Conservative local discontinuous Galerkin method for compressible miscible displacements in porous media *

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Abstract: In [H. Guo, Q. Zhang, J. Wang, Applied Mathematics and Computation, 259 (2015), 88-105], a nonconservative local discontinuous Galerkin (LDG) method for both flow and transport equations was introduced for the one-dimensional coupled system of compressible miscible displacement problem. In this paper, we will continue our effort and develop a conservative LDG method for the problem in two space dimensions. Optimal error estimates in $L^\infty(0,T;L^2)$ norm for not only the solution itself but also the auxiliary variables will be derived. The main difficulty is how to treat the inter-element discontinuities of two independent solution variables (one from the flow equation and the other from the transport equation) at cell interfaces. Numerical experiments will be given to confirm the accuracy and efficiency of the scheme.

Keywords: local discontinuous Galerkin method; error estimate; compressible miscible displacement
AMS(2000) Subject Classifications: 65M15, 65M60

1 Introduction

Numerical modeling of miscible displacements in porous media is important and interesting in oil recovery and enviromental pollution problem. The miscible displacement problem is described by a coupled system of nonlinear partial differential equations. The need for accurate solutions to the coupled equations challenges numerical analysts to design new methods.

The compressible miscible displacements have been studied intensively in the literature. In [9, 10] , Douglas and Roberts presented the mixed finite element method for

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miscible displacement problem. A variety of numerical techniques have been introduced to obtain better approximations, such as the modified method of characteristic finite element method (MMOC) [11, 12, 33], characteristic finite element method [32], high-order Godunov scheme [3], streamline diffusion method [18], and Mass-conservative characteristic finite element method [19]. Recently, discontinuous Galerkin (DG) for miscible displacement has been investigated by numerical experiments and was reported to exhibit good numerical performance [1, 21]. In [23, 24, 7], primal semi-discrete discontinuous Galerkin methods with interior penalty are proposed to solve the coupled system of flow and reactive transport in porous media.

The DG method gained even greater popularity recently for good stability, high order accuracy, and flexibility on h-p adaptivity and on complex geometry. But, it is difficult to apply the DG method directly to the equations with higher order derivatives. The idea of the local discontinuous Galerkin (LDG) method is to rewrite the equations with higher order derivatives into a first order system, then apply the DG method to the system. As an extension of DG schemes for hyperbolic conservation laws, the LDG methods share the advantages of the DG methods. Besides, a key advantage of this scheme is the local solvability, i.e. the auxiliary variables approximating the gradient of the solution can be locally eliminated. The first LDG method was introduced by Cockburn and Shu in [6] for solving nonlinear convection diffusion equations containing second order spatial derivatives. Their work was motivated by the successful numerical experiments of Bassi and Rebay [2] for the compressible Navier-Stokes equations. The methods were further developed in [28, 29, 30] for solving many nonlinear wave equations with higher order derivatives.

In our previous work [14], we have used the LDG method to the one-dimensional coupled system of compressible miscible displacement problem. But the method in [14] is not conservative. Recently, we [15] applied the LDG methods to solve incompressible miscible displacements in porous media. In this paper we continue our works to develop a conservative LDG method for compressible miscible displacements in two space dimensions. The main difficulty is how to treat the inter-element discontinuities of two independent solution variables (one from the flow equation and the other from the transport equation) at cell interfaces. More precisely, in this problem, the approximations of $\mathbf{u}$ in the convection term in (2.1) is discontinuous across the cell interfaces and it is difficult to obtain error estimates if we analyze the convection and diffusion terms separately. To explain this point, let us consider the following hyperbolic equation

$$u_t + (a(x)u)_x = 0,$$

where $a(x)$ is discontinuous at $x = x_0$. In [13, 16], the authors studied such a problem and defined

$$Q = \frac{a(x_0 + b) - a(x_0)}{b}.$$

If Q is bounded from below for all b, then the solution exists, but may not be unique. If Q is bounded from above for all b, we can guarantee the uniqueness, but the solution may not exist. Recently, Wang et al. [25, 26] obtained optimal error estimates of the LDG methods with IMEX time marching for linear and nonlinear convection-diffusion problems. The key idea is to explore an important relationship between the gradient and interface jump of the numerical solution polynomial with the numerical approximation of auxiliary variable for the gradient in the LDG methods, which is stated in Lemma 4.4. Moreover, the systems are coupled together. Therefore, we will derive four energy inequalities to obtain optimal error estimates in $L^\infty(0,T;L^2)$ for concentration c, in $L^2(0,T;L^2)$ for $\mathbf{s} = -\nabla c$ and $L^\infty(0,T;L^2)$ for velocity $\mathbf{u}$. Here we should mention the difference between our LDG method and the DG method in [7], where the interior penalty discontinuous Galerkin (IPDG) method was introduced and optimal error estimates in $L^2(0,T;H^1)$ norm for concentration c were given. In our proof, induction hypothesis is used as a tool, instead of the cut-off operator proposed in [24]. Therefore, it is not necessary to choose the sufficiently large positive constant M, and the possibility of infinite times of loops for extreme cases can be avoided.

The paper is organized as follows. In Section 2, we demonstrate the governing equations of the compressible miscible displacements in porous media. In Section 3, we present some preliminaries, including the basic notations and norms to be used throughout the paper, the LDG spatial discretization and the error equations. Section 4 is the main body of the paper where we present the projections and some essential properties of the finite element spaces, error equations and the details of the optimal error estimates for compressible miscible displacement problem. Then numerical results are given to demonstrate the accuracy and capability of the method in Section 5. We will end in Section 6 with some concluding remarks.

2 Compressible miscible displacement problem

In this section, we demonstrate the governing equations of the compressible miscible displacements in porous. Detailed discussion on physical theories can be found in [8]. Let Ω be a rectangular domain. The classical equations governing the compressible miscible displacement in porous media in two space dimensions are as follows:

$$\begin{array}{ll} d(c)\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = q, & (x,y) \in \Omega, 0 < t \le T, \\ \mathbf{u} = \frac{-\kappa(x,y)}{\mu(c)} \nabla p, & (x,y) \in \Omega, 0 < t \le T, \\ \phi \frac{\partial c}{\partial t} + b(c)\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla c = \nabla \cdot (\mathbf{D} \nabla c) + (\tilde{c} - c) q, & (x,y) \in \Omega, 0 < t \le T, \end{array} \tag{2.1}$$

where the dependent variables p, $\mathbf{u}$ and c are the pressure in the fluid mixture, the Darcy velocity of the mixture (volume flowing across a unit across-section per unit time), and the concentration of interested species measured in amount of species per unit volume of the fluid mixture, respectively. ϕ and κ are the porosity and the permeability of the rock, respectively. μ is the concentration-dependent viscosity. q is the external volumetric flow rate, and $\tilde{c}$ is the concentration of the fluid in the external flow. $\tilde{c}$ must be specified at points at which injection ($q > 0$) takes place, and is assumed to be equal to c at production points ($q < 0$). We shall also consider only molecular diffusion, so that $\mathbf{D} = \phi(x, y)d_m I$ with I being the identity matrix. In this paper the tensor matrix $\mathbf{D}$ is assumed to be positive definite. Moreover, the pressure is uniquely determined up to a constant, thus we assume $\int_\Omega p dx dy = 0$ at $t = 0$. For simplicity, we confine ourselves to a two component displacement problem. The numerical method can be applied to the multi-component model. The coefficients can be stated as follows:

$$
\begin{aligned}
&c = c_1 = 1 - c_2, \\
&a(c) = a(x, y, c) = \frac{\kappa(x, y)}{\mu(c)}, \\
&b(c) = b(x, y, c) = \phi(x, y)c_1\{m_1 - \sum_{j=1}^{2} m_j c_j\}, \\
&d(c) = d(x, y, c) = \phi(x, y)\sum_{j=1}^{2} m_j c_j,
\end{aligned}
$$

with c_i being the concentration of i th component of the fluid mixture, and m_i being the "constant compressibility" factor. In this problem, the initial concentration are pressure are given as

$$c(x, y, 0) = c_0(x, y), \quad p(x, y, 0) = p_0(x, y), \quad (x, y) \in \Omega.$$

Finally, we make the following hypotheses (H) for (2.1).

1. $0 < \kappa_* \le \kappa(x, y) \le \kappa^*$, $0 < \mu_* \le \mu(c) \le \mu^*$, $0 < \phi_* \le \phi(x, y) \le \phi^*$, $0 < d_* \le d(c) \le d^*$, $|q| \le C$, $|b(c)| \le C$, $|\mu'(c)| \le C$ and $|d'(c)| \le C$.

2. $d(c)$, $\mu'(c)$ and $d'(c)$ are uniformly Lipschtiz continuous with respect to c, respectively.

3. $\mathbf{D}$ is uniformly Lipschtiz continuous, and for any $\mathbf{v}$, $\mathbf{w} \in R^2$ there exist two positive constants D_*, D^* such that $\mathbf{v}^T\mathbf{D}\mathbf{v} \ge D_*\mathbf{v}^T\mathbf{v} = D_*\|\mathbf{v}\|^2$ and $\mathbf{v}^T\mathbf{D}\mathbf{w} \le D^*\|\mathbf{v}\|\|\mathbf{w}\|$.

4. $\mathbf{u}, \mathbf{u}_t, c, \nabla c, c_t, p_t$ and p_{tt} are uniformly bounded in R^2.

3 Preliminaries

In this section, we will demonstrate some preliminary results that will be used through out the paper.

3.1 Basic notations

In this section, we present the notations. Let $0 = x_{\frac{1}{2}} < \cdots < x_{N_x+\frac{1}{2}} = 1$ and $0 = y_{\frac{1}{2}} < \cdots < y_{N_y+\frac{1}{2}} = 1$ be the grid points in the x and y directions, respectively. Define $I_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $J_j = (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$. Let $K = I_i \times J_j$, $i = 1, \cdots, N_x$, $j = 1, \cdots, N_y$, be a partition of Ω and denote $\Omega_h = \{K\}$. The mesh sizes in the x and y directions are given as $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $\Delta y_j = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$, respectively and $h = \max(\max_i \Delta x_i, \max_j \Delta y_j)$. Moreover, we assume the partition is quasi-uniform. The finite element space is chosen as

$$W_h^k = \{z : z|_K \in Q^k(K), \forall K \in \Omega_h\},$$

where $Q^k(K)$ denotes the space of tensor product polynomials of degrees at most k in K. Note that functions in W_h^k are discontinuous across element interfaces. This is one of the main differences between the DG method and traditional finite element methods. We choose $\boldsymbol{\beta} = (1, 1)^T$ to be a fixes vector that is not parallel to any normals of the element interfaces. We denote Γ_h be the set of all element interfaces and $\Gamma_0 = \Gamma_h \backslash \partial\Omega$. Let $e \in \Gamma_0$ be an interior edge shared by elements K_ℓ and K_r, where $\boldsymbol{\beta} \cdot \mathbf{n}_\ell > 0$, and $\boldsymbol{\beta} \cdot \mathbf{n}_r < 0$, respectively, with $\mathbf{n}_\ell$ and $\mathbf{n}_r$ being the outward normal of K_ℓ and K_r, respectively. For any $z \in W_h^k$, we define $z^- = z|_{\partial K_\ell}$ and $z^+ = z|_{\partial K_r}$, respectively. The jump is given as $[z] = z^+ - z^-$. Moreover, for $\mathbf{s} \in \mathbf{W}_h^k = W_h^k \times W_h^k$, we define $\mathbf{s}^+$ and $\mathbf{s}^-$ and $[\mathbf{s}]$ analogously. We also define $\partial\Omega_- = \{e \in \partial\Omega | \boldsymbol{\beta} \cdot \mathbf{n} < 0\}$, where $\mathbf{n}$ is the outer normal of e, and $\partial\Omega_+ = \partial\Omega \backslash \partial\Omega_-$. For any $e \in \partial\Omega_-$, there exists $K \in \Omega_h$ such that $e \in \partial K$, we define $z^+|_e = z|_{\partial K}$, and define z^- on $\partial\Omega_+$ analogously. For simplicity, given $e = \{x_{\frac{1}{2}}\} \times J_j \in \partial\Omega_-$ and $\tilde{e} = \{x_{N_x+\frac{1}{2}}\} \times J_j \in \partial\Omega_+$, by periodic boundary condition, we define

$$z^-|_e = z^-|_{\tilde{e}}, \quad \text{and} \quad z^+|_{\tilde{e}} = z^+|_e.$$

Similarly, given $e = I_i \times \{y_{\frac{1}{2}}\} \in \partial\Omega_-$ and $\tilde{e} = I_i \times \{y_{N_y+\frac{1}{2}}\} \in \partial\Omega_+$, we define

$$z^-|_e = z^-|_{\tilde{e}}, \quad \text{and} \quad z^+|_{\tilde{e}} = z^+|_e.$$

Throughout this paper, the symbol C is used as a generic constant which may appear differently at different occurrences.

3.2 Norms

In this subsection, we define several norms that will be used throughout the paper.

Denote $\|u\|_{0,K}$ to be the standard L^2 norm of u in cell K. For any natural number ℓ, we consider the norm of the Sobolev space $H^\ell(K)$, defined by

$$\|u\|_{\ell,K} = \left\{ \sum_{0 \le \alpha+\beta \le \ell} \left\| \frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial y^\beta} \right\|_{0,K}^2 \right\}^{\frac{1}{2}}.$$

Moreover, we define the norms on the whole computational domain as

$$\|u\|_\ell = \left(\sum_{K \in \Omega_h} \|u\|_{\ell,K}^2 \right)^{\frac{1}{2}}.$$

For convenience, if we consider the standard L^2 norm, then the corresponding subscript will be omitted.

Let Γ_K be the edges of K, and we define

$$\|u\|_{\Gamma_K}^2 = \int_{\partial K} u^2 ds.$$

We also define

$$\|u\|_{\Gamma_h}^2 = \sum_{K \in \Omega_h} \|u\|_{\Gamma_K}^2.$$

Moreover, we define the standard L^∞ norm of u in K as $\|u\|_{\infty,K}$, and define the L^∞ norm on the whole computational domain as

$$\|u\|_\infty = \max_{K \in \Omega_h} \|u\|_{\infty,K}.$$

Finally, we define similar norms for vector $\mathbf{u} = (u_1, u_2)^T$ as

$$\|\mathbf{u}\|_{\ell,K}^2 = \|u_1\|_{\ell,K}^2 + \|u_2\|_{\ell,K}^2, \quad \|\mathbf{u}\|_{\Gamma_K}^2 = \|u_1\|_{\Gamma_K}^2 + \|u_2\|_{\Gamma_K}^2, \quad \|\mathbf{u}\|_{\infty,K} = \max\{\|u_1\|_{\infty,K}, \|u_2\|_{\infty,K}\}.$$

Similarly, the norms on the whole computational domain are given as

$$\|\mathbf{u}\|_\ell^2 = \sum_{K \in \Omega_h} \|\mathbf{u}\|_\ell^2, \quad \|\mathbf{u}\|_{\Gamma_h}^2 = \sum_{K \in \Omega_h} \|\mathbf{u}\|_{\Gamma_K}^2, \quad \|\mathbf{u}\|_\infty = \max_{K \in \Omega_h} \|\mathbf{u}\|_{\infty,K}.$$

3.3 LDG scheme and the main theorem

To construct the LDG scheme, we introduce some auxiliary variables to approximate the derivatives of the solution which further yields a first order system:

$$\phi\frac{\partial c}{\partial t}+B(c)\frac{\partial p}{\partial t}+\nabla\cdot(\mathbf{u}c)+\nabla\cdot\mathbf{z}=\tilde{c}q, \tag{3.2}$$

$$\mathbf{s}=-\nabla c, \tag{3.3}$$

$$\mathbf{z}=\mathbf{D}\mathbf{s}, \tag{3.4}$$

$$A(c)\mathbf{u}+\nabla p=0, \tag{3.5}$$

$$d(c)\frac{\partial p}{\partial t}+\nabla\cdot\mathbf{u}=q, \tag{3.6}$$

where $A(c)=\mu(c)\kappa(x,y)^{-1}, B(c)=cd(c)+b(c)=c\phi(x,y)m_1$. We multiply (3.2)-(3.6) by test functions $v,\zeta\in W_h^k$, $\boldsymbol{\theta},\mathbf{w},\boldsymbol{\psi}\in\mathbf{W}_h^k$, respectively. Formally integrate by parts in K to get

$$\begin{aligned}
(\phi c_t,v)_K+(B(c)p_t,v)_K &= (\mathbf{u}c+\mathbf{z},\nabla v)_K-\langle(\mathbf{u}c+\mathbf{z})\cdot\boldsymbol{\nu}_K,v\rangle_{\partial K}+(\tilde{c}q,v)_K,\\
(\mathbf{s},\mathbf{w})_K &= (c,\nabla\cdot\mathbf{w})_K-\langle c,\mathbf{w}\cdot\boldsymbol{\nu}_K\rangle_{\partial K},\\
(\mathbf{z},\boldsymbol{\psi})_K &= (\mathbf{D}\mathbf{s},\boldsymbol{\psi})_K,\\
(A(c)\mathbf{u},\boldsymbol{\theta})_K &= (p,\nabla\cdot\boldsymbol{\theta})_K-\langle p,\boldsymbol{\theta}\cdot\boldsymbol{\nu}_K\rangle_{\partial K},\\
(d(c)p_t,\zeta)_K &= (\mathbf{u},\nabla\zeta)_K-\langle\mathbf{u}\cdot\boldsymbol{\nu}_K,\zeta\rangle_{\partial K}+(q,\zeta)_K,
\end{aligned}$$

where $(u,v)_K=\int_K uvdxdy$, $(\mathbf{u},\mathbf{v})_K=\int_K\mathbf{u}\cdot\mathbf{v}dxdy$, $\langle u,v\rangle_{\partial K}=\int_{\partial K}uvds$ and $\boldsymbol{\nu}_K$ is the outer normal of K. Replacing the exact solutions $c,\ p,\ \mathbf{s},\ \mathbf{z},\ \mathbf{u}$ in the above equations by their numerical approximations $c_h,\ p_h\in W_h^k$ and $\mathbf{s}_h,\ \mathbf{z}_h,\ \mathbf{u}_h\in\mathbf{W}_h^k$, respectively and using numerical fluxes at the cell interfaces to obtain the LDG scheme:

$$(\phi c_{ht},v)_K+(B(c_h)p_{ht},v)_K = \mathcal{L}_K^c(\mathbf{u}_h,c_h,v)+\mathcal{L}_K^d(\mathbf{z}_h,v)+(\tilde{c_h}q,v)_K, \tag{3.7}$$

$$(\mathbf{s}_h,\mathbf{w})_K = \mathcal{D}_K(c_h,\mathbf{w}), \tag{3.8}$$

$$(\mathbf{z}_h,\boldsymbol{\psi})_K = (\mathbf{D}\mathbf{s}_h,\boldsymbol{\psi})_K, \tag{3.9}$$

$$(A(c_h)\mathbf{u}_h,\boldsymbol{\theta})_K = \mathcal{D}_K(p_h,\boldsymbol{\theta}), \tag{3.10}$$

$$(d(c_h)p_{ht},\zeta)_K = \mathcal{L}_K^d(\mathbf{u}_h,\zeta)+(q,\zeta)_K, \tag{3.11}$$

where

$$\mathcal{L}_K^c(\mathbf{s},c,v) = (\mathbf{s}c,\nabla v)_K-\langle\widehat{\mathbf{s}c}\cdot\boldsymbol{\nu}_K,v\rangle_{\partial K}, \tag{3.12}$$

$$\mathcal{L}_K^d(\mathbf{s},v) = (\mathbf{s},\nabla v)_K-\langle\widehat{\mathbf{s}_h}\cdot\boldsymbol{\nu}_K,v\rangle_{\partial K}, \tag{3.13}$$

$$\mathcal{D}_K(c,\mathbf{w}) = (c,\nabla\cdot\mathbf{w})_K-\langle\widehat{c},\mathbf{w}\cdot\boldsymbol{\nu}_K\rangle_{\partial K}. \tag{3.14}$$

We use alternating fluxes for the diffusion term and take

$$\widehat{\mathbf{z}_h} = \mathbf{z}_h^-, \quad \widehat{c_h} = c_h^+, \quad \widehat{\mathbf{u}_h} = \mathbf{u}_h^-, \quad \widehat{p_h} = p_h^+.$$

For the convection term, we consider Lax-Friedrichs flux

$$\widehat{\mathbf{u}_h c_h} = \frac{1}{2}(\mathbf{u}_h^+ c_h^+ + \mathbf{u}_h^- c_h^- - \alpha \boldsymbol{\nu}_e (c_h^+ - c_h^-)),$$

where $\alpha > 0$ can be chosen as any constant and $\boldsymbol{\nu}_e$ is the unit normal of the $e \in \Gamma_0$ such that $\boldsymbol{\beta} \cdot \boldsymbol{\nu}_e > 0$. Moreover, we define

$$(u, v) = \sum_{K \in \Omega_h} (u, v)_K, \qquad (\mathbf{u}, \mathbf{v}) = \sum_{K \in \Omega_h} (\mathbf{u}, \mathbf{v})_K,$$

and

$$\mathcal{L}^c(\mathbf{s}, c, v) = \sum_{K \in \Omega_h} \mathcal{L}_K^c(\mathbf{s}, c, v), \qquad \mathcal{L}^d(\mathbf{s}, v) = \sum_{K \in \Omega_h} \mathcal{L}_K^d(\mathbf{s}, v), \qquad \mathcal{D}(c, \mathbf{w}) = \sum_{K \in \Omega_h} \mathcal{D}_K(c, \mathbf{w}).$$

It is easy to check the following identities by integration by parts on each cell

Lemma 3.1 *For any functions v and $\mathbf{w}$,*

$$\mathcal{L}^d(\mathbf{w}, v) + \mathcal{D}(v, \mathbf{w}) = 0. \tag{3.15}$$

Now we state the main theorem.

Theorem 3.1 *Let $c \in H^{k+3}, \boldsymbol{s} \in (H^{k+2})^2, \boldsymbol{u} \in (H^{k+1})^2$ be the exact solutions of the problem* (3.2)-(3.6), *and let $\mathbf{u}_h, p_h, c_h, \mathbf{s}_h, \mathbf{z}_h$ be the numerical solutions of the semi-discrete LDG scheme* (3.7)-(3.11) *with initial discretization given as* (4.4). *If the finite element space is the piecewise tensor product polynomials of degree $k \geq 1$ and h is sufficiently small, then we have the error estimate*

$$\begin{aligned} &\|c - c_h\|_{L^\infty(0,T;L^2)} + \|\mathbf{s} - \mathbf{s}_h\|_{L^\infty(0,T;L^2)} \\ &+ \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(0,T;L^2)} + \|p - p_h\|_{L^\infty(0,T;L^2)} + \|(p - p_h)_t\|_{L^\infty(0,T;L^2)} \\ &+ \|(c - c_h)_t\|_{L^2(0,T;L^2)} + \|(\mathbf{u} - \mathbf{u}_h)_t\|_{L^2(0,T;L^2)} \leq C h^{k+1}, \end{aligned} \tag{3.16}$$

where the constant C is independent of h.

4 The proof of the main theorem

In this section, we proceed to the proof of Theorem 3.1. We first introduce several projections and present some auxiliary results. Subsequently, we make an a priori error estimate which provides the boundedness of the numerical approximations. Then we construct the error equations which further yield five main energy inequalities and complete the proof of (3.16). Finally, we verify the a priori error estimate at the end of this section.

4.1 Projections and interpolation properties

In this section, we will demonstrate the projections and several useful lemmas. Let us start with the classical inverse properties [5].

Lemma 4.1 *Assuming $u \in W_h^k$, there exists a positive constant C independent of h and u such that*

$$h\|u\|_{\infty,K} + h^{1/2}\|u\|_{\Gamma_K} \leq C\|u\|_K.$$

We will use several special projections in this paper. Firstly, we define P^+ into W_h^k which is, for each cell K

$$(P^+u - u, v)_K = 0, \forall v \in Q^{k-1}(K), \int_{J_j} (P^+u - u)(x_{i-\frac{1}{2}}, y)v(y)dy = 0, \forall v \in P^{k-1}(J_j),$$
$$(P^+u - u)(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}) = 0, \int_{I_i} (P^+u - u)(x, y_{j-\frac{1}{2}})v(x)dx = 0, \forall v \in P^{k-1}(I_i),$$

where P^k denotes the polynomials of degree k. Moreover, we also define Π_x^- and Π_y^- into W_h^k which are, for each cell K,

$$(\Pi_x^- u - u, v_x)_K = 0, \forall v \in Q^k(K), \int_{J_j} (\Pi_x^- u - u)(x_{i+\frac{1}{2}}, y)v(y)dy = 0, \forall v \in P^k(J_j),$$
$$(\Pi_y^- u - u, v_y)_K = 0, \forall v \in Q^k(K), \int_{I_i} (\Pi_y^- u - u)(x, y_{j+\frac{1}{2}})v(x)dx = 0, \forall v \in P^k(I_i),$$

as well as a two-dimensional projection $\mathbf{\Pi}^- = \Pi_x^- \otimes \Pi_y^-$. Finally, we also use the L^2-projection P_k into W_h^k which is, for each cell K

$$(P_k u - u, v)_K = 0, \forall v \in Q^k(K), \tag{4.1}$$

and its two dimensional version $\mathbf{P}_k = P_k \otimes P_k$. For the special projections mentioned above, we give the following lemma by the standard approximation theory [5].

Lemma 4.2 *Suppose $w \in H^{k+1}(\Omega)$, then for any project P_h, which is either P^+, Π_x^-, Π_y^- or P_k, we have*

$$\|w - P_h w\| + h^{1/2}\|w - P_h w\|_{\Gamma_h} \leq Ch^{k+1}.$$

Moreover, the projection P^+ on the Cartesian meshes has the following superconvergence property [4].

Lemma 4.3 *Suppose $w \in H^{k+2}(\Omega)$, then for any $\boldsymbol{\rho} \in \mathbf{W}_h$ we have*

$$|\mathcal{D}(w - P^+w, \boldsymbol{\rho})| \leq Ch^{k+1}\|w\|_{k+2}\|\boldsymbol{\rho}\|. \tag{4.2}$$

In this paper, we use e to denote the error between the exact and numerical solutions, i.e. $e_c = c - c_h,\ e_p = p - p_h,\ \mathbf{e}_u = \mathbf{u} - \mathbf{u}_h,\ \mathbf{e}_s = \mathbf{s} - \mathbf{s}_h,\ \mathbf{e}_z = \mathbf{z} - \mathbf{z}_h$. As the general treatment of the finite element methods, we split the errors into two terms as

$$\begin{array}{lll} e_c = \eta_c - \xi_c, & \eta_c = c - P^+c, & \xi_c = c_h - P^+c, \\ e_p = \eta_p - \xi_p, & \eta_p = p - P^+p, & \xi_p = p_h - P^+p, \\ \mathbf{e}_u = \boldsymbol{\eta}_u - \boldsymbol{\xi}_u, & \boldsymbol{\eta}_u = \mathbf{u} - \boldsymbol{\Pi}^-\mathbf{u}, & \boldsymbol{\xi}_u = \mathbf{u}_h - \boldsymbol{\Pi}^-\mathbf{u}, \\ \mathbf{e}_s = \boldsymbol{\eta}_s - \boldsymbol{\xi}_s, & \boldsymbol{\eta}_s = \mathbf{s} - \mathbf{P}_k\mathbf{s}, & \boldsymbol{\xi}_s = \mathbf{s}_h - \mathbf{P}_k\mathbf{s}, \\ \mathbf{e}_z = \boldsymbol{\eta}_z - \boldsymbol{\xi}_z, & \boldsymbol{\eta}_z = \mathbf{z} - \boldsymbol{\Pi}^-\mathbf{z}, & \boldsymbol{\xi}_z = \mathbf{z}_h - \boldsymbol{\Pi}^-\mathbf{z}. \end{array}$$

Based on the above, it is easy to see that

$$\mathcal{L}^d(\boldsymbol{\eta}_u, v) = \mathcal{L}^d(\boldsymbol{\eta}_z, v) = 0. \tag{4.3}$$

Following [25, 26, 27, 31] with some minor changes, we have the following lemma

Lemma 4.4 *Suppose ξ_c and $\boldsymbol{\xi}_s$ are defined above, we have*

$$\|\nabla\xi_c\| \le C(\|\boldsymbol{\xi}_s\| + h^{k+1}), \qquad h^{-\frac{1}{2}}\|[\xi_c]\|_{\Gamma_h} \le C(\|\boldsymbol{\xi}_s\| + h^{k+1}).$$

The proof of the main error estimate requires the following initial discretization, whose detailed construction will be given in the appendix.

Lemma 4.5 *We choose the initial solution*

$$c_h^0 = P^+c_0, \quad \mathbf{u}_h^0 = \boldsymbol{\Pi}^-\mathbf{u}_0, \tag{4.4}$$

where $\mathbf{u}_0 = -a(c_0)\nabla p_0$, Then we have

$$\|c(x,0) - c_h(x,0)\| \le Ch^{k+1}, \tag{4.5}$$

$$\|\mathbf{u}(x,0) - \mathbf{u}_h(x,0)\| \le Ch^{k+1}, \tag{4.6}$$

$$\|\mathbf{s}(x,0) - \mathbf{s}_h(x,0)\| \le Ch^{k+1}, \tag{4.7}$$

$$\|p_t(x,0) - p_{ht}(x,0)\| \le Ch^{k+1}, \tag{4.8}$$

$$\|p(x,0) - p_h(x,0)\| \le Ch^{k+1}. \tag{4.9}$$

The proof of this lemma will also be given in the appendix.

4.2 A priori error estimates

In this subsection, we would like to make an a priori error estimate assumption that

$$\|c - c_h\| + \|\mathbf{u} - \mathbf{u}_h\| + \|p_t - p_{ht}\| \le h, \tag{4.10}$$

which further implies

$$\|c_h\|_\infty + \|\mathbf{u}_h\|_\infty + \|p_{ht}\|_\infty \le C \tag{4.11}$$

by hypothesis 4.

4.3 Error equations

In this section, we proceed to construct the error equations. From (3.7)-(3.11), we have the following error equations

$$
\begin{aligned}
(B(c)p_t - B(c_h)p_{ht} + \phi e_{ct}, v) &= \mathcal{L}^c(\mathbf{u}, c, v) - \mathcal{L}^c(\mathbf{u}_h, c_h, v) + \mathcal{L}^d(\mathbf{e}_z, v) + (\tilde{e_c} q, v) && (4.12)\\
(\mathbf{e}_s, \mathbf{w}) &= \mathcal{D}(e_c, \mathbf{w}), && (4.13)\\
(\mathbf{e}_z, \boldsymbol{\psi}) &= (\mathbf{D}(\mathbf{s} - \mathbf{s}_h), \boldsymbol{\psi}), && (4.14)\\
((A(c)\mathbf{u} - A(c_h)\mathbf{u}_h), \boldsymbol{\theta}) &= \mathcal{D}(e_p, \boldsymbol{\theta}), && (4.15)\\
(d(c)p_t - d(c_h)p_{ht}, \zeta) &= \mathcal{L}^d(\mathbf{e}_u, \zeta), && (4.16)
\end{aligned}
$$

$\forall v, \zeta \in W_h^k, \mathbf{w}, \boldsymbol{\psi}, \boldsymbol{\theta} \in \mathbf{W}_h^k$, where

$$
\tilde{e_c} = \begin{cases} 0, & q > 0, \\ e_c, & q < 0. \end{cases}
$$

4.4 The first energy inequality

Taking the test functions $v = \xi_c$, $\mathbf{w} = \boldsymbol{\xi}_z$, and $\boldsymbol{\psi} = -\boldsymbol{\xi}_s$ in (4.12), (4.13) and (4.14), respectively, and use Lemma 3.1 and (4.3) to obtain

$$
(\phi \frac{\partial \xi_c}{\partial t}, \xi_c) + (\mathbf{D}\boldsymbol{\xi}_s, \boldsymbol{\xi}_s) = R_1 + R_2 - R_3 - R_4 + R_5, \tag{4.17}
$$

where

$$
\begin{aligned}
R_1 &= (\phi \frac{\partial \eta_c}{\partial t}, \xi_c) + (\mathbf{D}\boldsymbol{\eta}_s, \boldsymbol{\xi}_s)\,,\\
R_2 &= (B(c)p_t - B(c_h)p_{ht}, \xi_c),\\
R_3 &= (\mathbf{u}c - \mathbf{u}_h c_h, \nabla \xi_c) + \sum_{e \in \Gamma_e} \langle (\mathbf{u}c - \widehat{\mathbf{u}_h c_h}) \cdot \boldsymbol{\nu}_e, [\xi_c] \rangle_e,\\
R_4 &= \mathcal{D}(\eta_c, \boldsymbol{\xi}_z),\\
R_5 &= (\boldsymbol{\eta}_s, \boldsymbol{\xi}_z) - (\boldsymbol{\eta}_z, \boldsymbol{\xi}_s) - (\tilde{e_c} q, \xi_c),
\end{aligned}
$$

with $\Gamma_e = \Gamma_0 \cup \partial\Omega_-$ and $\langle u, v \rangle_e = \int_e uv ds$. Now, we estimate $R_i' s$ term by term. Using hypotheses 1 and 3, Lemma 4.2 and the Schwarz inequality, we can get

$$
R_1 \leq C\|\eta_{ct}\|\|\xi_c\| + C\|\boldsymbol{\eta}_s\|\|\boldsymbol{\xi}_s\| \leq Ch^{k+1}\left(\|\xi_c\| + \|\boldsymbol{\xi}_s\|\right), \tag{4.18}
$$

For R_2, we have

$$
\begin{aligned}
R_2 &= \Big[\Big(B(c)(p - p_h)_t, \xi_c\Big) + \Big((B(c) - B(c_h))p_{ht}, \xi_c\Big)\Big]\\
&\leq C\|(p - p_h)_t\|\|\xi_c\| + C\|c - c_h\|\|\xi_c\|\\
&\leq C\|\xi_c\|\|\xi_{p_t}\| + \|\xi_c\| + h^{k+1}), && (4.19)
\end{aligned}
$$

where in the second step we use Schwarz inequality and hypothesis 1 and (4.11), and the last step requires Lemma 4.2. We estimate R_3 by dividing it into three parts

$$R_3 = R_{31} + R_{32} - R_{33}, \tag{4.20}$$

where

$$\begin{aligned} R_{31} &= (\mathbf{u}c - \mathbf{u}c_h, \nabla\xi_c) + (\mathbf{u}c_h - \mathbf{u}_h c_h, \nabla\xi_c), &(4.21)\\ R_{32} &= \frac{1}{2}\sum_{e\in\Gamma_e} \langle (2\mathbf{u}c - \mathbf{u}_h^+ c_h^+ - \mathbf{u}_h^- c_h^-)\cdot \boldsymbol{\nu}_e, [\xi_c]\rangle_e, &(4.22)\\ R_{33} &= \frac{1}{2}\sum_{e\in\Gamma_e} \langle \alpha[\eta_c - \xi_c], [\xi_c]\rangle_e. &(4.23)\end{aligned}$$

Using hypothesis 4 and (4.11), we have

$$\begin{aligned} R_{31} &\le C\left(\|c - c_h\| + \|\mathbf{u} - \mathbf{u}_h\|\right)\|\nabla\xi_c\| \\ &\le C\left(h^{k+1} + \|\boldsymbol{\xi}_u\| + \|\xi_c\|\right)\left(\|\boldsymbol{\xi}_s\| + h^{k+1}\right), \end{aligned} \tag{4.24}$$

where in the first step, we use Schwarz inequality while the second step follows from Lemmas 4.2 and 4.4. C depends on $\|u\|_\infty$ and $\|c_h\|_\infty$. The estimate of R_{32} also requires hypothesis 4 and (4.11),

$$\begin{aligned} R_{32} &= \frac{1}{2}\sum_{e\in\Gamma_e} \langle \left(\mathbf{u}(c - c_h^+) + (\mathbf{u} - \mathbf{u}_h^+)c_h^+ + \mathbf{u}(c - c_h^-) + (\mathbf{u} - \mathbf{u}_h^-)c_h^-\right)\cdot\boldsymbol{\nu}_e, [\xi_c]\rangle_e \\ &\le C\left(\|c - c_h\|_{\Gamma_h} + \|\mathbf{u} - \mathbf{u}_h\|_{\Gamma_h}\right)\|[\xi_c]\|_{\Gamma_h} \\ &\le Ch^{\frac{1}{2}}(\|\eta_c\|_{\Gamma_h} + \|\xi_c\|_{\Gamma_h} + \|\boldsymbol{\eta}_u\|_{\Gamma_h} + \|\boldsymbol{\xi}_u\|_{\Gamma_h})(\|\boldsymbol{\xi}_s\| + h^{k+1}) \\ &\le C\left(h^{k+1} + \|\boldsymbol{\xi}_u\| + \|\xi_c\|\right)\left(\|\boldsymbol{\xi}_s\| + h^{k+1}\right), \end{aligned} \tag{4.25}$$

where in the second step we use Schwarz inequality, the third step follows from Lemma 4.4, the last one requires Lemmas 4.1 and 4.2. C depends on $\|\mathbf{u}\|_\infty$ and $\|c_h\|_\infty$. Now we proceed to the estimate of R_{33},

$$\begin{aligned} R_{33} &\le C(\|\eta_c\|_{\Gamma_h} + \|\xi_c\|_{\Gamma_h})\|[\xi_c]\|_{\Gamma_h} \\ &\le Ch^{\frac{1}{2}}(\|\eta_c\|_{\Gamma_h} + \|\xi_c\|_{\Gamma_h})(\|\boldsymbol{\xi}_s\| + h^{k+1}) \\ &\le C\left(h^{k+1} + \|\xi_c\|\right)\left(\|\boldsymbol{\xi}_s\| + h^{k+1}\right), \end{aligned} \tag{4.26}$$

where the first step follows from Schwarz inequality, the second step is based on Lemma 4.4, the third one requires Lemma 4.2. Plug (4.24), (4.25) and (4.26) into (4.20) to obtain

$$R_3 \le C\left(h^{k+1} + \|\boldsymbol{\xi}_u\| + \|\xi_c\|\right)\left(\|\boldsymbol{\xi}_s\| + h^{k+1}\right). \tag{4.27}$$

The estimate of R_4 follows from Lemma 4.3

$$R_4 \leq Ch^{k+1}\|c\|_{k+2}\|\boldsymbol{\xi}_z\|. \tag{4.28}$$

Now we begin to deal with R_5. Using Lemma 4.2 and the Schwartz inequality, we easily obtain

$$\begin{aligned} R_5 &\leq \|\boldsymbol{\eta}_s\|\|\boldsymbol{\xi}_z\| + \|\boldsymbol{\eta}_z\|\|\boldsymbol{\xi}_s\| + C\|\tilde{e_c}\|\|\xi_c\| \\ &\leq Ch^{k+1}\left(\|\boldsymbol{\xi}_z\| + \|\boldsymbol{\xi}_s\|\right) + Ch^{k+1}\|\xi_c\| + C\|\xi_c\|^2. \end{aligned} \tag{4.29}$$

Substituting the estimation (4.18), (4.19), (4.27), (4.28), (4.29) into (4.17) and use hypothesis 3, we obtain

$$\begin{aligned} \frac{d}{dt}\|\phi^{\frac{1}{2}}\xi_c\|^2 + \|\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\xi}_s\|^2 \leq C \Big[&\left(h^{k+1} + \|\boldsymbol{\xi}_u\| + \|\xi_c\|\right)\left(\|\boldsymbol{\xi}_s\| + h^{k+1}\right) \\ &+ h^{k+1}\|\boldsymbol{\xi}_z\| + h^{2(k+1)} + \|\xi_c\|^2 + \|\xi_{p_t}\|^2 \Big]. \end{aligned} \tag{4.30}$$

Integrating with the equation with respect to time between 0 and t, we obtain

$$\begin{aligned} &\|\xi_c\|^2 + \int_0^t \|\boldsymbol{\xi}_s\|^2 dt \\ \leq\, &C\int_0^t (\|\xi_c\|^2 + \|\boldsymbol{\xi}_u\|^2 + \|\xi_{p_t}\|^2 + \|\boldsymbol{\xi}_z\|^2 + \|\boldsymbol{\xi}_s\|^2)dt + Ch^{2(k+1)}. \end{aligned} \tag{4.31}$$

We take the time derivative in equation (4.13), we have

$$(\mathbf{e}_{st}, \mathbf{w}) = \mathcal{D}(e_{ct}, \mathbf{w}), \tag{4.32}$$

Taking the test functions $v = \xi_{ct}$, $\mathbf{w} = \boldsymbol{\xi}_z$, and $\boldsymbol{\psi} = -\xi_{\mathbf{s}t}$ in (4.12), (4.32) and (4.14), respectively, and use (3.15) and (4.3) to obtain

$$(\phi\xi_{ct}, \xi_{ct}) + \frac{1}{2}\frac{d}{dt}(\mathbf{D}\boldsymbol{\xi}_s, \boldsymbol{\xi}_s) = \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 + \tilde{R}_4 + \tilde{R}_5 + \tilde{R}_6, \tag{4.33}$$

where

$$\begin{aligned} \tilde{R}_1 &= (\phi\eta_{ct}, \xi_{ct}), \\ \tilde{R}_2 &= (\mathbf{D}\boldsymbol{\eta}_s, \boldsymbol{\xi}_{st}), \\ \tilde{R}_3 &= (B(c)p_t - B(c_h)p_{ht}, \xi_{ct}), \\ \tilde{R}_4 &= -(\mathbf{u}c - \mathbf{u}_h c_h, \nabla\xi_{ct}) - \sum_{e\in\Gamma_e}\langle(\mathbf{u}c - \widehat{\mathbf{u}_h c_h})\cdot\boldsymbol{\nu}_e, [\xi_{ct}]\rangle_e, \\ \tilde{R}_5 &= -\mathcal{D}(\eta_{ct}, \boldsymbol{\xi}_z), \\ \tilde{R}_6 &= (\boldsymbol{\eta}_{st}, \boldsymbol{\xi}_z) - (\boldsymbol{\eta}_z, \boldsymbol{\xi}_{st}) - (\tilde{e}_c q, \xi_{ct}), \end{aligned}$$

Now, we estimate $\tilde{R}_i^{\prime}s$ term by term. Using the projection and the Schwartz inequality, we can get

$$\tilde{R}_1 \le C\|\eta_{ct}\|^2 + C\|\xi_{ct}\|^2 \le Ch^{2(k+1)} + \epsilon\|\xi_{ct}\|^2, \tag{4.34}$$

$$\begin{aligned}\tilde{R}_2 &= \frac{d}{dt}(\mathbf{D}\boldsymbol{\eta}_s, \boldsymbol{\xi}_s) - (\mathbf{D}\boldsymbol{\eta}_{st}, \boldsymbol{\xi}_s)\\ &\le \frac{d}{dt}(\mathbf{D}\boldsymbol{\eta}_s, \boldsymbol{\xi}_s) + C\|\boldsymbol{\xi}_s\|^2 + Ch^{2(k+1)},\end{aligned} \tag{4.35}$$

$$\begin{aligned}\tilde{R}_3 &= \Big[\Big(B(c)(p-p_h)_t, \xi_{ct}\Big) + \Big((B(c)-B(c_h))p_{ht}, \xi_{ct}\Big)\Big]\\ &\le C\|(p-p_h)_t\|\|\xi_{ct}\| + C\|c-c_h\|\|\xi_{ct}\|\\ &\le C\|\xi_{p_t}\|^2 + C\|\xi_c\|^2 + \epsilon\|\xi_{ct}\|^2 + Ch^{2(k+1)},\end{aligned} \tag{4.36}$$

where in the second step we use Schwarz inequality and hypothesis 1, and the last step requires Lemma 4.2. We estimate R_4 by dividing it into three parts

$$\tilde{R}_4 = \tilde{R}_{41} + \tilde{R}_{42} + \tilde{R}_{43}, \tag{4.37}$$

where

$$\begin{aligned}\tilde{R}_{41} &= -(\mathbf{u}c - \mathbf{u}_h c_h, \nabla\xi_{ct}),\\ R_{42} &= -\frac{1}{2}\sum_{e\in\Gamma_e}\langle(2\mathbf{u}c - \mathbf{u}_h^+ c_h^+ - \mathbf{u}_h^- c_h^-)\cdot\boldsymbol{\nu}_e, [\xi_{ct}]\rangle_e,\\ R_{43} &= \frac{1}{2}\sum_{e\in\Gamma_e}\langle\alpha[\eta_c - \xi_c], [\xi_{ct}]\rangle_e.\end{aligned}$$

Using hypothesis 4 and (4.11), we have

$$\begin{aligned}\tilde{R}_{41} &= \frac{d}{dt}\Big(\mathbf{u}_h c_h - \mathbf{u}c, \nabla\xi_c\Big) + \Big((\mathbf{u}c - \mathbf{u}_h c_h)_t, \nabla\xi_c\Big)\\ &= \frac{d}{dt}\Big(\mathbf{u}_h c_h - \mathbf{u}c, \nabla\xi_c\Big) + \Big(\mathbf{u}_t c - \mathbf{u}_{ht} c_h, \nabla\xi_c\Big) + \Big(\mathbf{u}c_t - \mathbf{u}_h c_{ht}, \nabla\xi_c\Big)\\ &= \frac{d}{dt}\Big(\mathbf{u}_h c_h - \mathbf{u}c, \nabla\xi_c\Big) + \Big(\mathbf{u}_t(c-c_h), \nabla\xi_c\Big) + \Big((\mathbf{u}-\mathbf{u}_h)_t c_h, \nabla\xi_c\Big)\\ &\quad + (c_t(\mathbf{u}-\mathbf{u}_h), \nabla\xi_c) + ((c-c_h)_t\mathbf{u}_h, \nabla\xi_c)\\ &\le \frac{d}{dt}\Big(\mathbf{u}_h c_h - \mathbf{u}c, \nabla\xi_c\Big) + C\|c-c_h\|^2 + \epsilon\|(\mathbf{u}-\mathbf{u}_h)_t\|^2\\ &\quad + C\|\mathbf{u}-\mathbf{u}_h\|^2 + \epsilon\|(c-c_h)_t\|^2 + C\|\nabla\xi_c\|^2\\ &\le \frac{d}{dt}\Big(\mathbf{u}_h c_h - \mathbf{u}c, \nabla\xi_c\Big) + Ch^{2(k+1)} + C\|\xi_c\|^2 + \epsilon\|\boldsymbol{\xi}_{ut}\|^2\\ &\quad + C\|\boldsymbol{\xi}_u\|^2 + \epsilon\|\xi_{ct}\|^2 + C\|\boldsymbol{\xi}_s\|^2,\end{aligned} \tag{4.38}$$

where in the forth step, we use Schwarz inequality while the last step follows from Lemmas 4.2 and 4.4. The estimate of $\tilde{R}_{42}$ also requires hypothesis 4 and (4.11),

$$
\begin{aligned}
\tilde{R}_{42} =& -\frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle(2\mathbf{u}c-\mathbf{u}_h^+c_h^+-\mathbf{u}_h^-c_h^-)\cdot\boldsymbol{\nu}_e,[\xi_c]\rangle_e \\
&+\sum_{e\in\Gamma_e}\langle(\frac{\mathbf{u}^+c^++\mathbf{u}^-c^-}{2}-\frac{\mathbf{u}_h^+c_h^++\mathbf{u}_h^-c_h^-}{2})_t\cdot\boldsymbol{\nu}_e,[\xi_c]\rangle_e \\
\le& \frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle(\mathbf{u}_h^+c_h^++\mathbf{u}_h^-c_h^--2\mathbf{u}c)\cdot\boldsymbol{\nu}_e,[\xi_c]\rangle_e+C\|(\mathbf{u}c-\mathbf{u}_hc_h)_t\|_{\Gamma_h}\|[\xi_c]\|_{\Gamma_h} \\
\le& \frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle(\mathbf{u}_h^+c_h^++\mathbf{u}_h^-c_h^--2\mathbf{u}c)\cdot\boldsymbol{\nu}_e,[\xi_c]\rangle_e \\
&+Ch^{\frac{1}{2}}(\|c_t(\mathbf{u}-\mathbf{u}_h)\|_{\Gamma_h}+\|(c-c_h)_t\mathbf{u}_h\|_{\Gamma_h})(\|\xi_{\mathbf{s}}\|+h^{k+1}) \\
&+Ch^{\frac{1}{2}}(\|\mathbf{u}_t(c-c_h)\|_{\Gamma_h}+\|(\mathbf{u}-\mathbf{u}_h)_tc_h\|_{\Gamma_h})(\|\xi_{\mathbf{s}}\|+h^{k+1}) \\
\le& \frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle(\mathbf{u}_h^+c_h^++\mathbf{u}_h^-c_h^--2\mathbf{u}c)\cdot\boldsymbol{\nu}_e,[\xi_c]\rangle_e \\
&+Ch^{2(k+1)}+C\|\boldsymbol{\xi}_u\|^2+\epsilon\|\xi_{ct}\|^2+C\|\xi_c\|^2+\epsilon\|\boldsymbol{\xi}_{ut}\|^2+C\|\boldsymbol{\xi}_s\|^2, \qquad (4.39)
\end{aligned}
$$

where in the second step we use Schwarz inequality, the third step follows from and Lemma 4.4, the last one requires Lemmas 4.1 and 4.2. Now we proceed to the estimate of $\widetilde{R}_{43}$,

$$
\begin{aligned}
\tilde{R}_{43} =& \frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle\alpha[\eta_c-\xi_c],[\xi_c]\rangle_e-\frac{1}{2}\sum_{e\in\Gamma_e}\langle\alpha[\eta_{ct}-\xi_{ct}],[\xi_c]\rangle_e \\
\le& \frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle\alpha[\eta_c-\xi_c],[\xi_c]\rangle_e+C(\|\eta_{ct}\|_{\Gamma_h}+\|\xi_{ct}\|_{\Gamma_h})\|[\xi_c]\|_{\Gamma_h} \\
\le& \frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle\alpha[\eta_c-\xi_c],[\xi_c]\rangle_e+Ch^{\frac{1}{2}}(\|\eta_{ct}\|_{\Gamma_h}+\|\xi_{ct}\|_{\Gamma_h})(\|\boldsymbol{\xi}_s\|+h^{k+1}) \\
\le& \frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle\alpha[\eta_c-\xi_c],[\xi_c]\rangle_e+Ch^{2(k+1)}+\epsilon\|\xi_{ct}\|^2+C\|\boldsymbol{\xi}_s\|^2, \qquad (4.40)
\end{aligned}
$$

where the second step follows from Schwarz inequality, the third one is based on Lemma 4.4, the last one requires Lemmas 4.1 and 4.2. Plug (4.38), (4.39) and (4.40) into (4.37) to obtain

$$
\begin{aligned}
\tilde{R}_4 \le& \frac{d}{dt}\Big(\mathbf{u}_hc_h-\mathbf{u}c,\nabla\xi_c\Big)+\frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle(\mathbf{u}_h^+c_h^++\mathbf{u}_h^-c_h^--2\mathbf{u}c)\cdot\boldsymbol{\nu}_e,[\xi_c]\rangle_e \\
&+\frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle\alpha[\eta_c-\xi_c],[\xi_c]\rangle_e+C(h^{2(k+1)}+\|\boldsymbol{\xi}_u\|^2+\|\xi_c\|^2+\|\boldsymbol{\xi}_s\|^2) \\
&+\epsilon(\|\xi_{ct}\|^2+\|\boldsymbol{\xi}_{ut}\|^2). \qquad (4.41)
\end{aligned}
$$

The estimate of $\tilde{R}_5$ follows from Lemma 4.3

$$\tilde{R}_5 \leq Ch^{k+1}\|c\|_{k+2}\|\boldsymbol{\xi}_z\|. \tag{4.42}$$

Now we begin to deal with $\tilde{R}_6$. Using Lemma 4.2 and the Schwartz inequality, we easily obtain

$$\begin{aligned}
\tilde{R}_6 &= (\boldsymbol{\eta}_{st}, \boldsymbol{\xi}_z) - \frac{d}{dt}(\boldsymbol{\eta}_z, \boldsymbol{\xi}_s) + (\boldsymbol{\eta}_{zt}, \boldsymbol{\xi}_s) - (\tilde{e_c}q, \xi_{ct}) \\
&\leq \|\boldsymbol{\eta}_{st}\|\|\boldsymbol{\xi}_z\| - \frac{d}{dt}(\boldsymbol{\eta}_z, \boldsymbol{\xi}_s) + \|\boldsymbol{\eta}_{zt}\|\|\boldsymbol{\xi}_s\| + C\|\tilde{e_c}\|\|\xi_{ct}\| \\
&\leq -\frac{d}{dt}(\boldsymbol{\eta}_z, \boldsymbol{\xi}_s) + C\left(h^{2(k+1)} + \|\boldsymbol{\xi}_z\|^2 + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2\right) + \epsilon\|\xi_{ct}\|^2. \qquad (4.43)
\end{aligned}$$

Substituting the estimation (4.34)-(4.36) and (4.41)-(4.43)into (4.33) and use hypothesis 3, we obtain

$$\begin{aligned}
&\|\phi^{\frac{1}{2}}\xi_{ct}\|^2 + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\xi}_s\|^2 \\
\leq\ & \frac{d}{dt}(\mathbf{D}\boldsymbol{\eta}_s, \boldsymbol{\xi}_s) - \frac{d}{dt}(\boldsymbol{\eta}_z, \boldsymbol{\xi}_s) + \frac{d}{dt}\Big(\mathbf{u}_h c_h - \mathbf{u}c, \nabla\xi_c\Big) \\
&+\frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle(\mathbf{u}_h^+ c_h^+ + \mathbf{u}_h^- c_h^- - 2\mathbf{u}c)\cdot\boldsymbol{\nu}_e, [\xi_c]\rangle_e + \frac{1}{2}\sum_{e\in\Gamma_e}\frac{d}{dt}\langle\alpha[\eta_c - \xi_c], [\xi_c]\rangle_e \\
&+C(h^{2(k+1)} + \|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 + \|\xi_{p_t}\|^2 + \|\boldsymbol{\xi}_z\|^2) \\
&+\epsilon(\|\xi_{ct}\|^2 + \|\boldsymbol{\xi}_{ut}\|^2). \qquad (4.44)
\end{aligned}$$

Noticing that

$$(\mathbf{D}\boldsymbol{\eta}_s, \boldsymbol{\xi}_s) - (\boldsymbol{\eta}_z, \boldsymbol{\xi}_s) \leq C\|\boldsymbol{\eta}_s\|^2 + \|\boldsymbol{\eta}_z\|^2 + \epsilon\|\boldsymbol{\xi}_s\|^2 \leq Ch^{2(k+1)} + \epsilon\|\boldsymbol{\xi}_s\|^2. \tag{4.45}$$

and

$$\begin{aligned}
\Big(\mathbf{u}_h c_h - \mathbf{u}c, \nabla\xi_c\Big) &= (c(\mathbf{u}_h - \mathbf{u}), \nabla\xi_c) + (\mathbf{u}_h(c_h - c), \nabla\xi_c) \\
&\leq C\|\mathbf{u} - \mathbf{u}_h\|^2 + C\|c - c_h\|^2 + C\|\nabla\xi_c\|^2 \\
&\leq Ch^{2(k+1)} + C\|\boldsymbol{\xi}_u\|^2 + C\|\boldsymbol{\xi}_c\|^2 + \epsilon\|\boldsymbol{\xi}_s\|^2, \qquad (4.46)
\end{aligned}$$

where the last one requires Lemmas 4.2 and 4.4.

$$\begin{aligned}
&\frac{1}{2}\sum_{e\in\Gamma_e}\langle(\mathbf{u}_h^+ c_h^+ + \mathbf{u}_h^- c_h^- - 2\mathbf{u}c)\cdot\boldsymbol{\nu}_e, [\xi_c]\rangle_e + \frac{1}{2}\sum_{e\in\Gamma_e}\langle\alpha[\eta_c - \xi_c], [\xi_c]\rangle_e \\
\leq\ & C(\|\mathbf{u}c - \mathbf{u}_h c_h\|_{\Gamma_h} + \|\eta_c\|_{\Gamma_h} + \|\xi_c\|_{\Gamma_h})\|[\xi_c]\|_{\Gamma_h} \\
\leq\ & Ch^{\frac{1}{2}}(\|\mathbf{u}c - \mathbf{u}_h c\|_{\Gamma_h} + \|\mathbf{u}_h c - \mathbf{u}_h c_h\|_{\Gamma_h} + \|\eta_c\|_{\Gamma_h} + \|\xi_c\|_{\Gamma_h})(\|\boldsymbol{\xi}_s\| + h^{k+1}) \\
\leq\ & Ch^{2(k+1)} + C\|\boldsymbol{\xi}_u\|^2 + C\|\xi_c\|^2 + \epsilon\|\boldsymbol{\xi}_s\|^2, \qquad (4.47)
\end{aligned}$$

where the second step follows from Schwarz inequality, the third one is based on Lemma 4.4, the last one requires Lemmas 4.1 and 4.2. Integrating (4.44) with respect to time between 0 and t, then applying (4.45)-(4.47), we obtain

$$\begin{aligned}\int_0^t \|\xi_{ct}\|^2 dt + \|\boldsymbol{\xi}_s\|^2 \le\; & C\int_0^t (\|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 + \|\xi_{p_t}\|^2 + \|\boldsymbol{\xi}_z\|^2)dt \\ & + \epsilon\int_0^t (\|\xi_{ct}\|^2 + \|\boldsymbol{\xi}_{ut}\|^2)dt + Ch^{2(k+1)} \\ & + C\|\boldsymbol{\xi}_u\|^2 + C\|\xi_c\|^2 + \epsilon\|\boldsymbol{\xi}_s\|^2. \end{aligned} \tag{4.48}$$

Combining (4.48) and (4.31), we obtain

$$\begin{aligned}\int_0^t \|\xi_{ct}\|^2 dt + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 \le\; & C\int_0^t (\|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 + \|\xi_{p_t}\|^2 + \|\boldsymbol{\xi}_z\|^2)dt \\ & + \epsilon\int_0^t (\|\xi_{ct}\|^2 + \|\boldsymbol{\xi}_{ut}\|^2)dt + Ch^{2(k+1)} + C\|\boldsymbol{\xi}_u\|^2 + \epsilon\|\boldsymbol{\xi}_s\|^2. \end{aligned}$$

which further yields

$$\begin{aligned}\int_0^t \|\xi_{ct}\|^2 dt + \|\boldsymbol{\xi}_{\mathbf{s}}\|^2 + \|\xi_c\|^2 \le\; & C\int_0^t (\|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 + \|\xi_{p_t}\|^2 + \|\boldsymbol{\xi}_z\|^2)dt \\ & + \epsilon\int_0^t \|\boldsymbol{\xi}_{ut}\|^2 dt + Ch^{2(k+1)} + C\|\boldsymbol{\xi}_u\|^2. \end{aligned} \tag{4.49}$$

Now, we proceed to eliminate $\|\boldsymbol{\xi}_z\|$ on the right-hand side to the above equation. Setting $\boldsymbol{\psi} = \boldsymbol{\xi}_z$ in (4.14) to obtain

$$(\boldsymbol{\xi}_z, \boldsymbol{\xi}_z) = (\boldsymbol{\eta}_z, \boldsymbol{\xi}_z) - (\mathbf{D}(\mathbf{s} - \mathbf{s}_h), \boldsymbol{\xi}_z).$$

Then we have

$$\|\boldsymbol{\xi}_z\|^2 \le \|\boldsymbol{\eta}_z\|\|\boldsymbol{\xi}_z\| + C\left(\|\boldsymbol{\eta}_s\| + \|\boldsymbol{\xi}_s\|\right)\|\boldsymbol{\xi}_z\| \le C(\|\boldsymbol{\xi}_s\|^2 + h^{2(k+1)}) + \epsilon\|\boldsymbol{\xi}_z\|^2,$$

where in the first step we use Schwarz inequality and hypothesis 3, the second step follows from Lemma 4.2. We can cancel $\|\boldsymbol{\xi}_z\|$ in the above equation to obtain

$$\|\boldsymbol{\xi}_z\|^2 \le C(\|\boldsymbol{\xi}_s\|^2 + h^{2(k+1)}). \tag{4.50}$$

Combining (4.49) and (4.50), we obtain the first energy Inequality

$$\begin{aligned}\int_0^t \|\xi_{ct}\|^2 dt + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 \le\; & C\int_0^t (\|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 + \|\xi_{p_t}\|^2)dt \\ & + \epsilon\int_0^t \|\boldsymbol{\xi}_{ut}\|^2 dt + Ch^{2(k+1)} + C\|\boldsymbol{\xi}_u\|^2. \end{aligned} \tag{4.51}$$

4.5 The second energy inequality

We start from an easier case. Take $\boldsymbol{\theta} = \boldsymbol{\xi}_u$ and $\zeta = \xi_p$ in (4.15) and (4.16), respectively and use Lemma 3.1 and (4.3) to obtain

$$(A(c)\boldsymbol{\xi}_u, \boldsymbol{\xi}_u) + \frac{1}{2}\frac{d}{dt}(d(c)\xi_p, \xi_p) = T_1 + T_2 + T_3 + T_4 + T_5 - T_6, \tag{4.52}$$

where

$$\begin{aligned} T_1 &= (A(c)\boldsymbol{\eta}_u, \boldsymbol{\xi}_u), \\ T_2 &= ((A(c) - A(c_h))\mathbf{u}_h, \boldsymbol{\xi}_u), \\ T_3 &= \frac{1}{2}(d(c)_t \xi_p, \xi_p), \\ T_4 &= (d(c)\eta_{p_t}, \xi_p), \\ T_5 &= ((d(c) - d(c_h))p_{h_t}, \xi_p), \\ T_6 &= \mathcal{D}(\eta_p, \boldsymbol{\xi}_u). \end{aligned}$$

Now, we estimate $T_i's$ term by term. Using Lemma 4.2 and Schwarz inequality, we can get

$$T_1 \le C\|\boldsymbol{\eta}_u\|^2 + \epsilon\|\boldsymbol{\xi}_u\|^2 \le Ch^{2(k+1)} + \epsilon\|\boldsymbol{\xi}_u\|^2, \tag{4.53}$$

where we use hypothesis 1 to obtain $|A(c)| = |\frac{\mu(c)}{\kappa(x,y)}| \le \frac{\mu^*}{\kappa_*}$. Using 4.11, we have

$$\begin{aligned} T_2 &\le C\|A(c) - A(c_h)\|^2 + \epsilon\|\boldsymbol{\xi}_u\|^2 \le C\|A_c^{'}(c - c_h)\|^2 + \epsilon\|\boldsymbol{\xi}_u\|^2 \\ &\le Ch^{2(k+1)} + C\|\xi_c\|^2 + \epsilon\|\boldsymbol{\xi}_u\|^2, \end{aligned} \tag{4.54}$$

where in the first step we use Schwarz inequality, the second step follows from hypothesis 1, and the last step requires Lemma 4.2. Moreover, $A_c^{'}$ is the mean value given by $A_c^{'} = A^{'}(\lambda_c c + (1 - \lambda_c)c_h)$ with $0 \le \lambda_c \le 1$.

$$T_3 = \frac{1}{2}(d^{'}(c)c_t\xi_p, \xi_p) \le C\|\xi_p\|^2, \tag{4.55}$$

where we use hypothesis 1.

$$T_4 \le C\|\eta_{p_t}\|^2 + C\|\xi_p\|^2 \le Ch^{2(k+1)} + C\|\xi_p\|^2, \tag{4.56}$$

$$\begin{aligned} T_5 &\le C\|d(c) - d(c_h)\|^2 + C\|\xi_p\|^2 \le C\|d_c^{'}(c - c_h)\|^2 + C\|\xi_p\|^2 \\ &\le Ch^{2(k+1)} + C\|\xi_c\|^2 + C\|\xi_p\|^2, \end{aligned} \tag{4.57}$$

where in the first step we use (4.11), the second step follows from hypothesis 1 with d_c' being the mean value given by $d_c' = d'(\lambda_c c + (1 - \lambda_c)c_h)$ with $0 \le \lambda_c \le 1$. For T_6, we use Lemma 4.3 and Schwarz inequality to obtain

$$T_6 \le Ch^{2(k+1)} + \epsilon\|\boldsymbol{\xi}_u\|^2. \tag{4.58}$$

Substituting (4.53)-(4.58) into (4.52), we have the estimate

$$\|A^{\frac{1}{2}}(c)\boldsymbol{\xi}_u\|^2 + \frac{1}{2}\frac{d}{dt}\|d^{\frac{1}{2}}(c)\xi_p\|^2 \le Ch^{2(k+1)} + C\|\xi_p\|^2 + C\|\xi_c\|^2 + \epsilon\|\boldsymbol{\xi}_u\|^2. \tag{4.59}$$

Integrating (4.59) with respect to time between 0 and t and using the hypothesis 1 , we obtain the second energy Inequality

$$\int_0^t \|\boldsymbol{\xi}_u\|^2 dt + \|\xi_p\|^2 \le Ch^{2(k+1)} + C\int_0^t (\|\xi_p\|^2 + \|\xi_c\|^2)dt. \tag{4.60}$$

4.6 The third energy inequality

We take the time derivative in equation (4.15), we have

$$((A(c)\mathbf{u} - A(c_h)\mathbf{u}_h)_t, \boldsymbol{\theta}) = \mathcal{D}(e_{p_t}, \boldsymbol{\theta}), \tag{4.61}$$

Take $\boldsymbol{\theta} = \boldsymbol{\xi}_u$ and $\zeta = \xi_{p_t}$ in (4.61) and (4.16), respectively and use (3.15) and (4.3) to obtain

$$\frac{1}{2}\frac{d}{dt}(A(c)\boldsymbol{\xi}_u, \boldsymbol{\xi}_u) + (d(c)\xi_{p_t}, \xi_{p_t}) = \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3 + \tilde{T}_4 + \tilde{T}_5 - \tilde{T}_6, \tag{4.62}$$

where

$$\begin{aligned}
\tilde{T}_1 &= -\frac{1}{2}((A(c))_t\boldsymbol{\xi}_u, \boldsymbol{\xi}_u), \\
\tilde{T}_2 &= ((A(c)\boldsymbol{\eta}_u)_t, \boldsymbol{\xi}_u), \\
\tilde{T}_3 &= (((A(c) - A(c_h))\mathbf{u}_h)_t, \boldsymbol{\xi}_u), \\
\tilde{T}_4 &= (d(c)\eta_{p_t}, \xi_{p_t}), \\
\tilde{T}_5 &= ((d(c) - d(c_h))p_{h t}, \xi_{p_t}), \\
\tilde{T}_6 &= \mathcal{D}(\eta_{p_t}, \boldsymbol{\xi}_u).
\end{aligned}$$

Now, we estimate $\tilde{T}_i's$ term by term. Using hypothesis 1 and Schwarz inequality, we can get

$$\tilde{T}_1 = -\frac{1}{2}(A^{'}(c)c_t\boldsymbol{\xi}_u, \boldsymbol{\xi}_u) \le C\|\boldsymbol{\xi}_u\|^2, \tag{4.63}$$

and

$$\begin{aligned}
\tilde{T}_2 &= (A'(c)c_t\boldsymbol{\eta}_u, \boldsymbol{\xi}_u) + (A(c)\boldsymbol{\eta}_{ut}, \boldsymbol{\xi}_u) \\
&\le C\|\boldsymbol{\xi}_u\|^2 + C\|\boldsymbol{\eta}_u\|^2 + C\|\boldsymbol{\eta}_{ut}\|^2 \\
&\le C\|\boldsymbol{\xi}_u\|^2 + Ch^{2(k+1)}.
\end{aligned} \tag{4.64}$$

The estimate of $\tilde{T}_3$ is slightly complicated,

$$\begin{aligned}
\tilde{T}_3 &= ((A(c)-A(c_h))_t \boldsymbol{u}_h, \boldsymbol{\xi}_u) - ((A(c)-A(c_h))(\mathbf{u}-\mathbf{u}_h)_t, \boldsymbol{\xi}_u) \\
&\quad +((A(c)-A(c_h))\boldsymbol{u}_t, \boldsymbol{\xi}_{\mathbf{u}}) \\
&= ((A'(c)-A'(c_h))c_t \mathbf{u}_h, \boldsymbol{\xi}_u) + (A'(c_h)(c-c_h)_t \mathbf{u}_h, \boldsymbol{\xi}_u) \\
&\quad -(A'_c(c-c_h)(\mathbf{u}-\mathbf{u}_h)_t, \boldsymbol{\xi}_u) + (A'_c(c-c_h)\mathbf{u}_t, \boldsymbol{\xi}_u) \\
&\leq C\|c-c_h\|\|\boldsymbol{\xi}_u\| + C\|(c-c_h)_t\|\|\boldsymbol{\xi}_u\| \\
&\quad +C\|\boldsymbol{\xi}_u\|_\infty \|c-c_h\|\|(\mathbf{u}-\mathbf{u}_h)_t\| + C\|c-c_h\|\|\boldsymbol{\xi}_u\| \\
&\leq C\|c-c_h\|^2 + C\|\boldsymbol{\xi}_u\|^2 + \epsilon\|(c-c_h)_t\|^2 + \epsilon\|(\mathbf{u}-\mathbf{u}_h)_t\|^2 \\
&\leq C\|\xi_c\|^2 + C\|\boldsymbol{\xi}_u\|^2 + \epsilon\|\xi_{ct}\|^2 + \epsilon\|\boldsymbol{\xi}_{ut}\|^2 + Ch^{2(k+1)},
\end{aligned} \tag{4.65}$$

where in the third step we use Schwarz inequality and hypotheses 1 and 2, and the last step requires Lemma 4.2. Applying the Schwarz inequality, we have

$$\tilde{T}_4 \leq C\|\eta_{p_t}\|^2 + \epsilon\|\xi_{p_t}\|^2 \leq Ch^{2(k+1)} + \epsilon\|\xi_{p_t}\|^2, \tag{4.66}$$

$$\begin{aligned}
\tilde{T}_5 &\leq C\|d(c)-d(c_h)\|^2 + \epsilon\|\xi_{p_t}\|^2 \leq C\|d_c^{'}(c-c_h)\|^2 + \epsilon\|\xi_{p_t}\|^2 \\
&\leq Ch^{2(k+1)} + C\|\xi_c\|^2 + \epsilon\|\xi_{p_t}\|^2,
\end{aligned} \tag{4.67}$$

For $\tilde{T}_6$, we use Lemma 4.3 to obtain

$$\tilde{T}_6 \leq Ch^{k+1}\|p\|_{k+2}\|\boldsymbol{\xi}_u\|. \tag{4.68}$$

Substituting (4.63)-(4.68) into (4.62), we have the estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}(c)\boldsymbol{\xi}_u\|^2 + \|d^{\frac{1}{2}}(c)\xi_{p_t}\|^2 \\
&\leq Ch^{2(k+1)} + C\|\boldsymbol{\xi}_u\|^2 + C\|\xi_c\|^2 + \epsilon\|\xi_{p_t}\|^2 + \epsilon\|\xi_{ct}\|^2 + \epsilon\|\boldsymbol{\xi}_{ut}\|^2.
\end{aligned} \tag{4.69}$$

Integrating (4.69) with respect to time between 0 and t and using the hypothesis 1 , we obtain the third energy Inequality

$$\begin{aligned}
&\|\boldsymbol{\xi}_u\|^2 + \int_0^t \|\xi_{p_t}\|^2 dt \\
&\leq Ch^{2(k+1)} + C\int_0^t (\|\boldsymbol{\xi}_u\|^2 + \|\xi_c\|^2)dt + \epsilon\int_0^t (\|\xi_{ct}\|^2 + \|\boldsymbol{\xi}_{ut}\|^2)dt.
\end{aligned} \tag{4.70}$$

4.7 The fourth energy inequality

We take the time derivative in equation (4.16), we have

$$((d(c)p_t - d(c_h)p_{ht})_t, \zeta) = \mathcal{L}^d(\mathbf{e}_{ut}, \zeta), \tag{4.71}$$

Take $\boldsymbol{\theta} = \boldsymbol{\xi}_{ut}$ and $\zeta = \xi_{p_t}$ in (4.61) and (4.71), respectively and use (3.15) and (4.3) to obtain

$$(A(c)\boldsymbol{\xi}_{ut}, \boldsymbol{\xi}_{ut}) + \frac{1}{2}\frac{d}{dt}(d(c_h)\xi_{p_t}, \xi_{p_t}) = \tilde{\tilde{T}}_1 + \tilde{\tilde{T}}_2 + \tilde{\tilde{T}}_3 - \tilde{\tilde{T}}_4 + \tilde{\tilde{T}}_5 + \tilde{\tilde{T}}_6 - \tilde{\tilde{T}}_7, \tag{4.72}$$

where

$$\begin{aligned}
\tilde{\tilde{T}}_1 &= -((A(c))_t\boldsymbol{\xi}_u, \boldsymbol{\xi}_{ut}),\\
\tilde{\tilde{T}}_2 &= ((A(c)\boldsymbol{\eta}_u)_t, \boldsymbol{\xi}_{ut}),\\
\tilde{\tilde{T}}_3 &= (((A(c) - A(c_h))\mathbf{u}_h)_t, \boldsymbol{\xi}_{ut}),\\
\tilde{\tilde{T}}_4 &= -\frac{1}{2}((d(c_h))_t\xi_{p_t}, \xi_{p_t}),\\
\tilde{\tilde{T}}_5 &= ((d(c_h)\eta_{p_t})_t, \xi_{p_t}),\\
\tilde{\tilde{T}}_6 &= (((d(c) - d(c_h))p_t)_t, \xi_{p_t}),\\
\tilde{\tilde{T}}_7 &= \mathcal{D}(\eta_{p_t}, \boldsymbol{\xi}_{ut}).
\end{aligned}$$

Now, we estimate $\tilde{\tilde{T}}_i's$ term by term. Using hypothesis 1 and Schwarz inequality, we can get

$$\tilde{\tilde{T}}_1 = -\frac{1}{2}(A'(c)c_t\boldsymbol{\xi}_u, \boldsymbol{\xi}_{ut}) \le C\|\boldsymbol{\xi}_u\|^2 + \epsilon\|\boldsymbol{\xi}_{ut}\|^2, \tag{4.73}$$

and

$$\begin{aligned}
\tilde{\tilde{T}}_2 &= (A'(c)c_t\boldsymbol{\eta}_u, \boldsymbol{\xi}_{ut}) + (A(c)\boldsymbol{\eta}_{ut}, \boldsymbol{\xi}_{ut})\\
&\le \epsilon\|\boldsymbol{\xi}_{ut}\|^2 + C\|\boldsymbol{\eta}_u\|^2 + C\|\boldsymbol{\eta}_{ut}\|^2\\
&\le \epsilon\|\boldsymbol{\xi}_{ut}\|^2 + Ch^{2(k+1)}.
\end{aligned} \tag{4.74}$$

Now, we estimate $\tilde{\tilde{T}}_3$,

$$\begin{aligned}
\tilde{\tilde{T}}_3 &= ((A(c) - A(c_h))_t\mathbf{u}_h, \boldsymbol{\xi}_{ut}) - ((A(c) - A(c_h))(\mathbf{u} - \mathbf{u}_h)_t, \boldsymbol{\xi}_{ut})\\
&\quad + ((A(c) - A(c_h))u_t, \boldsymbol{\xi}_{ut})\\
&= ((A'(c) - A'(c_h))c_t\mathbf{u}_h, \boldsymbol{\xi}_{ut}) + (A'(c_h)(c - c_h)_t\mathbf{u}_h, \boldsymbol{\xi}_{ut})\\
&\quad + ((A(c) - A(c_h))\boldsymbol{\xi}_{ut}, \boldsymbol{\xi}_{ut}) - ((A(c) - A(c_h))\boldsymbol{\eta}_{ut}, \boldsymbol{\xi}_{ut}) + (A'_c(c - c_h)\mathbf{u}_t, \boldsymbol{\xi}_{ut})\\
&\le C\|c - c_h\|\|\boldsymbol{\xi}_{ut}\| + C\|(c - c_h)_t\|\|\boldsymbol{\xi}_{ut}\|\\
&\quad + \|A^{\frac{1}{2}}(c)\boldsymbol{\xi}_{ut}\|^2 - \|A^{\frac{1}{2}}(c_h)\boldsymbol{\xi}_{ut}\|^2 + C\|\boldsymbol{\eta}_{ut}\|\|\boldsymbol{\xi}_{ut}\|\\
&\le \|A^{\frac{1}{2}}(c)\boldsymbol{\xi}_{ut}\|^2 - \|A^{\frac{1}{2}}(c_h)\boldsymbol{\xi}_{ut}\|^2 + C\|\xi_c\|^2\\
&\quad + C\|\xi_{ct}\|^2 + \epsilon\|\boldsymbol{\xi}_{ut}\|^2 + Ch^{2(k+1)},
\end{aligned} \tag{4.75}$$

where in the third step we use Schwarz inequality and hypotheses 1,2, and the last step requires Lemma 4.2.

$$\begin{aligned}\tilde{\tilde{T}}_4 &= \frac{1}{2}\Big(d'(c_h)(c-c_h)_t\xi_{p_t},\xi_{p_t}\Big) - \frac{1}{2}\Big(d'(c_h)c_t\xi_{p_t},\xi_{p_t}\Big)\\ &\le C\|\xi_{p_t}\|_\infty\|(c-c_h)_t\|\|\xi_{p_t}\| + C\|\xi_{p_t}\|^2\\ &\le C\|\xi_{ct}\|^2 + C\|\xi_{p_t}\|\|^2 + Ch^{2(k+1)},\end{aligned} \tag{4.76}$$

where in the second step we use Schwarz inequality and hypothesis 1, and the last step requires Lemma 3.2. C depends on $\|c_t\|_\infty$. Similarly, we can estimate $\tilde{\tilde{T}}_5$ and $\tilde{\tilde{T}}_6$

$$\begin{aligned}\tilde{\tilde{T}}_5 &= -(d'(c_h)(c-c_h)_t\eta_{p_t},\xi_{p_t}) + (d'(c_h)c_t\eta_{p_t},\xi_{p_t}) + (d(c_h)\eta_{p_{tt}},\xi_{p_t})\\ &\le C\|\xi_{p_t}\|_\infty\|(c-c_h)_t\|\|\eta_{p_t}\| + C\|\eta_{p_t}\|\|\xi_{p_t}\| + C\|\eta_{p_{tt}}\|\|\xi_{p_t}\|\\ &\le C\|\xi_{ct}\|^2 + C\|\xi_{p_t}\|^2 + Ch^{2(k+1)},\end{aligned} \tag{4.77}$$

$$\begin{aligned}\tilde{\tilde{T}}_6 &= ((d'(c)-d'(c_h))c_tp_t,\xi_{p_t}) + (d'(c_h)(c-c_h)_tp_t,\xi_{p_t}) + ((d(c)-d(c_h))p_{tt},\xi_{p_t})\\ &\le C\|c-c_h\|^2 + C\|(c-c_h)_t\|^2 + C\|\xi_{p_t}\|^2\\ &\le C\|\xi_c\|^2 + C\|\xi_{ct}\|^2 + C\|\xi_{p_t}\|^2 + Ch^{2(k+1)}.\end{aligned} \tag{4.78}$$

For $\tilde{\tilde{T}}_7$, we use Lemma 4.3 to obtain

$$\tilde{\tilde{T}}_7 \le Ch^{k+1}\|p\|_{k+2}\|\boldsymbol{\xi}_{ut}\|. \tag{4.79}$$

Substituting (4.73)-(4.79) into (4.72), we have the estimate

$$\begin{aligned}&\|A^{\frac{1}{2}}(c_h)\boldsymbol{\xi}_{ut}\|^2 + \frac{1}{2}\frac{d}{dt}\|d^{\frac{1}{2}}(c_h)\xi_{p_t}\|^2\\ &\le Ch^{2(k+1)} + C(\|\boldsymbol{\xi}_u\|^2 + C\|\xi_c\|^2 + \|\xi_{p_t}\|^2 + \|\xi_{ct}\|^2) + \epsilon\|\boldsymbol{\xi}_{ut}\|^2.\end{aligned} \tag{4.80}$$

Integrating (4.80) with respect to time between 0 and t and using the hypothesis 1 , we obtain the fourth energy Inequality

$$\begin{aligned}&\int_0^t \|\boldsymbol{\xi}_{ut}\|^2 dt + \|\xi_{p_t}\|^2\\ &\le Ch^{2(k+1)} + C\int_0^t (\|\boldsymbol{\xi}_u\|^2 + \|\xi_c\|^2 + \|\xi_{p_t}\|^2 + \|\xi_{ct}\|^2)dt.\end{aligned} \tag{4.81}$$

4.8 Proof of Theorem 3.1

Now we are ready to combine the four energy inequalities and finish the proof of Theorem 3.1. Firstly, combing (4.51) with (4.70), we obtain

$$\begin{aligned}&\int_0^t \|\xi_{ct}\|^2 dt + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 \\ \leq\ & C\int_0^t (\|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 + \|\xi_{p_t}\|^2)dt + \epsilon\int_0^t \|\boldsymbol{\xi}_{ut}\|^2 dt + Ch^{2(k+1)}. \quad (4.82)\end{aligned}$$

Secondly,combing (4.81) with (4.82), we obtain

$$\begin{aligned}&\int_0^t \|\boldsymbol{\xi}_{ut}\|^2 dt + \|\xi_{p_t}\|^2 \\ \leq\ & C\int_0^t (\|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\xi}_s\|^2 + \|\xi_c\|^2 + \|\xi_{p_t}\|^2)dt + Ch^{2(k+1)}. \quad (4.83)\end{aligned}$$

Then, adding (4.60), (4.70), (4.82) and (4.83), we obtain

$$\begin{aligned}&\|\boldsymbol{\xi}_u\|^2 + \|\xi_p\|^2 + \|\xi_{p_t}\|^2 + \|\xi_c\|^2 + \|\boldsymbol{\xi}_s\|^2 + \int_0^t (\|\boldsymbol{\xi}_{ut}\|^2 + \|\xi_{ct}\|^2)dt \\ \leq\ & Ch^{2(k+1)} + C\int_0^t (\|\boldsymbol{\xi}_u\|^2 + \|\xi_p\|^2 + \|\xi_{p_t}\|^2 + \|\xi_c\|^2 + \|\boldsymbol{\xi}_s\|^2)dt \\ & +\epsilon\int_0^t (\|\boldsymbol{\xi}_{ut}\|^2 + \|\xi_{ct}\|^2)dt. \quad (4.84)\end{aligned}$$

Employing Gronwall's lemma, we obtain

$$\|\boldsymbol{\xi}_u\|^2 + \|\xi_p\|^2 + \|\xi_{p_t}\|^2 + \|\xi_c\|^2 + \|\boldsymbol{\xi}_s\|^2 + \int_0^t (\|\boldsymbol{\xi}_{ut}\|^2 + \|\xi_{ct}\|^2)dt \leq Ch^{2(k+1)}. \quad (4.85)$$

Finally, by using the standard approximation result, we obtain (3.16).To complete the proof, let us verify the a priori assumption (4.10). For $k \geq 1$, we can consider h small enough so that $Ch^{k+1} < \frac{1}{2}h$, where C is the constant determined by the final time T. Then if $t^* = \inf\{t : \|c - c_h\| + \|\mathbf{u} - \mathbf{u}_h\| + \|p_t - p_{ht}\| \geq h\}$, we should have $\|c - c_h\| + \|\mathbf{u} - \mathbf{u}_h\| + \|p_t - p_{ht}\| = h$ by continuity in time at $t = t^*$. However, if $t^* < T$, theorem 3.1 implies that $\|c - c_h\| + \|\mathbf{u} - \mathbf{u}_h\| + \|p_t - p_{ht}\| \leq Ch^{k+1}$ for $t \leq t^*$, in particular $h = \|(c-c_h)(t^*)\| + \|(\mathbf{u}-\mathbf{u}_h)(t^*)\| + \|(p_t-p_{ht})(t^*)\| \leq Ch^{k+1} < \frac{1}{2}h$, which is a contradiction. Therefore, there always holds $t^* \geq T$, and thus the a priori assumption (4.10) is justified.

5 Numerical example

In this section we provide numerical examples to illustrate the accuracy and capability of the method. Time discretization is given as the third order strong-stability-preserving Runge-Kutta method [22]. We take the time step to be sufficiently small such that the error in time is negligible compared to spatial error. In the scheme, the numerical flux in the convection term is taken as $\widehat{\mathbf{u}_h c_h} = \frac{1}{2}(\mathbf{u}_h^+ c_h^+ + \mathbf{u}_h^- c_h^-)$. Moreover, other parameters are taken as follows

- The solution domain $\Omega = [0,1] \times [0,1]$, $T = 0.01$, $\Delta t = r * h^2$, here r denotes the grid ratio and r depends on the polynomial degree.
- We take $\phi(x,y) = 1$, $\kappa(x,y) = 1$, $\mu(c) = 1$, for simplicity.

Example 5.1 *We first consider the problem with the constant matrix $\mathbf{D}(\mathbf{u}) = \alpha \mathbf{I}$, where α is a constant, in addition, we take the initial and boundary condition $c_0 = sin(2\pi(x+y))$, $p_0 = -2\pi(x^2+y^2)$, $c(0,t) = c(2\pi,t)$, and the parameters $b(c) = 0$, $d(c) = 1$ and the source term*

$$f = 2\pi\cos(2\pi(x+y+t))(4\pi(x+y+t)+1) + 8\alpha\pi^2\sin(2\pi(x+y+t)) - 2\pi,$$

the exact solution is

$$c = \sin(2\pi(x+y+t)), \mathbf{u} = (4\pi x + 2\pi t, 4\pi y + 2\pi t),$$

The L^2 error and the numerical orders of accuracy at time $t = 0.01$ with uniform meshes are contained in Tables 1 and 2. We can see that the method with Q^k elements gives $(k+1)-$th order of accuracy in L^2 norm.

Table 1: The numerical results for c with $\alpha = 1$

N	$Q^1/r = 0.01$		$Q^2/r = 0.01$		$Q^3/r = 0.001$	
	L^2 error	order	L^2 error	order	L^2 error	order
10	2.3021e-02	–	8.0016e-04	–	2.0744e-04	–
20	5.8006e-03	1.99	9.9746e-05	3.00	1.3097e-05	3.99
40	1.4512e-03	2.00	1.2417e-05	3.01	8.1846e-07	4.00
80	3.6279e-04	2.00	1.5521e-06	3.00	5.1097e-08	4.00
160	9.0695e-05	2.00	1.9400e-07	3.00	3.1875e-09	4.00

Example 5.2 *Next we consider the problem with matrix $\mathbf{D}(\mathbf{u}) = \mathbf{u} \otimes \mathbf{u} + \mathbf{I}$, in addition, we take the initial and boundary condition $c_0 = \sin(2\pi(x+y))$, $p_0 = -2\pi(x^2+y^2)$, $c(0,t) = c(2\pi,t)$, and the parameters $b(c) = 0$, $d(c) = 1$ and the source term*

$$f = 2\pi\cos(2\pi(x+y+t))(4\pi(x+y+t))(1-12\pi^2) - 2\pi + 4\pi^2(16\pi^2(x+y+t)^2+2)\sin(2\pi(x+y+t)),$$

the exact solution is

$$c = \sin(2\pi(x+y+t)), \mathbf{u} = (4\pi x + 2\pi t, 4\pi y + 2\pi t),$$

Table 2: The numerical results for c with $\alpha = 0.01$

N	$Q^1/r = 0.01$		$Q^2/r = 0.01$		$Q^3/r = 0.001$	
	L^2 error	order	L^2 error	order	L^2 error	order
10	2.3021e-02	–	7.9917e-04	–	2.0744e-04	–
20	5.8006e-03	1.99	9.9612e-05	3.00	1.3097e-05	3.99
40	1.4501e-03	2.00	1.2450e-05	3.00	8.1796e-07	4.00
80	3.6247e-04	2.00	1.5524e-06	3.00	5.1100e-08	4.00
160	9.0603e-05	2.00	1.9355e-07	3.00	3.1875e-09	4.00

The L^2 error and the numerical orders of accuracy at time $t = 0.01$ with uniform meshes is contained in Tables 3. We can see that the method with Q^k elements gives $(k+1)-$th order of accuracy in L^2 norm.

Table 3: The numerical results for c

N	$Q1/r = 0.01$		$Q2/r = 0.01$		$Q3/r = 0.001$	
	L^2 error	order	L^2 error	order	L^2 error	order
10	2.3022e-02	–	7.9948e-04	–	2.0756e-04	–
20	5.8006e-03	1.99	9.9643e-05	3.00	1.3104e-05	3.99
40	1.4492e-03	2.00	1.2393e-05	3.01	8.2105e-07	4.00
80	3.6223e-04	2.00	1.5477e-06	3.00	5.1348e-08	4.00
160	9.0551e-05	2.00	1.9308e-07	3.00	3.2097e-09	4.00

Example 5.3 *We choose the initial condition as*

$$c_0 = \frac{1}{2}(1 + \cos(2\pi x)\cos(2\pi y)), \qquad p_0 = \cos(2\pi x)\cos(2\pi y) - 1.$$

Other parameters are taken as

$$q(x, y, 0) = 0, m_1 = 0.35, m_2 = 1, \phi(x) = 1, \mathbf{D}(\mathbf{u}) = \begin{pmatrix} |u| & 0 \\ 0 & |u| \end{pmatrix}$$

We choose $\Delta t = 0.01 \min\{\Delta x^2, \Delta y^2\}$ with final time $T = 0.1$, and the numerical approximation of c is given in Figure 1.

Example 5.4 *We change the initial condition in Example 5.3 to*

$$c_0 = \begin{cases} 0.001, & (x-0.5)^2 + (y-0.5)^2 < 0.09, \\ 0, & \textit{otherwise}, \end{cases} \qquad p_0 = \sin(\pi x)\sin(\pi y).$$

Other parameters are taken as

$$q(x, y, 0) = 0, m_1 = 1, m_2 = 1, \phi(x) = 1, \mathbf{D}(\mathbf{u}) = \mathbf{I}$$

and the numerical approximation of c is given in Figure 2.

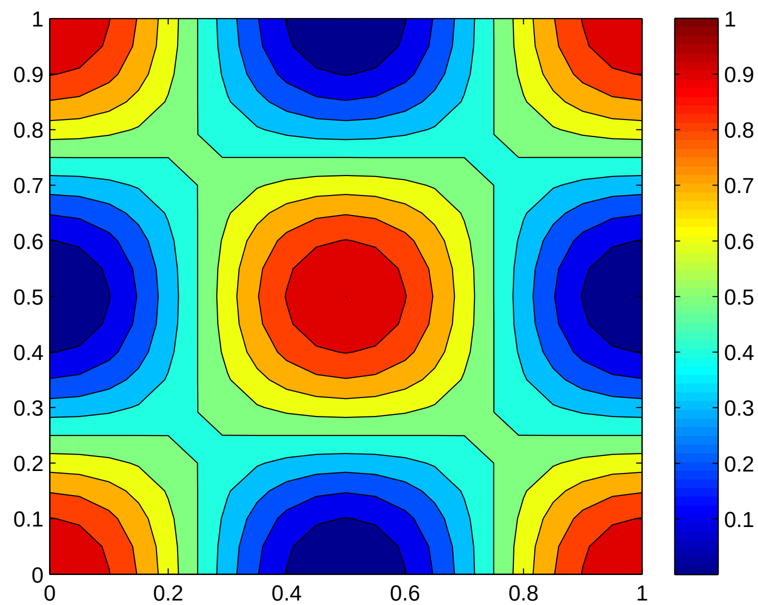

Figure 1: Numerical approximations of c at $t = 0.1$ with $Nx = Ny = 40$ in Example 5.3.

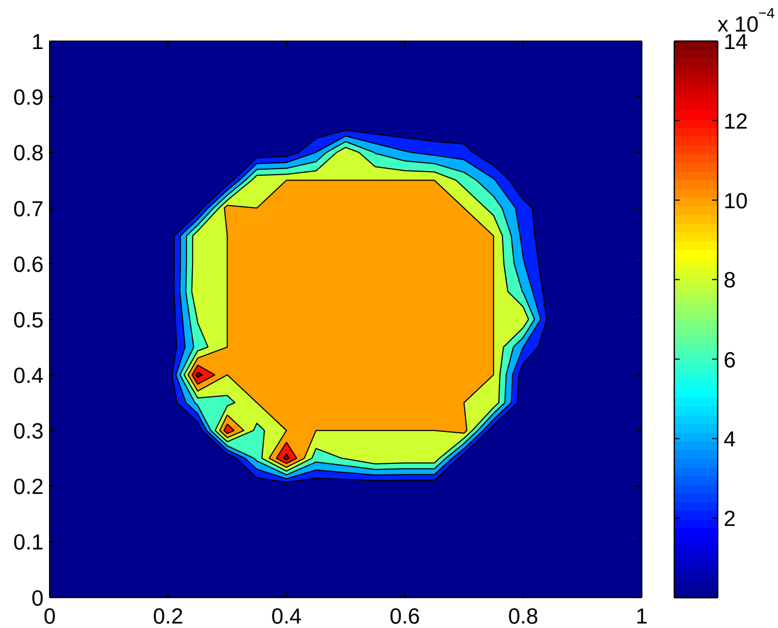

Figure 2: Numerical approximations of c at $t = 0.1$ with $Nx = Ny = 40$ in Example 5.4.

6 Concluding remarks

In this paper, the conservative LDG method for both flow and transport equations is introduced for the coupled system of compressible miscible displacement problem. The optimal order of error estimates hold not only for the solution itself but also for the auxiliary variables. Special projections and a priori assumption help to eliminate the jump terms at the cell interfaces which arise from the discontinuity nature of the numerical method, the nonlinearity and coupling of the model.

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A Proof of Lemma 4.5

Recall that we have chosen the initial condition $c_h^0 = P^+c_0, \mathbf{u}_h^0 = \mathbf{\Pi}^-\mathbf{u}_0$, where $\mathbf{u}_0 = -a(c_0)\nabla p_0$, and $\widehat{p_h} = p_h^+, \widehat{\mathbf{u}_h} = \mathbf{u}_h^-, \widehat{\mathbf{z}_h} = \mathbf{z}_h^-, \widehat{c_h} = c_h^+$. For simplicity, we will drop the 0 in the superscripts and subscripts in this section. It is clear that (4.5) and (4.6) hold. Taking the test function $\zeta = \xi_{p_t}$ and summing over K in (4.16), we have

$$\Big(d(c)\xi_{p_t}, \xi_{p_t}\Big) = \Big(d(c)\eta_{p_t}, \xi_{p_t}\Big) + \Big(p_{ht}(d(c) - d(c_h)), \xi_{p_t}\Big), \tag{A.1}$$

where we have used $\mathbf{u}_h = \mathbf{\Pi}^-\mathbf{u}, \widehat{\mathbf{u}_h} = \mathbf{u}_h^-$ and the property of the projection (4.3) . Using the Schwartz inequality, we can get

$$\|d^{\frac{1}{2}}(c)\xi_{p_t}\|^2 \le C\|\eta_{p_t}\|\|\xi_{p_t}\| + C\|c - c_h\|\|\xi_{p_t}\|, \tag{A.2}$$

By Lemma 4.2 and (4.5), we easily prove

$$\|\xi_{p_t}\| \le Ch^{k+1}. \tag{A.3}$$

Similarly, taking the test function $\mathbf{w} = \boldsymbol{\xi}_s$ and summing over K in (4.13), we have

$$(\boldsymbol{\xi}_s, \boldsymbol{\xi}_s) = (\boldsymbol{\eta}_s, \boldsymbol{\xi}_s) - \mathcal{D}(\eta_c, \boldsymbol{\xi}_s), \tag{A.4}$$

where we have used $c_h = P^+c$. Using the Schwartz inequality and the Lemma 4.3, we can get

$$\|\boldsymbol{\xi}_s\|^2 \le \|\boldsymbol{\xi}_s\|\|\boldsymbol{\eta}_s\| + Ch^{k+1}\|c\|_{k+2}\|\boldsymbol{\xi}_s\|. \tag{A.5}$$

By Lemma 4.2, we easily prove

$$\|\boldsymbol{\xi}_s\| \le Ch^{k+1}, \tag{A.6}$$

By the standard approximation results, (4.7) and (4.8) hold. At last we estimate $p - p_h$, following the technique in [17]. By (3.10) the initial data p_h is the solution of the following equations

$$(A(c_h)\mathbf{u}_h, \boldsymbol{\theta})_K - (p_h, \nabla \cdot \boldsymbol{\theta})_K + \langle \widehat{p_h}, \boldsymbol{\theta} \cdot \boldsymbol{\nu}_K \rangle_{\partial_K} = 0, \tag{A.7}$$

and also satisfies

$$(p - p_h, 1) = 0. \tag{A.8}$$

From (4.15), we have

$$(A(c)\mathbf{u} - A(c_h)\mathbf{u}_h, \boldsymbol{\theta})_K - (p - p_h, \nabla \cdot \boldsymbol{\theta})_K + \langle p - \widehat{p_h}, \boldsymbol{\theta} \cdot \boldsymbol{\nu}_K \rangle_{\partial_K} = 0. \tag{A.9}$$

We use $\mathbf{u}_h$ to find a well-defined p_h, and we only need to prove the uniqueness. If there are two solutions p_1 and p_2 satisfying (A.7) and (A.8), then we can easily get

$$(p_1 - p_2, \nabla \cdot \boldsymbol{\theta})_K - \langle \widehat{p_1} - \widehat{p_2}, \boldsymbol{\theta} \cdot \boldsymbol{\nu}_K \rangle_{\partial_K} = 0, \tag{A.10}$$

$$(p_1 - p_2, 1) = 0. \tag{A.11}$$

We consider the elliptic linear problem

$$-\boldsymbol{\zeta}^* = \nabla \xi^*, \text{in} \quad \Omega, \tag{A.12}$$

$$\eta^* = \nabla \cdot \boldsymbol{\zeta}^*, \text{in} \quad \Omega, \tag{A.13}$$

subject to periodic boundary conditions. To make the problem well-defined, we assume that the average of ξ^* on Ω is a given constant and that of η^* is zero. We have the elliptic regularity result

$$\|\boldsymbol{\zeta}^*\|_{H^1(\Omega)} + \|\xi^*\|_{H^2(\Omega)} \le C\|\eta^*\|. \tag{A.14}$$

Taking $\eta^* = p_1 - p_2$ and $\widehat{p_i} = p_i^+, i = 1, 2$, we get

$$\begin{aligned}
&(p_1 - p_2, p_1 - p_2)_K \\
=\ &(p_1 - p_2, \nabla \cdot \boldsymbol{\zeta}^*)_K \\
=\ &(p_1 - p_2, \nabla \cdot (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*))_K + (p_1 - p_2, \nabla \cdot \Pi\boldsymbol{\zeta}^*)_K \\
=\ &(p_1 - p_2, \nabla \cdot (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*))_K - \langle \widehat{p_1} - \widehat{p_2}, (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \cdot \boldsymbol{\nu}_K \rangle_{\partial_K} + \langle \widehat{p_1} - \widehat{p_2}, \boldsymbol{\zeta}^* \cdot \boldsymbol{\nu}_K \rangle_{\partial_K} \\
=\ &-(\nabla(p_1 - p_2), \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*)_K + \langle p_1 - p_2, (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \cdot \boldsymbol{\nu}_{\boldsymbol{K}} \rangle_{\partial_K} \\
&-\langle \widehat{p_1} - \widehat{p_2}, (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \cdot \boldsymbol{\nu}_K \rangle_{\partial_K} + \langle \widehat{p_1} - \widehat{p_2}, \boldsymbol{\zeta}^* \cdot \boldsymbol{\nu}_K \rangle_{\partial_K}
\end{aligned} \tag{A.15}$$

where the third step follows from (A.10) and the last equality is based on integration by parts. We take $\Pi\boldsymbol{\zeta}^* = \Pi^-\boldsymbol{\zeta}^*$ and sum over K. By the continuity of $\boldsymbol{\zeta}^*$ and the definition of the projection Π^-, we obtain

$$(p_1 - p_2, p_1 - p_2) = 0 \tag{A.16}$$

Then we get $p_1 = p_2$. We have proved that p_h is well-defined. In the following, we estimate $\|p - p_h\|$. We use the same technique above and take $\eta^* = p - p_h$ to obtain

$$\begin{aligned}
&(p - p_h, p - p_h)_K \\
=\ & (p - p_h, \nabla \cdot \boldsymbol{\zeta}^*)_K \\
=\ & (p - p_h, \nabla \cdot (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*))_K + (p - p_h, \nabla \cdot \Pi\boldsymbol{\zeta}^*)_K \\
=\ & (p - p_h, \nabla \cdot (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*))_K - (A(c)\mathbf{u} - A(c_h)\mathbf{u}_h, \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*)_K \\
& -\langle p - \widehat{p}_h, (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \cdot \boldsymbol{\nu_K}\rangle_{\partial K} + (A(c)\mathbf{u} - A(c_h)\mathbf{u}_h, \boldsymbol{\zeta}^*)_K + \langle p - \widehat{p}_h, \boldsymbol{\zeta}^* \cdot \boldsymbol{\nu_K}\rangle_{\partial K} \\
=\ & -(\nabla(p - p_h), \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*)_K + \langle p - p_h, (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \cdot \boldsymbol{\nu_K}\rangle_{\partial K} \\
& -(A(c)\mathbf{u} - A(c_h)\mathbf{u}_h, \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*)_K - \langle p - \widehat{p}_h, (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \cdot \boldsymbol{\nu_K}\rangle_{\partial K} \\
& +(A(c)\mathbf{u} - A(c_h)\mathbf{u}_h, \boldsymbol{\zeta}^*)_K + \langle p - \widehat{p}_h, \boldsymbol{\zeta}^* \cdot \boldsymbol{\nu_K}\rangle_{\partial K} \\
=\ & -(\nabla(p - p_h), \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*)_K + \langle \widehat{p}_h - p_h, (\boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \cdot \boldsymbol{\nu_K}\rangle_{\partial K} \\
& -(A(c)\mathbf{u} - A(c_h)\mathbf{u}_h, \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*)_K + (A(c)\mathbf{u} - A(c_h)\mathbf{u}_h, \boldsymbol{\zeta}^*)_K \\
& +\langle p - \widehat{p}_h, \boldsymbol{\zeta}^* \cdot \boldsymbol{\nu_K}\rangle_{\partial K}
\end{aligned} \tag{A.17}$$

where the third one follows from (A.9) and the fourth equality is based on the integrate by parts. Recalling that $\widehat{p}_h = p_h^+$, we take $\Pi\boldsymbol{\zeta}^* = \Pi^-\boldsymbol{\zeta}^*$ and sum over K. By the continuity of $\boldsymbol{\zeta}^*$ and the definition of the projection $\boldsymbol{\Pi}^-$, we obtain

$$\begin{aligned}
\|p - p_h\|^2 &= -(\nabla\eta_p, \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) - (A(c)\mathbf{u} - A(c_h)\mathbf{u}_h, \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \\
&\quad +(A(c)\mathbf{u} - A(c_h)\mathbf{u}_h, \boldsymbol{\zeta}^*) \\
&= -(\nabla\eta_p, \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) - (A(c)(\mathbf{u} - \mathbf{u}_h), \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \\
&\quad -((A(c) - A(c_h))\mathbf{u}_h, \boldsymbol{\zeta}^* - \Pi\boldsymbol{\zeta}^*) \\
&\quad +(A(c_0)(\mathbf{u} - \mathbf{u}_h), \boldsymbol{\zeta}^*) + ((A(c) - A(c_h))\mathbf{u}_h, \boldsymbol{\zeta}^*) \\
&\le Ch^{k+1}\|\boldsymbol{\zeta}^*\|_{H^1(\Omega)} + Ch^{k+2}\|\boldsymbol{\zeta}^*\|_{H^1(\Omega)} + Ch^{k+1}\|\boldsymbol{\zeta}^*\| \\
&\le Ch^{k+1}\|\boldsymbol{\zeta}^*\|_{H^1(\Omega)} \\
&\le Ch^{k+1}\|p - p_h\|,
\end{aligned} \tag{A.18}$$

which further implies

$$\|p - p_h\| \le Ch^{k+1}. \tag{A.19}$$

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