

Isogeny orbits in a family of abelian varieties

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Abstract

We prove that if a curve of a non-isotrivial family of abelian varieties over a curve contains infinitely many isogeny orbits of a finitely generated subgroup of a simple abelian variety then it is special. This result fits into the context of Zilber-Pink conjecture and partially generalizes a result of Faltings. Moreover by using the polyhedral reduction theory we give a new proof of a result of Bertrand.

1 Introduction

In this paper we are interested in a curve of an abelian scheme that contains infinitely many isogeny orbits of a finitely generated group of a simple abelian variety. We prove that it is special. Zilber-Pink have conjectured, roughly speaking, that a subvariety containing many special points must be special. This generalizes many well-known problems including conjectures of Mordell-Lang, Manin-Mumford and André-Oort. Special points considered in this paper are isogeny orbits which are closely related to the generalized Hecke orbits considered by Zilber-Pink. Therefore our result fits into the context of their conjectures.

Let S be a smooth irreducible abstract algebraic curve over $\overline{\mathbb{Q}}$, and $\pi : A \rightarrow S$ be an abelian scheme. An abelian scheme $A \rightarrow S$ refers to a smooth proper group scheme with geometrically connected fibers. Then A can be regarded as a smooth family of abelian varieties over S . Take an abelian variety A' defined over $\overline{\mathbb{Q}}$ and a finitely generated group $\Gamma \subset A'(\overline{\mathbb{Q}})$. We call a point $q \in A_t(\overline{\mathbb{Q}})$, where $t \in S(\overline{\mathbb{Q}})$, special if there exist an isogeny $\phi : A' \rightarrow A_t$ and $\gamma \in \Gamma$ with $\phi(\gamma) = q$. In this paper we prove

Theorem 1.1. *Assume that A' is simple and A is non-isotrivial. If an irreducible Zariski closed algebraic curve X of A , over $\overline{\mathbb{Q}}$, contains infinitely many special points, then either X is a subtorus of A_q for some $q \in S(\overline{\mathbb{Q}})$ or there exists $n \in \mathbb{N}$ such that $[n]X(\overline{\mathbb{Q}}) = 0$.*

In the next section we prove a partially stronger result in case of a family of elliptic curves, and indicate two major obstructions preventing that argument from

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working in case of families of abelian varieties. In Section 3 we use the polyhedral reduction theory to give a new proof of a result of Bertrand, which is crucial for this paper. In Section 4 we present the proof of the main theorem. The basic strategy of this paper is to compare different heights in number theory, including geometric Faltings height, Neron-Tate height and Weil height.

Throughout this paper $h_F(A)$ refers to the geometric Faltings height of an abelian variety A over $\overline{\mathbb{Q}}$, $h_{X,D} : X(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ refers to a Weil height function of a variety over $\overline{\mathbb{Q}}$ with respect to the divisor D and $\hat{h}_{A,D} : A(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}_{\geq 0}$ refers to the canonical height function of an abelian variety over $\overline{\mathbb{Q}}$ with respect to a symmetric divisor D . Two linearly equivalent divisors respectively isomorphic line bundles are connected by \sim respectively \cong . The set of linear equivalence of divisors or line bundles of X is $\text{Pic}(X)$. For a complete nonsingular curve X over $\overline{\mathbb{Q}}$, we have a canonical surjective homomorphism $\text{deg} : \text{Pic}(X) \rightarrow \mathbb{Z}$. Let \mathcal{L} be an invertible sheaf of an abelian variety A , $\chi(\mathcal{L})$ be its Euler characteristic and $\lambda_{\mathcal{L}}$ be the morphism from $A(\overline{\mathbb{Q}})$ to $\text{Pic}(A)$. The subgroup $\text{Pic}^0(A)$ of $\text{Pic}(A)$ consists of invertible sheaves \mathcal{L} for which $\lambda_{\mathcal{L}} \equiv 0$. The Neron-Severi group of A is denoted by $\text{NS}(A)$. Points of the dual abelian or Picard variety A^\vee of A parametrize the elements of $\text{Pic}^0(A)$. A homomorphism of abelian varieties $\phi : A \rightarrow B$ gives rise to a dual $\phi^\vee : B^\vee \rightarrow A^\vee$. Write $\text{End}(A) = \text{End}_{\overline{\mathbb{Q}}}(A)$ and $\text{End}^0(A) = \text{End}(A) \otimes \mathbb{Q}$, on which we have another deg function. The set of isomorphism classes of pairs (A, λ) with A an abelian variety of dimension g and λ a polarization of A of degree $d = \chi(\lambda)$ is parameterized by the Siegel modular variety $M_{g,d}$, in particular $M_{1,1} = \mathbb{A}^1$. If Γ is a finitely generated abelian group then we let Γ_t respectively Γ_{nt} to be the torsion subgroup respectively its complement.

2 Isogeny orbits in a family of elliptic curves

In this section we let S be a smooth irreducible abstract algebraic curve over $\overline{\mathbb{Q}}$, and $\pi : A \rightarrow S$ an abelian scheme of dimension one. Then A can be regarded as a smooth family of elliptic curves over S . Take an elliptic curve A' defined over $\overline{\mathbb{Q}}$ and a $p \in A'(\overline{\mathbb{Q}})$. We call $q \in A_t(\overline{\mathbb{Q}})$, where $t \in S(\overline{\mathbb{Q}})$, special if there exists either an isogeny $\phi : A' \rightarrow A_t$ with $\phi(p) = q$, or an isogeny $\phi : A_t \rightarrow A'$ with $\phi(q) = p$. We prove

Proposition 2.1. *Assume that A is non-isotrivial. If an irreducible Zariski closed algebraic curve X of A , over $\overline{\mathbb{Q}}$, contains infinitely many special points, then either X is some special fiber A_t that is isogenous to A' or there exists $n \in \mathbb{N}$ such that $[n]X(\overline{\mathbb{Q}}) = 0$.*

Proof. Firstly we assume that X is not any fiber A_t , otherwise there is nothing to prove. Secondly we notice that it suffices to prove the result under the assumption that X is a section $s : S \rightarrow A$ of $\pi : A \rightarrow S$. Indeed in the general case if let X'

be a smooth resolution of X , then $A \times_S X' \rightarrow X'$ is also a smooth family of elliptic curves over X' and we write $f : X' \rightarrow A$ to be the natural morphism.

$$\begin{array}{ccccc}
 X' & & & & \\
 \swarrow \text{dotted } s & \searrow \text{solid } id & & & \\
 & A \times_S X' & \xrightarrow{pr_2} & X' & \\
 \swarrow \text{solid } f & \downarrow pr_1 & & \downarrow \pi \circ f & \\
 & A & \xrightarrow{\pi} & S &
 \end{array}$$

The above commutative diagram provides a section $s : X' \rightarrow A \times_S X'$ of $A \times_S X'/X'$. Moreover it is easy to check that $s(X') \subset A \times_S X'$ contains infinitely special points if and only if $X \subset A$ does and that $[n]X = 0$ if and only if $([n] \circ s)X = 0$. Therefore the general case reduces to a special case that X comes from a section.

If p is a torsion point and A' has complex multiplication, then our assertion is a special case of a result of André [2]. If p is a torsion point and A' has no complex multiplication, then by a lemma of Habegger [6, Lemma 5.8] there are only finitely many elliptic curves isogenous to A' with bounded height (The proof of Habegger's statement relies heavily on the work of Szpiro and Ullmo). This makes André's argument valid line by line after replacing only Poonen's lemma [8] by Habegger's lemma. From here on we assume that p is not torsion.

Given an elliptic curve E over $\overline{\mathbb{Q}}$ we write $\hat{h}_E : E(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}_{\geq 0}$ for the canonical height function with respect to the divisor given by the zero section (0) of $E/\overline{\mathbb{Q}}$. For any other symmetric divisor D of E , the associated canonical height function satisfies

$$\hat{h}_{E, D} = \deg D \cdot \hat{h}_E,$$

as $D \sim \deg D(0)$ for any symmetric divisor D of E . Before proceeding we notice that if $\phi : E' \rightarrow E$ is an isogeny of elliptic curves over $\overline{\mathbb{Q}}$ with $\phi(a) = b$, where a, b are closed points, then we have

$$\hat{h}_E(b) = \hat{h}_{E', \phi^*(0)}(a) = \deg(\phi^*(0))\hat{h}_{E'}(a) = \deg \phi \cdot \hat{h}_{E'}(a).$$

Let D be the divisor given by the zero section of the abelian scheme A/S , then the canonical height function on $A_t(\overline{\mathbb{Q}})$ with respect to $\deg D_t$ is simply \hat{h}_{A_t} for any $t \in S(\overline{\mathbb{Q}})$. According to the non-isotriviality of A , the modular map $j : S(\overline{\mathbb{Q}}) \rightarrow \mathbb{A}^1(\overline{\mathbb{Q}})$ is non-constant. Without ambiguity we write $h : \mathbb{A}^1(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ for the standard Weil height function on \mathbb{A}^1 and $h : s \in S(\overline{\mathbb{Q}}) \rightarrow h(j(s)) \in \mathbb{R}$ for the Weil height function on S with respect to $j^*((0))$.

There are two types of special points. The back orbit of p and the forward orbit of p . We first assume that X contains infinitely many back orbits $q_i (i = 1, 2, \dots)$ of

p . If $q_i \in A_{t_i}$, where $t_i \in S(\overline{\mathbb{Q}})$, then $q_i = s(t_i)$. Let $\phi_i : A_{t_i} \rightarrow A'$ be the isogeny that satisfies $\phi_i(q_i) = p$ then we have

$$\hat{h}_{A_{t_i}}(q_i) = \frac{\hat{h}_{A'}(p)}{\deg \phi_i}. \quad (1)$$

Using the lemma of Habegger [6, Lemma 5.8], there are only finitely many elliptic curves over $\overline{\mathbb{Q}}$ within the isogeny class of E' with bounded Weil height. Using the non-isotriviality of A , given any elliptic curve E_1 there are only finitely many i such that E_{t_i} is isomorphic to E_1 over $\overline{\mathbb{Q}}$. These two facts clearly lead to

$$\lim_{i \rightarrow \infty} h(t_i) \rightarrow \infty. \quad (2)$$

By Silverman's specialization theorem [10] there is a constant C such that

$$\lim_{h(t) \rightarrow \infty} \frac{\hat{h}_{A_t}(s(t))}{h(t)} = C. \quad (3)$$

Because of (2) we can apply (3) to t_i and obtain

$$\lim_{i \rightarrow \infty} \frac{\hat{h}_{A_{t_i}}(q_i)}{h(t_i)} = C.$$

Using (1) we have

$$\lim_{i \rightarrow \infty} \frac{\hat{h}_{A'}(p)}{\deg \phi_i \cdot h(t_i)} = C.$$

As p is not torsion, we have $h_{A'}(p) > 0$ and therefore the above identity gives $C = 0$. Recall in Silverman's specialization theorem [10], the constant $C = 0$ means that the canonical height of X regarded as a point in the abelian variety A_η over the generic point is zero. According to the non-isotriviality of A , the $\overline{\mathbb{Q}}(S)/\overline{\mathbb{Q}}$ trace of A is trivial. This implies that X is a torsion of the abelian variety over the generic fiber. There exists $n \in \mathbb{N}$ such that $[n]X(\overline{\mathbb{Q}}) = 0$, a contradiction to our assumption that p is not torsion.

Now we assume that X contains infinitely many forward orbits $\{q_i\}_{i=1}^\infty$ of p . Let $\phi_i : A' \rightarrow A_{t_i}$ be the isogeny that satisfies $\phi_i(p) = q_i$, then

$$\hat{h}_{A_{t_i}}(q_i) = \deg \phi_i \cdot \hat{h}_{A'}(p). \quad (4)$$

The inequality of Faltings [4] implies that

$$h_F(A_{t_i}) \leq h_F(A') + \log \deg \phi_i / 2. \quad (5)$$

We claim that $\lim_{i \rightarrow \infty} \deg \phi_i = \infty$. Otherwise there are infinitely many i such that A_{t_i} are isomorphic to each other over $\overline{\mathbb{Q}}$, a contradiction to the fact that A is non-isotrivial. Therefore (4), (5) and $\hat{h}_{A'}(p) > 0$ lead to

$$\lim_{i \rightarrow \infty} \frac{\hat{h}_{A_{t_i}}(s(t_i) = q_i)}{h(t_i)} = \infty.$$

The identity (2) is still valid. This makes the above equality a contradiction to Silverman's specialization theorem (3). \square

The above argument does not work in the context of families of abelian varieties for the following reasons. Firstly the relation between canonical heights of points on isogenous abelian varieties is not as simple as in (1). Secondly the lemma of Habegger is not known in higher dimensional case. More precisely we do not know whether within an isogeny class of abelian varieties there are only finitely many ones with bounded height. Without this result we have no validity of (2) in general yet, which is essential if we want to directly apply Silverman's specialization theorem.

3 Polyhedral reduction theory and canonical heights

In this section we give a proof of Lemma 3.1 below, based on the polyhedral reduction theory [1]. Actually this lemma is not new. G. Rémond pointed out to us that it is equivalent to the main theorem of Bertrand [3] in case of simple abelian varieties, linked by the theorem of Mordell-Weil. We still present the proof here, as our approach is rather distinct from Bertrand's original one.

Throughout this section A is an abelian variety over $\overline{\mathbb{Q}}$. Any symmetric line bundle \mathcal{L} of A defines a canonical height function $\hat{h}_{A,\mathcal{L}}$ that is quadratic on $A(\overline{\mathbb{Q}})$. We remark that $\hat{h}_{A,\mathcal{L}}$ depends only on the class of \mathcal{L} in $\text{NS}(A)$. Indeed if a symmetric line bundle \mathcal{L} maps to zero in the short exact sequence

$$0 \rightarrow \text{Pic}^0(A) \rightarrow \text{Pic}(A) \rightarrow \text{NS}(A) \rightarrow 0,$$

then we have $\mathcal{L} \in \text{Pic}^0(A)$. This leads to $[-1]^*\mathcal{L} = \mathcal{L}^{-1}$. Together with $[-1]^*\mathcal{L} = \mathcal{L}$ we have $\mathcal{L}^2 = 0$ and therefore $\hat{h}_{A,\mathcal{L}} \equiv 0$. Hence there will be no confusion when we label the canonical height function of a symmetric line bundle to its image in the Neron-Severi group.

When A is a simple abelian variety, a recent result of Kawaguchi and Silverman [5] tells us that for any nonzero nef symmetric $\mathcal{L} \in \text{Pic}(A) \otimes \mathbb{R}$ the canonical height $\hat{h}_{A,\mathcal{L}}(x) = 0$ if and only if $x \in A(\overline{\mathbb{Q}})_t$.

The endomorphism algebra $\text{End}^0(A)$ is semi-simple and contains $\text{End}(A)$ as a lattice. The unit $(\text{End}^0(A) \otimes \mathbb{R})^\times$ is reductive, and $\text{Aut}(A)$ is an arithmetic group. The function deg extends to a homogeneous function of degree $2g$ on $\text{End}^0(A) \otimes \mathbb{R}$.

For any line bundle \mathcal{L} the theorem of square leads to a group homomorphism

$$\lambda_{\mathcal{L}} : A(\overline{\mathbb{Q}}) \rightarrow A^\vee(\overline{\mathbb{Q}})$$

which takes x to $T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$. It is an isogeny if and only if \mathcal{L} is ample.

From here on we fix an ample line bundle \mathcal{N} , which defines a Rosati involution \dagger of $\phi \in \text{End}^0(A)$ by $\phi^\dagger = \lambda_{\mathcal{N}}^{-1} \circ \phi^\vee \circ \lambda_{\mathcal{N}}$. The map

$$\text{NS}(A)_{\mathbb{Q}} \rightarrow \text{End}^0(A)$$

defined by $\mathcal{L} \mapsto \lambda_N^{-1} \circ \lambda_{\mathcal{L}}$ identifies $\text{NS}(A)_{\mathbb{Q}}$ with the subset of $\text{End}^0(A)$ of elements fixed by \dagger . Given $\phi \in \text{Aut}(A)$ and $\mathcal{L} \in \text{Pic}(A)$ it is straightforward to check that $\lambda_{\phi^*(\mathcal{L})} = \phi^\vee \circ \lambda_{\mathcal{L}} \circ \phi$. Which extends to be an action of $\text{End}^0(A)$ on $\text{NS}(A)_{\mathbb{Q}} \subset \text{End}^0(A)$ by $\alpha^\phi = \phi^\dagger \circ \alpha \circ \phi$. The bilinear form

$$\langle \phi, \psi \rangle \mapsto \text{Tr}(\phi \circ \psi^\dagger).$$

on $\text{End}^0(A) \times \text{End}^0(A)$ is positive definite.

As a finite dimensional algebra over \mathbb{R} with a positive involution, $\text{End}^0(A) \otimes \mathbb{R}$ is isomorphic to $\prod_i M_{r_i}(\mathbb{R}) \times \prod_j M_{s_j}(\mathbb{C}) \times \prod_k M_{t_k}(\mathbb{H})$ where the involution is given by conjugations. Under this identification $N^1(A) = \text{NS}(A) \otimes \mathbb{R}$ respectively the ample cone $\text{Amp}(A)$ is isomorphic to $\prod_i H_{r_i}(\mathbb{R}) \times \prod_j H_{s_j}(\mathbb{C}) \times \prod_k H_{t_k}(\mathbb{H})$ respectively $\prod_i P_{r_i}(\mathbb{R}) \times \prod_j P_{s_j}(\mathbb{C}) \times \prod_k P_{t_k}(\mathbb{H})$, where H_r is the space of symmetric or Hermitian symmetric matrices and P_r consists of positive ones. Let $G(\text{Amp}(A))$ be the automorphism group of the cone $\text{Amp}(A)$ and $G(\text{Amp}(A))^0$ its identity component. The homomorphism $(\text{End}^0(A) \otimes \mathbb{R})^\times \rightarrow G(\text{Amp}(A))^0$ is surjective. According to Ash's main result of polyhedral reduction theory, there exists a rational polyhedral cone $F \subset \overline{\text{Amp}(A)}$ such that $(\text{Aut}(A) \cdot F) \cap \text{Amp}(A) = \text{Amp}(A)$. For more details of this paragraph we refer [1] and [9].

Using the theorem of Ash we give a new proof of

Lemma 3.1 (Bertrand). *Let A be a simple abelian variety defined over $\overline{\mathbb{Q}}$, and let $\Gamma \subset A(\overline{\mathbb{Q}})$ be a finitely generated group. Then there exists a constant $C > 0$ depending on Γ and A such that for any symmetric ample divisor $\mathcal{M} \in \text{Pic}(A)$ and nontorsion $x \in \Gamma$ we have*

$$\hat{h}_{A, \mathcal{M}}(x) \geq C(\chi(\mathcal{M}))^{1/g}.$$

Proof. It is well-known that $\text{End}(A)$ is of finite rank, hence $\text{End}(A)(\Gamma)$ is also a finitely generated group. Therefore it suffices to prove our lemma under the assumption that Γ is invariant under $\text{Aut}(A)$.

By the polyhedral reduction theory [1], there is a rational polyhedral fundamental domain $F \subset \overline{\text{Amp}(A)}$ under the action of $\text{Aut}(A)$. The rationality of F guarantees that there is a basis $\{v_1, \dots, v_t\} \subset \text{NS}(A) \cap \overline{\text{Amp}(A)}$ such that if $w \in F \cap \text{NS}(A)$ then there are nonnegative real numbers r_i with $w = \sum_{i=1}^t r_i v_i$.

We have $\Gamma = \Gamma_t + \Gamma_{nt}$. Because v_i are nef, a result of Kawaguchi-Silverman [5] tells us that \hat{h}_{A, v_i} is a positive bilinear function on the finitely generated abelian Γ_{nt} , therefore there exists a constant c_1 such that

$$\hat{h}_{A, v_i}(\gamma) \geq c_1 \tag{6}$$

for all $1 \leq i \leq t$ and all non-zero $\gamma \in \Gamma_{nt}$. Furthermore for any $\mathcal{M} \in \text{Amp}(A)$ and $x = x_1 + x_2 \in \Gamma$ we have $\hat{h}_{A, \mathcal{M}}(x) = \hat{h}_{A, \mathcal{M}}(x_2)$. Consequently the inequality (6) is valid for all $1 \leq i \leq t$ and non-torsion $\gamma \in \Gamma$.

Take a symmetric ample divisor with image $\mathcal{M} \subset F$, then we have nonnegative real numbers r_i such that $\mathcal{M} = \sum_{i=1}^t r_i v_i$. In particular for any non-torsion $\gamma \in \Gamma$ we have

$$\hat{h}_{A, \mathcal{M}}(\gamma) \geq c_1 \max_{i=1}^t \{r_i\} \quad (7)$$

The degree function $\deg : \text{End}^0(A) \otimes \mathbb{R} \rightarrow \mathbb{R}$ is homogeneous of degree $2g$. Take $\deg \mathcal{M}$ to be the degree of the image of \mathcal{M} in $\text{End}^0(A) \otimes \mathbb{R}$. Then we shall have

$$\deg \mathcal{M} = \deg(\lambda_{\mathcal{M}}) / \deg(\lambda_{\mathcal{N}}) = c_2 \chi(\mathcal{M})^2 \quad (8)$$

where $c_2 = 1 / \deg(\lambda_{\mathcal{N}})$. The homogeneity of $\deg : \text{End}^0(A) \otimes \mathbb{R} \rightarrow \mathbb{R}$ implies that there is a positive constant c_3 which depends only on v_i but not on \mathcal{M} such that

$$\deg \mathcal{M} \leq c_3 \sum_{i=1}^t r_i^{2g}. \quad (9)$$

The above (7), (8) and (9) obvious lead to a constant $C > 0$ such that

$$\hat{h}_{A, \mathcal{M}}(\gamma) \geq C(\chi(\mathcal{M}))^{1/g}$$

for all $\mathcal{M} \subset F \cap \text{Amp}(A)$ and non-torsion $\gamma \in \Gamma$.

For general $\mathcal{M} \in \text{Amp}(A)$, there exists $\sigma \in \text{Aut}(A)$ such that $\sigma^*(\mathcal{M}) \in F$. We have assumed that $\sigma^{-1}(x) \in \Gamma$ and it is clear that $\chi(\sigma^*(\mathcal{M})) = \chi(\mathcal{M})$. Therefore for any non-torsion $\gamma \in \Gamma$

$$\hat{h}_{A, \mathcal{M}}(\gamma) = \hat{h}_{A, \sigma^*(\mathcal{M})}(\sigma^{-1}(\gamma)) \geq C(\chi(\mathcal{M}))^{1/g},$$

which proves what has been claimed in the lemma. \square

4 Proof of the main theorem

It is unknown to us whether in an isogeny class of abelian varieties there are only finitely many ones with bounded height. Therefore we can not directly use Silverman's specialization theorem as before. Instead we shall use some arguments of [10] to prove our main theorem.

Proof of Theorem 1.1. Firstly we may assume that Γ is invariant under $\text{Aut}(A')$. Secondly by the same trick used in the proof of Proposition 2.1 we assume that X is a section $s : S \rightarrow A$ of $\pi : A \rightarrow S$. We write $\epsilon : S \rightarrow A$ for the zero section.

In the decomposition $\Gamma = \Gamma_t + \Gamma_{nt}$, Γ_t is finite, there exists a positive integer n such that $[n]\Gamma_t = 0$. If there are infinitely many $t \in S(\overline{\mathbb{Q}})$ such that there exists $\gamma_t \in \Gamma_t$ and isogeny $\phi_t : A' \rightarrow A_t$ with $\phi_t(x_t) = s(t)$, then we also have $[n]s(t) = 0$. This implies that $[n]X(\overline{\mathbb{Q}})$ intersects with the zero section infinitely many times. This leads to $[n]X(\overline{\mathbb{Q}}) = 0$.

Now we assume that there are infinitely many distinct $t_i \in S(\overline{\mathbb{Q}})$ ($i = 1, 2, 3, \dots$) such that exists $\gamma_i \in \Gamma \setminus \Gamma_t$ and isogenies $\phi_i : A' \rightarrow A_{t_i}$ with $\phi_i(\gamma_i) = s(t_i)$.

By theory of Theta functions, there is a smooth semiabelian scheme $\overline{A}/\overline{S}$ extending the family A/S (where \overline{S} is the smooth compactification of S) together with a symmetric very ample line bundle \mathcal{L}' of A/S that extends to a very ample one \mathcal{L} of $\overline{A}/\overline{S}$. Moreover \mathcal{L} is indeed very ample on $\overline{A}/\overline{\mathbb{Q}}$ and makes the total space a projective variety. For details of this paragraph we refer [7].

The Euler characteristic $\chi(\mathcal{L}_t)$ is a constant function of $t \in S(\overline{\mathbb{Q}})$, and we denote it by d . Because A is non-isotrivial, the modular map $j : S \rightarrow M_{g,d}$ is not constant. We claim that

$$\lim_{i \rightarrow \infty} \deg \phi_i = \infty.$$

Otherwise there are infinitely many $t \in S(\overline{\mathbb{Q}})$ such that A_t are all isomorphic to each other over $\overline{\mathbb{Q}}$. According to a well-known geometric finiteness theorem, given any abelian variety A^0 and $d \in \mathbb{N}$ there are only finitely many isomorphism classes of polarized abelian varieties (A^0, λ) with λ of degree d . These two facts together force the modular map $j : S \rightarrow M_{g,d}$ to be constant. This contradicts to the non-isotriviality of A and proves the claim.

Concerning isogenous ϕ_i of abelian varieties Faltings' inequality ([4]) gives

$$h_F(A_{t_i}) \leq h_F(A') + \log \deg \phi_i / 2. \quad (10)$$

Under the isogeny canonical heights satisfy $\hat{h}_{A_{t_i}, \mathcal{L}_{t_i}}(s(t_i)) = \hat{h}_{A', \phi_i^*(\mathcal{L}_{t_i})}(\gamma_i)$. The Euler characteristics satisfy $\chi(\phi_i^*(\mathcal{L}_{t_i})) = \deg \phi_i \cdot \chi(\mathcal{L}_{t_i}) = d \deg \phi_i$. By the lemma in the last section there exists a positive constant c_1 such that

$$\hat{h}_{A_{t_i}, \mathcal{L}_{t_i}}(s(t_i)) \geq c_1 (\deg \phi_i)^{1/g}. \quad (11)$$

The group morphism [2] : $A/S \rightarrow A/S$ extends to a morphism on the semiabelian scheme $\overline{A}/\overline{S}$. As mentioned before \mathcal{L} gives a projective embedding of \overline{A} . By this condition a theorem of Silverman-Tate [10, Theorem A] applies, and consequently there exists positive constants c_2 and c_3 such that

$$\left| \hat{h}_{A_{t_i}, \mathcal{L}_{t_i}}(s(t_i)) - h_{\overline{A}, \mathcal{L}}(s(t_i)) \right| < c_2 h_{\overline{S}, \epsilon^*(\mathcal{L})}(t_i) + c_3. \quad (12)$$

Because $h_{\overline{A}, \mathcal{L}}(s(t_i)) = h_{\overline{S}, s^*(\mathcal{L})}(t_i)$ and because both $\epsilon^*(\mathcal{L})$ and $s^*(\mathcal{L})$ are ample, there exists positive constants c_4 and c_5 such that

$$h_{\overline{A}, \mathcal{L}}(s(t_i)) \leq c_4 h_{\overline{S}, \epsilon^*(\mathcal{L})}(t_i) + c_5 \quad (13)$$

As indicated in [11], Zarhin's trick works for families and therefore $B = (A \times A^\vee)^4$ is an abelian scheme over S with principal polarization. Because a constant family of abelian varieties contains no nonconstant subfamily, B is also non-isotrivial. The modular map $J : S \rightarrow M_{g,1}$ attached to B with respect to this principle polarization

is nonconstant. Let \mathcal{N} be an ample line bundle of the Baily-Borel compactification of $M_{g,1}$. By another inequality of Faltings [4], by $h_F((A_t \times A_t^\vee)^4) = 8h_F(A_t)$ and by the fact that $h_{\overline{\mathcal{L}}, \epsilon^*(\mathcal{L})}$ is almost proportional to $h_{\overline{\mathcal{L}}, J^*(\mathcal{N})}$ there exist positive constants c_6, c_7 and c_8 such that

$$|h_F(A_{t_i}) - c_6 h_{\overline{\mathcal{L}}, \epsilon^*(\mathcal{L})}(t_i)| \leq c_7 + c_8 \log(\max(1, h_{\overline{\mathcal{L}}, \epsilon^*(\mathcal{L})}(t_i))). \quad (14)$$

Notice that although not explicitly mentioned in [4], one can check carefully (or see the lemma below) that c_8 is independent of the number field K . Combining (11), (12) and (13) there exist positive constants c_9 and c_{10} such that

$$c_9(\deg \phi_i)^{1/g} \leq h_{\overline{\mathcal{L}}, \epsilon^*(\mathcal{L})}(t_i) + c_{10}. \quad (15)$$

Combining (10) and (14) there exist positive constants c_{11}, c_{12} and c_{13} such that

$$h_{\overline{\mathcal{L}}, \epsilon^*(\mathcal{L})}(t_i) \leq c_{11} + c_{12} \log(\max(1, h_{\overline{\mathcal{L}}, \epsilon^*(\mathcal{L})}(t_i))) + c_{13} \log(\deg \phi_i). \quad (16)$$

It is clear that (15) contradicts to (16) as the degree of the isogenies $\deg \phi_i$ goes to the infinity. \square

Lastly we sketch a calculation to make sure that positive constants c_3, c_4 obtained in [4, p.356] are independent of number fields.

Lemma 4.1. *Let $X \subset \mathbb{P}_{\mathbb{Z}}^n$ be Zariski-closed, $Y \subset X$ closed, $\|\cdot\|$ a hermitian metric on $\mathcal{O}(1)|(X(\mathbb{C}) - Y(\mathbb{C}))$ with logarithmic singularities along Y and $\|\cdot\|_1$ a hermitian metric on $\mathcal{O}(1)|X(\mathbb{C})$. For $x \in X(\overline{\mathbb{Q}}) - Y(\overline{\mathbb{Q}})$ one defines $h(x)$ and $h_1(x)$ as in [4]. There exists positive constant c_3 and c_4 such that for all $x \in X(\overline{\mathbb{Q}}) - Y(\overline{\mathbb{Q}})$ we have*

$$|h(x) - h_1(x)| \leq c_3 + c_4 \log(\max(1, h_1(x))).$$

Proof. We may assume that Y is the intersection of X with a linear subspace and the set of common zeros of $f_1, \dots, f_r \in \Gamma(X/\mathbb{Z}, \mathcal{O}(1))$ is exactly Y . By multiplying the metric we assume $\|f_i\|_1 \leq 1$. A rational point $x \in X(K) - Y(K)$ corresponds to $\rho : \text{Spec}(R) \rightarrow X$, where R is the integer ring of the number field K . We assume $f_1(x) \neq 0$. By definition we have

$$[K : \mathbb{Q}]h_1(x) \geq \sum_{\sigma} -\log \|f_1\|_1(\sigma(x))$$

and $[K : \mathbb{Q}]|h(x) - h_1(x)| = \left| \sum_{\sigma} \log(\|f_1\|/\|f_1\|_1)(\sigma(x)) \right|$, where σ runs through all embeddings of $K \hookrightarrow \mathbb{C}$. According to logarithmic singularities of the metric there exist positive constants c_1, c_2 such that for all number field K and for K rational point x

$$[K : \mathbb{Q}]|h(x) - h_1(x)| \leq c_1[K : \mathbb{Q}] + c_2 \sum_{\sigma} \log(-\log \|f_1\|_1(\sigma(x))).$$

Furthermore we have

$$\begin{aligned} \sum_{\sigma} \log(-\log \|f_1\|_1(\sigma(x))) &\leq \log\left(\sum_{\sigma} -\log \|f_1\|_1(\sigma(x))/[K:\mathbb{Q}]\right)^{[K:\mathbb{Q}]} \\ &\leq [K:\mathbb{Q}] \log h_1(x). \end{aligned}$$

The above inequalities prove the desired claim. \square

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References

- [1] A. Ash, D. Mumford, M. Rapoport, Y. Tai, *Smooth compactification of locally symmetric varieties*, 2nd ed. Cambridge (2010).
- [2] Y. André, Shimura varieties, subvarieties, and CM points. Six lectures at the University of Hsinchu (Taiwan). August-September 2001 (with an appendix by C.L. Chai), <http://www.math.umd.edu/~yu/notes.shtml>.
- [3] D. Bertrand, Minimal heights and polarizations on group varieties, *Duke Math.J.* **80** (1995), 223–250.
- [4] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, *Invent. Math.* **73** (1983), 349–366.
- [5] S. Kawaguchi, J.H. Silverman, Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights of abelian varieties, arXiv:1301.4964.
- [6] P. Habegger, Special points on fibered powers of elliptic surfaces, *J. Reine Angew. Math.*, to appear.
- [7] Y. Namikawa, A new compactification of the Siegel space and degeneration of abelian varieties, I, *Math. Ann.* **221** (1976), 97–141; II, *ibid.*, 201–241.
- [8] B. Poonen, Spans of Hecke points on modular curves, *Math.Res.Lett.* **8** (2001), 767–770.
- [9] A. Prendergast-Smith, The cone conjecture for abelian varieties, *J.Math.Sci.Univ. Tokyo* **19** (2012), 243–261.
- [10] J.H. Silverman, Heights and the specialization map for families of abelian varieties. *J. Reine Angew. Math.* **342**(1983), 197–211.
- [11] Yu.G. Zarhin, A finiteness theorem for unpolarized abelian varieties over number fields with prescribed places of bad reduction, *Invent. Math.* **79** (1985), 309–321.