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Relation between large dimension operators and oscillator algebra of Young diagrams

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The operators with large scaling dimensions can be labeled by Young diagrams. Among other bases, the operators using restricted Schur polynomials have been known to have a large N but nonplanar limit under which they map to states of a system of harmonic oscillators. We analyze the oscillator algebra acting on pairs of long rows or long columns in the Young diagrams of the operators. The oscillator algebra can be reached by a Inonu–Wigner contraction of the $u(2)$ algebra inside of the $u(p)$ algebra of p giant gravitons. We present evidences that integrability in this case can persist at higher loops due to the presence of the oscillator algebra which is expected to be robust under loop corrections in the nonplanar large N limit.

Keywords: Gauge theory; permutation; Young diagram; large dimension operator; gauge/gravity correspondence.

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1. Introduction

The AdS/CFT duality provides insights for both gauge theory and gravity theory [1–3]. There are interesting maps between the gauge side and the gravity side. For example, the BMN operators [4] and spin chain operators [5, 6] are dual to string states on the gravity side.

The large dimension operators can describe giant gravitons which are branes on the gravity side [7–12]. The Schur polynomial operators [10] labeled by Young diagrams are dual to the giant graviton states on the gravity side [10]. For instance, long rows of the Young diagrams correspond to giant gravitons that grow in external spacetime directions. Similarly, long columns of the Young diagrams correspond to giant gravitons that grow in internal directions.

It is interesting that the operators with large scaling dimensions can take the form in the bases labeled by Young diagrams [10, 12–18]. These large dimension operators can be mapped to both giant gravitons and bubbling geometries on the gravity side [13, 19]. For example, a family of BPS operators can be labeled by the representations of Brauer algebras [15, 18] and are connected to the bubbling geometries [19]. The restricted Schur polynomial operators and the operators with global symmetry basis or flavor basis can conveniently describe the giant graviton states [14, 16, 17]. The Brauer basis, the restricted Schur basis, and the global symmetry basis or flavor basis, can be transformed between each other.

The spectra of the operators describing giant graviton excitations have been recently computed, for example [20–30]. In a large N but nonplanar limit, integrability in the nonplanar regime was observed. There have been many evidences of it at one- and two-loop in various sectors.

The operators can be diagonalized by harmonic oscillator states, for example [20, 22, 23], and by double coset ansatz [24]. The set of operators map to a system of harmonic oscillators. The harmonic oscillator dynamics can be interpreted as resulting from strings stretching between pairs of giant gravitons.

The picture of strings stretching between pairs of giant gravitons is reminiscent of the approach in the eigenvalue picture of [31], where the dynamics of background geometries and their string excitations can be treated in an eigenvalue basis.

In this paper we analyze an oscillator algebra and its role in integrability in the large N but nonplanar regime. We study the relation between the oscillator algebra and higher loop dilatation operators. We find that in the nonplanar large N limit the higher loop dilatation operators will not correct the diagonalization of the operators if they satisfy the oscillator algebra.

The organization of this paper is as follows. After introducing the relevant bases of the operators and their mixing in Sec. 2, we analyze in Sec. 3 the oscillator algebra associated to the spectra of the operators, and their relation by Inonu–Wigner contraction to an $u(2)$ algebra inside of an $u(p)$ algebra of the Young diagram operators with p long rows or long columns. In Sec. 4, we discuss the influence of the oscillator algebra on the action of the higher loop dilatation operators. We find that the h-loop dilatation operators preserve the integrability in the large N but nonplanar limit, if they satisfy the oscillator algebra. We discuss general number of pairs of giant gravitons or long rows in Sec. 5, and discuss the effective spring constants between them and the influence of the higher loop corrections on the spectra. Finally we briefly conclude in Sec. 6.

2. Bases of Operators and Mixing of Operators

We will analyze restricted Schur polynomials built from the fields in $\mathcal{N} = 4$ gauge theory. It contains various interesting sectors, such as the su(2) sector and the su$(2\,|\,3)$ sector. It can also be viewed as a $\mathcal{N} = 1$ gauge theory. It has several Higgs-like scalar fields, and the matrix scalars can be organized into complex fields, for example:

$$Z = \phi^1 + i\phi^2, \quad Y = \phi^3 + i\phi^4, \quad X = \phi^5 + i\phi^6.$$

We will focus on restricted Schur polynomials built using nZ and mY fields and will often refer to the Y fields as "impurities". The size $n+m$ is of order $O(N)$.

These operators that we study have a large scaling dimension. The restricted Schur polynomial in this case is, see for example [16],

$$\begin{aligned}\chi_{R,(r,s)\alpha\beta}&(Z^{\otimes n}, Y^{\otimes m})\\ &= \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \mathrm{Tr}(P_{R\to(r,s)\alpha\beta}\Gamma_R(\sigma))Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}.\end{aligned} \tag{1}$$

The label R is an irreducible representation of the symmetric group S_{n+m} in the form of a Young diagram with $n+m$ boxes. The labels r and s are Young diagrams with n and m boxes respectively. The r is an irreducible representation of the group S_n and the s is an irreducible representation of S_m. The group S_{n+m} has a subgroup $S_n \times S_m$ whose irreducible representations are labeled by (r,s). An irreducible representation R of S_{n+m} can subduce many different representations (r,s) of $S_n \times S_m$. The $\alpha\beta$ are multiplicity labels of the irreducible representations (r,s), which label different ways that (r,s) are subduced from R. The trace is realized by including a projector $P_{R\to(r,s)} = P_{R\to(r,s)\alpha\beta}$ and tracing over all of R, that is, $\mathrm{Tr}(P_{R\to(r,s)\alpha\beta}\Gamma_R(\sigma))$. This projector is from the carrier space of R to the carrier space of (r,s). We can also use a shorthand notation that

$$Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}} = \mathrm{Tr}[\sigma Z^{\otimes n}Y^{\otimes m}],$$

where σ is an element of S_{n+m}.

The normalization of the restricted Schur polynomial operator is, see for example [16],

$$\langle \chi_{R,(r,s)}(Z,Y)\chi_{R,(r,s)}(Z,Y)^\dagger\rangle = f_R\frac{\mathrm{hooks}_R}{\mathrm{hooks}_r\,\mathrm{hooks}_s},$$

where f_R is the product of the factors in Young diagram R and hooks_R is the product of the hook lengths of Young diagram R. The normalized operators $O_{R,(r,s)}(Z,Y)$ can be obtained by:

$$O_{R,(r,s)\alpha\beta}(Z,Y) = \sqrt{\frac{\mathrm{hooks}_r\,\mathrm{hooks}_s}{f_R\,\mathrm{hooks}_R}}\chi_{R,(r,s)\alpha\beta}(Z,Y). \tag{2}$$

In terms of the normalized operators, the action of the dilatation operator is, see for example [20, 21],

$$DO_{R,(r,s)\alpha\beta}(Z,Y) = \sum_{T,(t,u)} N_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta}O_{T,(t,u)\gamma\delta}(Z,Y).$$

For example at one loop,

$$N_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta} = -g_{\rm YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \,\text{hooks}_T \,\text{hooks}_r \,\text{hooks}_s}{f_R \,\text{hooks}_R \,\text{hooks}_t \,\text{hooks}_u}}$$
$$\times \operatorname{Tr}\Big(\big[\Gamma^{(R)}(m+1,1), P_{R\to(r,s)\alpha\beta}\big] I_{R'T'}$$
$$\times \big[\Gamma^{(T)}(m+1,1), P_{T\to(t,u)\delta\gamma}\big] I_{T'R'}\Big). \tag{3}$$

Here the $c_{RR'}$ is the weight of the corner box removed from Young diagram R to obtain Young diagram R', and similarly T' is a Young diagram obtained from T by removing a box. The d_u denotes the dimension of symmetric group irrep u. The intertwiner operator $I_{R_1T_1}$ is a map from the carrier space of irreducible representation R_1 to the carrier space of irreducible representation T_1. The $I_{R_1T_1}$ is nonzero if R_1 and T_1 are Young diagrams of the same shape. For the operators with p long rows in the Young diagram R, we remove m_i impurities from each ith-row to obtain the Young diagram r, and we may denote $\{m_i|_{i=1,\ldots,p}\}$ as $\vec{m}$. The $p=2$ case is relatively the most elementary situation in the discussions here.

After performing the trace we have

$$DO_{R,(r,s)\alpha\beta} = -g_{\rm YM}^2 \sum_{u,\gamma\delta} \sum_{1\le i<j\le p} M^{(ij)}_{s\alpha\beta;u\gamma\delta} \Delta_{ij} O_{R,(r,u)\gamma\delta}, \tag{4}$$

where Δ_{ij} acts only on the Young diagrams R, r. The $M^{(ij)}_{s\alpha\beta;u\gamma\delta}$ is a mixing matrix in the space of the Young diagrams of impurities. The action of the operator Δ_{ij} can be written as

$$\Delta_{ij} = \Delta^+_{ij} + \Delta^0_{ij} + \Delta^-_{ij}. \tag{5}$$

We denote the length of the ith-row of r by r_i. The Young diagram r^+_{ij} is obtained by removing a box from row j and then adding it to row i. The Young diagram r^-_{ij} is obtained by removing a box from row i and then adding it to row j. In terms of these Young diagrams we have that:

$$\begin{aligned}
\Delta^0_{ij} O_{R,(r,s)\alpha\beta} &= -(2N + r_i + r_j) O_{R,(r,s)\alpha\beta},\\
\Delta^+_{ij} O_{R,(r,s)\alpha\beta} &= \sqrt{(N+r_i)(N+r_j)}\, O_{R^+_{ij},\left(r^+_{ij},s\right)\alpha\beta},\\
\Delta^-_{ij} O_{R,(r,s)\alpha\beta} &= \sqrt{(N+r_i)(N+r_j)}\, O_{R^-_{ij},\left(r^-_{ij},s\right)\alpha\beta}.
\end{aligned} \tag{6}$$

The Δ_{ij} acts on R, r and on the Z's. Note that the R and r change in the same way. The $c_i = N + r_i$ is the factor of the corner box in the ith-row, while $c_j = N + r_j$ is the factor of the corner box in the jth-row, and they are of order N. The number of rows in the Young diagrams R, r is p, and the length of the p rows is long and of order N. The Young diagram s of impurities has no more than p rows. The Δ_{ij} acts on each pair of rows (i, j). For these operators with a very large dimension of order $O(N)$, the nonplanar diagrams already contribute in the leading order, and the spectra are obtained by summing over planar and nonplanar Feynman diagrams.

These expressions are evaluated in a large N but nonplanar limit. The expressions (6) are written in the case for the AdS giants. The expressions for the sphere giants are similar. The weights are now $c_i = N - r_i, c_j = N - r_j$, corresponding to the factors of the corner boxes on the ith- and jth-columns. In other words, we replace $N + r_i$ by $N - r_i$, and $N + r_j$ by $N - r_j$ in the above equations for the actions of $\Delta^0_{ij}, \Delta^+_{ij}$ and Δ^-_{ij}.

The construction of the restricted Schur polynomial operators in various sectors, such as su(2), su(2 | 3), sl(2) and su(3) sectors, and their anomalous dimensions have been considered in the recent work for example [20–30]. There are many other very interesting works on giant graviton excitations from various different perspectives, for example [33–35].

3. $su(2)$ Algebra and Oscillator Algebra

The p number of giant graviton D3-branes is expected to have an $u(p)$ symmetry. The $u(p)$ symmetry algebra, with $p \geq 2$, contains the $u(2)$ algebra as a subalgebra, which in turn, contains the $su(2)$ algebra as a subalgebra. The $u(2)$ algebra is the symmetry algebra acting on a pair of giant gravitons. The $su(2)$ algebra is embedded as

$$su(2) \subset u(2) \subseteq u(p). \tag{7}$$

We first review the construction of the $u(2)$ algebra from the $u(p)$ algebra that was performed in [23]. The fundamental representation of the $u(p)$ algebra represents the elements of the Lie algebra as $p \times p$ matrices. The generators $E_{ik} \in u(p)$ can be written as

$$(E_{ik})_{ab} = \delta_{ia}\delta_{kb}, \quad 1 \leq i, k, a, b \leq p.$$

The p operators E_{ii} commute with each other so we can choose a basis in which they are diagonal at the same time. The restricted Schur polynomial labeled by the Young diagrams is identified with the state with $\{E_{ii}\}$. The Young diagrams r with p rows are the irreducible representations of the symmetric group and also the irreducible representations of the $u(p)$ group. There is a map $\frac{1}{2}E_{ii} \mapsto c_i$, for $i = 1, \ldots, p$, where c_i are the factors of the corner boxes of each row i. The $\{\frac{1}{2}E_{ii} \,|\, i = 1, \ldots, p\}$ corresponds to the $\{c_i \,|\, i = 1, \ldots, p\}$. The description for the case of p long columns is similar, where the c_i are the factors of the corner boxes of each column i, for $i = 1, \ldots, p$. The $p = 2$ case is relatively the simplest case in this construction.

We consider the generators:

$$Q_{ij} = \frac{1}{2}(E_{ii} - E_{jj}), \quad Q^+_{ij} = E_{ij}, \quad Q^-_{ij} = E_{ji},$$

which obey the $su(2)$ algebra of angular momentum, raising and lowering operators [23],

$$\left[Q_{ij}, Q^+_{ij}\right] = Q^+_{ij}, \quad \left[Q_{ij}, Q^-_{ij}\right] = -Q^-_{ij}, \quad \left[Q^+_{ij}, Q^-_{ij}\right] = 2Q_{ij}. \tag{8}$$

We can also define

$$Q^+_{ij} = Q^1_{ij} + iQ^2_{ij}, \quad Q^-_{ij} = Q^1_{ij} - iQ^2_{ij}, \quad Q_{ij} = Q^3_{ij}$$

and span$\{Q^1_{ij}, Q^2_{ij}, Q^3_{ij}\}$ is the $su(2)$ algebra.

The representations of these $su(2)$ subalgebras can be labeled with the eigenvalue of $\vec{Q}^2_{ij} = \frac{1}{2}(Q^+_{ij}Q^-_{ij} + Q^-_{ij}Q^+_{ij}) + (Q_{ij})^2$ and the eigenvalue of $Q_{ij} = Q^3_{ij}$. The η and $\Lambda(\Lambda+1)$ are defined as the eigenvalues of the operators Q_{ij} and $\vec{Q}^2_{ij}$ respectively. The states are labeled by $|\eta, \Lambda\rangle$ and:

$$Q^+_{ij}|\eta, \Lambda\rangle = \sqrt{(\Lambda + \eta + 1)(\Lambda - \eta)}|\eta + 1, \Lambda\rangle,$$
$$Q^-_{ij}|\eta, \Lambda\rangle = \sqrt{(\Lambda + \eta)(\Lambda - \eta + 1)}|\eta - 1, \Lambda\rangle,$$

where $-\Lambda \leq \eta \leq \Lambda$. We have so far reviewed the construction of the $su(2)$ algebra $\{Q^1_{ij}, Q^2_{ij}, Q^3_{ij}\}$ from the $u(p)$ algebra performed in [23].

Let us now turn to the action of the dilatation operator Δ_{ij} on these Young diagram operators. We focus on a pair of giant gravitons labeled by i and j, which also correspond to a pair of long rows labeled by i and j. In particular, the operators Δ_{ij} are, as according to Eqs. (5) and (6), for example [20, 22, 24],

$$\Delta_{ij} = -\frac{1}{2}(E_{ii} + E_{jj}) + Q^-_{ij} + Q^+_{ij}, \tag{9}$$

so we can write it in the form

$$\Delta_{ij} = 2A^3_{ij} - (c_i + c_j)I_{ij}, \tag{10}$$

where we also define an abelian generator $I_{ij} = \frac{1}{2(c_i+c_j)}(E_{ii} + E_{jj})$. The eigenvalue of the Q_{ij} is $\frac{1}{2}(c_i - c_j)$ and the $\Lambda = \frac{1}{2}\max|c_i - c_j|$.

We can define a different set of $su(2)$ generators which will be convenient for the evaluation of the Δ_{ij}. These can be defined as:

$$\begin{aligned}
A^+ &= \frac{1}{2}(E_{ii} - E_{jj}) + \frac{1}{2}(E_{ij} - E_{ji}) = Q^3_{ij} + iQ^2_{ij},\\
A^- &= \frac{1}{2}(E_{ii} - E_{jj}) - \frac{1}{2}(E_{ij} - E_{ji}) = Q^3_{ij} - iQ^2_{ij},\\
A^3 &= \frac{1}{2}(E_{ij} + E_{ji}) = \frac{1}{2}\left(Q^+_{ij} + Q^-_{ij}\right) = Q^1_{ij}.
\end{aligned} \tag{11}$$

The algebra

$$\mathrm{span}\{A^+, A^-, A^3\}$$

is the $su(2)$ algebra with commutation relations:

$$[A^-, A^+] = 2A^3, \quad [A^3, A^+] = -A^+, \quad [A^3, A^-] = A^-. \tag{12}$$

It is transformed from span$\{Q^1_{ij}, Q^2_{ij}, Q^3_{ij}\}$ by an automorphism of $su(2)$. In this paper, we mainly work with the new basis $\{A^+, A^-, A^3\}$.

The linear span of

$$\mathrm{span}\{A^+, A^-, A^3, I\}, \tag{13}$$

forms the $su(2) \times u(1)$ algebra, where I is the generator of the $u(1)$ algebra that we also include. The $su(2) \times u(1)$ algebra is also the $u(2)$ algebra. We now denote Δ_{ij} as $\Delta_{(1)ij}$. We can conveniently express

$$\Delta_{(1)ij} = 2A^3 - (c_i + c_j)I,$$

where we have suppressed the ij indices. In particular $\Delta_{(1)ij}$ is a linear combination of A^3 and I. This expression is according to the findings in, for example, [20, 22, 24].

We can perform a Inonu–Wigner contraction of this $su(2) \times u(1)$ algebra, by a linear transformation:

$$\begin{bmatrix} a^\dagger \\ a \\ \frac{1}{2}\Delta_{(1)ij} \\ 1 \end{bmatrix} = \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2}\xi^{-2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A^+ \\ A^- \\ A^3 \\ I \end{bmatrix}, \tag{14}$$

where $\xi = \frac{1}{\sqrt{c_i + c_j}}$ and is of order $O(\frac{1}{\sqrt{N}})$. The inverse transformation is:

$$\begin{bmatrix} A^+ \\ A^- \\ A^3 \\ I \end{bmatrix} = \begin{bmatrix} \xi^{-1} & 0 & 0 & 0 \\ 0 & \xi^{-1} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\xi^{-2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a^\dagger \\ a \\ \frac{1}{2}\Delta_{(1)ij} \\ 1 \end{bmatrix}. \tag{15}$$

This Inonu–Wigner contraction corresponds to the limit $\xi^2 \Delta_{(1)ij} \ll 1$, in other words the eigenvalue of $\frac{1}{2}\Delta_{(1)ij}$, which is the oscillator level, is much smaller than N. The above $su(2) \times u(1)$ algebra, which is also an $u(2)$ algebra, via the Inonu–Wigner contraction, becomes the harmonic oscillator algebra,

$$\mathrm{span}\big\{a^\dagger, a, \Delta_{(1)ij}, 1\big\}, \tag{16}$$

with commutation relations

$$[a, a^\dagger] = 1, \quad \big[\Delta_{(1)ij}, a^\dagger\big] = -2a^\dagger, \quad \big[\Delta_{(1)ij}, a\big] = 2a. \tag{17}$$

This is a harmonic oscillator algebra obeyed by the creation and annihilation operators of this oscillator.

The $u(p)$ symmetry algebra always includes $u(2)$ symmetry algebra as a subalgebra. From the point of view of the operators given by Young diagrams, the $u(2)$ symmetry algebra is the symmetry algebra acting on two long rows, the ith-row and the jth-row. From the point of view of the gravity side, the $u(2)$ symmetry algebra is the symmetry algebra of two giant graviton branes labeled by i and j.

The existence of $u(p)$ symmetry is also from the Young diagrams with p long rows, which also label the irreducible representations of the symmetric algebras

[20, 22]. This involves the duality between representations of symmetric groups and unitary groups that we have discussed.

The $u(p)$ symmetry can also be observed from the symmetry of p giant gravitons. The $u(p)$ symmetry can also be seen from the bubbling geometries in [13] with a white disk of area p inside the black disk of area N. The white disk with area p is the geometric dual of p giant gravitons. If we zoom in near the white disk it approaches another AdS space and is dual to p number of three-branes with $u(p)$ symmetry.

4. Higher Loop Anomalous Dimensions and Oscillator Algebra

We now discuss relation between the oscillator algebra and higher loop dilatation operators. The anomalous dimension $\gamma(g)$ expanded at one- and two-loops is the eigenvalue of

$$\hat{D} = \hat{D}_2 + \hat{D}_4,$$

with the one-loop dilatation operator

$$\hat{D}_2 = -2g : \mathrm{Tr}([Z,Y][\partial_Z,\partial_Y]) : \tag{18}$$

and the two-loop dilatation operator [6]:

$$\begin{aligned}\hat{D}_4 = -2g^2 : \mathrm{Tr}([[Z,Y],\partial_Z][[\partial_Z,\partial_Y],Z]) : -2g^2 : \mathrm{Tr}([[Z,Y],\partial_Y][[\partial_Z,\partial_Y],Y]) : \\ -2g^2 : \mathrm{Tr}\big([[Z,Y],T^a]\big[[\partial_Z,\partial_Y],T^a\big]\big) :\end{aligned} \tag{19}$$

where in the convention of [6] $g = \frac{\tilde{g}^2_{\mathrm{YM}}}{16\pi^2}$. The normalization for the $\hat{D}_2$ and $\hat{D}_4$ is in the convention of [6]. The normalization in this convention for $\hat{D}_2$ and $\hat{D}_4$ is a factor of two larger than the normalization used in the convention of [20–24, 26, 29], for example. Here we denote D_2 and D_4 for the convention in [20–24, 26, 29].

The action of the two-loop dilatation operator on the restricted Schur polynomials has been evaluated by [25], and it is given by,

$$\hat{D}_4 O_{R,(r,s)\alpha\beta} = -2g^2 \sum_{u,\,\gamma\delta}\sum_{i<j} M^{(ij)}_{s\alpha\beta;u\gamma\delta}\Delta_{(2)ij} O_{R,(r,u)\gamma\delta}, \tag{20}$$

where

$$\begin{aligned}M^{(ij)}_{s\alpha\beta;u\gamma\delta} = \frac{m}{\sqrt{d_s d_u}}\big(&\langle\vec{m},s,\beta;a\big|E^{(1)}_{ii}\big|\vec{m},u,\delta;b\rangle\langle\vec{m},u,\gamma;b\big|E^{(1)}_{jj}\big|\vec{m},s,\alpha;a\rangle \\ &+\langle\vec{m},s,\beta;a\big|E^{(1)}_{jj}\big|\vec{m},u,\delta;b\rangle\langle\vec{m},u,\gamma;b\big|E^{(1)}_{ii}\big|\vec{m},s,\alpha;a\rangle\big),\end{aligned}$$

in which a and b are summed. The a labels states in irreducible representation s and the b labels states in irreducible representation u.

We denote $\Delta_{(1)ij} = \Delta_{ij}$ for the one-loop dilatation operator and $\Delta_{(2)ij}$ for the two-loop dilatation operator. The $\Delta_{(2)ij}$ has been computed by [25], and it can be written as a sum of two terms. We have defined that:

$$\begin{aligned}
\Delta^0_{ij} O_{R,(r,s)\alpha\beta} &= -(2N + r_i + r_j) O_{R,(r,s)\alpha\beta},\\
\Delta^+_{ij} O_{R,(r,s)\alpha\beta} &= \sqrt{(N + r_i)(N + r_j)}\, O_{R^+_{ij},(r^+_{ij},s)\alpha\beta},\\
\Delta^-_{ij} O_{R,(r,s)\alpha\beta} &= \sqrt{(N + r_i)(N + r_j)}\, O_{R^-_{ij},\left(r^-_{ij},s\right)\alpha\beta}.
\end{aligned} \tag{21}$$

The $\Delta_{(2)ij} = \Delta^{(1)}_{ij} + \Delta^{(2)}_{ij}$ and can be written as:

$$\Delta^{(1)}_{ij} = n\big(\Delta^+_{ij} + \Delta^0_{ij} + \Delta^-_{ij}\big), \tag{22}$$

$$\Delta^{(2)}_{ij} = \big(\Delta^+_{ij}\big)^2 + \Delta^0_{ij}\Delta^+_{ij} + 2\Delta^+_{ij}\Delta^-_{ij} + \Delta^0_{ij}\Delta^-_{ij} + \big(\Delta^-_{ij}\big)^2. \tag{23}$$

As the same as $\Delta_{(1)ij}$, the $\Delta_{(2)ij}$ acts on each pair of rows (i, j). The c_i, c_j and n are of order N.

We can simplify the two-loop dilatation operator $\Delta_{(2)ij}$ as

$$\Delta_{(2)ij} = (\Delta_{(1)ij} + (c_i + c_j + n)I)\Delta_{(1)ij}, \tag{24}$$

when acting on these Young diagram operators. These expressions are evaluated in the large N but nonplanar limit. From this expression, we see that the operators that are eigenstates of $\Delta_{(1)ij}$, are also eigenstates of $\Delta_{(2)ij}$. Since $\Delta_{(1)ij}$ mixes the operators that differ by moving at most one box in the (R, r) irreducible representations, $\Delta_{(2)ij}$ mixes the operators that differ by moving at most two boxes in the (R, r) irreducible representations. The action of $(\Delta^+_{ij})^h, (\Delta^-_{ij})^h$ mix the operators that differ by moving at most h boxes in the (R, r) irreducible representations, and therefore $(\Delta_{(1)ij})^h$ mixes the operators that differ by moving at most h boxes in the (R, r) irreducible representations.

The eigenstates in the oscillator basis are states of finite harmonic oscillators, see for example [22, 20, 24]. These operators can be written as

$$O_{q_{ij}}(\sigma) = \sum_{R,r} \tilde{f}^{R,r}_{q_{ij}} O_{R,r}(\sigma) = \sum_{R,r}\sum_{s,\alpha\beta} \tilde{f}^{R,r}_{q_{ij}} C^s_{\alpha\beta}(\sigma) O_{R,(r,s)\alpha\beta}, \tag{25}$$

where $\tilde{f}^{R,r}_{q_{ij}}$ are the wave functions of the discrete harmonic oscillator. The Young diagrams (R, r) both have p long rows, and $R \vdash n + m, r \vdash n$. Those functions also appear in the study of models of finite harmonic oscillators, for example [32]. The $C^s_{\alpha\beta}(\sigma)$ are group theoretic coefficients and $O_{R,r}(\sigma)$ are the operators labeled by a permutation graph σ that maps [24] to the data $\{n_{ij}\}$ via the function $n_{ij}(\sigma)$, where n_{ij} are the number of strings stretching between the two branes labeled by i and j.

The eigenvalues of Δ_{ij} acting on $O_{q_{ij}}(\sigma)$ is $4q_{ij}$, where q_{ij} is an integer [20, 22] denoting the level of the harmonic oscillator for the pair of giant gravitons i and j, that is,

$$\Delta_{(1)ij} O_{q_{ij}}(\sigma) = 4q_{ij} O_{q_{ij}}(\sigma) \tag{26}$$

and

$$\hat{D}_2 O_{q_{ij}}(\sigma) = -2g \sum_{R,r} \tilde{f}_{q_{ij}}^{R,r} n_{ij} \Delta_{ij} O_{R,r}(\sigma) = -8g q_{ij} n_{ij} O_{q_{ij}}(\sigma). \tag{27}$$

The q_{ij} is bounded above due to the discrete finite harmonic oscillator. Therefore we consider the simple case in which $q_{ij} \ll N$. We see that

$$\Delta_{(2)ij} O_{q_{ij}}(\sigma) = 4q_{ij}(c_i + c_j + n) O_{q_{ij}}(\sigma) = (c_i + c_j + n) \Delta_{(1)ij} O_{q_{ij}}(\sigma), \tag{28}$$

in the large N limit, and

$$\hat{D}_4 O_{q_{ij}}(\sigma) = -2g^2 \sum_{R,r} \tilde{f}_{q_{ij}}^{R,r} n_{ij} \Delta_{(2)ij} O_{R,r}(\sigma) = -8g^2 q_{ij} n_{ij} (c_i + c_j + n) O_{q_{ij}}(\sigma). \tag{29}$$

In this case, the anomalous dimension $\gamma(\lambda)$ expanded at one- and two-loop orders is

$$\gamma_{(1)} = \frac{8q_{ij} n_{ij}}{N} \lambda, \qquad \gamma_{(2)} = \frac{8q_{ij} n_{ij}}{N} \left(\frac{c_i + c_j + n}{N} \right) \lambda^2, \tag{30}$$

where $\frac{c_i+c_j+n}{N} = \frac{r_i+r_j+2N+n}{N}$ is of order 1, $q_{ij} = 0, 1, 2, \ldots, q_{\max}$, and the λ denotes gN. This is the simple case that there are equal number of strings emanating from brane i to brane j, and from brane j to brane i, and these numbers are n_{ij}^+ and n_{ij}^- respectively. In this case $n_{ij} = n_{ij}^+ + n_{ij}^- = 2n_{ij}^+$, where n_{ij}^+ is a non-negative integer. If $|c_i - c_j| \ll c_i + c_j$, then $r_i + r_j$ is approximately $2r_j$ [25]. Note that in this paper we also assumed the special case that we are looking at the operators whose $q_{\max}$ is much smaller than N.

From Eq. (28), the commutation relations of $\Delta_{(2)ij}$ are

$$[a, a^\dagger] = 1, \quad \left[\Delta_{(2)ij}, a^\dagger\right] = -2(c_i + c_j + n) a^\dagger, \quad \left[\Delta_{(2)ij}, a\right] = 2(c_i + c_j + n) a. \tag{31}$$

So the two-loop dilatation operator $\Delta_{(2)ij}$ also satisfies the oscillator algebra. This relation was observed in [25].

We can write them in terms of polynomials of A^3. Using the relation

$$\Delta_{(1)ij} = 2A^3 - (c_i + c_j) I, \tag{32}$$

the $\Delta_{(2)ij}$ can be rewritten as:

$$\begin{aligned} \Delta_{(2)ij} &= \big(2A^3 - (c_i + c_j) I\big)(2A^3 + nI) \\ &= 4(A^3)^2 + 2(n - c_i - c_j) A^3 - n(c_i + c_j) I. \end{aligned} \tag{33}$$

So $\Delta_{(2)ij}$ is a polynomial of A^3 of degree 2, that is, $P_2(A^3)$. The $\Delta_{(1)ij}$ is a polynomial of A^3 of degree 1, that is, $P_1(A^3)$. By the structure of the dilatation operators at higher loops, the higher loop dilatation operators $\Delta_{(h)ij}$, with h the loop order, is a polynomial of A^3 of degree h, that is, $P_h(A^3)$,

$$\Delta_{(h)ij} = P_h(A^3),$$

where

$$A^3 = \frac{1}{2}[\Delta_{(1)ij} + (c_i + c_j)I] \tag{34}$$

and they can mix operators that differ by moving at most h boxes in the (R, r) irreducible representations in the large N limit that we consider. The $P_h(A^3)$ is also a polynomial of Δ^+ and Δ^-. These dilatation operators form a polynomial ring $\mathbb{R}[A^3]$ over A^3.

The operators with $q_{ij} = 0$ should correspond to the BPS states, and they are the states of the giant gravitons without the non-BPS excitations. These states are thus the eigenstates of the anomalous piece of the dilatation operator with the eigenvalues being zero. Since both $\Delta_{(1)ij}$ and $\Delta_{(2)ij}$ contain an overall factor of $(2A^3 - (c_i + c_j)I)$, this factor acting on the BPS states is zero. As analyzed above, if $\Delta_{(h)ij}$, with h the loop order, is a polynomial of A^3 of degree h, then it is expected to contain an overall factor of $(2A^3 - (c_i + c_j)I)$. It may be written as:

$$\begin{aligned}\Delta_{(h)ij} &= \left(2A^3 - (c_i + c_j)I\right)\left[\sum_{l=1,\ldots,h} a_l (A^3)^{h-l}\right] \\ &= \Delta_{(1)ij}\left[N^{h-1}F_h + u_{h-2}\right],\end{aligned} \tag{35}$$

when acting on the operators, where F_h is order $O(1)$ coefficient, and $u_{h-2} \leq O(N^{h-2})$. The F_h is of order $O(1)$ because the eigenvalue of $(A^3)^{h-1}$ is of order $O(N^{h-1})$. In the derivation from the first line to the second line of Eq. (35), we used that $A^3 O_{q_{ij}}(\sigma) = \frac{1}{2}(c_i + c_j + \Delta_{(1)ij})O_{q_{ij}}(\sigma) = [O(N)]O_{q_{ij}}(\sigma)$, which means that the eigenvalue of A^3 on the diagonalized operators is of order $O(N)$. The subleading terms in Eq. (35) are subleading in the large N limit.

So we have the relation

$$\Delta_{(h)ij} O_{q_{ij}}(\sigma) = F_h N^{h-1} \Delta_{(1)ij} O_{q_{ij}}(\sigma), \tag{36}$$

in the large N limit. In this relation, we have also assumed that we are considering the oscillator level to be much smaller than N, that is, we are giving an additional condition $\frac{q_{ij}}{N} \ll 1$ to make simplifications. The examples of the $h = 1$ and $h = 2$ are $F_1 = 1$ and $F_2 = \frac{c_i + c_j + n}{N}$, and are both of order $O(1)$. So we have:

$$\begin{aligned}\Delta_{(1)ij} O_{q_{ij}}(\sigma) &= 4q_{ij} O_{q_{ij}}(\sigma) = F_1 \Delta_{(1)ij} O_{q_{ij}}(\sigma), \\ \Delta_{(2)ij} O_{q_{ij}}(\sigma) &= 4q_{ij}(c_i + c_j + n) O_{q_{ij}}(\sigma) = F_2 N \Delta_{(1)ij} O_{q_{ij}}(\sigma).\end{aligned}$$

In the large N limit, from the structure of the dilatation operator,

$$\Delta_{(h)ij} O_{q_{ij}}(\sigma) = 4q_{ij} F_h N^{h-1} O_{q_{ij}}(\sigma) = F_h N^{h-1} \Delta_{(1)ij} O_{q_{ij}}(\sigma), \tag{37}$$

where F_h is order $O(1)$.

In the large N limit, as long as the higher loop dilatation operator $\Delta_{(h)ij}$ is a polynomial of A^3, it will satisfy the following oscillator algebra,

$$[a, a^\dagger] = 1, \quad \left[\Delta_{(h)ij}, a^\dagger\right] = -2F_h N^{h-1} a^\dagger, \quad [\Delta_{(h)ij}, a] = 2F_h N^{h-1} a. \tag{38}$$

We refer to Eq. (38) as the oscillator algebra satisfied by $\Delta_{(h)ij}$. If the oscillator algebra is satisfied by $\Delta_{(l)ij}$, with $l = 1, \ldots, h$, then the integrability is preserved at the h-loop order. The oscillator algebra at h-loop means that, among other aspects, $\Delta_{(h)ij}$ will not change the q_{ij} in Eq. (26).

Assuming that the oscillator algebra is satisfied in the large N limit at all loops, if we sum over $\sum_{h=1}^{\infty} g^h \Delta_{(h)ij} = \Delta$, we have that:

$$\left[\sum_{h=1}^{\infty} g^h \Delta_{(h)ij}, a^\dagger\right] = \sum_{h=1}^{\infty} g^h \left[\Delta_{(h)ij}, a^\dagger\right] = -2\sum_{h=1}^{\infty} g^h F_h N^{h-1} a^\dagger = -2f(\lambda) a^\dagger,$$

where we define an interpolating function $f(\lambda)$,

$$\sum_{h=1}^{\infty} g^h F_h N^{h-1} = \sum_{h=1}^{\infty} \lambda^h F_h / N = \frac{1}{N} f(\lambda). \tag{39}$$

The $f(\lambda)$ is a function of λ, and its coefficients in λ expansions are also functions of $\frac{c_i}{N}, \frac{c_j}{N}$. Since $F_1 = 1$, the expansion of $f(\lambda)$ is $f(\lambda) = \lambda + \sum_{h=2}^{\infty} F_h \lambda^h$. The oscillator algebra satisfied by the Δ is hence,

$$[a, a^\dagger] = 1, \quad [\Delta, a^\dagger] = -2\frac{f(\lambda)}{N} a^\dagger, \quad [\Delta, a] = 2\frac{f(\lambda)}{N} a. \tag{40}$$

The dilatation operator at all loops can be expanded as $\hat{D}(g) = \sum_{h=0}^{\infty} \hat{D}_{2h}$, where $\hat{D}_{2h}$ is at order g^h, with h the loop order. In the large N limit,

$$g^h \Delta_{(h)ij} O_{q_{ij}}(\sigma) = F_h g^h N^{h-1} \Delta_{(1)ij} O_{q_{ij}}(\sigma)$$

and hence the anomalous dimension is

$$\gamma = \sum_{h=1}^{\infty} \gamma_{(h)} = 8 q_{ij} n_{ij} \sum_{h=1}^{\infty} g^h F_h N^{h-1} = \frac{8 q_{ij} n_{ij}}{N} f(\lambda). \tag{41}$$

The one-loop anomalous dimension is $\gamma_{(1)} = 8\frac{\lambda}{N} q_{ij} n_{ij}$. As compared to the one-loop expression, the effect at higher loops is the renormalization of the coupling to an interpolating function, that is, $\lambda \to f(\lambda)$, while keeping the same $q_{ij} n_{ij}$ dependence which is protected by the oscillator algebra. The $f(\lambda)$ is also a function of $\frac{c_i}{N}, \frac{c_j}{N}$, and may be written also as $f(\lambda; \frac{c_i}{N}, \frac{c_j}{N})$.

We have focused on the su(2) sector of the operators and on a pair of long rows labeled by i and j in the Young diagram with total p rows. This sector has a harmonic oscillator algebra, for which we have presented evidence that they will persist at higher loops.

The diagonalized operators acted on by the one- and two-loop dilatation operators will not be changed by the h-loop dilatation operators if the oscillator algebra, Eq. (38), is preserved at loop order h. So, if the oscillator algebra is preserved at all loop orders, they would not be changed at all loops.

Our analysis indicates that the nonplanar integrability is protected in the large N limit by the oscillator algebra. This is an integrability in a large N but nonplanar limit. We give further evidence that this integrability in the nonplanar regime is preserved at higher loops and possibly at all loops. This is the case if higher loop and all loop dilatation operators in the large N but nonplanar limit satisfy the oscillator algebra. The oscillator algebra descends from the $u(2)$ symmetry algebra inside of $u(p)$ symmetry algebra of the system of p giant gravitons and such symmetry is expected to be robust under loop corrections. We have presented evidences that the higher loop dilatation operators in the large N limit can satisfy the oscillator algebra.

5. General Number of Pairs and Effective Spring Constants

We now consider pairs of giant gravitons. Each pair of long rows labeled by i and j corresponds to a pair of giant gravitons labeled by i and j. The numbers of strings between each pair of giant gravitons is n_{ij}. The total number of strings is m, which is divided into p integers $\{m_i \,|\, i = 1, \ldots, p\}$. The m_i denotes the number of strings emanating from the ith-brane. Because of charge conservation on each brane, the number of strings terminating on the ith-brane is also m_i. The string configurations are modding out by the permutation symmetry $H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p}$ on both their left open-ends and their right open-ends. The open string configurations are in one-to-one correspondence with elements of the double coset [24]:

$$H_L \backslash S_m / H_R, \tag{42}$$

with $m = \sum_i m_i$, and $H_L = H_R = H$ which is the subgroup $S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p}$. The H_L permutes the m left open-ends and the H_R permutes the m right open-ends. The relation $H_L = H_R$ is due to the charge conservation on each brane.

There is a map

$$\sigma \mapsto \{n_{ij} \,|\, 1 \leq i < j \leq p\}, \tag{43}$$

where σ is an element of the double coset [24]. The numbers $n_{ij}(\sigma)$ can be given from the element of the double coset σ. For a generic configuration of open strings on p giant gravitons, by distinguishing orientation, the number of strings emanating from brane i and terminating on brane j is n_{ij}^{+}, whereas the number of strings emanating from brane j and terminating on brane i is n_{ij}^{-}. The total number of strings between brane i and j, without distinguishing orientation, is thus $n_{ij} = n_{ij}^{+} + n_{ij}^{-}$. The permutation graph σ corresponds to $n_{ij}(\sigma) = n_{ij}^{+}(\sigma) + n_{ij}^{-}(\sigma)$ strings between the pairs of two branes labeled by i and j, and $m_i - \sum_{j\neq i} n_{ij}^{+} = n_{ii}$ strings emanating from and terminating on the same brane i. We also have that $n_{ij}^{+} \leq m_i \leq m$.

The operators correspond to the data $\{n_{ij}\}$ and its associated permutation graph is

$$O_{R,r}(\sigma) = \sum_{s\vdash m}\sum_{\alpha\beta} C^s_{\alpha\beta}(\sigma) O_{R,(r,s)\alpha\beta}, \tag{44}$$

where the group theoretic coefficients are [24]:

$$C^s_{\alpha\beta}(\sigma) = \frac{\sqrt{d_s}|H|}{\sqrt{m!}} \sum_{l,j} \Gamma^{(s)}(\sigma)_{lj} B^{s\to 1_H}_{l\alpha} B^{s\to 1_H}_{j\beta}.$$

The $B^{s\to 1_H}_{l\alpha}$ are the branching coefficients for the trivial irrep 1_H of H inside the representation s of S_m. The $B^{s\to 1_H}_{l\alpha}$ gives the expansion of the αth-occurrence of the identity irrep of H when irrep s of S_m is decomposed into irreps of the subgroup H, in terms of the states labeled l in s. The l, j here are labels of states in the irreducible representation s. Similar methods of representation theory used in this context were also used in defining the multi-matrix operators in the global symmetry basis or the flavor basis [14, 17].

Since the interaction between the p giant gravitons are pairwise, the most elementary situation is the interaction between two giant gravitons labeled by i and j, where $1 \le i < j \le p$. A simple situation is provided by a pair of giant gravitons with equal number of strings from each other, that is $n_{ij} = n^+_{ij} + n^-_{ij} = 2n^+_{ij}$. This also corresponds to a pair of long rows labeled by i and j.

For $p = 2$, which is the simplest and the most elementary case, H is $S_{m_1} \times S_{m_2}$. The $m_1 + m_2$ left open-ends of the strings are divided into m_1 and m_2 of them, respectively, on the two giant gravitons. Because of charge conservation, the number of right open-ends is also m_1 and m_2, respectively, on the two giant gravitons. The left open-ends and the right open-ends are related by a permutation in $S_{m_1+m_2}$. The open string configurations are thus in one-to-one correspondence with elements of the double coset

$$H\backslash S_{m_1+m_2}/H, \tag{45}$$

since the left coset corresponds to modding out by the $S_{m_1} \times S_{m_2}$ on the left open-ends of the strings, and the right coset corresponds to modding out by the $S_{m_1} \times S_{m_2}$ on the right open-ends of the strings. For the simplest case that $p = 2$, the permutation graph corresponds to $n_{12} = n^+_{12} + n^-_{12}$ strings between the two branes labeled by 1 and 2, $n_{11} = m_1 - n^+_{12}$ strings emanating and terminating on the same brane 1, and $n_{22} = m_2 - n^-_{12}$ strings emanating and terminating on the same brane 2.

The action of the dilatation operator on the operator $O_{R,r}(\sigma)$ is

$$DO_{R,r}(\sigma) = -g^2_{\mathrm{YM}} \sum_{1\le i<j\le p} n_{ij}(\sigma)\Delta_{ij} O_{R,r}(\sigma). \tag{46}$$

The p long rows correspond to the p giant gravitons. The $n_{ij}(\sigma)$ is a map $\sigma \mapsto \{n_{ij}\}$. The excitation energy of the system of giant gravitons δE is given by the eigenvalue of the operator $\sum g^2_{\mathrm{YM}} n_{ij}\Delta_{ij}$.

Since the operator $O_{q_{ij}}(\{n_{ij}\})$ in the oscillator basis [20, 22] is

$$O_{q_{ij}}(\{n_{ij}\}) = O_{q_{ij}}(\sigma) = \sum_{R,r} \tilde{f}^{R,r}_{q_{ij}} O_{R,r}(\sigma) \tag{47}$$

and

$$\Delta_{ij} O_{q_{ij}}(\sigma) = 4q_{ij} O_{q_{ij}}(\sigma),$$

where q_{ij} is a non-negative integer, we have that

$$D_2 O_{q_{ij}}(\sigma) = -g^2_{\mathrm{YM}} \sum_{R,r} \tilde{f}^{R,r}_{q_{ij}} n_{ij}(\sigma) \Delta_{ij} O_{R,r}(\sigma) = -4g^2_{\mathrm{YM}} q_{ij} n_{ij} O_{q_{ij}}(\sigma). \tag{48}$$

The action of the dilatation operator $\hat{D}_4$ on these operators is

$$\hat{D}_4 O_{R,r}(\sigma) = -2g^2 \sum_{i<j} n_{ij}(\sigma) \Delta_{(2)ij} O_{R,r}(\sigma), \tag{49}$$

which was obtained in [25], and:

$$\hat{D}_4 O_{q_{ij}}(\sigma) = -2g^2 \sum_{R,r} \tilde{f}^{R,r}_{q_{ij}} n_{ij}(\sigma) \Delta_{(2)ij} O_{R,r}(\sigma) = -8g^2 q_{ij} n_{ij} (c_i + c_j + n) O_{q_{ij}}(\sigma). \tag{50}$$

We focus on a pair of branes labeled by i and j. We consider the simple case that there are equal numbers of strings emanating from brane i to brane j, and from brane j to brane i. By charge conservation on each brane, in this case, $n^+_{ij} = n^-_{ij}$ and $n_{ij} = 2n^+_{ij} = 2n^-_{ij}$. The action of the dilatation operator is

$$D O_{R,r}(\sigma) = -g^2_{\mathrm{YM}} n_{ij}(\sigma) \Delta_{ij} O_{R,r}(\sigma). \tag{51}$$

The spectrum with general number of pairs of giant gravitons has been studied in [23]. Note that the notation $2n_{ij}$ used in [23] is the $2n^+_{ij} = n_{ij}$ in the discussion here. The frequency $2g^2_{\mathrm{YM}} n_{ij} = 4g^2_{\mathrm{YM}} n^+_{ij}$ is analogous [23] to the spring constant k_{ij} of the pair of giant gravitons (i, j),

$$k_{ij} = 4n^+_{ij} \frac{\lambda}{N}. \tag{52}$$

We see that k_{ij} is a function of the coupling constant, that is $k_{ij} = k_{ij}(\lambda)$. The excitation energy δE of the system of the giant gravitons is

$$\delta E = 2q_{ij} k_{ij} = 8g^2_{\mathrm{YM}} q_{ij} n^+_{ij} = 8\frac{\lambda}{N} q_{ij} n^+_{ij}. \tag{53}$$

Here we also recover the $p = 2$ case [20, 22]. Our convention here is $\delta E = \hat{\gamma} = \frac{1}{2}\gamma$, where normalization convention for $\hat{\gamma}$ was used in [20–24, 26, 29] and normalization convention for γ was used in [6].

Our analysis suggests that the higher loop dilatation operator in the large N limit will satisfy the oscillator algebra. So they will not change the $q_{ij} n_{ij}$ dependence of the spectra. We have provided evidence for this from the oscillator algebra. They will renormalize the effective coupling and the coefficient $\frac{\lambda}{N}$ to $\frac{f(\lambda)}{N}$. The $f(\lambda)$ is an effective coupling and also an interpolating function. So the effective spring constants become $k_{ij} = 4n_{ij}^{+}\frac{f(\lambda)}{N}$. In the derivation of this, we have also assumed that we are considering the oscillator level to be much smaller than N, in other words $\frac{q_{ij}}{N} \ll 1$. According to the renormalization of the effective coupling constant at higher loops, we have

$$\delta E = 2q_{ij}k_{ij} = 8\frac{f(\lambda)}{N}q_{ij}n_{ij}^{+}. \tag{54}$$

The effective Hamiltonian of this quantum mechanic system is a system of masses with kinetic energies and with pairwise potentials proportional to $k_{ij}|x_i - x_j|^2$ which are quadratic functions of the relative displacements between pairs of such masses. In other words, according to our analysis, the spring constants will also be renormalized.

6. Discussion

We have analyzed in detail the oscillator algebra of Young diagrams and its relation to the $u(2)$ algebra by Inonu–Wigner contraction for the large dimension operators with p long rows or p long columns in the Young diagrams. The existence of the harmonic oscillator algebra at higher loops is an important evidence for integrability in this nonplanar regime at higher loops.

The dependence of the spectra on the $q_{ij}n_{ij}$, where q_{ij} and n_{ij} label the oscillator level and the string number respectively, in the large N limit is protected by the oscillator algebra and is robust under loop corrections if the oscillator algebra persists at higher loops.

We have provided evidences that the oscillator algebra is preserved in the large N but nonplanar limit at higher loops. One evidence is that higher loop dilatation operators are polynomials of the Lie algebra generator which we call A^3, in the large N but nonplanar limit, and these operators will satisfy the oscillator algebra. If in the large N limit, the h-loop dilatation generator preserves the oscillator algebra, then the integrability in this nonplanar regime is preserved at h-loop.

The oscillator algebra itself is a Inonu–Wigner contraction of the $u(2)$ algebra inside of the $u(p)$ algebra of p giant gravitons. Since the $u(p)$ algebra of p number of branes is robust, such symmetry is expected to be robust under loop corrections.

The existence of BPS states as the lowest energy eigenstates is also one evidence. Those are the states that can be viewed as the eigenvectors with zero eigenvalues under the anomalous piece of the dilatation operator. They are also related to the root of the aforementioned polynomials.

The evidences of the integrability at higher loops in the nonplanar large N limit presented in this paper are in addition to other evidences of different reasons, which are related to double coset ansatz [24] or to global symmetry [30].

Under higher loop corrections, the spectra are expected to be given by an interpolating function, that is a function of the 't Hooft coupling as well as weights of the corner boxes of the Young diagrams. The spring constant of the oscillator between a pair of giant gravitons will also be an interpolating function of the coupling constant.

The operators in the oscillator basis can be diagonalized by the harmonic oscillator wavefunctions and labeled by oscillator levels. Two-loop diagonalizations have been also computed in [25]. We presented evidences that the diagonalization of these operators at higher loops are not corrected, in the large N limit that we consider, due to the robustness of both the double coset symmetry and the harmonic oscillator algebra. We have argued that the higher loop corrections do not change the basis of the diagonalization of the operators in the large N but nonplanar limit. They modify the spectra of these operators by corrections at higher orders of the coupling.

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