Optimal convergence and superconvergence of semi-Lagrangian discontinuous Galerkin methods for linear convection equations in one space dimension[1]

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Abstract

In this paper, we apply semi-Lagrangian discontinuous Galerkin (SLDG) methods for linear hyperbolic equations in one space dimension and analyze the error between the numerical and exact solutions under the L^2-norm. In all the previous works, the theoretical analysis of the SLDG method would suggest a suboptimal convergence rate due to the error accumulation over time steps. However, numerical experiments demonstrate an optimal convergence rate and, if the terminal time is large, a superconvergence rate. In this paper, we will prove optimal convergence and optimal superconvergence rates. There are three main difficulties: 1. The error analysis on overlapping meshes. Due to the nature of the semi-Lagrangian time discretization, we need to introduce the background Eulerian mesh and the shifted mesh. The two meshes are staggered, and it is not easy to construct local projections and to handle the error accumulation during time evolution. 2. The superconvergence of time dependent terms under the L^2-norm. The error of the numerical and exact solutions can be divided into two parts, the projection error and the time dependent superconvergence term. The projection strongly depends on the superconvergence rates. Therefore, we need to construct a sequence of projections and improve the superconvergence rates gradually. 3. The stopping criterion of the sequence of projections. The sequence of projections are basically of the same form. We need to show that the projections exist up to some certain order since the superconvergence rate cannot be infinity. Hence, we will seek some "hidden" condition for the existence of the projections. In this paper, we will solve all the three difficulties and construct several local projections to prove the optimal convergence and superconvergence rates. Numerical experiments verify the theoretical findings.

Keywords: Semi-Lagrangian methods; discontinous Galerkin (DG) method; optimal convergence; optimal superconvergence; overlapping meshes.

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1 Introduction

In this paper, we apply semi-Lagrangian discontinuous Galerkin (SLDG) methods for linear hyperbolic equations in one space dimension. SLDG methods with mass conservation were first proposed for Vlasov applications in [27, 26]. Since then, there have been a strong line of research development for theoretical understanding [16] and applications [20, 19, 4, 5] of SLDG methods. Usually, the convergence analysis of semi-Lagrangian methods [17, 26] suggest an error estimate of h^p/τ, where h and p are the mesh size and approximation order for spatial discretization, respectively; and the denominator with time step size τ indicates the error accumulation over time steps. Yet, for the SLDG method, such estimate is suboptimal [26]; furthermore, little is known on the theoretical level on the superconvergence of the method. In this paper, we investigate the optimal convergence and optimal superconvergence of the numerical approximations toward the exact solutions under the L^2-*norm*.

Superconvergence of discontinuous Galerkin methods for hyperbolic equations have been studied intensively. We refer the reader to [1, 2, 3, 11, 8, 6, 7, 9, 10, 12, 29, 13, 14, 24, 25, 23, 31, 28, 18] for an incomplete list of references on the superconvergence of DG methods for hyperbolic problems. It is not easy to obtain optimal superconvergence rates for time dependent hyperbolic equations. The first work in this direction was given in [31], where Fourier analysis was applied. In [31, 21], the authors used piecewise polynomials of degree k, with $k = 1, 2, 3$, as the numerical approximations and rewrote the DG methods into finite difference schemes. By doing so, the size of amplification matrices would depend on k, and it is extremely difficult to find the eigenvalues and eigenvectors when k is large. Subsequently, in [29], the authors first proved $k + 2$-th order superconvergence of the DG approximation towards a particular projection of the exact solution, and the same convergence rate at the downwind-biased Radau points. Different from the technique in [31], the energy analysis was developed in [29], and the proof works for all k. However, the superconvergence rates at the downwind point as well as the cell average are still not optimal. Recently, correction function technique was developed in [11] to obtain the $2k + 1$-th order superconvergence. The idea was further developed to obtain the optimal superconvergence rates for DG methods with upwind-biased numerical fluxes [6], and for problems with singular initial data [18]. The extension to problems in two space dimensions was also demonstrated in [8]. Moreover, the analysis for nonlinear equations was discussed in [9]. Some recent development of the superconvergence of DG methods can be found in the review papers [24, 10]. However, no previous works discussing fully discretized DG methods without Fourier analysis is available.

In this paper, we consider the SLDG methods for linear hyperbolic equations, and investigate the optimal convergence and optimal superconvergence of the error between the numerical and exact solutions under the L^2-norm, motivated by our numerical verification in Section 5. Below we summarize the main ideas of our analysis presented in the paper.

1. Due to the special time discretization, the SLDG scheme contains two sets of meshes: the background Eulerian mesh and the shifted mesh. After each time step, one has to perform a L^2 projection of the evolved numerical approximation from the shifted mesh to the background Eulerian mesh. In a standard semi-Lagrangian convergence analysis, the projection error accumulates during time evolution, leading to a suboptimal convergence rate in the theoretical analysis [17, 26]. However, numerical experiments demonstrate an optimal convergence rate. Therefore, we have to construct a special projection such that the projection error does not accumulate during time evolution and the rest high-order term would not affect the optimal

convergence rate. Indeed, it is not easy to prove the optimal convergence rate on overlapping meshes. To the best knowledge of the authors, the only work with optimal convergence rate for DG methods in this direction was given in [22], where the semi-discretized central DG methods were discussed. No previous works discussed the superconvergence of DG methods on overlapping meshes. However, for SLDG methods, the semi-discrete version does not exist. Hence the idea in [22] cannot be applied directly. The basic idea for optimal convergence contains two steps. We first assume the exact solution $u(x)$ to be a polynomial of degree $k+1$ and construct *a special local projection* such that the SLDG scheme maps the projection of $u(x)$ on the shifted mesh to the projection of $u(x)$ on the background Eulerian mesh, see (3.1). In the second step, we consider a general smooth solution, and project it to a polynomial of degree $k+1$ with projection error $\mathcal{O}(h^{k+2})$. The accumulation of the projection error is of $(k+1)$-th order accurate, hence it does not affect the optimal convergence rate during time evolution.

2. The superconvergence of time dependent terms under the L^2-norm. It is well known that by using piecewise polynomials of degree k, the error between the numerical and exact solutions under the L^2-norm is at most $(k+1)$-th order accurate. In [30], the authors applied Fourier analysis and studied the case of piecewise linear functions on uniform meshes. The leading error term is shown to be of a constant magnitude independent of time t. This motivates the division of the numerical error into two parts, one being the leading time independent term and the other being a time dependent superconvergent term, see (4.9). If the final time is large, the superconvergence term would dominate and we can observe the desired superconvergence rate under the L^2-norm. It is not easy to obtain the optimal superconvergence rate under the L^2-norm. To the best of our knowledge, no previous results on the optimal superconvergence for fully discretized DG methods are available. In this paper, we will construct a sequence of projections that can be used for monomials with different degrees. More precisely, we define projection P_ℓ such that if the exact solution $u(x) = x^{k+\ell}$, then the SLDG scheme maps the projection of $u(x)$ on the shifted mesh to the projection of $u(x)$ on the background Eulerian mesh. However, in the general treatment of DG method, only one projection can be used, hence the relationship among the projections have to be investigated.

3. The stopping criterion of the sequence of projections. Numerical experiments demonstrate that the superconvergence rate is $2k+1$, leading to $\ell \leq k+1$. Therefore, special stopping criterion in the construction of the projections has to be studied. In this paper, we will prove a special equality that serves as the stopping criterion, i.e. the projection exists if and only if the special equality is valid. Here we would like to point out that the stopping criterion given in the paper is quite different from those given by the correction function technique, e.g. [11]. In the correction function technique, we should apply Legendre expansion of the numerical approximations, and demonstrate that the numerical flux due to the correction function does not vanish after $k+1$ steps. However, in SLDG method no numerical fluxes available, hence we need to explore new stopping criterion.

Finally, we comment that the technique introduced in this paper has special mesh requirements. To construct local projections on overlapping meshes, we need to assume the spatial mesh to be uniformly distributed following [22]. Moreover, the SLDG method is a fully discretized DG scheme, hence the projection depends on the time step τ. To avoid projection change between different time steps, we need to assume the time mesh size to be a constant. Though the above assumptions are essential in the theoretical analysis, numerical experiments demonstrate the same convergence and superconvergence rates for general random meshes. Moreover, numerical experiments also indicate that the uniform mesh assumption is reasonable as the error would accumulate randomly during time evolution if such an assumption is missing. Some advanced techniques have to be developed to obtain the optimal convergence and superconvergence rates for general random meshes, and this will be discussed in the future.

The organization of this paper is as follows. In Section 2, we introduce the SLDG method. We will proceed to prove optimal convergence and superconvergence rates in Sections 3 and 4, respectively. Numerical experiments will be given in Section 5 to verify the theoretical results. We will end in Section 6 with concluding remarks.

2 The SLDG method for a linear convection problem

In this paper, we consider the SLDG method for the following 1D linear convection problem with constant speed on a bounded domain $[x_a, x_b]$ with periodic boundary conditions

$$\begin{cases} u_t + u_x = 0, \quad t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.1}$$

We discretize the computational domain $\Omega = [x_a, x_b]$ into N *uniform* elements:

$$x_a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = x_b, \tag{2.2}$$

with $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ denoting an element of length $h = \frac{x_b - x_a}{N}$ for $j = 1, 2, \cdots, N$. Let $\Delta = \cup_j I_j$ be the set of mesh, and $\tau = t^{n+1} - t^n$ represents the time discretization step, which is assumed to be constant. In the framework of DG method, we consider the finite dimensional approximation space

$$V_h^k = \{v_h : v_h|_{I_j} \in P^k(I_j), j = 1, 2, \cdots, N\}, \tag{2.3}$$

where $P^k(I_j)$ denotes the set of polynomials of degree at most k over I_j.

For the purpose of carrying out theoretical analysis for a simple linear equation (2.1), we introduce the SLDG scheme as a shifting and projection procedure [27]. In particular, to update SLDG solution $u_h^{n+1} \in V_h^k$ from $u_h^n \in V_h^k$, we follow the following two steps:

1. S_τ (forward shifting): $S_\tau(u_h^n(x)) = u_h^n(x - \tau)$.
2. P_Δ: L^2 projection on each computational cell base on the mesh Δ.

Thus, the SLDG scheme for a linear equation (2.1) can be written as

$$u_h^{n+1} = P_\Delta \circ S_\tau(u_h^n) \doteq \mathcal{G}(u_h^n). \tag{2.4}$$

We denote I_j to be the background Eulerian cell, and $\tilde{I}_j$ to be the upstream cell obtained by tracing I_j along characteristics from time level t^n to t^{n+1}.

3 Optimal convergence

In this section, we prove the optimal convergence rate of the SLDG scheme. We assume uniform mesh with $\tau = \lambda h$, where $\lambda < 1$ is a constant. When $\tau \geq h$, the situation can always be decomposed to the whole grid shifting of the DG solution on the uniform mesh, together with a SLDG scheme with $\tau < h$.

3.1 Norms and notations

In this subsection, we define several norms and introduce the notations for projection that will be used throughout the paper.

Norms. Denote $\|u\|_{0,I_j}$ to be the standard L^2 norm of u in cell I_j. For any natural number ℓ, we consider the norm of the Sobolev space $H^\ell(I_j)$, defined by

$$\|u\|_{\ell,I_j} = \left\{ \sum_{0\leq\alpha\leq\ell} \left\| \frac{\partial^\alpha u}{\partial x^\alpha} \right\|_{0,I_j}^2 \right\}^{\frac{1}{2}}.$$

Moreover, we define the norms on the whole computational domain as

$$\|u\|_\ell = \left(\sum_{1\leq j\leq N} \|u\|_{\ell,I_j}^2 \right)^{\frac{1}{2}}.$$

For convenience, if we consider the standard L^2 norm, then the corresponding subscript will be omitted. Moreover, we define the standard L^∞ norm of u in I_j as $\|u\|_{\infty,I_j}$, and define the L^∞ norm on the whole computational domain as

$$\|u\|_\infty = \max_{1\leq j\leq N} \|u\|_{\infty,I_j}.$$

Notations. We introduce the following notations for projections.

- P_Δ: L^2 projection onto the mesh Δ.
- P_Δ^*: the special projection onto the mesh Δ.
- P: the special projection on a *reference interval* I_j. The idea of introducing the notation P is to enable a representation of the special projection such that its uniqueness and existence can be proved, see Lemma 3.1.

3.2 Basic idea

We first demonstrate the basic idea assuming $u(x) \in P^{k+1}([x_a, x_b])$. The key point is to show that the leading term of the numerical error (of order h^{k+1}) will not accumulate with each time step evolution. There are two key components to show the optimal convergence rate.

1. **Find a special projection $P^*_\Delta u$.** More precisely, we write the error as $e = u - u_h = \eta - \xi$, where $\eta = u - P^*_\Delta u$ and $\xi = u_h - P^*_\Delta u$, with $P^*_\Delta u$ being a special projection on the mesh Δ, such that

$$\boxed{P^*_\Delta \circ S_\tau u = P_\Delta \circ S_\tau \circ (P^*_\Delta u) \overset{(2.4)}{=} \mathcal{G} \circ P^*_\Delta u.} \tag{3.1}$$

The L.H.S of the above equation is to firstly evolve the solution by applying the operator S_τ, followed by the special projection; while the R.H.S. of the equation is to firstly apply the special projection P^*_Δ, followed by the SLDG operator $\mathcal{G}$. The equality is essential to the no accumulation of error in time, as will be shown in (3.8). With the assumption of uniform mesh, (3.1) gives rise to the following identity, $\forall v \in V_h^k$

$$\int_{I_j} P^*_\Delta u(x, t^{n+1}) v(x)\, dx = \int_{I_j} S_\tau \circ P^*_\Delta (u(x, t^n)) v(x)\, dx = \int_{I_j} S_\tau \circ P^*_\Delta \circ S_{-\tau}(u(x, t^{n+1})) v(x)\, dx.$$

For simplicity of notation, let $u(x) = u(x, t^{n+1})$, then we have $\forall v \in V_h^k$

$$\int_{I_j} P^*_{\tilde{\Delta}} u(x) v(x)\, dx = \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} P^*_{\tilde{\Delta}} u(x) v(x)\, dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} P^*_{\tilde{\Delta}} u(x) v(x)\, dx, \tag{3.2}$$

where $P^*_{\tilde{\Delta}} = S_\tau \circ P^*_\Delta \circ S_{-\tau}$ is the special projection on the shifted cells $\tilde{I}_j$'s. Please see Figure 1 for illustration of these intervals. It is easy to check that

$$P^*_{\tilde{\Delta}}(u(x-\tau)) = S_\tau \circ P^*_\Delta \circ S_{-\tau} u(x-\tau) = S_\tau \circ P^*_\Delta u(x) = (P^*_\Delta u)(x-\tau), \tag{3.3}$$

where $(P^*_\Delta u)(x-\tau)$ is the right shift of $P^*_\Delta u(x)$ by τ. In the rest of the paper, we always use $P^*_\Delta u(x-\tau)$ to represent $(P^*_\Delta u)(x-\tau)$, the shift of the projection, and use $P^*_\Delta(u(x-\tau))$ to represent the projection of $u(x-\tau)$. In general, $P^*_\Delta(u(x-\tau)) \neq (P^*_\Delta u)(x-\tau)$. To prove the existence and uniqueness of the special projection, we let P be the special projection on a *reference cell* I_j,

$$\int_{I_j} Pu(x) v(x)\, dx \doteq \int_{I_j} P^*_\Delta u(x) v(x)\, dx,$$

and transform $P^*_{\tilde{\Delta}}$ (on mesh $\tilde{\Delta}$) into P (on the reference interval I_j) by coordinate transformation. In particular,

$$\begin{aligned}
\int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} P^*_{\tilde{\Delta}} u(x) v(x)\, dx &= \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} P^*_{\tilde{\Delta}}(u(x-\tau) + u(x) - u(x-\tau)) v(x)\, dx \\
&= \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [Pu(x-\tau) + P(u(x) - u(x-\tau))] v(x)\, dx \\
&= \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [Pu(x-\tau) + u(x) - u(x-\tau)] v(x)\, dx
\end{aligned} \tag{3.4}$$

where we applied (3.3) in the second step and the last step is due to the fact that $u(x) - u(x-\tau) \in P^k$ with $u(x) \in P^{k+1}[x_a, x_b]$. Similarly, we have

$$\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} P^*_{\tilde{\Delta}} u(x) v(x)\, dx = \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} [Pu(x-\tau+h) + u(x) - u(x-\tau+h)] v(x)\, dx, \tag{3.5}$$

where we introduced h on the right-hand side since the integral is in cell $\tilde{I}_{j-1}$, not in cell $\tilde{I}_j$ and we need to shift one cell. From (3.2), (3.4), (3.5), we have $\forall v \in V_h^k$

$$\int_{I_j} Pu(x)v(x)\ dx = \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} [Pu(x-\tau+h) + u(x) - u(x-\tau+h)]v(x)\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [Pu(x-\tau) + u(x) - u(x-\tau)]v(x)\ dx. \tag{3.6}$$

For example, in a simplified P^1 setting, we consider $u(x) = x^2$ and let $Pu = r_1 x + s_1$ on the reference cell I_j. Then (3.6) gives

$$\int_{I_j} (r_1 x + s_1)v(x)\ dx = \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} [(r_1(x-\tau+h)+s_1) + (x^2 - (x-\tau+h)^2)]v(x)\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [(r_1(x-\tau)+s_1) + (x^2 - (x-\tau)^2)]v(x)\ dx, \quad \forall v \in P^1(I_j).$$

One can show that such a special projection exists and is unique with the condition that P preserves the mass $\int_{I_j} Pu(x)dx = \int_{I_j} u(x)dx$. We use Mathematica to find

$$r_1 = \frac{1}{3}(h - 2\tau), \quad s_1 = h^2/12. \tag{3.7}$$

Lemma 3.1 in the next section establishes the existence and uniqueness of such a special projection in a general setting.

2. **The numerical error does not accumulate with $P_\Delta^* u$.** When (3.1) holds, then after applying SLDG operator $\mathcal{G}$ for n time steps, we have

$$\mathcal{G}^n \circ P_\Delta^* u = P_\Delta^* \circ S_\tau^n u = P_\Delta^* \circ S_{n\tau} u, \tag{3.8}$$

assuming $u(x) \in P^{k+1}([x_a, x_b])$. That is to say, applying SLDG operators $\mathcal{G}$ n times on the special projection, is the same as applying the special projection only once on the analytical solution $S_{n\tau}u$. This indicates that the numerical error does not accumulate with SLDG time steps. For a general result on error accumulation, please see Lemma 3.5.

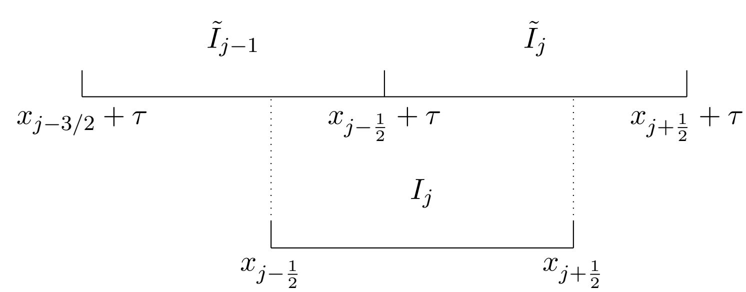


Figure 1: I_j is a cell of the Eulerian background mesh at t^n. $\tilde{I}_j$ is obtained from I_j by tracing along characteristics to t^{n+1}. I_j intersects with $\tilde{I}_{j-1}$ and $\tilde{I}_j$.

3.3 General proof of optimal convergence rate

We first define the projection to be used in this section. Given a function $f(x) \in C(\Omega)$, similar to (3.6), we define the projection Pf in cell I_j such that for any $v \in V_h^k$, we have

$$\begin{aligned}&\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} [Pf(x-\tau+h) + f(x) - f(x-\tau+h)]v(x)\ dx \\ +& \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [Pf(x-\tau) + f(x) - f(x-\tau)]v(x)\ dx = \int_{I_j} Pf(x)v(x)\ dx,\end{aligned} \tag{3.9}$$

and

$$\int_{I_j} Pf(x)\ dx = \int_{I_j} f(x)\ dx, \tag{3.10}$$

where $Pf(x-\tau)$ can be obtained by shifting the projection of $f(x)$ to the right by τ. Likewise for $Pf(x-\tau+h)$. Now, we can state several lemmas demonstrating the properties of the projection given above.

Lemma 3.1. The projection given above is well-defined i.e. the projection exists and it is unique.

Proof. We first demonstrate that the projection (3.9)-(3.10) is not over-determined. Actually, we take $v = 1$ in (3.9), then the left-hand side of (3.9) turns out to be

$$\begin{aligned}&\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} [Pf(x-\tau+h) + f(x) - f(x-\tau+h)]\ dx \\ &+ \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [Pf(x-\tau) + f(x) - f(x-\tau)]\ dx \\ =& \int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} [Pf(x) - f(x)]\ dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} [Pf(x) - f(x)]\ dx + \int_{I_j} f(x)\ dx \\ =& \int_{I_j} Pf(x)\ dx,\end{aligned}$$

where in the first step, we applied substitutions. Therefore, the condition (3.9) with $v = 1$ is redundant. Next, we show the uniqueness and existence of the projection. We rewrite (3.9) as

$$\begin{aligned}&\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} Pf(x-\tau+h)v(x)\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} Pf(x-\tau)v(x)\ dx - \int_{I_j} Pf(x)v(x)\ dx \\ =& \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} [f(x-\tau+h) - f(x)]v(x)\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [f(x-\tau) - f(x)]v(x)\ dx.\end{aligned} \tag{3.11}$$

It is easy to see that (3.11) and (3.10) form a linear system. The system has a unique solution if and only if the homogeneous system has zero solution only. Therefore, we set the right-hand sides of (3.10) and (3.11) to be zero, and prove that Pf is also zero. To do that we assume

$$\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} Pf(x-\tau+h)v(x)\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} Pf(x-\tau)v(x)\ dx - \int_{I_j} Pf(x)v(x)\ dx = 0, \tag{3.12}$$

and

$$\int_{I_j} Pf(x)\ dx = 0. \tag{3.13}$$

We take $v(x) = Pf(x)$ and apply Cauchy-Schwarz inequality and substitutions to obtain

$$\begin{aligned}
&\int_{I_j} [Pf(x)]^2\ dx \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} Pf(x-\tau+h)Pf(x)\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} Pf(x-\tau)Pf(x)\ dx \\
&\leq \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \frac{1}{2}[Pf(x-\tau+h)]^2 + \frac{1}{2}[Pf(x)]^2\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \frac{1}{2}[Pf(x-\tau)]^2 + \frac{1}{2}[Pf(x)]^2\ dx \\
&= \int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \frac{1}{2}[Pf(x)]^2\ dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \frac{1}{2}[Pf(x)]^2 + \int_{I_j} \frac{1}{2}[Pf(x)]^2\ dx \\
&= \int_{I_j} [Pf(x)]^2\ dx.
\end{aligned}$$

Therefore, the "$\leq$" in step two should be "$=$", and hence we have $Pf(x) = Pf(x-\tau+h)$, $\forall\, x \in (x_{j-\frac{1}{2}}, x_{j-\frac{1}{2}}+\tau)$ and $Pf(x) = Pf(x-\tau)$, $\forall\, x \in (x_{j-\frac{1}{2}}+\tau, x_{j+\frac{1}{2}})$. Then $Pf(x) = Pf(x-\tau+h) = Pf(x-\tau)$ in the complex plane. If $Pf(x)$ is not a constant, then it has k roots in the complex plane, denoted as $z_1, \cdots, z_k$. Since $Pf(x) = Pf(x-\tau)$, then $\tilde{z}_1 = z_1+\tau, \cdots, \tilde{z}_k = z_k+\tau$ are also the roots of $Pf(x)$. Hence $\{z_1, \cdots, z_k\} = \{\tilde{z}_1, \cdots, \tilde{z}_k\}$. This is a contradiction, since $\sum_i z_i \neq \sum_i \tilde{z}_i$ as $\tau \neq 0$. Now we conclude that $Pf(x)$ must be a constant. Then by (3.13), we have $Pf(x) = 0$, and finish the proof. $\square$

Moreover, it is easy to check that the projection (3.9)-(3.10) is local, which further yields the following lemma [15].

Lemma 3.2. The projection (3.9)-(3.10) satisfies

$$\|f - Pf\|_{I_j} \leq Ch^{k+1}\|f\|_{k+1,I_j},$$

where the constant C is independent of h.

In addition, we also have the following estimate of the special projection.

Lemma 3.3. The projection (3.9)-(3.10) satisfies

$$\|Pf\|_{\infty,I_j} \leq C\|f\|_{\infty,I_j}$$

for any continuous function f in I_j, and the constant C is independent of h.

Proof. We define a special norm in $P^k(I_j)$ as

$$|||f|||_j = \max_{0\leq i\leq k}\left\{\left|\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} f(x-\tau+h)x^i dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} f(x-\tau)x^i dx - \int_{I_j} f(x)x^i dx\right|, \left|\int_{I_j} f(x)dx\right|\right\}. \tag{3.14}$$

By the proof of Lemma 3.1, it is easy to verify that $|||\cdot|||_j$ is indeed a norm in $P^k(I_j)$. By the norm equivalence in finite dimensional spaces and scaling argument, there exists a positive constant C independent of h, such that $h\|v\|_{\infty,I_j} \le C|||v|||_j$ for any $v \in P^k(I_j)$. Hence

$$\|Pf\|_{\infty,I_j} \le Ch^{-1}|||Pf|||_j = Ch^{-1}|||f|||_j \le C\|f\|_{\infty,I_j},$$

where the second step follows from (3.9). $\square$

Next, we demonstrate why the projection (3.9)-(3.10) is useful, and the result is given below.

Lemma 3.4. Suppose the exact solution at time level n is $f(x+\tau) \in P^{k+1}(I_{j-1} \cup I_j)$, and the numerical solution is given as $u_h^n = P(f(x+\tau))$ in cell I_{j-1} and I_j, then $u_h^{n+1} = Pf(x)$ at time level $n+1$ in cell I_j.

Proof. The proof would be basically the same as that given in Subsection 3.2, hence we skip most of the details.

Define $\tilde{P} = S_\tau \circ P \circ S_{-\tau}$, then following the analysis in (3.3), (3.4) and (3.5), we have

$$\int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \tilde{P}f(x)v(x)\ dx = \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [Pf(x-\tau) + f(x) - f(x-\tau)]v(x)\ dx,$$

and

$$\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \tilde{P}f(x)v(x)\ dx = \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} [Pf(x-\tau+h) + f(x) - f(x-\tau+h)]v(x)\ dx.$$

Therefore, by (3.9), we have

$$\begin{aligned}
\int_{I_j} Pf(x)v(x)\ dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \tilde{P}f(x)v(x)\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \tilde{P}f(x)v(x)\ dx \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} S_\tau \circ P \circ S_{-\tau}f(x)v(x)\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} S_\tau \circ P \circ S_{-\tau}f(x)v(x)\ dx \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} S_\tau \circ P(f(x+\tau))v(x)\ dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} S_\tau \circ P(f(x+\tau))v(x)\ dx,
\end{aligned}$$

which further yields

$$Pf(x) = \mathcal{G}(P(f(x+\tau))) = u_h^{n+1} \tag{3.15}$$

by the definition of the SLDG scheme (2.4). $\square$

As the general treatment of finite element methods, we split the error between the exact and numerical solutions as $e = u - u_h = \eta - \xi$, where $\eta = u - Pu$ and $\xi = u_h - Pu$. In Lemma 3.4, we considered some special exact solutions. The following lemma is used for general exact solutions.

Lemma 3.5. Suppose the exact solution at time level n is $u(x) \in C^{k+2}(\Omega)$, then

$$\|\xi^{n+1}\| \le \|\xi^n\| + \mathcal{O}(h^{k+2}).$$

Proof. We first consider the error in cell I_j and introduce some notations. We rewrite the exact solution at time level n ($t = t^n$) as $u(x, t^n) = u_{k+1}(x) + r(x)$, where $u_{k+1}(x) \in P^{k+1}(I_{j-1} \cup I_j)$ and $r(x) = u(x) - u_{k+1}(x)$ with $r(x) = \mathcal{O}(h^{k+2})$ in $I_{j-1} \cup I_j$. It is easy to see, at time level $n+1$ ($t = t^{n+1}$), the exact solution is $u(x, t^{n+1}) = u(x - \tau, t^n)$, then

$$\begin{aligned}
\xi^{n+1}(x) &= u_h^{n+1} - Pu(x, t^{n+1}) = \mathcal{G}(u_h^n) - P(u_{k+1}(x-\tau)) - P(r(x-\tau)) \\
&= \mathcal{G}(u_h^n) - \mathcal{G}P(u_{k+1}(x)) - P(r(x-\tau)) \\
&= \mathcal{G}(\xi^n) + \mathcal{G}P(r(x)) - P(r(x-\tau)),
\end{aligned}$$

where in line 2, we used Lemma 3.4. Therefore,

$$\begin{aligned}
\|\xi^{n+1}(x)\|_{I_j} &\leq \|\mathcal{G}(\xi^n)\|_{I_j} + \|\mathcal{G}P(r(x))\|_{I_j} + \|P(r(x-\tau))\|_{I_j} \\
&\leq \|\mathcal{G}(\xi^n)\|_{I_j} + \|Pr(x-\tau)\|_{I_j} + \|P(r(x-\tau))\|_{I_j} \\
&\leq \|\mathcal{G}(\xi^n)\|_{I_j} + Ch^{1/2}\|r(x)\|_{\infty, I_{j-1}\cup I_j} \\
&\leq \|\mathcal{G}(\xi^n)\|_{I_j} + Ch^{k+2+1/2},
\end{aligned}$$

where in the second step, we used the definition of the SLDG scheme (2.4) and the property of the L^2 projection in the SLDG scheme, the third step follows from Lemma 3.3. The above inequality further yields

$$\begin{aligned}
\|\xi^{n+1}(x)\|^2 &= \sum_{j=1}^{N} \|\xi^{n+1}(x)\|_{I_j}^2 \\
&\leq \sum_{j=1}^{N} \left(\|\mathcal{G}(\xi^n)\|_{I_j} + Ch^{k+2+1/2}\right)^2 \\
&= \sum_{j=1}^{N} \|\mathcal{G}(\xi^n)\|_{I_j}^2 + 2\sum_{j=1}^{N} Ch^{k+2+1/2}\|\mathcal{G}(\xi^n)\|_{I_j} + \sum_{j=1}^{N}(Ch^{k+2+1/2})^2 \\
&\leq \sum_{j=1}^{N} \|\mathcal{G}(\xi^n)\|_{I_j}^2 + 2\sqrt{\sum_{j=1}^{N}(Ch^{k+2+1/2})^2 \sum_{j=1}^{N} \|\mathcal{G}(\xi^n)\|_{I_j}^2} + \sum_{j=1}^{N}(Ch^{k+2+1/2})^2 \\
&\leq \left(\|\mathcal{G}(\xi^n)\| + Ch^{k+2}\right)^2 \leq \left(\|\xi^n\| + Ch^{k+2}\right)^2,
\end{aligned}$$

where in the last step, we applied the property of the L^2 projection in the SLDG scheme. Finally, take square roots on both sides of the above inequality, we finish the proof. □

Remark 3.6. In the proof of Lemma 3.5, we want $r(x) = \mathcal{O}(h^{k+2})$ in $I_{j-1} \cup I_j$. This is a reasonable assumption, and there are several ways to obtain this condition. For example, one may choose $u_{k+1}(x)$ as the L^2 projection of $u(x)$ over $I_{j-1} \cup I_j$ or as the truncated Taylor expansion of $u(x)$ at $x = x_{j-\frac{1}{2}}$, etc.

Remark 3.7. The $r(x)$ in the proof given above may different in the estimates of different cells. For example, if we consider the estimate of ξ in cell I_j, the $r(x)$ given in I_{j-1} might be totally different from the one if we consider the estimate in cell I_{j-1}.

Finally, we can state the main result in this section.

Theorem 3.8. *Suppose $u \in C^{k+2}(\Omega)$ and the numerical approximations $u_h \in V_h^k$ over a uniform mesh. The initial discretization is given as the special projection* (3.9)-(3.10)*. Then we have*

$$\|(u-u_h)(t)\| \leq C(1+t)h^{k+1}, \tag{3.16}$$

where the positive constant C does not depend on h.

Proof. We apply Lemma 3.5 recurrently to obtain

$$\|\xi^n\| \leq \|\xi^0\| + Cnh^{k+2} \leq Cth^{k+1}.$$

Therefore, by Lemma 3.2, we have

$$\|e^n\| = \|\eta^n\| + \|\xi^n\| \leq C(1+t)h^{k+1}.$$

□

4 Superconvergence

In this section, we proceed to investigate the superconvergence of the SLDG scheme, i.e. the time dependent part of the L^2 error is of order $2k+1$. We first present the basic idea and then use a special example to demonstrate the difficulties. Finally, we prove the $(2k+1)$-th order superconvergence. We will define several new projections as corrections of the special projection given in Section 3. For simplicity of presentation, we denote the projection P given in (3.9)-(3.10) as P_1. Moreover, we also assume $\tau = \lambda h$ with $\lambda < 1$ in this section.

4.1 Basic idea

We first demonstrate the basic idea to obtain the optimal superconvergence rate. Following the same idea in Section 3, we first consider $u(x) \in P^{2k+1}$ for simplicity. We would like to construct a special projection P_Δ^* such that (3.1) holds for $u(x) \in P^{2k+1}$. If such a projection exists, the error can then be decomposed into two parts: the leading projection error (of order h^{k+1}) together with the time dependent superconvergence term which may accumulate during time evolution (of order h^{2k+1}), see Theorem 4.12. To find the special projection, we consider a sequence of monomials $x^{k+1}, x^{k+2}, \cdots, x^{2k+1}$ and look for special projections denoted as P_i $(1 \leq i \leq k+1)$ act on them, i.e., P_i is used to project x^{k+i} to V_h^k. Subsequently, we will show that $P_i x^{k+j} = P_j x^{k+j}$ for $i \geq j$. Due to the linearity of the SLDG algorithm, the special projection for $u(x) \in P^{2k+1}$ is exactly P_{k+1}. For a general smooth function $u(x)$, we can write $u(x) = u_{2k+1}(x) + r(x)$, where $u_{2k+1}(x) \in P^{2k+1}(I_{j-1} \cup I_j)$ and $r(x) = \mathcal{O}(h^{2k+2})$. By the same analysis in Section 3, only the high-order term $r(x)$ contributes to the error accumulation, leading to the optimal superconvergence rate.

4.2 A special example

Before we provide the general proof of the optimal superconvergence, we would like to use the special case of $k = 1$ to demonstrate the basic idea. In particular, when $k = 1$, we will show below that the special projection satisfying (3.1) exists for $u = x^3$, but it does not exists for $u = x^4$, i.e. P_2 exists but P_3 does not. Note that the existence of P_1 has been shown in Lemma 3.1.

Theorem 4.1. *Consider $k = 1$, we can find a unique special projection P such that $\forall u \in P^3$*

$$P \circ S_\tau(u) = \mathcal{G} \circ P(u); \tag{4.1}$$

together with the condition that $\int_{I_j} Pu(x)dx = \int_{I_j} u(x)dx$. However, we cannot find such a special projection such that (4.1) is satisfied $\forall u \in P^4$.

Proof. In our proof, we always refer to Figure 1 for the notation of intervals, and also define $\tilde{P} = S_\tau \circ P \circ S_{-\tau}$ as the projection on cell $\tilde{I}_j$ in this section. Let $u_i(x) = x^{k+i}$, and assume $P(x^{k+i}) = r_i x + s_i$ on a reference cell I_j, then

$$\tilde{P}u_i(x-\tau) = r_i(x-\tau) + s_i, \quad x \in \tilde{I}_j,$$

$$\tilde{P}u_i(x-\tau+h) = r_i(x-\tau+h) + s_i, \quad x \in \tilde{I}_{j-1},$$

following the analysis in (3.3).

- The case of $i = 2$ and $u(x) = x^3$. On $\tilde{I}_j$,

$$\begin{aligned}\tilde{P}(x^3) &= \tilde{P}((x-\tau)^3 + 3\tau(x-\tau)^2 + 3\tau^2(x-\tau) + \tau^3) \\ &= (r_2(x-\tau) + s_2) + 3\tau(r_1(x-\tau) + s_1) + 3\tau^2(x-\tau) + \tau^3.\end{aligned}$$

  Similarly, on $\tilde{I}_{j-1}$,

$$\begin{aligned}\tilde{P}(x^3) &= \tilde{P}((x-\tau+h)^3 + 3(\tau-h)(x-\tau+h)^2 + 3(\tau-h)^2(x-\tau+h) + (\tau-h)^3) \\ &= (r_2(x-\tau+h) + s_2) + 3(\tau-h)(r_1(x-\tau+h) + s_1) + 3(\tau-h)^2(x-\tau+h) + (\tau-h)^3.\end{aligned}$$

  Analogy to (3.6), but for $u(x) = x^3$, we have $\forall v \in V_h^k$

$$\begin{aligned}\int_{I_j} (r_2 x + s_2)v(x)\, dx = &\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} [(r_2(x-\tau+h) + s_2) \\ &+ 3(\tau-h)(r_1(x-\tau+h) + s_1) + 3(\tau-h)^2(x-\tau+h) + (\tau-h)^3]v(x)\, dx \\ &+ \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [(r_2(x-\tau) + s_2) + 3\tau(r_1(x-\tau) + s_1) + 3\tau^2(x-\tau) + \tau^3]v(x)\, dx,\end{aligned}$$

  We use Mathematica to compute r_2 and s_2. We apply (3.7) and take $v = 1$ to find the above identify is automatically satisfied. Therefore, the condition for $v = 1$ cannot provide any information of r_2 and s_2. Then we take $v = x$ to obtain

$$r_2 = \frac{5}{12}h^2 - \frac{2}{3}h\tau + \frac{2}{3}\tau^2. \tag{4.2}$$

  Also from the condition $\int_{I_j} Pu(x)dx = \int_{I_j} u(x)dx$, we have

$$\int_{I_j} (r_2 x + s_2)dx = \int_{I_j} x^3 dx \quad \Rightarrow \quad s_2 = 0. \tag{4.3}$$

- The case of $i=3$ and $u(x)=x^4$. On $\tilde{I}_j$,

$$\begin{aligned} P_3(x^4) &= P((x-\tau)^4+4\tau(x-\tau)^3+6\tau^2(x-\tau)^2+4\tau^3(x-\tau)+\tau^4) \\ &= (r_3(x-\tau)+s_3)+4\tau(r_2(x-\tau)+s_2)+6\tau^2(r_1(x-\tau)+s_1)+4\tau^3(x-\tau)+\tau^4. \end{aligned}$$

Similarly, on $\tilde{I}_{j-1}$,

$$\begin{aligned} P_3(x^4) &= (r_3(x-\tau+h)+s_3)+4(\tau-h)(r_2(x-\tau+h)+s_2) \\ &\quad +6(\tau-h)^2(r_1(x-\tau+h)+s_1)+4(\tau-h)^3(x-\tau+h)+(\tau-h)^4. \end{aligned}$$

Analogy to (3.6), but for $u(x)=x^4$, we have $\forall v\in V_h^k$

$$\begin{aligned} \int_{I_j}(r_3x+s_3)v(x)\ dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau}[(r_3(x-\tau+h)+s_3)+4(\tau-h)(r_2(x-\tau+h)+s_2) \\ &\quad +6(\tau-h)^2(r_1(x-\tau+h)+s_1)+4(\tau-h)^3(x-\tau+h)+(\tau-h)^4]v(x)\ dx \\ &\quad +\int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}}[(r_3(x-\tau)+s_3)+4\tau(r_2(x-\tau)+s_2)+6\tau^2(r_1(x-\tau)+s_1) \\ &\quad +4\tau^3(x-\tau)+\tau^4]v(x)\ dx, \end{aligned}$$

We also use Mathematica to find the values of r_3 and s_3. We apply (3.7),(4.2),(4.3) and take $v=1$ in the above equation to obtain

$$h^2-h\tau+\tau^2=0.$$

Clearly, this is a contradiction.

□

We can see that P_2 exists since the equation for $v=1$ is redundant while P_3 does not exist since the equation for $v=1$ yields a contradiction. We can use this observation to prove the optimal superconvergence.

4.3 Optimal superconvergence

In this subsection, we proceed to prove the optimal superconvergence of the SLDG scheme.

We first define the projections to be used in this subsection. In cell I_j, for any $f\in C^k(I_j)$, we define a sequence of projections $P_{\ell+1}f$, $\ell=0,2,\cdots,k$, such that for any $v\in V_h^k$, we have

$$\begin{aligned} &\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau}\left[\sum_{i=0}^{\ell}\frac{(\tau-h)^i}{i!}P_{\ell+1-i}f^{(i)}(x-\tau+h)+f(x)-\sum_{i=0}^{\ell}\frac{(\tau-h)^i}{i!}f^{(i)}(x-\tau+h)\right]v(x)\ dx \\ &+\int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}}\left[\sum_{i=0}^{\ell}\frac{\tau^i}{i!}P_{\ell+1-i}f^{(i)}(x-\tau)+f(x)-\sum_{i=0}^{\ell}\frac{\tau^i}{i!}f^{(i)}(x-\tau)\right]v(x)\ dx \qquad (4.4) \\ &=\int_{I_j}P_{\ell+1}f(x)v(x)\ dx. \end{aligned}$$

and

$$\int_{I_j}P_{\ell+1}f(x)\ dx=\int_{I_j}f(x)\ dx, \qquad (4.5)$$

where $f^{(i)}(x)$ is the ith derivative of $f(x)$. Before we demonstrate the existence and uniqueness of the above projections, we define $P^{\perp}f(x) = f(x) - Pf(x)$ and would like to present the following lemma.

Lemma 4.2. If $P_1, \cdots, P_\ell$ are well-defined, then we have

$$\int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \sum_{i=1}^{\ell} P^{\perp}_{\ell-i+1} f^{(i)}(x) \frac{(\tau-h)^i}{i!}\ dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=1}^{\ell} P^{\perp}_{\ell-i+1} f^{(i)}(x) \frac{\tau^i}{i!}\ dx = 0. \tag{4.6}$$

Proof. For any $0 \le s \le \ell - 1$, the projection P_{s+1} satisfies

$$\begin{aligned}&\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \sum_{i=0}^{s} \frac{(\tau-h)^i}{i!} P^{\perp}_{s+1-i} f^{(i)}(x-\tau+h) v(x)\ dx \\ &+ \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \sum_{i=0}^{s} \frac{\tau^i}{i!} P^{\perp}_{s+1-i} f^{(i)}(x-\tau) v(x)\ dx = \int_{I_j} P^{\perp}_{s+1} f(x) v(x)\ dx.\end{aligned}$$

Applying substitution, we have

$$\begin{aligned}&\int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \sum_{i=0}^{s} \frac{(\tau-h)^i}{i!} P^{\perp}_{s+1-i} f^{(i)}(x) v(x+\tau-h)\ dx \\ &+ \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=0}^{s} \frac{\tau^i}{i!} P^{\perp}_{s+1-i} f^{(i)}(x) v(x+\tau)\ dx = \int_{I_j} P^{\perp}_{s+1} f(x) v(x)\ dx.\end{aligned}$$

We replace $f(x)$ by $f^{(\ell-s)}(x)$ and take $v(x) = x^{\ell-s}$ to obtain

$$\begin{aligned}&\int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \sum_{i=0}^{s} \frac{(\tau-h)^i}{i!} P^{\perp}_{s+1-i} f^{(\ell-s+i)}(x)(x+\tau-h)^{\ell-s}\ dx \\ &+ \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=0}^{s} \frac{\tau^i}{i!} P^{\perp}_{s+1-i} f^{(\ell-s+i)}(x)(x+\tau)^{\ell-s}\ dx = \int_{I_j} P^{\perp}_{s+1} f^{(\ell-s)}(x) x^{\ell-s}\ dx.\end{aligned}$$

We replace i by $i - \ell + s + i$, then the above equation is equivalent to

$$\begin{aligned}&\int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \sum_{i=\ell-s}^{\ell} \frac{(\tau-h)^{i-\ell+s}}{(i-\ell+s)!} P^{\perp}_{\ell-i+1} f^{(i)}(x)(x+\tau-h)^{\ell-s}\ dx \\ &+ \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=\ell-s}^{\ell} \frac{\tau^{i-\ell+s}}{(i-\ell+s)!} P^{\perp}_{\ell-i+1} f^{(i)}(x)(x+\tau)^{\ell-s} = \int_{I_j} P^{\perp}_{s+1} f^{(\ell-s)}(x) x^{\ell-s}\ dx.\end{aligned}$$

Next, we replace s by $\ell - s$ in the above equation to obtain

$$\begin{aligned}&\int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \sum_{i=s}^{\ell} \frac{(\tau-h)^{i-s}}{(i-s)!} P^{\perp}_{\ell-i+1} f^{(i)}(x)(x+\tau-h)^{s}\ dx \\ &+ \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=s}^{\ell} \frac{\tau^{i-s}}{(i-s)!} P^{\perp}_{\ell-i+1} f^{(i)}(x)(x+\tau)^{s} - \int_{I_j} P^{\perp}_{\ell-s+1} f^{(s)}(x) x^{s}\ dx = 0,\end{aligned} \tag{4.7}$$

for $1 \le s \le \ell$. Denote the left-hand side of the above equation to be $A_s + B_s$, where

$$A_s = \int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \sum_{i=s}^{\ell} \frac{(\tau-h)^{i-s}}{(i-s)!} P^{\perp}_{\ell-i+1} f^{(i)}(x)(x+\tau-h)^s \ dx - \int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} P^{\perp}_{\ell-s+1} f^{(s)}(x) x^s \ dx,$$

and

$$B_s = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=s}^{\ell} \frac{\tau^{i-s}}{(i-s)!} P^{\perp}_{\ell-i+1} f^{(i)}(x)(x+\tau)^s \ dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} P^{\perp}_{\ell-s+1} f^{(s)}(x) x^s \ dx.$$

Then by (4.7) $A_s + B_s = 0$, and

$$\begin{aligned}
&\sum_{s=1}^{\ell} \frac{(-1)^{s+1}}{s!} B_s \\
&= \sum_{s=1}^{\ell} \frac{(-1)^{s+1}}{s!} \left[\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=s}^{\ell} \frac{\tau^{i-s}}{(i-s)!} P^{\perp}_{\ell-i+1} f^{(i)}(x)(x+\tau)^s \ dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} P^{\perp}_{\ell-s+1} f^{(s)}(x) x^s \ dx \right] \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \left[\sum_{s=1}^{\ell} \sum_{i=s}^{\ell} \frac{(-1)^{s+1}}{s!} \frac{\tau^{i-s}}{(i-s)!} P^{\perp}_{\ell-i+1} f^{(i)}(x)(x+\tau)^s - \sum_{s=1}^{\ell} \frac{(-1)^{s+1}}{s!} P^{\perp}_{\ell-s+1} f^{(s)}(x) x^s \right] \ dx \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \left[\sum_{i=1}^{\ell} \sum_{s=1}^{i} \frac{(-1)^{s+1}}{s!} \frac{\tau^{i-s}}{(i-s)!} P^{\perp}_{\ell-i+1} f^{(i)}(x)(x+\tau)^s - \sum_{i=1}^{\ell} \frac{(-1)^{i+1}}{i!} P^{\perp}_{\ell-i+1} f^{(i)}(x) x^i \right] \ dx \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \left[\sum_{i=1}^{\ell} \sum_{s=1}^{i} \binom{i}{s} (x+\tau)^s (-\tau)^{i-s} \frac{(-1)^{i+1}}{i!} P^{\perp}_{\ell-i+1} f^{(i)}(x) - \sum_{i=1}^{\ell} \frac{(-1)^{i+1}}{i!} P^{\perp}_{\ell-i+1} f^{(i)}(x) x^i \right] \ dx \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \left[\sum_{i=1}^{\ell} \sum_{s=0}^{i} \binom{i}{s} (x+\tau)^s (-\tau)^{i-s} \frac{(-1)^{i+1}}{i!} P^{\perp}_{\ell-i+1} f^{(i)}(x) - \sum_{i=1}^{\ell} \frac{(-1)^{i+1}}{i!} P^{\perp}_{\ell-i+1} f^{(i)}(x) x^i \right] \ dx \\
&\quad - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=1}^{\ell} (-\tau)^i \frac{(-1)^{i+1}}{i!} P^{\perp}_{\ell-i+1} f^{(i)}(x) \ dx \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \left[\sum_{i=1}^{\ell} \frac{(-1)^{i+1}}{i!} P^{\perp}_{\ell-i+1} f^{(i)}(x) x^i - \sum_{i=1}^{\ell} \frac{(-1)^{i+1}}{i!} P^{\perp}_{\ell-i+1} f^{(i)}(x) x^i \right] \ dx \\
&\quad + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=1}^{\ell} P^{\perp}_{\ell-i+1} f^{(i)}(x) \frac{\tau^i}{i!} \ dx \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=1}^{\ell} P^{\perp}_{\ell-i+1} f^{(i)}(x) \frac{\tau^i}{i!} \ dx.
\end{aligned}$$

Similarly, we can also prove that

$$\sum_{s=1}^{\ell} \frac{(-1)^{s+1}}{s!} A_s = \int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \sum_{i=1}^{\ell} P^{\perp}_{\ell-i+1} f^{(i)}(x) \frac{(\tau-h)^i}{i!} \ dx.$$

Finally, by using the fact that $A_s + B_s = 0$, we have (4.6) and complete the proof. $\square$

Now, we can state several lemmas demonstrating the properties of the projections $P_\ell' s$.

Lemma 4.3. If $P_1, \cdots, P_\ell$ are well defined, then so is $P_{\ell+1}$, $0 \le \ell \le k$.

Proof. Following the same proof in Lemma 3.1, we only need to show that the projection (4.4)-(4.5) is not over-determined, i.e. the condition with $v = 1$ in (4.4) is redundant. Actually, we take $v = 1$ in (4.4) to obtain

$$\begin{aligned}
&\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \left[\sum_{i=0}^{\ell} \frac{(\tau-h)^i}{i!} P_{\ell+1-i} f^{(i)}(x-\tau+h) + f(x) - \sum_{i=0}^{\ell} \frac{(\tau-h)^i}{i!} f^{(i)}(x-\tau+h)\right] dx \\
&+ \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \left[\sum_{i=0}^{\ell} \frac{\tau^i}{i!} P_{\ell+1-i} f^{(i)}(x-\tau) + f(x) - \sum_{i=0}^{\ell} \frac{\tau^i}{i!} f^{(i)}(x-\tau)\right] dx \\
&= \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}}} P_{\ell+1} f(x)\, dx.
\end{aligned}$$

With suitable substitutions, we can obtain

$$\begin{aligned}
&\int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \sum_{i=0}^{\ell} \frac{(\tau-h)^i}{i!} [P_{\ell+1-i} f^{(i)}(x) - f^{(i)}(x)]\, dx \\
&+ \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=0}^{\ell} \frac{\tau^i}{i!} [P_{\ell+1-i} f^{(i)}(x) - f^{(i)}(x)]\, dx = \int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{1}{2}}} P_{\ell+1} f(x) - f(x)\, dx,
\end{aligned}$$

which further yields

$$\int_{x_{j+\frac{1}{2}}-\tau}^{x_{j+\frac{1}{2}}} \sum_{i=1}^{\ell} \frac{(\tau-h)^i}{i!} [P_{\ell+1-i} f^{(i)}(x) - f^{(i)}(x)]\, dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}-\tau} \sum_{i=1}^{\ell} \frac{\tau^i}{i!} [P_{\ell+1-i} f^{(i)}(x) - f^{(i)}(x)]\, dx = 0.$$

The above identity has been given in (4.6). Therefore, $P_{\ell+1}$ exists and it is unique. ☐

Remark 4.4. In the proof of (4.6), we took $v(x) = x^\ell$ (i.e. $s = 0$). Therefore, we need to assume $\ell \le k$, and this is the reason why we cannot obtain (2k+2)th order superconvergence rate.

The above lemma has a straightforward corollary, hence we skip the proof.

Corollary 4.5. The projections $P_1, \cdots, P_{k+1}$ are well defined, i.e. each projection exists and is unique.

Moreover, it is easy to check that the projection (4.4)-(4.5) is local. Therefore, we have the following lemma [15].

Lemma 4.6. The projection (4.4)-(4.5) satisfies

$$\|f - P_{\ell+1} f\|_{I_j} \le C h^{k+1} \|f\|_{k+1, I_j}$$

for any $0 \le \ell \le k$, and the constant C is independent of h.

In addition, we also have the following estimate of the special projections $P_{\ell+1}$, $0 \le \ell \le k$.

Lemma 4.7. The projection (4.4)-(4.5) satisfies

$$\|P_{\ell+1}f\|_{\infty,I_j} \le C\|f\|_{\infty,I_j} + \sum_{i=1}^{\ell} Ch^{k+i+1}\|f^{(i+k+1)}(x)\|_{\infty,I_j}$$

for any function $f \in C^{\ell+k+1}(I_j)$ and $0 \le \ell \le k$, and the constant C is independent of h.

Proof. We can rewrite (4.4) as

$$\int_{x_{j-\frac12}}^{x_{j-\frac12}+\tau} P_{\ell+1}f(x-\tau+h)v(x)\,dx + \int_{x_{j-\frac12}+\tau}^{x_{j+\frac12}} P_{\ell+1}f(x-\tau)v(x)\,dx - \int_{x_{j-\frac12}}^{x_{j+\frac12}} P_{\ell+1}f(x)v(x)\,dx$$
$$= \int_{x_{j-\frac12}}^{x_{j-\frac12}+\tau} f(x-\tau+h)v(x)\,dx + \int_{x_{j-\frac12}+\tau}^{x_{j+\frac12}} f(x-\tau)v(x)\,dx - \int_{x_{j-\frac12}}^{x_{j+\frac12}} f(x)v(x)\,dx$$
$$+ \int_{x_{j-\frac12}}^{x_{j-\frac12}+\tau} \sum_{i=1}^{\ell} \frac{(\tau-h)^i}{i!} P^{\perp}_{\ell+1-i} f^{(i)}(x-\tau+h)v(x)\,dx + \int_{x_{j-\frac12}+\tau}^{x_{j+\frac12}} \sum_{i=1}^{\ell} \frac{\tau^i}{i!} P^{\perp}_{\ell+1-i} f^{(i)}(x-\tau)v(x)\,dx.$$

Therefore, define

$$I_j^1 = [x_{j-\frac12}, x_{j-\frac12}+\tau], \quad I_j^2 = [x_{j-\frac12}+\tau, x_{j+\frac12}], \quad \tilde{I}_j^1 = [x_{j-\frac12}, x_{j+\frac12}-\tau], \quad \tilde{I}_j^2 = [x_{j+\frac12}-\tau, x_{j+\frac12}],$$

we have

$$\left|\int_{x_{j-\frac12}}^{x_{j-\frac12}+\tau} P_{\ell+1}f(x-\tau+h)v(x)\,dx + \int_{x_{j-\frac12}+\tau}^{x_{j+\frac12}} P_{\ell+1}f(x-\tau)v(x)\,dx - \int_{x_{j-\frac12}}^{x_{j+\frac12}} P_{\ell+1}f(x)v(x)\,dx\right|$$
$$- \left|\int_{x_{j-\frac12}}^{x_{j-\frac12}+\tau} f(x-\tau+h)v(x)\,dx + \int_{x_{j-\frac12}+\tau}^{x_{j+\frac12}} f(x-\tau)v(x)\,dx - \int_{x_{j-\frac12}}^{x_{j+\frac12}} f(x)v(x)\,dx\right|$$
$$\le \sum_{i=1}^{\ell} Ch^i \left(\|P^{\perp}_{\ell+1-i} f^{(i)}(x-\tau+h)\|_{I_j^1}\|v(x)\|_{I_j^1} + \|P^{\perp}_{\ell+1-i} f^{(i)}(x-\tau)\|_{I_j^2}\|v(x)\|_{I_j^2}\right)$$
$$= \sum_{i=1}^{\ell} Ch^i \left(\|P^{\perp}_{\ell+1-i} f^{(i)}(x)\|_{\tilde{I}_j^2}\|v(x)\|_{I_j^1} + \|P^{\perp}_{\ell+1-i} f^{(i)}(x)\|_{\tilde{I}_j^1}\|v(x)\|_{I_j^2}\right)$$
$$\le \sum_{i=1}^{\ell} Ch^i \|P^{\perp}_{\ell+1-i} f^{(i)}(x)\|_{I_j}\|v(x)\|_{I_j}$$
$$\le \sum_{i=1}^{\ell} Ch^{k+i+1} \|f^{(i+k+1)}(x)\|_{I_j}\|v(x)\|_{I_j}$$
$$\le \sum_{i=1}^{\ell} Ch^{k+i+2} \|f^{(i+k+1)}(x)\|_{\infty,I_j},$$

where in the second inequality from the bottom, we applied Lemma 4.6. We also use the definition of the special norm given in (3.14), following the same analysis in Lemma 3.3, we have

$$\|P_{\ell+1}f\|_{\infty,I_j} \le Ch^{-1}|||P_{\ell+1}f|||_j \le Ch^{-1}|||f|||_j + \sum_{i=1}^{\ell} Ch^{k+i+1}\|f^{(i+k+1)}(x)\|_{\infty,I_j}$$
$$\le C\|f\|_{\infty,I_j} + \sum_{i=1}^{\ell} Ch^{k+i+1}\|f^{(i+k+1)}(x)\|_{\infty,I_j}.$$

$\square$

The next lemma shows the relationship among $P_\ell' s$, $1 \le \ell \le k+1$.

Lemma 4.8. If $f(x) \in P^{k+s}(I_j)$, then $P_s f(x) = P_{\ell+1} f(x)$, for all $1 \le s \le k$ and $s \le \ell \le k$.

Proof. We use mathematical induction. Suppose $f(x) \in P^{k+1}(I_j)$, then $f^{(i)}(x) \in P^{k+1-i}(I_j)$, leading to $P_{\ell+1-i} f^{(i)}(x) = f^{(i)}(x)$ for all $i = 1, \cdots, \ell$. Therefore, (4.4) turns out to be

$$\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} [P_{\ell+1} f(x-\tau+h) + f(x) - f(x-\tau+h)] v(x)\, dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} [P_{\ell+1} f(x-\tau) + f(x) - f(x-\tau)] v(x)\, dx = \int_{I_j} P_{\ell+1} f(x) v(x)\, dx.$$

which is exactly (3.9). By Lemma 3.1, $P_1 f(x) = P_{\ell+1} f(x)$ for all $1 \le \ell \le k$. Now we make the following assumption:
(A) If $f(x) \in P^{k+m}(I_j)$ then $P_m f(x) = P_{\ell+1} f(x)$ for all $1 \le m \le s-1$ and $m \le \ell \le k$.
We want to show $P_s f(x) = P_{\ell+1} f(x)$ for all $s \le \ell \le k$ and $f(x) \in P^{k+s}(I_j)$.

Assume $f(x) \in P^{k+s}(I_j)$, then $f^{(i)}(x) \in P^{k+s-i}(I_j)$, leading to $P_{\ell+1-i} f^{(i)}(x) = f^{(i)}(x)$ for all $i = s, \cdots, \ell$. Therefore, (4.4) turns out to be

$$\begin{aligned}
&\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} P_{\ell+1} f(x-\tau+h) v(x)\, dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} P_{\ell+1} f(x-\tau) v(x)\, dx \\
&+ \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \left[\sum_{i=1}^{s-1} \frac{(\tau-h)^i}{i!} P_{\ell+1-i} f^{(i)}(x-\tau+h) + f(x) - \sum_{i=0}^{s-1} \frac{(\tau-h)^i}{i!} f^{(i)}(x-\tau+h) \right] v(x)\, dx \\
&+ \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \left[\sum_{i=1}^{s-1} \frac{\tau^i}{i!} P_{\ell+1-i} f^{(i)}(x-\tau) + f(x) - \sum_{i=0}^{s-1} \frac{\tau^i}{i!} f^{(i)}(x-\tau) \right] v(x)\, dx \\
&= \int_{I_j} P_{\ell+1} f(x) v(x)\, dx.
\end{aligned}$$

By using assumption (A) with $m = s-i$ and the fact that $\ell+1-i > s-i$, we have

$$\begin{aligned}
&\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} P_{\ell+1} f(x-\tau+h) v(x)\, dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} P_{\ell+1} f(x-\tau) v(x)\, dx \\
&+ \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \left[\sum_{i=1}^{s-1} \frac{(\tau-h)^i}{i!} P_{s-i} f^{(i)}(x-\tau+h) + f(x) - \sum_{i=0}^{s-1} \frac{(\tau-h)^i}{i!} f^{(i)}(x-\tau+h) \right] v(x)\, dx \\
&+ \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \left[\sum_{i=1}^{s-1} \frac{\tau^i}{i!} P_{s-i} f^{(i)}(x-\tau) + f(x) - \sum_{i=0}^{s-1} \frac{\tau^i}{i!} f^{(i)}(x-\tau) \right] v(x)\, dx \\
&= \int_{I_j} P_{\ell+1} f(x) v(x)\, dx.
\end{aligned}$$

We can see that $P_{\ell+1}$ and P_s satisfy the same condition given in (4.4). By Corollary 4.5, we have $P_s f(x) = P_{\ell+1} f(x)$ and finish the proof. $\square$

Following the analysis in Section 3, we demonstrate why the projection P_{k+1} is useful, and the result is given below.

Lemma 4.9. Suppose the exact solution at time level n is $f(x+\tau) \in P^{2k+1}(I_{j-1} \cup I_j)$, and the numerical solution is given as $u_h^n = P_{k+1}(f(x+\tau))$, then $u_h^{n+1} = P_{k+1} f(x)$.

Proof. We define $\tilde{P}_{k+1} = S_\tau \circ P_{k+1} \circ S_{-\tau}$. Notice the fact that $f(x) - \sum_{i=0}^{k} \frac{\tau^i}{i!} f^{(i)}(x-\tau) \in V_h^k$, then

$$\begin{aligned}\int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \tilde{P}_{k+1} f(x) v(x)\, dx &= \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \left[\sum_{i=0}^{k} \frac{\tau^i}{i!} P_{k+1} f^{(i)}(x-\tau) + f(x) - \sum_{i=0}^{k} \frac{\tau^i}{i!} f^{(i)}(x-\tau)\right] v(x)\, dx \\ &= \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \left[\sum_{i=0}^{k} \frac{\tau^i}{i!} P_{k+1-i} f^{(i)}(x-\tau) + f(x) - \sum_{i=0}^{k} \frac{\tau^i}{i!} f^{(i)}(x-\tau)\right] v(x)\, dx,\end{aligned}$$

where we applied Lemma 4.8 in the second step. Similarly,

$$\begin{aligned}&\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \tilde{P}_{k+1} f(x) v(x)\, dx \\ &\quad = \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \left[\sum_{i=0}^{k} \frac{(\tau-h)^i}{i!} P_{k+1-i} f^{(i)}(x-\tau+h) + f(x) - \sum_{i=0}^{k} \frac{(\tau-h)^i}{i!} f^{(i)}(x-\tau+h)\right] v(x)\, dx.\end{aligned}$$

By (4.4) with $\ell = k$, we have

$$\begin{aligned}\int_{I_j} P_{k+1} f(x) v(x)\, dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} \tilde{P}_{k+1} f(x) v(x)\, dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} \tilde{P}_{k+1} f(x) v(x)\, dx \\ &= \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} S_\tau \circ P_{k+1} \circ S_{-\tau} f(x) v(x)\, dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} S_\tau \circ P_{k+1} \circ S_{-\tau} f(x) v(x)\, dx \\ &= \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\tau} S_\tau \circ P_{k+1}(f(x+\tau)) v(x)\, dx + \int_{x_{j-\frac{1}{2}}+\tau}^{x_{j+\frac{1}{2}}} S_\tau \circ P_{k+1}(f(x+\tau)) v(x)\, dx,\end{aligned}$$

which further yields

$$P_{k+1} f(x) = \mathcal{G}(P_{k+1}(f(x+\tau))) = u_h^{n+1}$$

by the definition of the SLDG scheme (2.4). $\square$

In this section, we redefine η and ξ as $\eta = u - P_{k+1} u$ and $\xi = u_h - P_{k+1} u$, then we still have $e = u - u_h = \eta - \xi$. In the above lemma, we considered some special exact solutions. The following lemma is used for general exact solutions.

Lemma 4.10. Suppose the exact solution at time level n is $u(x, t^n) \in C^{2k+2}(\Omega)$, then

$$\|\xi^{n+1}\| \leq \|\xi^n\| + \mathcal{O}(h^{2k+2}).$$

Proof. We first introduce some notations. We rewrite the exact solution $u(x, t^n) = u_{2k+1} + r(x)$, $u_{2k+1} \in P^{k+1}(I_{j-1} \cup I_j)$ and $r(x) = u(x) - u_{2k+1}(x)$. Then at time level $n+1$, the exact solution is $u(x, t^{n+1}) = u_{2k+1}(x-\tau) + r(x-\tau)$. Moreover, we assume $r(x)$ satisfies

$$\|r^{(s)}(x)\|_{\infty, I_{j-1} \cup I_j} \leq C h^{2k+2-s}, \quad s = 1, 2, \cdots, 2k+1. \tag{4.8}$$

We consider the error in cell I_j,

$$\begin{aligned}
\xi^{n+1}(x) &= u_h^{n+1} - Pu(x,t^{n+1}) = \mathcal{G}(u_h^n) - P_{k+1}(u_{2k+1}(x-\tau)) - P_{k+1}(r(x-\tau)) \\
&= \mathcal{G}(u_h^n) - \mathcal{G}P_{k+1}(u_{2k+1}(x)) - P_{k+1}(r(x-\tau)) \\
&= \mathcal{G}(\xi^n) + \mathcal{G}(P_{k+1}r(x)) - P_{k+1}(r(x-\tau)).
\end{aligned}$$

where in line 2, we used Lemma 4.9. Therefore,

$$\begin{aligned}
\|\xi^{n+1}(x)\|_{I_j} &\le \|\mathcal{G}(\xi^n)\|_{I_j} + \|\mathcal{G}(P_{k+1}r(x))\|_{I_j} + \|P_{k+1}(r(x-\tau))\|_{I_j} \\
&\le \|\mathcal{G}(\xi^n)\|_{I_j} + \|P_{k+1}r(x-\tau)\|_{I_j} + \|P_{k+1}(r(x-\tau))\|_{I_j} \\
&\le \|\mathcal{G}(\xi^n)\|_{I_j} + Ch^{1/2}\|P_{k+1}r(x)\|_{\infty,I_{j-1}\cup I_j} + Ch^{1/2}\|P_{k+1}(r(x-\tau))\|_{\infty,I_j} \\
&\le \|\mathcal{G}(\xi^n)\|_{I_j} + Ch^{1/2}\left(\|r(x)\|_{\infty,I_{j-1}\cup I_j} + \sum_{i=1}^{k} h^{k+i+1}\|r^{(k+i+1)}(x)\|_{\infty,I_{j-1}\cup I_j}\right) \\
&\le \|\mathcal{G}(\xi^n)\|_{I_j} + Ch^{1/2}\left(h^{2k+2} + h^{2k+2} + \sum_{i=1}^{k} h^{k+i+1}h^{k+1-i}\right) \\
&= \|\mathcal{G}(\xi^n)\|_{I_j} + Ch^{2k+2+1/2},
\end{aligned}$$

where in the second line, we applied the property of the L^2 projection in the SLDG scheme, the fourth line requires Lemma 4.7 and the fifth line follows from (4.8). The above inequality further yields

$$\begin{aligned}
\|\xi^{n+1}(x)\|^2 &= \sum_{j=1}^{N} \|\xi^{n+1}(x)\|_{I_j}^2 \\
&\le \sum_{j=1}^{N} \left(\|\mathcal{G}(\xi^n)\|_{I_j} + Ch^{2k+2+1/2}\right)^2 \\
&\le \sum_{j=1}^{N} \|\mathcal{G}(\xi^n)\|_{I_j}^2 + 2\sum_{j=1}^{N} \|\mathcal{G}(\xi^n)\|_{I_j} Ch^{2k+2+1/2} + \sum_{j=1}^{N} (Ch^{2k+2+1/2})^2 \\
&\le \sum_{j=1}^{N} \|\mathcal{G}(\xi^n)\|_{I_j}^2 + 2\sqrt{\sum_{j=1}^{N} \|\mathcal{G}(\xi^n)\|_{I_j}^2 \sum_{j=1}^{N} (Ch^{2k+2+1/2})^2} + \sum_{j=1}^{N} (Ch^{2k+2+1/2})^2 \\
&= \|\mathcal{G}(\xi^n)\|^2 + 2\sqrt{\|\mathcal{G}(\xi^n)\|^2 (Ch^{2k+2})^2} + (Ch^{2k+2})^2 \\
&\le \left(\|\mathcal{G}(\xi^n)\| + Ch^{2k+2}\right)^2.
\end{aligned}$$

Take square roots on both sizes of the above equation, we have

$$\|\xi^{n+1}(x)\| \le \|\mathcal{G}(\xi^n)\| + Ch^{2k+2} \le \|\xi^n\| + Ch^{2k+2},$$

where in the last step, we applied the property of the L^2 projection in the SLDG scheme. □

Remark 4.11. Similar to Remark 3.6, the assumption (4.8) is reasonable, we can simply choose u_{2k+1} as the L^2 projection of $u(x,t^n)$ in $I_{j-1} \cup I_j$ or as the truncated Taylor expansion of $u(x,t^n)$ at $x = x_{j-\frac{1}{2}}$, etc. Moreover, similarly to Remark 3.7, the $r(x)$ in the proof given above may different in the estimates of different cells.

Finally, we can state the main result in this section.

Theorem 4.12. *Suppose $u \in C^{2k+2}(\Omega)$ and the numerical approximations $u_h \in V_h^k$ over a uniform mesh. The initial discretization is given as the special projection P_{k+1}. Then we have*

$$\|(u - u_h)(t)\| \le Ch^{k+1} + Cth^{2k+1}, \tag{4.9}$$

where the positive constant C does not depend on h or t.

Proof. We apply Lemma 4.10 recurrently to obtain

$$\|\xi^n\| \le \|\xi^0\| + Cnh^{2k+2} \le Cth^{2k+1}.$$

Therefore,

$$\|e^n\| = \|\eta^n\| + \|\xi^n\| \le Ch^{k+1} + Cth^{2k+1},$$

where the estimate of η has been given in Lemma 4.6. □

5 Numerical experiments

In this section, we use numerical experiments to verify our theoretical findings. We solve (2.1) with $u_0(x) = \sin(x)$. We first choose uniform meshes in both space and time, and take $N = 20, 30, 40$, $k = 1, 2$ in all the tests. The results are given in Figure 2. The following observations are made:

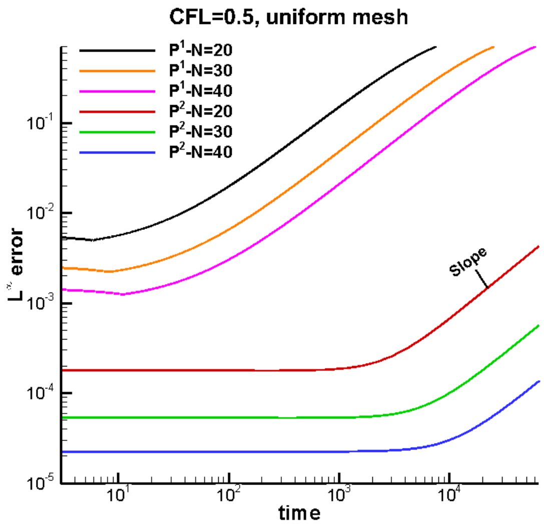
- The projection error dominates for a long time initially, i.e. error does not grow in time until much later.
- When the error starts to grow with time, the slope of such growth is measured to be of order h^{2k+1}. This is verified by the table on the right of Figure 2. We can observe $2k+1$-th order superconvergence rate if the final time is large, i.e. when the time dependent error in (4.9) starts to dominate.

Next, we consider nonuniform meshes and study the following three cases: (1) Random mesh in space but uniform mesh in time; (2) Random mesh in time but uniform mesh in space; (3) Random mesh in both space and time. In all the numerical experiments, the nonuniform meshes in space are obtained by randomly and independently perturbing each node in a uniform mesh by up to 10%. The nonuniform meshes in time are obtained by randomly and independently perturbing the CFL number 0.4 by up to 10%. The L^∞-norm of the errors were given in Figures 3-5, respectively. We can also observe $2k + 1$th order superconvergence rate if the final time is large.

However, from Figures 3-5, we can observe severe oscillations of the error during time evolution while the curves for uniform mesh are quite smooth. Therefore, the uniform mesh assumption is reasonable in the theoretical analysis. To obtain the proof for nonuniform meshes, some advanced techniques that can be used to handle the oscillations should be introduced. This work will be discussed in the future.

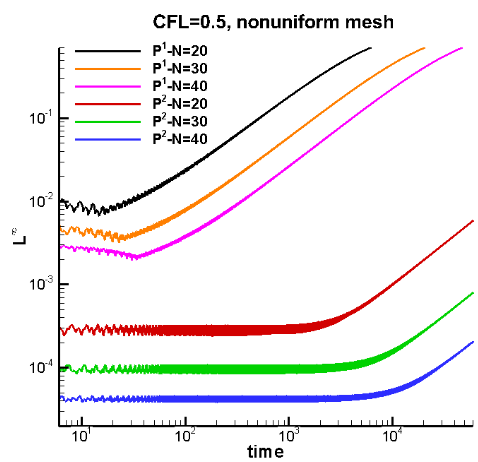
6 Conclusions

In this paper, we proved the optimal convergence and optimal superconvergence rates for the SLDG method for linear hyperbolic equations in one space dimension. Numerical experiments verify our theoretical findings. The proof for nonuniform meshes will be discussed in the future.



N	Slope	Order
	P^1	
20	1.53×10^{-4}	
30	4.73×10^{-5}	2.90
40	2.01×10^{-5}	2.97
	P^2	
20	6.59×10^{-8}	
30	8.62×10^{-9}	5.02
40	1.98×10^{-9}	5.11

Figure 2: Linear advection $u_t + u_x = 0$ with initial condition $u(x,0) = \sin(x)$. Slopes of L^∞ error vs. time for long time simulations. Uniform meshes have $N = 20, 30, 40$ elements.



N	Slope	Order
	P^1	
20	1.88×10^{-4}	
30	5.97×10^{-5}	2.83
40	2.45×10^{-5}	3.09
	P^2	
20	9.76×10^{-8}	
30	1.30×10^{-8}	4.97
40	3.21×10^{-9}	4.86

Figure 3: Linear advection $u_t + u_x = 0$ with initial condition $u(x,0) = \sin(x)$. Slopes of L^∞ error vs. time for long time simulations. Random nonuniform meshes have $N = 20, 30, 40$ elements. For random nonuniform meshes, the length of an element is randomly perturbed by $0.1\Delta x$.

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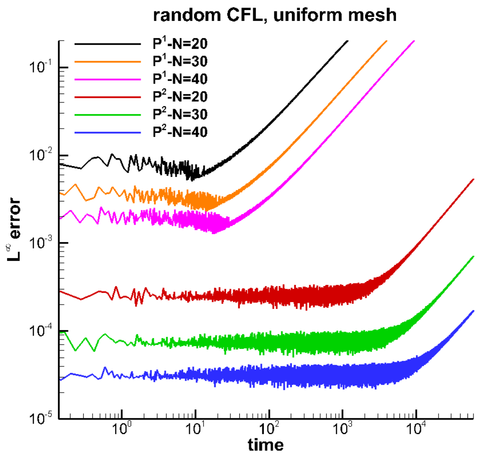
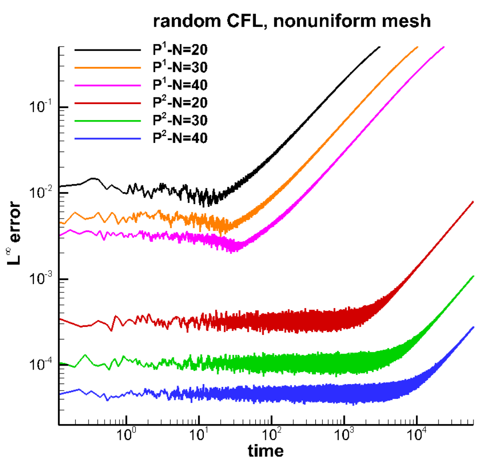


N	Slope	Order
	P^1	
20	1.83×10^{-4}	
30	5.53×10^{-5}	2.95
40	2.30×10^{-5}	3.05
	P^2	
20	8.91×10^{-8}	
30	1.17×10^{-8}	5.02
40	2.74×10^{-9}	5.03

Figure 4: Linear advection $u_t + u_x = 0$ with initial condition $u(x, 0) = \sin(x)$. Slopes of L^∞ error vs. time for long time simulations. Uniform meshes have $N = 20, 30, 40$ elements. Random CFL. The CFL is a 10% random perturbation of 0.4.



N	Slope	Order
	P^1	
20	2.10×10^{-4}	
30	6.82×10^{-5}	2.78
40	2.87×10^{-5}	3.01
	P^2	
20	1.33×10^{-7}	
30	1.78×10^{-8}	4.95
40	4.25×10^{-9}	4.97

Figure 5: Linear advection $u_t + u_x = 0$ with initial condition $u(x, 0) = \sin(x)$. Slopes of L^∞ error vs. time for long time simulations. Random nonuniform meshes have $N = 20, 30, 40$ elements. The random nonuniform mesh is a 10% random perturbation of the uniform mesh. Random CFL. The CFL is a 10% random perturbation of 0.4.

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