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Optimal penalty parameter for C^0 IPDG

Xia Ji [a], Jiguang Sun [b], Yang Yang [b,*]

[a] LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, China
[b] Department of Mathematical Sciences, Michigan Technological University, United States

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ABSTRACT

We derive the lower bound of the penalty parameter in the C^0 IPDG for the bi-harmonic equation. Based on the bound, we propose a pre-processing algorithm. Numerical examples are shown to support the theory. In addition, we found that an *optimal* penalty does exist.

1. Introduction

We study the problem of optimal penalty parameter for the C^0 IPDG (interior penalty discontinuous Galerkin) which has been proposed for the bi-harmonic equation (see [1] and the references therein). It has the practical value in the sense that the penalty parameter has an impact on the error and the linear system.

The choice of the penalty parameter for interior penalty methods has been considered by many researchers. The idea is to sharpen the inequalities in the proof of the ellipticity of the operator and the major tool is the trace inverse inequalities [2]. Shahbazi [3] considered the symmetric IPDG for the Poisson equation with Dirichlet boundary conditions and derived an explicit expression for the penalty parameter. It is shown that the penalty parameter depends on the polynomial basis and the quality of the mesh. Epshteyn and Riviére [4] performed a detailed analysis on the symmetric IPDG and provide ample numerical examples. In particular, they showed that the parameter depends on the smallest $\cot\theta$ over all angles of the triangle in 2D or over all dihedral angles in the tetrahedron in 3D. For further study on the penalty problem, we refer the readers to [5–11].

In this paper, we consider the estimation of the penalty parameter for the C^0 IPDG for the bi-harmonic equation following the spirit of [3,4]. The C^0 IPDG has a simpler formulation since there is no need to penalize the value of the function across the elements. We found that the error increases as the penalty parameter passes certain optimal value if a uniform penalty is used. To further optimize the numerical method, we propose a pre-processing algorithm to compute the penalty parameters, in particular, when the mesh is unstructured. This is also useful for the $h-p$ adaptive IPDG. The rest of the paper is arranged as follows. In Section 2, we introduce the C^0 IPDG for the bi-harmonic equation [12,1]. Analysis of the optimal parameter is contained in Section 3. We present the algorithm for the pre-processing C^0 IPDG in Section 4. In Section 5, we present numerical examples to support our analysis.

* Corresponding author. Tel.: +1 9064873039.
E-mail address: yyang7@mtu.edu (Y. Yang).

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2. C^0 IPDG

Let Ω denote a bounded polygonal Lipschitz domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, and let n denote the unit outward normal. The appropriate solution space of the bi-harmonic equation is

$$H_0^2(\Omega) = \{u \in H^2(\Omega) \mid u = \partial u/\partial n = 0 \text{ on } \partial\Omega\}.$$

We also need the dual space $(H_0^2(\Omega))' = H^{-2}(\Omega)$ as well as spaces $H^{-2+\alpha}(\Omega)$ for $\alpha > 0$. Given a function $f \in H^{-2}(\Omega)$, the bi-harmonic equation is to seek a function $u \in H_0^2(\Omega)$ such that

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{2.1a}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{2.1b}$$

Following [1,12], we define

$$(u, v) = \int_\Omega uv \,\mathrm{d}x \quad \text{and} \quad a(u, v) = (D^2u : D^2v),$$

where $D^2u : D^2v = \sum_{i,j=1}^2 u_{x_ix_j} v_{x_ix_j}$. A weak formulation for (2.1) is: For $f \in H^{-2}(\Omega)$, find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^2(\Omega). \tag{2.2}$$

Now we give a brief introduction to the C^0 IPDG for the bi-harmonic equation and refer the readers to [13,12,1] for more details. Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω with mesh size h and $V_h \subset H_0^1(\Omega)$ be the $\mathbb{P}_k$ Lagrange finite element space ($k \geq 2$) associated with $\mathcal{T}_h$. The space V_h is a subspace of $C^0(\bar{\Omega}) \cap H^2(\Omega, \mathcal{T}_h)$ where

$$H^2(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega) : v_T = v|_T \in H^2(T)\ \forall T \in \mathcal{T}_h\}.$$

Let $\mathcal{E}_h$ be the set of edges in $\mathcal{T}_h$, define $\mathcal{E}_h^B = \mathcal{E}_h \cap \partial\Omega$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \mathcal{E}_h^B$. For $e \in \mathcal{E}_h^0$, the common edge of two adjacent triangles $T^\pm \in \mathcal{T}_h$, and $v \in H^2(\Omega, \mathcal{T}_h)$, we define the jump in the flux to be

$$[\![\partial v/\partial n]\!] = \left.\frac{\partial v_{T^+}}{\partial n_e}\right|_e - \left.\frac{\partial v_{T^-}}{\partial n_e}\right|_e.$$

For simplicity, we use $v^\pm$ to denote $v_{T^\pm}$. Moreover, we let

$$\frac{\partial^2 v}{\partial n_e^2} = n_e \cdot (\nabla^2 v) n_e,$$

and define the average normal–normal component to be

$$\left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v^+}{\partial n_e^2} + \frac{\partial^2 v^-}{\partial n_e^2}\right),$$

where n_e is the unit normal pointing from T^- to T^+. When $e \in \mathcal{E}_h^B$, n_e is the unit outward normal and we define

$$[\![\partial v/\partial n]\!] = -\frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} = \frac{\partial^2 v_T}{\partial n_e^2},$$

where T is the triangle with edge e.

Following [12], the discrete form for the bi-harmonic equation can be written as follows: For $f \in H^{-2+\alpha}(\Omega)$, for some $\alpha > 1/2$, find $u_h \in V_h$ such that

$$a_h(u_h, v) = (f, v) \quad \forall v \in V_h, \tag{2.3}$$

where

$$a_h(w, v) = \mathcal{A}_h(w, v) + b_h(w, v) + c_h(w, v), \tag{2.4}$$

and

$$\mathcal{A}_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T D^2 w : D^2 v \,\mathrm{d}x,$$

$$b_h(w, v) = \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] + \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n_e} \right]\!\!\right] \mathrm{d}s,$$

$$c_h(w, v) = \sum_{e \in \mathcal{E}_h} \frac{\sigma_e}{|e|} \int_e \left[\!\!\left[\frac{\partial w}{\partial n_e} \right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] \mathrm{d}s.$$

Here $\sigma_e > 0$ is the penalty parameter, which may take different values on different edges.

3. Optimizing the penalty parameter

In this section, we proceed to find an optimal parameter σ_e, whose estimation relies on the following trace inverse inequalities [2]:

$$\int_e v^2 \mathrm{d}s \le \frac{(k+1)(k+d)}{d} \frac{\mathcal{A}(e)}{\mathcal{V}(T)} \int_T v^2 \mathrm{d}x,$$

where $\mathcal{A}$, $\mathcal{V}$ denote the length of e and the area of T, respectively.

We define the mesh dependent norm $\|\cdot\|_h$ on V_h as follows

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} \frac{\sigma_e}{|e|} \|[\![\partial v/\partial n]\!]\|^2_{L^2(e)}. \tag{3.5}$$

The penalty needs to be large enough to guarantee the ellipticity of

$$a_h(v,v) = \sum_{T\in\mathcal{T}_h} \int_T D^2 v : D^2 v \,\mathrm{d}x + 2\sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] \mathrm{d}s + \sum_{e\in\mathcal{E}_h} \frac{\sigma_e}{|e|} \int_e \left| \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] \right|^2 \mathrm{d}s. \tag{3.6}$$

Let us consider the second term on the right-hand side. Note that

$$\frac{\partial^2 v}{\partial n_e^2} = n_e \cdot \begin{pmatrix} \dfrac{\partial^2 v}{\partial x^2} & \dfrac{\partial^2 v}{\partial x \partial y} \\ \dfrac{\partial^2 v}{\partial x \partial y} & \dfrac{\partial^2 v}{\partial y^2} \end{pmatrix} n_e = n_1^2 \frac{\partial^2 v}{\partial x^2} + 2n_1 n_2 \frac{\partial^2 v}{\partial x \partial y} + n_2^2 \frac{\partial^2 v}{\partial y^2}, \quad n_e = (n_1, n_2)^T$$

and thus by Cauchy–Schwarz inequality

$$\left(\frac{\partial^2 v}{\partial n_e^2}\right)^2 \le \left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2.$$

As a consequence, we have for $e \in \mathcal{E}_h^0$

$$\left\| \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \right\|_{L^2(e)} \le \frac{1}{2} \left\| D^2 v^+ \right\|_{L^2(e)} + \frac{1}{2} \left\| D^2 v^- \right\|_{L^2(e)}, \tag{3.7}$$

and for $e \in \mathcal{E}_h^B$

$$\left\| \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \right\|_{L^2(e)} \le \left\| D^2 v \right\|_{L^2(e)}, \tag{3.8}$$

where

$$\left\| D^2 w \right\|^2_{L^2(e)} = \int_e \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \right] \mathrm{d}x\mathrm{d}y,$$

with $w = v,\ v^+$ and v^-. To estimate the second normal derivative, we employ the following result (Eqs. (36) and (37) of [4]): For any $v \in \mathbb{P}_{k^T}(T)$, and e is an edge of T,

$$\|v\|_{L^2(e)} \le \sqrt{\frac{2(k^T+1)(k^T+2)\cot\theta_T}{|e|}} \|v\|_{L^2(T)}, \tag{3.9}$$

where θ_T and k^T are the smallest angle and the degree of polynomial approximation in the triangle T, respectively. With the above result, we have for $e \in \mathcal{E}_h^0$

$$\left\| \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \right\|_{L^2(e)} \le \sqrt{\frac{C_{T^+}}{2|e|} |v|^2_{H^2(T^+)}} + \sqrt{\frac{C_{T^-}}{2|e|} |v|^2_{H^2(T^-)}},$$

and $e \in \mathcal{E}_h^B$

$$\left\| \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \right\|_{L^2(e)} \le \sqrt{\frac{2C_T}{|e|} |v|^2_{H^2(T)}},$$

where

$$C_E = (k^E - 1)k^E \cot\theta_E, \quad \text{with } E = T,\ T^+ \text{ and } T^-.$$

Here we have to use the fact that $\frac{\partial^2 v}{\partial n_e^2}$ is a polynomial of degree $k^E - 2$ to obtain C_E. Therefore, by Cauchy–Schwarz inequality

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n_e^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \mathrm{d}s \\
&\quad\le \sum_{e\in\mathcal{E}_h} \left\| \left\{\!\!\left\{\frac{\partial^2 v}{\partial n_e^2}\right\}\!\!\right\} \right\|_{L^2(e)} \left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L^2(e)} \\
&\quad\le \sum_{e\in\mathcal{E}_h^B} \sqrt{\frac{2C_T}{|e|}|v|_{H^2(T)}^2} \left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L^2(e)} + \sum_{e\in\mathcal{E}_h^0} \left(\sqrt{\frac{C_{T^+}}{2|e|}|v|_{H^2(T^+)}^2} + \sqrt{\frac{C_{T^-}}{2|e|}|v|_{H^2(T^-)}^2}\right) \left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L^2(e)} \\
&\quad\le \left(3\sum_{T\in\mathcal{T}_h} S_T\, |v|_{H^2(T)}^2\right)^{\frac12} \left(\sum_{e\in\mathcal{E}_h^B} \frac{2C_T}{S_e} \frac{\left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L^2(e)}^2}{|e|} + \sum_{e\in\mathcal{E}_h^0} \frac{\left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L^2(e)}^2}{2|e|} \left(\frac{C_{T^+}}{S_e} + \frac{C_{T^-}}{S_e}\right)\right)^{\frac12},
\end{aligned}$$

where S_e are positive constants which depend on edge e, and

$$S_T = \sum_{i=1}^{3} \frac{S_{e_i}}{3}$$

with e_i being the edges of the triangle T. By Young's inequality, we have that

$$\begin{aligned}
2\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n_e^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \mathrm{d}s \le{}& \sum_{T\in\mathcal{T}_h} S_T\, |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h^B} \frac{6(k^T-1)k^T\cot\theta_T}{S_e} \frac{\left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L^2(e)}^2}{|e|} \\
&+ \sum_{e\in\mathcal{E}_h^0} \frac{3\left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L^2(e)}^2}{2|e|} \left(\frac{(k^{T^+}-1)k^{T^+}\cot\theta_{T^+}}{S_e} + \frac{(k^{T^-}-1)k^{T^-}\cot\theta_{T^-}}{S_e}\right).
\end{aligned}$$

Therefore, we obtain that

$$a_h(v,v) \ge \sum_{T\in\mathcal{T}_h}(1-S_T)|v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma_e - C_e}{|e|} \left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L^2(e)}^2$$

where

$$C_e = \begin{cases} \dfrac{3(k^{T^+}-1)k^{T^+}\cot\theta_{T^+}}{2S_e} + \dfrac{3(k^{T^-}-1)k^{T^-}\cot\theta_{T^-}}{2S_e}, & e\in\mathcal{E}_h^0, \\ \dfrac{6(k^T-1)k^T\cot\theta_T}{S_e}, & e\in\mathcal{E}_h^B. \end{cases}$$

Thus to guarantee the coercivity of a_h, $S_T < 1$ is required. Then the penalty needs to satisfy

$$\sigma_e > C_e. \tag{3.10}$$

We note that the above analysis can be directly applied to the three dimensional case using the three-dimensional inverse trace inequality. For simplicity, we only consider the two dimensional case in this short paper.

4. A pre-processing algorithm

In this section, we illustrate a pre-processing algorithm to compute σ_e. For simplicity, we choose S_e to be a constant independent on the edge e. Therefore, we have $S_e < 1$. Since the penalty parameter σ_e depends on the edge, we can first sweep the mesh and compute σ_e by using (3.10) for each edge. The algorithm is given below.

1. For each triangle $T \in \mathcal{T}_h$, find out the smallest angle and the degree of polynomial approximation, denoted as θ_T and k^T, respectively.

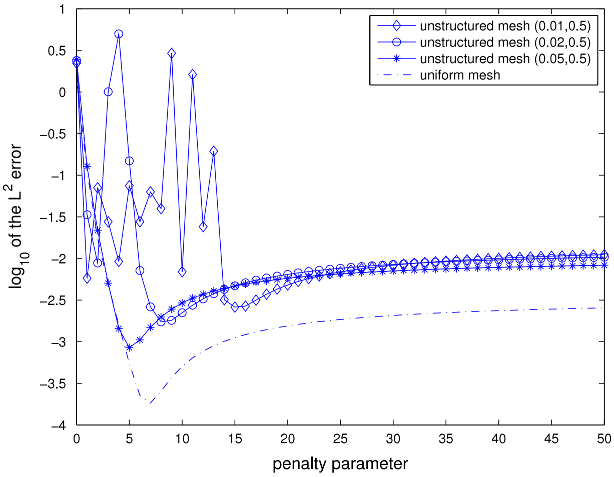

Fig. 1. The L^2 error vs. the penalty parameter for an unstructured mesh with $k = 2$.

Table 1
The minimum and maximum of the penalty parameters. The fourth column is the error using the pre-processing step. The fifth column is the error using a uniform penalty, i.e., the maximum value of penalty parameters.

k	$\min \sigma_e$	$\max \sigma_e$	err (LP)	err (UP)
2	8.8479	1.800e+3	5.2558e−04	1.3379e−3
3	26.5491	5.400e+3	2.1591e−05	9.2242e−4

2. For each edge $e \in \mathcal{E}_h^B$, denote T as the triangle with edge e. Take

$$\sigma_e > 6(k^T - 1)k^T \cot \theta_T.$$

3. For each edge $e \in \mathcal{E}_h^0$, assume it is shared by two triangles T^+ and T^-. Take

$$\sigma_e > \frac{3}{2}\left((k^{T^+} - 1)k^{T^+} \cot \theta_{T^+} + (k^{T^-} - 1)k^{T^-} \cot \theta_{T^-}\right).$$

5. Numerical examples

We consider a simple domain in 2D. Let $\Omega = [0, 1] \times [0, 1]$ and $u(x, y) = \sin^2(\pi x)\sin^2(\pi y)$. It is easy to check that u solves the bi-harmonic equation with

$$f(x, y) = 8\pi^4 \cos^2(\pi x)\cos^2(\pi y) - 16\pi^4 \cos^2(\pi x)\sin^2(\pi y) - 16\pi^4 \sin^2(\pi x)\cos^2(\pi y) + 24\pi^4 \sin^2(\pi x)\sin^2(\pi y).$$

We consider three unstructured meshes and a uniform mesh for the unit square. We generate the unstructured meshes as follows. We choose a point $(0.01, 0.5)$ in the unit square to obtain the initial mesh with 4 triangles. One of the triangle is given by $(0, 0)$, $(0.01, 0.5)$, $(0, 1)$. Then we uniformly refine the mesh into 512 triangles. We generate other two meshes by choosing the points at $(0.02, 0.5)$ and $(0.05, 0.5)$, respectively. The uniform mesh also contains 512 triangles. For the unstructured meshes, the theory predicts a larger penalty parameter due to the quality of the triangle.

We first compare the performance of the C^0 IPDG using a uniform penalty on two meshes. In Fig. 1, we show the L^2 error vs. the uniform penalty. It can be seen that for small penalty, the L^2 error is unstable for the unstructured meshes. In addition, the optimal penalty increases as the mesh quality gets worse. An interesting observation is that the error gets larger as the penalty parameter passes the *optimal value*. Thus an *optimal* penalty seems to exist. This is consistent with the claim by Brenner that a large penalty adversely affects the accuracy [12]. While the optimal value cannot be obtained analytically, it is always possible to test on a coarse mesh and make a good guess of it.

Secondly, we employ the pre-processing as above on the unstructured mesh with the point at $(0, 01, 0.5)$. In Table 1, we show the maximum and minimum of the penalty parameters. The last two columns are the error with the pre-processing and the error with a uniform penalty parameter, respectively. Numerically, we see that it is worth doing a pre-processing since the error is smaller and it is computationally very cheap.

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