(S) \cong T(S) \tag{3.1}$$

as lattices, where U stands for the hyperbolic lattice ($NS(S)$ and $T(S)$ are given by M_G and N_G respectively in this case). Thus it is no wonder that the corresponding Enriques Calabi–Yau threefold (Example 2.4) is an example of a self-mirror threefold. For $G \not\cong C_2$, it turns out that the corresponding K3 surface cannot be self-mirror symmetric as $\mathrm{rank}(T(S)) < 12$. In general M_G and N_G do not manifest symmetry over integers $\mathbb{Z}$, but the duality

$$U \oplus M_G \cong N_G. \tag{3.2}$$

still holds over rational numbers $\mathbb{Q}$ (or some extension of $\mathbb{Z}$) [9].

G	M_G	N_G
C_2	$U(2) \oplus E_8(-2)$	$U \oplus U(2) \oplus E_8(-2)$
$C_2 \times C_2$	$U(2) \oplus D_4(-2)$	$U(2)^{\oplus 2} \oplus D_4(-2)$
$C_2 \times C_2 \times C_2$	$U(2) \oplus \langle -4 \rangle^{\oplus 2}$	$U(2)^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$
D_6	$U(2) \oplus A_2(-2)$	$U(3) \oplus U(6) \oplus A_2(-2)$
D_8	$U(2) \oplus \langle -4 \rangle$	$U(4)^{\oplus 2} \oplus \langle -4 \rangle$
D_{10}	$U(2)$	$U(5) \oplus U(10)$
D_{12}	$U(2)$	$U(6)^{\oplus 2}$
$C_2 \times D_8$	$U(2)$	$U(4) \oplus \langle 4 \rangle \oplus \langle -4 \rangle$

It is worth noting that M_G is not equal to $NS(S)$ but to the G-invariant part $NS(S)^G$, while $T(S) = N_G$ always holds. The duality (3.2) can be thought of as an H-equivariant version of the duality (3.1) since we have $M_G = (H^2(S,\mathbb{Z})^H)^\iota$ and $N_G = (H^2(S,\mathbb{Z})^H)_\iota$. Based on this new duality, we showed in [9] that Calabi–Yau threefolds of type K are self-mirror symmetric while the corresponding K3 surfaces are not in general. We also obtained several results parallel to what is known for Borcea–Voisin threefolds: Voisin's work on Yukawa couplings [15], and Gross and Wilson's work on special Lagrangian fibrations [7].

Theorem 3.1 ([9]). *Let X be a Calabi–Yau threefold of type K. The asymptotic behavior of the A-Yukawa coupling Y_A^X around the large volume limit coincides with that of the B-Yukawa coupling Y_B^X around a large complex structure limit.*

Theorem 3.2 ([9]). *Any Calabi–Yau threefold X of type K admits a special Lagrangian T^3-fibration $\pi : X \to B$, where B is topologically either S^3 or an S^1-bundle over $\mathbb{RP}^2$.*

Another important result obtained in [9] is the computation of the Brauer groups $\mathrm{Br}(X) := \mathrm{Tor}(H^2(X, \mathcal{O}_X^\times))$, which are topological invariants for the Calabi–Yau threefolds.

Theorem 3.3 ([9]). *Let X be a Calabi–Yau threefold of type K with Galois group G. Then we have $\mathrm{Br}(X) \cong \mathbb{Z}_2^{\oplus m}$, where m is given by the following.*

G	C_2	$C_2 \times C_2$	$C_2 \times C_2 \times C_2$	D_6	D_8	D_{10}	D_{12}	$C_2 \times D_8$
m	1	2	3	1	2	1	2	3

The Brauer groups are believed to play an important role in mirror symmetry but very little is known at this point. We hope that our work provides new examples of interesting mirror symmetry with non-trivial Brauer groups.

There are a number of natural questions that arise from [8, 9]. Higher genus mirror symmetry of Calabi–Yau threefolds of type K would be very interesting to investigate. The simplest example (Enriques Calabi–Yau threefold) has previously been worked out both in physics and mathematics [10, 11]. In fact, the rich fibration structure makes the study of the special Kähler geometry of the complex moduli space particularly promising.

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