AN APPLICATION OF FREE TRANSPORT TO MIXED q-GAUSSIAN ALGEBRAS

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Abstract. We consider the mixed q-Gaussian algebras introduced by Speicher which are generated by the variables $X_i = l_i + l_i^*, i = 1, \dots, N$, where $l_i^* l_j - q_{ij} l_j l_i^* = \delta_{i,j}$ and $-1 < q_{ij} = q_{ji} < 1$. Using the free monotone transport theorem of Guionnet and Shlyakhtenko, we show that the mixed q-Gaussian von Neumann algebras are isomorphic to the free group von Neumann algebra $L(\mathbb{F}_N)$, provided that $\max_{i,j} |q_{ij}|$ is small enough. Similar results hold in the reduced C^*-algebra setting. The proof relies on some estimates which are generalizations of Dabrowski's results for the special case $q_{ij} \equiv q$.

1. Introduction

A fundamental problem in the theory of operator algebras is whether two algebras are isomorphic. The operator algebra (both the (reduced) C^*-algebra and von Neumann algebra) of the free group $\mathbb{F}_N$ with N generators has been a central object to study. In particular, these algebras are isomorphic to the algebras generated by N free semi-circular variables due to Voiculescu; see [VDN92]. Motivated from mathematical physics, Bożejko and Speicher introduced the q-Gaussian variables [BS91], which can be regarded as a deformation of the free semi-circular system. Since then, the q-Gaussian algebras have been extensively studied. For an incomplete list of results, see [BKS97, Shl04, Nou04, Śni04, Ric05, KN11, Avs11] among others. More recently, Dabrowski [Dab14], Guionnet and Shlyakhtenko [GS14] have shown that the q-Gaussian algebras are isomorphic to the algebras generated from the free groups for $|q|$ small enough. This result was proved using the powerful free monotone transport theorem. The first named author [Nel15] adapted this to the non-tracial setting and showed that the finitely generated q-deformed free Araki-Woods algebras are isomorphic to the finitely generated free Araki-Woods factor for $|q|$ small enough (*cf.* [Shl97], [Hia03]). In this paper, we give another application of Guionnet and Shlyakhtenko's theory.

The q-Gaussian variables and q-commutation relations were further generalized with the motivation from physics. In [Spe93], Speicher introduced the commutation relation

$$l_i^* l_j - q_{ij} l_j l_i^* = \delta_{i,j} \tag{1}$$

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where $Q = (q_{ij})_{i,j=1}^N$ is a symmetric matrix with $|q_{ij}| \leq 1$, and $\delta_{i,j}$ is the Kronecker delta function. In this paper, we call the operator algebras generated by $X_i = l_i + l_i^*$ the mixed q-Gaussian algebras and call X_i's the mixed q-Gaussian variables. In fact, the so-called braid relations (a.k.a. Yang–Baxter equation), which are more general than (1), were also studied by Bożejko, Speicher, Nou, and Kròlak in [BS94, Nou04, Krò00, Kró05], among others. As for (1), Lust-Piquard [LP99] showed the L^p boundedness of the Riesz transforms associated to the number operator of the system. More recently, Junge and the second named author [JZ15] studied various properties of the mixed q-Gaussian von Neumann algebras and in particular proved that they have the complete metric approximation property and are strongly solid in the sense of Ozawa and Popa [OP10] as long as $\max_{1\leq i,j\leq N} |q_{ij}| < 1$.

In the present paper, we show that if $\max_{1\leq i,j\leq N} |q_{ij}|$ is small enough then the mixed q-Gaussian algebras are isomorphic to the algebras generated from $\mathbb{F}_N$. To state the result precisely, let us denote by $\Gamma_q(\mathbb{R}^N)$ the q-Gaussian von Neumann algebra of N generators, $C_r^*(\mathbb{F}_N)$ (resp. $L(\mathbb{F}_N)$) the reduced C^*-algebra (resp. von Neumann algebra) generated from $\mathbb{F}_N$.

Theorem 1. *Let $Q = (q_{ij})$ be a symmetric $N \times N$ matrix with $N \in \{2, 3, \dots\}$ and $q_{ij} \in (-1, 1)$. Let Γ_Q be the von Neumann algebra generated by the mixed q-Gaussian variables $X_1, \dots, X_N$. Then there exists a $q_0 = q_0(N) > 0$ depending only on N such that $\Gamma_Q \cong \Gamma_0(\mathbb{R}^N) \cong L(\mathbb{F}_N)$ and $C^*(X_1, \dots, X_N) \cong C_r^*(\mathbb{F}_N)$ for all Q satisfying $\max_{i,j} |q_{ij}| < q_0$.*

The proof of this theorem relies on the construction of the conjugate variables and potentials for Γ_Q. To this end, we follow the idea of Dabrowski [Dab14] and obtain some estimates which are generalized from similar ones for the $q_{ij} \equiv q$ case.

In the final section of the paper, we discuss how the same methods (along with those present in [Nel15]) can be used to handle a generalization of (1) which falls into the type III setting.

2. The Mixed q-Gaussian Algebra

We refer the readers to [BS94, LP99, JZ15] for unexplained preliminary facts for the mixed q-Gaussian variables. Let $(e_i)_{i=1}^N$ be an orthonormal basis of $\mathbb{R}^N$. The Fock space associated with the mixed q-Gaussian variables is defined as $\mathcal{F}_Q = \oplus_{n=0}^\infty H_Q^n$, where H_Q^n is isomorphic to $(\mathbb{C}^N)^{\otimes n}$ as a vector space and $H_Q^0 = \mathbb{C}\Omega$ with Ω being the vacuum state. Let S_n denote the symmetric group on n elements and write $\underline{i} = (i_1, \dots, i_n)$ for a vector in $[N]^n := \{1, \dots, N\}^n$. The inner product of $\mathcal{F}_Q$ is given by

$$\langle e_{i_1} \otimes \cdots \otimes e_{i_m}, e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle_Q = \delta_{m,n} \sum_{\sigma \in S_n} a(\sigma, \underline{j}) \langle e_{i_1}, e_{j_{\sigma^{-1}(1)}} \rangle \cdots \langle e_{i_m}, e_{j_{\sigma^{-1}(n)}} \rangle.$$

Here $a(\sigma, \underline{j})$ is a product of (q_{kl}) defined as follows: We write $\tau_1 = (12), \tau_2 = (23), \dots, \tau_n = (n1)$ for transpositions. It is well known that $(\tau_i)_{i=1}^n$ is a generating set of S_n and that the number of inversions of $\sigma \in S_n$ is given by

$$|\sigma| = \min\{k \in \mathbb{N} : \sigma = \tau_{i_1} \cdots \tau_{i_k}\}.$$

For $\sigma \in S_n$, assume $|\sigma| = k$ and $\sigma = \tau_{m_1} \cdots \tau_{m_k}$. Then (see [BS94, LP99])

$$a(\sigma, \underline{i}) = \prod_{j=1}^{k-1} q(i_{\sigma_j(m_{k-j})}, i_{\sigma_j(m_{k-j}+1)}) q(i_{m_k}, i_{m_k+1}),$$

where $\sigma_j = \tau_{m_{k-j+1}} \cdots \tau_{m_k}$ and we have written $q_{i_1 i_2} = q(i_1, i_2)$. Let $X_i = l_i + l_i^*$ be the mixed q-Gaussian variables. Here $l_i = l(e_i)$ is the left creation operator and l_i^* the left annihilation operator. One can check that l_i^* is the adjoint operator of l_i with respect to the inner product $\langle \cdot, \cdot \rangle_Q$ of $L^2(\Gamma_Q, \tau_Q)$. Similarly, we can define r_i and r_i^*. By definition,

$$l_i(e_{j_1} \otimes \cdots \otimes e_{j_n}) = e_i \otimes e_{j_1} \otimes \cdots \otimes e_{j_n},$$

$$r_i(e_{j_1} \otimes \cdots \otimes e_{j_n}) = e_{j_1} \otimes \cdots \otimes e_{j_n} \otimes e_i,$$

$$l_i^*(e_{j_1} \otimes \cdots \otimes e_{j_n}) = \sum_{k=1}^{n} \delta_{i,j_k} q_{ij_1} \cdots q_{ij_{k-1}} e_{j_1} \otimes \cdots \otimes e_{j_{k-1}} \otimes e_{j_{k+1}} \otimes \cdots \otimes e_{j_n}.$$

Let Γ_Q denote the mixed q-Gaussian von Neumann algebra generated by $X_i, i = 1, \ldots, N$. By [BS94], there is a normal faithful tracial state τ_Q on Γ_Q defined as $\tau_Q(X) = \langle X\Omega, \Omega \rangle_Q$ for $X \in \Gamma_Q$. If $\max_{ij} |q_{ij}| < 1$, then there is a canonical unitary isomorphism between $L^2(\Gamma_Q, \tau_Q)$ and $\mathcal{F}_Q$ given by

$$X \mapsto X\Omega, \text{ for } X \in \Gamma_Q,$$

which extends continuously to $L^2(\Gamma_Q)$. From time to time this identification will be used implicitly in the following and we write $\langle \cdot, \cdot \rangle_{\tau_Q}$ for the inner product of $L^2(\Gamma_Q, \tau_Q)$. Given a finite-length tensor $\xi \in \mathcal{F}_Q$, there is a unique element $W(\xi)$ in Γ_Q such that $W(\xi)\Omega = \xi$, and $W(e_{i_1} \otimes \cdots \otimes e_{i_n})$ is called the Wick word (a.k.a. Wick product in the literature) of $e_{i_1} \otimes \cdots \otimes e_{i_n}$.

Following [GS14, Dab14], we consider $\mathbb{C}\langle Y_1, \ldots, Y_N \rangle$, the algebra of noncommutative polynomials in N self-adjoint variables. Given a noncommutative power series

$$F(Y_1, \ldots, Y_N) = \sum_{\underline{i}, p} a_{\underline{i}, p} Y_{i_1} \cdots Y_{i_p} \otimes Y_{i_{p+1}} \cdots Y_{i_n}$$

whose radius of convergence is greater than $R > 1$, we define the norm $\|F\|_R = \sum_{\underline{i}, p} |a_{\underline{i}, p}| R^n$. Similarly, for

$$F(Y_1, \ldots, Y_N) = \sum_{\underline{i}} a_{\underline{i}} Y_{i_1} \cdots Y_{i_n},$$

with radius of convergence greater than $R > 1$ we define $\|F\|_R = \sum_{\underline{i}} |a_{\underline{i}}| R^n$. For an algebra $\mathcal{A}$, we write $\mathcal{A}^{op}$ for the opposite algebra of $\mathcal{A}$, and write $a^\circ \in \mathcal{A}^{op}$ whenever $a \in \mathcal{A}$.

3. The Derivation $\partial_j^{(Q)}$ and Ξ_j

Consider the linear map

$$\partial_j^{(Q)}: \mathbb{C}\langle X_1,\dots,X_N\rangle \to \mathcal{B}(L^2(\Gamma_Q)), \quad \partial_j^{(Q)}(X) = [X, r_j] := Xr_j - r_jX.$$

For $i = 1,\dots,N$, define

$$\Xi_i : \mathcal{F}_Q \to \mathcal{F}_Q, \quad \Xi_i(e_{j_1}\otimes\cdots\otimes e_{j_n}) = q_{ij_1}\cdots q_{ij_n} e_{j_1}\otimes\cdots\otimes e_{j_n}.$$

We also write $q_i(\underline{j}) = q_{ij_1}\cdots q_{ij_n}$ for short.

For each $n\geq 1$, we consider the following equivalence relation on $[N]^n$: $\underline{i}\sim\underline{j}$ if $\exists\sigma\in S_n$ such that

$$\underline{i} = \sigma\cdot\underline{j} = (j_{\sigma(1)},\dots,j_{\sigma(n)}).$$

Let $[\underline{i}]$ denote the equivalence class of $\underline{i}\in[N]^n$. Note that $q_k(\underline{j}) = q_k(\underline{i})$ for each $\underline{j}\in[\underline{i}]$ and each $k=1,\dots,N$; consequently, we may at times denote $q_k(\underline{i})$ by $q_k([\underline{i}])$. For each equivalence class $[\underline{i}]$ we define the subspace

$$\mathcal{F}_{[\underline{i}]} := \operatorname{span}\left\{e_{j_1}\otimes\cdots\otimes e_{j_n} : \underline{j}\in[\underline{i}]\right\},$$

and denote by $p_{[\underline{i}]}$ the orthogonal projection onto $\mathcal{F}_{[\underline{i}]}$. It is easy to see that H_Q^0 along with the subspaces $\mathcal{F}_{[\underline{i}]}$ (ranging over all equivalence classes and all $n\geq 1$) offers an orthogonal decomposition of $\mathcal{F}_Q$, and consequently

$$p_\Omega + \sum_{n\geq 1}\sum_{[\underline{i}]\in[N]^n/\sim} p_{[\underline{i}]} = 1,$$

where p_Ω is the projection onto the vacuum vector. For notational consistency, we will often denote $p_\Omega = p_{[(\emptyset)]}\in[N]^0/\sim$.

For each $j=1,\dots,N$ it follows that

$$\Xi_j = \sum_{n\geq 0}\sum_{[\underline{i}]\in[N]^n/\sim} q_j(\underline{i})p_{[\underline{i}]}. \tag{2}$$

Moreover, if $q := \max_{1\leq i,j\leq N}|q_{ij}|$ satisfies $q^2N < 1$ then $\Xi_j\in HS(\mathcal{F}_Q)$, the Hilbert–Schmidt operators on $\mathcal{F}_Q$, since for each $n\geq 1$

$$\sum_{[\underline{i}]\in[N]^n/\sim}\|p_{[\underline{i}]}\|_{HS}^2 = \sum_{k_1+\cdots+k_N=n}\binom{n}{k_1,\dots,k_N} = N^n.$$

Noting that $[l_i, r_j] = 0$, we see that

$$\partial_j^{(Q)}(X_i)(e_{i_1}\otimes\cdots\otimes e_{i_n}) = \delta_{i,j}q_{ii_1}\cdots q_{ii_n}e_{i_1}\otimes\cdots\otimes e_{i_n},$$

and hence $\partial_j^{(Q)}(X_i) = \delta_{i,j}\Xi_j$. As the space of Hilbert–Schmidt operators is a two-sided ideal in $\mathcal{B}(\mathcal{F}_Q)$, the Leibniz rule implies $\partial_j^{(Q)}$ maps $\mathbb{C}\langle X_1,\dots,X_N\rangle$ into $HS(\mathcal{F}_Q)$ for each

$j = 1, \ldots, N$ whenever $\Xi_j \in HS(\mathcal{F}_Q)$. When this is the case, we think of $\partial_j^{(Q)}$ as a densely defined derivation

$$\partial_j^{(Q)} \colon L^2(\Gamma_Q, \tau_Q) \to HS(\mathcal{F}_Q).$$

Recall that $L^2(\Gamma_Q \bar{\otimes} \Gamma_Q^{op}, \tau_Q \otimes \tau_Q^{op})$ is isomorphic to $HS(\mathcal{F}_Q)$ via the map

$$a \otimes b^\circ \mapsto \langle \cdot, b^*\Omega \rangle a\Omega.$$

In particular, $1 \otimes 1^\circ \mapsto p_\Omega$. We will usually think of $\partial_j^{(Q)}$ as having range $L^2(\Gamma_Q \bar{\otimes} \Gamma_Q^{op}, \tau_Q \otimes \tau_Q^{op})$.

Proposition 2. *Suppose $\Xi_j \in HS(\mathcal{F}_Q)$. Then $\partial_j^{(Q)*}(1 \otimes 1^\circ) = X_j$.*

Proof. Fix $\underline{i} \in [N]^n$ and let $\pi_1 \in \mathcal{B}(\mathcal{F}_Q)$ denote the projection onto tensors of length one. Then there exist scalars $c_1, \ldots, c_n$ such that

$$\pi_1 X_{i_1} \cdots X_{i_n} \Omega = \sum_{t=1}^n c_t e_{i_t},$$

where we are summing over which operator $X_{i_1}, \ldots, X_{i_n}$ created the vector e_{i_t}. We claim

$$\begin{aligned} c_t &= \sum_{d \geq 0} \sum_{[\underline{j}] \in [N]^d/\sim} q_{i_t}(\underline{j}) \langle X_{i_1} \cdots X_{i_{t-1}} p_{[\underline{j}]} X_{i_{t+1}} \cdots X_{i_n} \Omega, \Omega \rangle_Q \\ &= \langle X_{i_1} \cdots X_{i_{t-1}} \Xi_{i_t} X_{i_{t+1}} \cdots X_{i_n} \Omega, \Omega \rangle_Q. \end{aligned}$$

First note that the second equality is immediate from (2). Now, the only terms from $\pi_1 X_{i_1} \cdots X_{i_n} \Omega$ which contribute to c_t are those where X_{i_t} creates e_{i_t}; that is, ones where the creation operator rather than the annihilation operator in X_{i_t} acts. Hence towards computing c_t we may replace X_{i_t} with l_{i_t} and compute

$$\pi_1 X_{i_1} \cdots X_{i_{t-1}} l_{i_t} X_{i_{t+1}} \cdots X_{i_n} \Omega.$$

Recall that we have the partition of unity $\{p_{[\underline{j}]} \colon d \geq 0,\ [\underline{j}] \in [N]^d/\sim\}$. For each $d \geq 0$ and $[\underline{j}] \in [N]^d/\sim$, let $\{\zeta_\ell^{[\underline{j}]}\}$ be an orthonormal basis for $\mathcal{F}_{[\underline{j}]}$. Then we have

$$\begin{aligned} &\pi_1 X_{i_1} \cdots X_{i_{t-1}} l_{i_t} X_{i_{t+1}} \cdots X_{i_n} \Omega \\ &\qquad = \sum_{d \geq 0} \sum_{[\underline{j}] \in [N]^d/\sim} \pi_1 X_{i_1} \cdots X_{i_{t-1}} l_{i_t} p_{[\underline{j}]} X_{i_{t+1}} \cdots X_{i_n} \Omega \\ &\qquad = \sum_{d \geq 0} \sum_{[\underline{j}] \in [N]^d/\sim} \sum_{\ell} \pi_1 X_{i_1} \cdots X_{i_{t-1}} e_{i_t} \otimes \zeta_\ell^{[\underline{j}]} \left\langle X_{i_{t+1}} \cdots X_{i_n} \Omega, \zeta_\ell^{[\underline{j}]} \right\rangle_Q. \end{aligned}$$

Furthermore, of the above terms the only ones which contribute to c_t are those where e_{i_t} survives; that is, where none of the operators $X_{i_1}, \ldots, X_{i_{t-1}}$ annihilate e_{i_t}. And yet, to survive the action of π_1, $\zeta_\ell^{[\underline{j}]}$ must be completely annihilated by $X_{i_1} \cdots X_{i_{t-1}}$. The annihilation operators from $X_{i_1} \cdots X_{i_{t-1}}$ tasked with this must each skip over e_{i_t} at a scalar cost $q_{i_t k}$ for some $k \in [N]$. Since $\zeta_\ell^{[\underline{j}]}$ is a linear combination $e_{k_1} \otimes \cdots \otimes e_{k_d}$, $\underline{k} \in [\underline{j}]$, the total scalar cost will be $q_{i_t}(\underline{j})$. The remaining actions of $X_{i_1} \cdots X_{i_{t-1}}$ (any creation operators and any annihilation operators acting on vectors left of e_{i_t} in the tensor product) are unaffected by the presence of e_{i_t}. In summary, the contribution to c_t from the terms in the sum above is as follows:

$$\sum_{d\geq 0} \sum_{[\underline{j}]\in[N]^d/\sim} \sum_{\ell} q_{i_t}(\underline{j}) e_{i_t} \left\langle X_{i_1} \cdots X_{i_{t-1}} \zeta_\ell^{[\underline{j}]}, \Omega \right\rangle_Q \left\langle X_{i_{t+1}} \cdots X_{i_n} \Omega, \zeta_\ell^{[\underline{j}]} \right\rangle_Q .$$

Noting that

$$\sum_{\ell} \left\langle X_{i_1} \cdots X_{i_{t-1}} \zeta_\ell^{[\underline{j}]}, \Omega \right\rangle_Q \left\langle X_{i_{t+1}} \cdots X_{i_n} \Omega, \zeta_\ell^{[\underline{j}]} \right\rangle_Q = \left\langle X_{i_1} \cdots X_{i_{t-1}} p_{[\underline{j}]} X_{i_{t+1}} \cdots X_{i_n} \Omega, \Omega \right\rangle_Q ,$$

we see that c_t has the claimed value.

Thus for $s \in [N]$ we have

$$\begin{aligned} \langle X_s, X_{i_1} \cdots X_{i_n} \rangle_{\tau_Q} &= \langle e_s, \pi_1 X_{i_1} \cdots X_{i_n} \Omega \rangle_Q \\ &= \sum_{t=1}^n \langle e_s, e_{i_t} \rangle_Q \langle \Omega, X_{i_1} \cdots X_{i_{t-1}} \Xi_{i_t} X_{i_{t+1}} \cdots X_{i_n} \Omega \rangle_Q \\ &= \langle p_\Omega, \partial_s^{(Q)}(X_{i_1} \cdots X_{i_n}) \rangle_{HS} \\ &= \langle 1 \otimes 1^\circ, \partial_s^{(Q)}(X_{i_1} \cdots X_{i_n}) \rangle_{\tau_Q \otimes \tau_Q^{op}} . \end{aligned}$$

Extending this via linearity from monomials to the dense subset $\mathbb{C}\langle X_1, \ldots, X_N \rangle \subset L^2(\Gamma_Q, \tau_Q)$ concludes the proof. □

Corollary 3. *Suppose* $\Xi_j \in HS(\mathcal{F}_Q)$. *Then*

$$\mathbb{C}\langle X_1, \ldots, X_N \rangle \otimes \mathbb{C}\langle X_1, \ldots, X_N \rangle^{op} \subset \operatorname{Dom} \partial_j^{(Q)*}.$$

In particular, for $a, b \in \mathbb{C}\langle X_1, \ldots, X_N \rangle$

$$\partial_j^{(Q)*}(a \otimes b^\circ) = aX_j b - m \circ (1 \otimes \tau_Q \otimes 1) \circ (1 \otimes \partial_j^{(Q)} + \partial_j^{(Q)} \otimes 1)(a \otimes b^\circ), \tag{3}$$

where $m(a \otimes b^\circ) = ab$. *Consequently,* $\partial_j^{(Q)}$ *is closable.*

Proof. The formula is a simple computation (*cf.* Proposition 4.1 in [Voi98], the proof of Theorem 34 in [Dab14], or Corollary 2.4 in [Nel15]). The closability of $\partial_j^{(Q)}$ then follows because this formula holds on the dense subset $\mathbb{C}\langle X_1, \ldots, X_N \rangle \otimes \mathbb{C}\langle X_1, \ldots, X_N \rangle^{op} \subset L^2(\Gamma_Q \bar{\otimes} \Gamma_Q^{op}, \tau_Q \otimes \tau_Q^{op})$. □

Let us update the notation $\partial_j^{(Q)}$ so that from now on it denotes the closure of this derivation.

Let $\phi : S_n \to \mathcal{B}(H_Q^n)$ be the quasi-multiplicative function defined in [BS94] and define $P^{(n)} = \sum_{\sigma \in S_n} \phi(\sigma)$. According to [BS94], we have

$$\langle \xi, \eta \rangle_Q = \delta_{n,m} \langle \xi, P^{(n)} \eta \rangle_0, \text{ for } \xi \in H_Q^n, \eta \in H_Q^m.$$

Here $\langle \cdot, \cdot \rangle_0$ is the inner product associated to $(\Gamma_0(\mathbb{R}^N), \tau_0)$. Let $q = \max_{1 \le i,j \le N} |q_{ij}|$. Assume $q < 1$. By [Boż98, Theorem 2], we find

$$\|(P^{(n)})^{-1}\| \le \Big[(1-q) \prod_{k=1}^{\infty} \frac{1+q^k}{1-q^k}\Big]^n.$$

Using the Gauss identity, we have the estimate

$$\|(P^{(n)})^{-1}\| \le \Big[(1-q)\Big(\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2}\Big)^{-1}\Big]^n \le \Big(\frac{1-q}{1-2q}\Big)^n. \tag{4}$$

Lemma 4. *If $\varepsilon > 0$ and $q(3 - 2q + (3+\varepsilon)^2 N^2) < 1$, then there exists a noncommutative power series representation of Ξ_i with radius of convergence greater than $R = \frac{2+\varepsilon}{1-q} > \|X_i\|$ such that*

$$\|\Xi_i - 1 \otimes 1^\circ\|_R \le \frac{qN^2(3+\varepsilon)^2}{1 - q(3 - 2q + (3+\varepsilon)^2 N^2)} =: \pi(q, N)$$

for $i = 1, \ldots, N$.

Proof. Following the argument of [Dab14], let G_n denote the Gram matrix of the inner product on (Γ_Q, τ_Q) from the natural basis $(e_{i_1} \otimes \cdots \otimes e_{i_n})$ of H_Q^n, where $\underline{i} \in [N]^n$. Namely, G_n is the matrix of $P^{(n)}$ in the basis $(e_{i_1} \otimes \cdots \otimes e_{i_n})$. We write $\psi_{\underline{i}} = W(e_{i_1} \otimes \cdots \otimes e_{i_n})$ for the Wick word. From the isomorphism $L^2(\Gamma_Q, \tau_Q) \cong \mathcal{F}_Q$, we can also write

$$(G_n)_{\underline{i}\underline{j}} = \langle e_{i_1} \otimes \cdots \otimes e_{i_n}, e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle_Q = \langle \psi_{\underline{i}}, \psi_{\underline{j}} \rangle_{\tau_Q}.$$

Let us define inductively the noncommutative polynomials, $\psi_\varepsilon = 1$ for the empty word ε and

$$\psi_{i_1,\ldots,i_n}(Y_1, \ldots, Y_N) = Y_{i_1} \psi_{i_2,\ldots,i_n} - \sum_{j=2}^{n} \delta_{i_1,i_j} \prod_{k=2, k\ge 2}^{j-1} q_{i_1 i_k} \psi_{i_2,\ldots,i_{j-1},i_{j+1},\ldots,i_n}(Y_1, \ldots, Y_N), \tag{5}$$

where the product over empty set is understood to be 1. It can be checked that $\psi_{\underline{i}} = \psi_{\underline{i}}(X_1, \ldots, X_N)$; *cf.* [Krò00]. Let us define $B = G_n^{-1/2}$. Note that B is a positive-definite symmetric $N^n \times N^n$ matrix and that $B_{\underline{i}\underline{j}} = 0$ unless $\underline{i} \sim \underline{j}$. For each $|\underline{i}| = n$ let

$$p_{\underline{i}}(Y_1, \ldots, Y_N) = \sum_{|\underline{j}|=n} B_{\underline{i}\underline{j}} \psi_{\underline{j}}(Y_1, \ldots, Y_N). \tag{6}$$

Then $\{p_{\underline{i}}(X_1, \ldots, X_N)\Omega\}_{|\underline{i}|=n}$ is an orthonormal basis of H_Q^n, and $\{p_{\underline{k}}(X_1, \ldots, X_N)\Omega\}_{\underline{k} \in [\underline{i}]}$ is an orthonormal basis of $\mathcal{F}_{[\underline{i}]}$. We want to write Ξ_i as a sum of tensors. Unlike the $q_{ij} \equiv q$

case considered in [Dab14], Ξ_i behaves more like a multiplier instead of a projection. Consider

$$\Xi_j(Y_1,\dots,Y_N)=\sum_{n=0}^{\infty}\sum_{|\underline{i}|=n} q_j(\underline{i})p_{\underline{i}}(Y_1,\dots,Y_N)\otimes p_{\underline{i}}^*(Y_1,\dots,Y_N).$$

One can check that

$$\Xi_j(X_1,\dots,X_N)\psi_{\underline{i}}=q_j(\underline{i})\psi_{\underline{i}},$$

which means that Ξ_j can be identified as $\Xi_j(X_1,\dots,X_N)$ via the isomorphism $\mathcal{F}_Q\cong L^2(\Gamma_Q,\tau_Q)$. By the change of basis formula (6), writing $w_{\underline{j}}=\psi_{\underline{j}}(Y_1,\dots,Y_N)$, we have

$$\begin{aligned}\Xi_j(Y_1,\dots,Y_N)&=\sum_{n=0}^{\infty}\sum_{|\underline{i}|=n} q_j(\underline{i})\sum_{|\underline{j}|,|\underline{k}|=n} B_{\underline{i}\underline{j}}B_{\underline{k}\underline{i}}w_{\underline{j}}\otimes w_{\underline{k}}^*\\&=\sum_{n=0}^{\infty}\sum_{|\underline{j}|,|\underline{k}|=n} q_j(\underline{k})(B^2)_{\underline{k}\underline{j}}w_{\underline{j}}\otimes w_{\underline{k}}^*,\end{aligned}$$

where we have used in the second equality that $B_{\underline{k}\underline{i}}=0$ unless $\underline{k}\sim\underline{i}$, in which case $q_j(\underline{i})=q_j(\underline{k})$. Taking the norm, we have

$$\Big\|\sum_{|\underline{j}|,|\underline{k}|=n} q_j(\underline{k})(B^2)_{\underline{k}\underline{j}}w_{\underline{j}}\otimes w_{\underline{k}}^*\Big\|_R\le q^n\sum_{|\underline{k}|=n}\|w_{\underline{k}}\|_R\Big\|\sum_{|\underline{j}|=n}(B^2)_{\underline{k}\underline{j}}w_{\underline{j}}\Big\|_R$$

By (5), we find in the same way as the proof of [Dab14, Corollary 29] that

$$\sup_{|\underline{i}|=n}\|w_{\underline{i}}\|_R\le\left(R+\frac{1}{1-q}\right)^n.$$

Using the triangle inequality, we have

$$\Big\|\sum_{|\underline{j}|=n}(B^2)_{\underline{k}\underline{j}}w_{\underline{j}}\Big\|_R\le\sum_{|\underline{j}|=n}|(G_n^{-1})_{\underline{k}\underline{j}}|\sup_{|\underline{i}|=n}\|w_{\underline{i}}\|_R\le N^n\|(P^{(n)})^{-1}\|\left(R+\frac{1}{1-q}\right)^n.$$

Combining with (4), we have

$$\Big\|\sum_{|\underline{j}|,|\underline{k}|=n} q_j(\underline{k})(B^2)_{\underline{k}\underline{j}}w_{\underline{j}}\otimes w_{\underline{k}}^*\Big\|_R\le q^nN^{2n}\left(R+\frac{1}{1-q}\right)^{2n}\left(\frac{1-q}{1-2q}\right)^n.$$

Plugging in $R=\frac{2+\varepsilon}{1-q}$ and summing over all $n\ge1$, we complete the proof. $\square$

4. Proof of the Main Theorem

Let us write $\mathcal{A}=\mathbb{C}\langle Y_1,\dots,Y_N\rangle$. Suppose Ξ_i is invertible. Let $\partial_j:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}^{op}$ denote the j-th free difference quotient with the property $\partial_jP=\sum_{P=AY_jB}A\otimes B$ for a monomial $P\in\mathcal{A}$. Since $\partial_jX_i=\delta_{i,j}\Xi_j\#\Xi_j^{-1}=\delta_{i,j}1\otimes1^\circ$, we have $\partial_j=\partial_j^{(Q)}\#\Xi_j^{-1}$, where $\#$ is the multiplication in $\Gamma_Q\bar{\otimes}\Gamma_Q^{op}$.

Proposition 5. *Assume $\pi(q,N)<1$. Then we have:*

(i) *There exist noncommutative power series $\xi_j(Y_1,\dots,Y_N)$ of convergence radius $R=\frac{2+\varepsilon}{1-q}>\|X_i\|$ such that $\{\xi_j(X_1,\dots,X_N)\}_{j=1}^N$ are the conjugate variables of $X_1,\dots,X_N$.*

(ii) *There exists a self-adjoint potential $V(Y_1,\dots,Y_N)$ which is also a noncommutative power series of convergence radius R such that $D_iV(Y_1,\dots,Y_N)=\xi_i(Y_1,\dots,Y_N)$ where D_i is the cyclic gradient, i.e., $D_iP=\sum_{P=AY_iB}BA$ for $P\in\mathcal{A}$.*

(iii) $\lim_{q\to 0}\|\xi_i(Y_1,\dots,Y_N)-Y_i\|_R=0$ *for $i=1,\dots,N$.*

Proof. By Lemma 4, $\Xi_j^{-1}=\Xi_j^{-1}(X_1,\dots,X_N)$ for a noncommutative power series $\Xi_j^{-1}(Y_1,\dots,Y_N)$ and we can define a noncommutative power series

$$\begin{aligned}\xi_j(Y_1,\dots,Y_N):=&\ \Xi_j^{-1}(Y_1,\dots,Y_N)\#Y_j\\ &-m\circ(1\otimes\tau_Q\otimes 1)\circ(1\otimes\partial_j^{(Q)}+\partial_j^{(Q)}\otimes 1)(\Xi_j^{-1}(Y_1,\dots,Y_N))\in\mathcal{A},\end{aligned}$$

where $(a\otimes b^\circ)\#x=axb$ and $m(a\otimes b^\circ)=ab$. Then by (3) we have

$$\xi_j:=\xi_j(X_1,\dots,X_N)=\partial_j^{(Q)*}(\Xi_j^{-1}).$$

Consequently for $P\in\mathbb{C}\langle X_1,\dots,X_N\rangle$ we have

$$\langle\xi_j,P\rangle_{\tau_Q}=\langle\Xi_j^{-1},\partial_j^{(Q)}(P)\rangle_{HS}=\langle 1\otimes 1^\circ,\partial_j(P)\rangle_{\tau_Q\otimes\tau_Q^{op}};$$

that is, ξ_j is a conjugate variable.

Let $\mathcal{N}$ be the number operator acting on $\mathbb{C}\langle Y_1,\dots,Y_N\rangle$; that is, $\mathcal{N}$ is defined by $\mathcal{N}P=dP$ for any monomial P of degree d. Let Σ denote the inverse of $\mathcal{N}$ restricted to non-scalar polynomials. Define

$$V(Y_1,\dots,Y_N)=\Sigma\left(\frac{1}{2}\sum_{i=1}^N\xi_i(Y_1,\dots,Y_N)Y_i+Y_i\xi_i(Y_1,\dots,Y_N)\right).$$

Then by precisely the same arguments as in Step 4 of the proof of Theorem 34 in [Dab14], one can see that $D_iV(Y_1,\dots,Y_N)=\xi_i(Y_1,\dots,Y_N)$. Indeed, thanks to Proposition 2 and part (i) above, Lemma 36 in [Dab14] can be verified using Lemma 12 in [Dab14] in our setting. The rest argument of Step 4 is algebraic, and does not use our particular inner product of $\mathcal{F}_Q$.

Finally, Lemma 4 implies that $\Xi_j^{-1}(Y_1,\dots,Y_N)$ converges to $1\otimes 1^\circ$ with respect to the R-norm as $q\to 0$. By an argument similar to that of Lemma 4.3 in [Nel15], it is easy to see that this implies $\lim_{q\to 0}\|\xi_j(Y_1,\dots,Y_N)-Y_j\|_R=0$. □

Proof of Theorem 1. This follows from Proposition 5 and the free monotone transport result of Guionnet and Shlyakhtenko [GS14, Corollary 4.3]. □

5. The Type III Case

One could consider the following more general commutation relations:

$$l_i^* l_j - q_{ij} l_j l_i^* = a_{ij}, \tag{7}$$

for some $\{a_{ij}\}_{1\leq i,j\leq N} \subset \mathbb{C}$. In [Shl97], Shlyakhtenko considered the case

$$\begin{cases} q_{ij} = 0 & 1 \leq i,j \leq N, \\ a_{ii} = 1 & 1 \leq i \leq N, \\ a_{ij} = \overline{a_{ji}} \in i\mathbb{R} & 1 \leq i,j \leq N,\ i \neq j. \end{cases}$$

These commutation relations are satisfied by creation and annihilation operators on a Fock space corresponding to a normalized basis of vectors with purely imaginary covariance. Shlyakhtenko showed that the von Neumann algebra generated by the semi-circular operators $\{l_i + l_i^*\}_{1\leq i\leq N}$, called a *free Araki-Woods factor*, is a full factor with type depending on (a_{ij}). In fact, provided the off-diagonal coefficients a_{ij}, $i \neq j$, are not all zero, the factor is of type III.

For $-1 < q < 1$, the commutation relations for the case

$$\begin{cases} q_{ij} = q & 1 \leq i,j \leq N, \\ a_{ii} = 1 & 1 \leq i \leq N, \\ a_{ij} = \overline{a_{ji}} \in i\mathbb{R} & 1 \leq i,j \leq N,\ i \neq j, \end{cases}$$

are once again satisfied by creation and annihilation operators on a now q-Fock space, and the von Neumann algebra generated by their corresponding semi-circular operators is called a *q-deformed free Araki-Woods algebra*. These were originally defined by Hiai in [Hia03], who established factoriality and a type classification in the situation that $N = \infty$ along with other technical conditions on (a_{ij}).

Using non-tracial free transport, it was shown in [Nel15] that for finite N and sufficiently small q, the q-deformed free Araki-Woods algebras are isomorphic to the corresponding free Araki-Woods factor at $q = 0$. By combining the methods of [Nel15] with those present in this paper, it is easy to check that the same holds true in the mixed q case. That is, there exists some sufficiently small parameter $q_0 > 0$ such that the semi-circular operators arising from the case

$$\begin{cases} |q_{ij}| \leq q_0 & 1 \leq i,j \leq N, \\ a_{ii} = 1 & 1 \leq i \leq N, \\ a_{ij} = \overline{a_{ji}} \in i\mathbb{R} & 1 \leq i,j \leq N,\ i \neq j, \end{cases} \tag{8}$$

generate a free Araki-Woods factor. We record this in the following theorem.

Theorem 6. *Let $Q = (q_{ij})$ be a symmetric $N \times N$ matrix with $N \in \{2, 3, \ldots\}$ and $q_{ij} \in (-1, 1)$. Let $A = (a_{ij})$ be an $N \times N$ matrix with $a_{ii} = 1$ and $a_{ij} = \overline{a_{ji}} \in i\mathbb{R}$ for all $1 \leq i,j \leq N$. Let $l_1, \ldots, l_N$ be creation operators satisfying the commutation relations (7) with constants determined by Q and A, and define $X_j = l_j + l_j^*$ for each $1 \leq j \leq N$. Then there exists $q_0 = q_0(N, A) > 0$ depending only on N and A such that for all Q satisfying*

$\max_{i,j} |q_{ij}| < q_0$ *the von Neumann algebra* $\Gamma_Q^A := W^*(X_1, \ldots, X_N)$ *is the free Araki-Woods factor corresponding to the orthogonal representation* $\mathbb{R} \ni t \mapsto (2A^{-1} - 1)^{it} \in M_N(\mathbb{C})$.

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