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# ASYMPTOTICALLY COMPATIBLE SCHEMES AND APPLICATIONS TO ROBUST DISCRETIZATION OF NONLOCAL MODELS*

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**Abstract.** Many problems in nature, being characterized by a parameter, are of interest both with a fixed parameter value and with the parameter approaching an asymptotic limit. Numerical schemes that are convergent in both regimes offer robust discretizations, which can be highly desirable in practice. The asymptotically compatible schemes studied in this paper meet such objectives for a class of parametrized problems. An abstract mathematical framework is established rigorously here together with applications to the numerical solution of both nonlocal models and their local limits. In particular, the framework can be applied to nonlocal diffusion models and a general state-based peridynamic system parametrized by the horizon radius. Recent findings have exposed the risks associated with some discretizations of nonlocal models when the horizon radius is proportional to the discretization parameter. Thus, it is desirable to develop asymptotically compatible schemes for such models so as to offer robust numerical discretizations to problems involving nonlocal interactions on multiple scales. This work provides new insight in this regard through a careful analysis of related conforming finite element discretizations and the finding is valid under minimal regularity assumptions on exact solutions. It reveals that as long as the finite element space contains continuous piecewise linear functions, the Galerkin finite element approximation is always asymptotically compatible. For piecewise constant finite element, whenever applicable, it is shown that a correct local limit solution can also be obtained as long as the discretization (mesh) parameter decreases faster than the modeling (horizon) parameter does. These results can be used to guide future computational studies of nonlocal problems.

**Key words.** nonlocal diffusion, peridynamic models, state-based model, local limit, finite element, convergence analysis, asymptotically compatible scheme

**AMS subject classifications.** 65J10, 49M25, 65N30, 65R20, 82C21, 46N40, 45A05

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## 1. Introduction.

This paper is motivated by the study of numerical methods for nonlocal models and their local limits. Nonlocal phenomena are ubiquitous in nature and nonlocal models have appeared in many subjects, from physics and biology to materials and social sciences. For example, there has been a great deal of interest recently in the nonlocal peridynamic (PD) continuum theory introduced first by Silling in [32]. PD models avoid the explicit use of spatial derivatives and provide alternatives to the classical partial differential equation (PDE) based continuum mechanics, especially when dealing with cracks and materials failure [4, 23, 28, 36, 37]. Mathematical analysis of PD models and other related nonlocal models, such as nonlocal diffusion, can be found in [1, 3, 8, 16, 18, 20]. Numerical approximations have also been studied in [6, 10, 17, 23, 24, 31, 34, 38, 39, 40].

A common feature of PD models is the introduction of the horizon parameter $\delta$ that characterizes the range (radius) of nonlocal interactions [16, 32]. As $\delta \to 0$, nonlocal effect diminishes and the zero-horizon limit of nonlocal PD models becomes a

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classical local differential equation model when the latter is well-defined. Such limiting behavior provides connections and consistencies between nonlocal and local models, and has immense practical significance especially for multiscale modeling and simulations. A natural question leading to the work here is how such limiting behaviors can be preserved in various discrete approximations. This is a critical issue in the applications of PD like models to problems involving possibly different scales, given the popularity and practicality to perform PD simulations with a coupled horizon $\delta$ and mesh spacing $h$. Recently, we have showed that some standard numerical methods for nonlocal diffusion (ND) and PD models may approximate the wrong local limit when the ratio of $\delta$ and $h$ is kept constant, while convergence to the desired local limit can also be established for some other discretizations [38]. To keep the discussion relatively simple, the results presented in [38] have mostly been confined to one-dimensional models. Still, they have clearly exposed the risks involved in some popular practices for dealing with nonlocal models and exemplified the need for more comprehensive numerical analysis of the relevant issues.

In this work, a major contribution is to introduce asymptotically compatible schemes and the corresponding abstract mathematical framework for their rigorous numerical analysis with respect to certain classes of parametrized problems and their asymptotic limits. This allows us to go beyond the discussion on approximations of nonlocal models and their local limits to establish a more general mathematical theory with a much broader perspective. Indeed, the vanishing nonlocality of the ND/PD models as $\delta$ tends to zero reminds us of many classical problems with a changing parameter. For instance, the vanishing viscosity limit of the Navier–Stokes equations to the Euler equations, the convergence of phase field models to their sharp interface limits, as well as the linear elasticity problem as the Lamé constant tends to infinity, etc. All these problems share a common feature that properties of the underlying equations change significantly in the limit process, so that it is not at all obvious what numerical methods may be effective for a vast range of parameter values and in some limiting cases. It is interesting and challenging to develop numerical methods that behave as desired while taking limits of the problems, and we consider such methods here and name them as asymptotically compatible schemes. While it is perhaps impossible to develop a theory that would encompass problems of many different types, we attempt to develop an abstract framework that can be applied to linear ND and PD models and their local limits. This may offer new insights into the study of other problems involving both a modeling parameter and a discretization parameter. Moreover, the abstraction also illustrates the key features that distinguish the ND and PD models considered here from other parametrized problems.

An immediate consequence of the abstract framework, which is also presented here, is the identification of asymptotically compatible finite element methods for the robust discretization of linear ND/PD models, as summarized later by a commutative diagram in Figure 1. The results are for models with very general nonlocal interaction kernels and solutions with minimal regularity assumptions, as well as general geometric meshes with no restrictions on the space dimension. They significantly generalize earlier exploratory findings [38] and offer another major contribution of this paper that will be of particular interest to people working on numerical simulations of ND/PD models. Furthermore, the concept of asymptotically compatible schemes is applicable to not only Galekin approximation but also other discretizations such as finite difference and collocations methods using quadratures [38].

The paper is organized as follows. In section 2, we introduce the asymptotically compatible scheme and an abstract framework for the convergence analysis.

In section 3, we consider the application to linear ND problems and characterize asymptotically compatible finite element methods. We also show that the discontinuous piecewise constant finite element, which is not reliable if the ratio $\delta/h$ is fixed [38], may be conditionally asymptotically compatible. Section 4 contains an application to state-based PD models. Results of numerical experiments are reported in section 5 to complement the theoretical analysis and to illustrate the order of accuracy of numerical schemes.

## 2. An abstract framework.

In this section, we introduce the notion of asymptotically compatible schemes and propose an abstract framework for their numerical analysis when they are applied to a special class of parametrized problems.

### 2.1. Notation and assumptions.

Before stating the main results, let us introduce notation and state the main assumptions. The assumptions are given in the order of (infinite-dimensional) function spaces, then bilinear forms, followed by induced linear operators, and finally the approximations.

We begin by considering a decreasing family of Hilbert spaces $\{\mathcal{T}_\sigma, \sigma \in [0,\infty]\}$ over $\mathbb{R}$ in the sense that $\mathcal{T}_{\sigma_2}$ is a dense subspace of $\mathcal{T}_{\sigma_1}$ for any $0 \le \sigma_1 \le \sigma_2 \le \infty$.

Let $(\cdot,\cdot)_{\mathcal{T}_\sigma}$ and $\|\cdot\|_{\mathcal{T}_\sigma}$ denote the corresponding inner product and norm on $\mathcal{T}_\sigma$ and denote the dual space of $\mathcal{T}_\sigma$ by $\mathcal{T}_{-\sigma} = \mathcal{T}_\sigma^*$.

We note that both spaces $\mathcal{T}_0$ and $\mathcal{T}_\infty$ are of particular interest to our discussions here. Indeed, we identify the dual space of $\mathcal{T}_0$ with itself, $\mathcal{T}_0^* = \mathcal{T}_0$. A typical example is given by the standard $L^2$ function space in applications we consider later. Moreover, we assume that $\mathcal{T}_0$ serves as the pivot space between $\mathcal{T}_{-\sigma}$ and $\mathcal{T}_\sigma$ so that a realization of the duality pairing $\langle\cdot,\cdot\rangle$ between $\mathcal{T}_{-\sigma}$ and $\mathcal{T}_\sigma$ is given as the extension of the inner product on $\mathcal{T}_0$. In other words, for any $\sigma \in [0,\infty]$, and for $w \in \mathcal{T}_0 \subseteq \mathcal{T}_{-\sigma}$,

$$\langle w, v\rangle = (w,v)_{\mathcal{T}_0} \quad \forall v \in \mathcal{T}_\sigma, \tag{2.1}$$

Thus, we do not specify any subscript related to $\sigma$ to distinguish the duality pairing.

Assumptions given above on the spaces $\{\mathcal{T}_\sigma\}$ are very generic so far. To discuss the special class of variational problems defined on spaces $\{\mathcal{T}_\sigma\}$, we state the following assumptions, which are crucial to the problems under consideration.

*Assumption* 2.1. The spaces $\{\mathcal{T}_\sigma\}$ are assumed to satisfy the properties below.

(i) *Uniform embedding* property: there are positive constants $M_1$ and $M_2$, independent of $\sigma \in (0,\infty)$, such that

$$M_1\|u\|_{\mathcal{T}_0} \le \|u\|_{\mathcal{T}_\sigma} \quad \forall u \in \mathcal{T}_\sigma \quad \text{and} \quad \|u\|_{\mathcal{T}_\sigma} \le M_2\|u\|_{\mathcal{T}_\infty} \quad \forall u \in \mathcal{T}_\infty.$$

(ii) *Asymptotically compact embedding* property: for any sequence $(u_n \in \mathcal{T}_n)$, if there is a constant $C > 0$ independent of $n$ such that

$$\|u_n\|_{\mathcal{T}_n} \le C \quad \forall n,$$

then the sequence $(u_n)$ is relatively compact in $\mathcal{T}_0$ and each limit point is in $\mathcal{T}_\infty$.

With spaces $\{\mathcal{T}_\sigma\}$ given, we now consider some parametrized bilinear forms.

*Assumption* 2.2. Let $a_\sigma\colon \mathcal{T}_\sigma \times \mathcal{T}_\sigma \to \mathbb{R}$ be a symmetric bilinear form, $\sigma \in [0,\infty]$.

(i) $a_\sigma$ is *bounded*: there exists a constant $C_2 > 0$ such that

$$a_\sigma(u,v) \le C_2\|u\|_{\mathcal{T}_\sigma}\|v\|_{\mathcal{T}_\sigma} \quad \forall u, v \in \mathcal{T}_\sigma.$$

(ii) $a_\sigma$ is *coercive*: there exists a constant $C_1 > 0$ such that

$$a_\sigma(u,u) \geq C_1 \|u\|^2_{\mathcal{T}_\sigma} \quad \forall u \in \mathcal{T}_\sigma.$$

Moreover, we assume that $C_1$ and $C_2$ are constants *independent of* $\sigma$.

Given the above assumption on the bilinear forms, for any $\sigma \in [0,\infty]$, we see that $a_\sigma(u,\cdot)$ defines a bounded linear functional for any $u \in \mathcal{T}_\sigma$. Moreover, by the Lax–Milgram theorem, it induces naturally a bounded linear operator, denoted by $\mathcal{A}_\sigma$, from $\mathcal{T}_\sigma$ to its dual $\mathcal{T}_{-\sigma}$, with a bounded inverse $\mathcal{A}_\sigma^{-1} : \mathcal{T}_{-\sigma} \to \mathcal{T}_\sigma$. In other words, using the notation given above, we have

$$\big\langle \mathcal{A}_\sigma u, v \big\rangle = a_\sigma(u,v) \quad \forall u,v \in \mathcal{T}_\sigma. \tag{2.2}$$

By the symmetry of $a_\sigma$, we easily see that $\mathcal{A}_\sigma$ is self-adjoint,

$$\big\langle \mathcal{A}_\sigma u, v \big\rangle = \big\langle u, \mathcal{A}_\sigma v \big\rangle \quad \forall u,v \in \mathcal{T}_\sigma, \tag{2.3}$$

and thus positive definite. We next give some assumptions on $\mathcal{A}_\sigma$.

*Assumption* 2.3. For $\mathcal{A}_\sigma$ defined by (2.2), we assume the following:

(i) A subspace $\mathcal{T}_*$ is dense in $\mathcal{T}_\infty$, and also dense in any $\mathcal{T}_\sigma$ with $\sigma \geq 0$, such that

$$\mathcal{A}_\sigma u \in \mathcal{T}_0 \quad \forall u \in \mathcal{T}_*.$$

(ii) $\mathcal{A}_\infty$ is the limit of $\mathcal{A}_\sigma$ in $\mathcal{T}_*$ in the sense that

$$\lim_{\sigma\to\infty} \|\mathcal{A}_\sigma u - \mathcal{A}_\infty u\|_{\mathcal{T}_{-\sigma}} = 0 \quad \forall u \in \mathcal{T}_*. \tag{2.4}$$

Since we are concerned with numerical approximations of the variational problems associated with the operators $\{\mathcal{A}_\sigma\}$ for $\sigma \in [0,\infty]$, we consider a family of closed subspaces $\{W_{\sigma,h} \subset \mathcal{T}_\sigma\}$ parametrized by an additional real parameter $h \in (0,h_0]$. The fact that we take $W_{\sigma,h}$ as a subspace of $\mathcal{T}_\sigma$ implies that we are effectively adopting a standard, internal, or equivalently conforming type Galerkin approximation approach.

Although in practice $W_{\sigma,h}$ is always finite dimensional with $h$ being the corresponding meshing parameter, it is not necessary to make such an assumption here for the theoretical analysis. Moreover, while $\{\mathcal{T}_\sigma\}$ is a decreasing family, for each $h > 0$, the family $\{W_{\sigma,h}\}$ does not have to be. All we need here are that $W_{\sigma,h}$ is closed in $\mathcal{T}_\sigma$ for each given $\sigma$ and $h$ and some basic assumptions on the approximation properties of $W_{\sigma,h}$ to $\mathcal{T}_\sigma$ as $h \to 0$ as stated below. The first part of these assumptions ensures the convergence of approximations to $\mathcal{T}_\sigma$ as $h \to 0$ for each $\sigma$, while the second part is concerned with the limiting behavior as both $h \to 0$ and $\sigma \to \infty$ at the same time.

*Assumption* 2.4. Assume that the family of subspaces $\{W_{\sigma,h} \subset \mathcal{T}_\sigma\}$ parametrized by $\sigma \in [0,\infty]$ and $h \in (0,h_0]$ satisfies the following properties:

(i) For each $\sigma \in [0,\infty]$, the family $\{W_{\sigma,h}, h \in (0,h_0]\}$ is dense in $\mathcal{T}_\sigma$ in the sense that $\forall v \in \mathcal{T}_\sigma$, there exists a sequence $\{v_n \in W_{\sigma,h_n}\}$ with $h_n \to 0$ as $n \to \infty$ such that

$$\|v - v_n\|_{\mathcal{T}_\sigma} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.5}$$

(ii) $\{W_{\sigma,h}, \sigma \in [0,\infty), h \in (0,h_0]\}$ is *asymptotically dense* in $\mathcal{T}_\infty$, i.e., $\forall v \in \mathcal{T}_\infty$, there exists a sequence $\{v_n \in W_{\sigma_n,h_n}\}_{h_n\to 0, \sigma_n\to\infty}$ as $n \to \infty$ such that

$$\|v - v_n\|_{\mathcal{T}_\infty} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.6}$$

**2.2. The parametrized variational problems and their approximations.** Consider a family of parametrized variational problems defined by the following: given $f \in \mathcal{T}_0$,

$$\text{find } u_\sigma \in \mathcal{T}_\sigma \text{ such that } \ a_\sigma(u_\sigma, v) = (f, v)_{\mathcal{T}_0} \quad \forall v \in \mathcal{T}_\sigma \tag{2.7}$$

for $\sigma \in [0, \infty]$. The approximation to $u_\sigma$ in a subspace $W_{\sigma,h}$ is defined by the following:

$$\text{find } u_{\sigma,h} \in W_{\sigma,h} \text{ such that } \ a_\sigma(u_{\sigma,h}, v) = (f, v)_{\mathcal{T}_0} \quad \forall v \in W_{\sigma,h}. \tag{2.8}$$

The existence and uniqueness of $u_\sigma$ and $u_{\sigma,h}$ follow from assumptions made earlier. We may also express (2.7) and (2.8) in strong forms as

$$\mathcal{A}_\sigma u_\sigma = f, \tag{2.9}$$

$$\mathcal{A}^h_\sigma u_{\sigma,h} = \pi^h_\sigma f, \tag{2.10}$$

where $\pi^h_\sigma$ is the $L^2$ projection operator onto the subspace $W_{\sigma,h}$ and $\mathcal{A}^h_\sigma : W_{\sigma,h} \to W^*_{\sigma,h}$ is the operator induced by the bilinear form $a_\sigma$ in $W_{\sigma,h}$ (or the solution operator of (2.8) in the specified subspace).

We are interested in establishing an abstract framework to analyze the various limits of $\{u_{\sigma,h}\}$ as we take limits in the parameters. We first state a convergence result for the solutions of the parametrized variational problems as $\sigma \to \infty$.

THEOREM 2.5 (convergence of variational solutions as $\sigma \to \infty$). *Given the assumptions on the family of spaces and the bilinear forms and operators, we have*

$$\|u_\sigma - u_\infty\|_{\mathcal{T}_0} \to 0 \quad \textit{as } \sigma \to \infty.$$

*Proof.* By (2.7) and the assumptions, we have

$$C_1 \|u_\sigma\|^2_{\mathcal{T}_\sigma} \le a_\sigma(u_\sigma, u_\sigma) = (f, u_\sigma)_{\mathcal{T}_0} \le \|f\|_{\mathcal{T}_{-\sigma}} \|u_\sigma\|_{\mathcal{T}_\sigma} \le M_1^{-1} \|f\|_{\mathcal{T}_0} \|u_\sigma\|_{\mathcal{T}_\sigma},$$

which leads to the uniform boundedness of $\{u_\sigma \in \mathcal{T}_\sigma\}$, and thus by the asymptotically compact embedding property, we get the convergence of a subsequence of $\{u_\sigma\}$ in $\mathcal{T}_0$ to a limit point $u_* \in \mathcal{T}_\infty$. For notational convenience, we use the same $\{u_\sigma\}$ to denote the subsequence. Now, taking $v \in \mathcal{T}_* \subset \mathcal{T}_\infty$, we have

$$\begin{aligned}(f - \mathcal{A}_\infty u_*, v)_{\mathcal{T}_0} &= (\mathcal{A}_\sigma u_\sigma - \mathcal{A}_\infty u_*, v)_{\mathcal{T}_0} = (u_\sigma, \mathcal{A}_\sigma v)_{\mathcal{T}_0} - (u_*, \mathcal{A}_\infty v)_{\mathcal{T}_0} \\ &= (u_\sigma, \mathcal{A}_\sigma v - \mathcal{A}_\infty v)_{\mathcal{T}_0} + (u_\sigma - u_*, \mathcal{A}_\infty v)_{\mathcal{T}_0}.\end{aligned}$$

We know that as $\sigma \to \infty$, $u_\sigma$ is uniformly bounded in $\mathcal{T}_\sigma$ (and consequently also in $\mathcal{T}_0$). Meanwhile, $\mathcal{A}_\infty v$ is in $\mathcal{T}_0$. Furthermore, we have that $u_\sigma - u_*$ goes to zero in $\mathcal{T}_0$ by the choice of $u_*$ and that $\|\mathcal{A}_\sigma v - \mathcal{A}_\infty v\|_{\mathcal{T}_{-\sigma}} \to 0$ by assumption (2.4). Together with the uniform boundedness of $\|u_\sigma\|_{\mathcal{T}_\sigma}$ and the assumption $\mathcal{A}_\infty v \in \mathcal{T}_0$, we thus arrive at

$$(f - \mathcal{A}_\infty u_*, v)_{\mathcal{T}_0} \to 0 \quad \text{as } \sigma \to \infty.$$

Moreover, since $\mathcal{T}_*$ is dense in $\mathcal{T}_\infty$, we see that $u_*$ is the unique solution $u_\infty$ of $\mathcal{A}_\infty u_\infty = f$ and the convergence of the whole sequence also follows. ∎

Next, we consider the convergence of approximations as $h \to 0$ for a given $\sigma$.

THEOREM 2.6 (convergence with a fixed $\sigma \in [0, \infty]$ as $h \to 0$). *For any given* $\sigma \in [0, \infty]$, *let* $u_\sigma$ *and* $u_{\sigma,h}$ *be defined by* (2.7) *and* (2.8). *Given the assumptions on* *the approximate spaces and the approximate bilinear forms, there exists a constant $C > 0$, independent of $h$, such that*

$$\|u_{\sigma,h} - u_\sigma\|_{\mathcal{T}_\sigma} \le C \inf_{v_{\sigma,h}\in W_{\sigma,h}} \|v_{\sigma,h} - u_\sigma\|_{\mathcal{T}_\sigma} \to 0 \quad \text{as } h \to 0.$$

*Proof.* The proof is similar to the standard best approximation property of the Ritz–Galerkin approximation. Given $\sigma \in [0,\infty]$, for any $v_{\sigma,h} \in W_{\sigma,h}$,

$$\begin{aligned} C_1\|u_\sigma - u_{\sigma,h}\|^2_{\mathcal{T}_\sigma} &\le a_\sigma(u_\sigma - u_{\sigma,h}, u_\sigma - u_{\sigma,h}) = a_\sigma(u_\sigma - u_{\sigma,h}, u_\sigma) \\ &= a_\sigma(u_\sigma - u_{\sigma,h}, u_\sigma - v_{\sigma,h}) \le C_2\|u_\sigma - u_{\sigma,h}\|_{\mathcal{T}_\sigma}\|u_\sigma - v_{\sigma,h}\|_{\mathcal{T}_\sigma}. \end{aligned}$$

So we have

$$\|u_\sigma - u_{\sigma,h}\|_{\mathcal{T}_\sigma} \le C \inf_{v_{\sigma,h}\in W_{\sigma,h}} \|u_\sigma - v_{\sigma,h}\|_{\mathcal{T}_\sigma} \to 0 \quad \text{as } h \to 0.$$

This proves the theorem. ∎

We now move on to an analogue of Theorem 2.5 for approximate problems, that is, we consider the convergence as $\sigma \to \infty$ but for a fixed $h > 0$. For this, we need a few additional assumptions on the approximation spaces.

THEOREM 2.7 (convergence of approximate solutions with $h > 0$ as $\sigma \to \infty$). *Given the assumptions on the family of spaces, bilinear forms, operators, and approximate spaces, and assume in addition that for a given $h > 0$, we have the following:*

(i) *Limit of approximate spaces:*

$$W_{\infty,h} = \mathcal{T}_\infty \cap \left(\bigcap_{\sigma\ge 0} W_{\sigma,h}\right). \tag{2.11}$$

(ii) *Approximation property of bilinear forms:*

$$\lim_{\sigma\to\infty} a_\sigma(u_h, v_h) = a_\infty(u_h, v_h) \quad \forall u_h, v_h \in W_{\infty,h}. \tag{2.12}$$

(iii) *A strengthened continuity property: for any sequence $(w_{\sigma,h} \in W_{\sigma,h})$ with uniformly bounded $(\|w_{\sigma,h}\|_{\mathcal{T}_\sigma})$ and satisfying $w_{\sigma,h} \to 0$ in $\mathcal{T}_0$ as $\sigma \to \infty$, we have*

$$\lim_{\sigma\to\infty} a_\sigma(w_{\sigma,h}, v_h) = 0 \quad \forall v_h \in W_{\infty,h}. \tag{2.13}$$

*Then, for the approximate solutions $u_{\sigma,h}$ of (2.8) with $\sigma \in (0,\infty)$, we have*

$$\|u_{\sigma,h} - u_{\infty,h}\|_{\mathcal{T}_0} \to 0 \quad \text{as } \sigma \to \infty. \tag{2.14}$$

*Proof.* Similar to the proof of Theorem 2.5, we have

$$C_1\|u_{\sigma,h}\|^2_{\mathcal{T}_\sigma} \le a_\sigma(u_{\sigma,h}, u_{\sigma,h}) = (f, u_{\sigma,h})_{\mathcal{T}_0} \le \|f\|_{\mathcal{T}_{-\sigma}}\|u_{\sigma,h}\|_{\mathcal{T}_\sigma} \le M_1^{-1}\|f\|_{\mathcal{T}_0}\|u_{\sigma,h}\|_{\mathcal{T}_\sigma}.$$

This leads to the uniform boundedness of $\{u_{\sigma,h} \in \mathcal{T}_\sigma\}$, and thus by the asymptotically compact embedding property, we get the convergence of a subsequence in $\mathcal{T}_0$ to a limit point $u_{*,h} \in \mathcal{T}_\infty$. By assumption (2.11), we have necessarily that $u_{*,h} \in W_{\infty,h}$. Using again the same $\{u_{\sigma,h}\}$ to denote the subsequence and taking $v_h \in W_{\infty,h} \subset W_{\sigma,h}$,

$$\begin{aligned} &(f, v_h)_{\mathcal{T}_0} - a_\infty(u_{*,h}, v_h) = a_\sigma(u_{\sigma,h}, v_h) - a_\infty(u_{*,h}, v_h) \\ &\quad = a_\sigma(u_{\sigma,h}, v_h) - a_\sigma(u_{*,h}, v_h) + a_\sigma(u_{*,h}, v_h) - a_\infty(u_{*,h}, v_h) \\ &\quad = a_\sigma(u_{\sigma,h} - u_{*,h}, v_h) + \left[a_\sigma(u_{*,h}, v_h) - a_\infty(u_{*,h}, v_h)\right] = \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

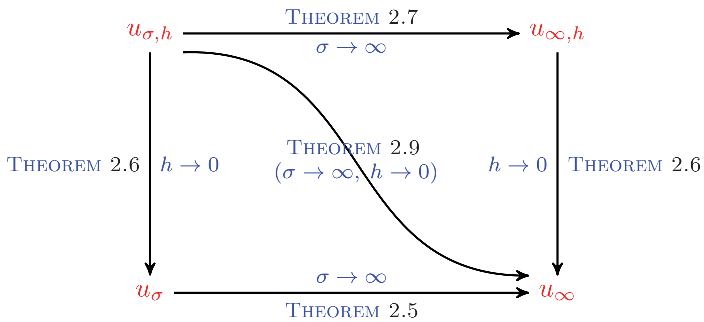


FIG. 1. *A diagram for asymptomatically compatible schemes and convergence results.*

Now, to estimate the first term, we let $w_{\sigma,h} = u_{\sigma,h} - u_{*,h} \in W_{\sigma,h}$ and apply the strengthened continuity property of $a_\sigma$ to get $|\mathrm{I}_1| \to 0$. Assumption (2.12) implies that $\mathrm{I}_2 \to 0$. Thus, $u_{*,h}$ is the unique solution of (2.8) with $\sigma = \infty$ and the unique limit point of the whole sequence $\{u_{\sigma,h}\}$. The theorem thus follows. □

**2.3. Asymptotically compatible schemes.** While we have the convergence of $\{u_{\sigma,h}\}$ for a given $\sigma$ as $h \to 0$, as well as the convergence of $\{u_\sigma\}$ to $u_\infty$ and $\{u_{\sigma,h}\}$ to $u_{\infty,h}$ as $\sigma \to \infty$, we are also interested in the behavior as both $\sigma \to \infty$ and $h \to 0$. We summarize this in Figure 1 and introduce the concept of asymptotically compatible schemes.

DEFINITION 2.8. *The family of convergent approximations* $\{u_{\sigma,h}\}$ *defined by* (2.8) *is said to be asymptotically compatible to the solution* $u_\infty$ *defined by* (2.7) *with* $\sigma = \infty$ *if for any sequence* $\sigma_n \to \infty$ *and* $h_n \to 0$*, we have* $\|u_{\sigma_n,h_n} - u_\infty\|_{\mathcal{T}_0} \to 0$.

Note that since $u_{\sigma_n,h_n}$ and $u_\infty$ may live in different spaces, the space $\mathcal{T}_0$ is the most natural space that contains all the elements involved. In cases where $u_{\sigma_n,h_n}$ represent discrete solutions, one may even use different meshes and basis functions. Nevertheless, additional compatibilities on the spaces are needed for the convergence of $u_{\sigma,h}$ and to $u_{\infty,h}$ as $\sigma \to \infty$, as suggested by (2.11), for example.

THEOREM 2.9 (asymptotically compatibility). *Under Assumptions* 2.1–2.4*, the family of approximations is asymptotically compatible.*

*Proof.* The first step is again similar to that in the proof of Theorems 2.5 and 2.7, that is, we can get $\|u_{\sigma,h}\|_{\mathcal{T}_\sigma}$ being uniformly bounded by some constant,

$$\|u_{\sigma,h}\|_{\mathcal{T}_\sigma} \le C. \tag{2.15}$$

Then for any sequences $\{\sigma_n\}$ and $\{h_n\}$, where $\sigma_n \to \infty$, $h_n \to 0$, the sequence $(u_{\sigma_n,h_n})_n$ is relatively compact in $\mathcal{T}_0$, and any limit point $u_*$ of the convergent subsequence in $\mathcal{T}_0$, still denoted by $(u_n = u_{\sigma_n,h_n})$ without loss of generality, is in $\mathcal{T}_\infty$. Let us show that $u_*$ solves (2.7) with $\sigma = \infty$ and therefore is unique so that the entire sequence actually converges to the unique solution $u_* = u_\infty$. That is, for $\|u_* - u_n\|_{\mathcal{T}_0} \to 0$ as $n \to \infty$, we need to prove for every $v \in \mathcal{T}_*$, $u_*$ satisfies (2.7).

By the asymptotically dense property (2.6) of $W_{\sigma,h}$ in $\mathcal{T}_\infty$, we can choose $v_n \in W_{\sigma_n,h_n}$ such that $\|v - v_n\|_{\mathcal{T}_\infty} \to 0$. Then we have the following equation:

$$\begin{aligned} a_\infty(u_*, v) - (f, v)_{\mathcal{T}_0} &= [a_\infty(u_*, v) - a_\infty(u_n, v)] + [a_\infty(u_n, v) - a_{\sigma_n}(u_n, v)] \\ &\quad + [a_{\sigma_n}(u_n, v) - a_{\sigma_n}(u_n, v_n)] + [(f, v_n)_{\mathcal{T}_0} - (f, v)_{\mathcal{T}_0}] \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned} \tag{2.16}$$

We now show that as $n \to \infty$, all four terms vanish. Now for the first part, since the operator $a_\infty$ is symmetric and $\mathcal{A}_\infty v \in \mathcal{T}_0$, we can rewrite I as

$$|\mathrm{I}| = |a_\infty(u_* - u_n, v)| = |(\mathcal{A}_\infty v, u_* - u_n)_{\mathcal{T}_0}| \leq \|\mathcal{A}_\infty v\|_{\mathcal{T}_0} \|u_* - u_n\|_{\mathcal{T}_0} \to 0.$$

Similarly we can rewrite the second part and use Assumption 2.3 to obtain

$$\begin{aligned}|\mathrm{II}| &= |(\mathcal{A}_\infty v - \mathcal{A}_{\sigma_n} v, u_n)_{\mathcal{T}_0}| \leq \|\mathcal{A}_\infty v - \mathcal{A}_{\sigma_n} v\|_{\mathcal{T}_{-\sigma_n}} \|u_n\|_{\mathcal{T}_{\sigma_n}} \\ &\leq C\|\mathcal{A}_\infty v - \mathcal{A}_{\sigma_n} v\|_{\mathcal{T}_{-\sigma_n}} \to 0.\end{aligned}$$

We then use the bound on the bilinear form $a_\sigma$ and the uniform embedding to get

$$\begin{aligned}|\mathrm{III}| &= |a_{\sigma_n}(u_n, v - v_n)| \leq C_2 \|u_n\|_{\mathcal{T}_{\sigma_n}} \|v - v_n\|_{\mathcal{T}_{\sigma_n}} \\ &\leq C_2 C \|v - v_n\|_{\mathcal{T}_{\sigma_n}} \leq C_2 C M_2 \|v - v_n\|_{\mathcal{T}_\infty} \to 0.\end{aligned}$$

Finally the last term can be estimated via the standard Cauchy–Schwartz inequality,

$$|\mathrm{IV}| \leq \|f\|_{\mathcal{T}_0} \cdot \|v - v_n\|_{\mathcal{T}_0} \to 0.$$

This shows that $u_*$ solves (2.7). This completes the proof of the theorem. ∎

**3. Applications to ND problem.** The first example we consider is a homogeneous Dirichlet volume constrained value problem associated with a linear ND model. We refer to [14, 25] for more discussions on related mathematical theory. These problems are nonlocal in nature and they can be cast into the parametrized form described in the above section since a parameter $\delta$ is often used in these models to denote the range of nonlocal interactions.

**3.1. Model equation.** Let $\Omega \subset \mathbb{R}^d$ denote a bounded, open domain with a piecewise planar boundary. The corresponding *interaction domain* is defined as

$$\Omega_{\mathcal{I}} = \{\mathbf{y} \in \mathbb{R}^d \backslash \Omega \text{ such that } \mathrm{dist}(\mathbf{y}, \partial\Omega) \leq 1\}. \tag{3.1}$$

Also we let $\Omega_w = \Omega \cup \Omega_{\mathcal{I}}$ be the domain containing both $\Omega$ and $\Omega_{\mathcal{I}}$. A nonlocal operator $\mathcal{L}$ is defined as, for any function $u(\mathbf{x}) : \Omega_w \to \mathbb{R}$,

$$\mathcal{L}u(\mathbf{x}) = -2 \int_\Omega (u(\mathbf{y}) - u(\mathbf{x}))\gamma(\mathbf{x}, \mathbf{y}) d\mathbf{y}, \tag{3.2}$$

where the kernel $\gamma = \gamma(\mathbf{x}, \mathbf{y})$ is nonnegative and symmetric, i.e., $\gamma(\mathbf{x}, \mathbf{y}) = \gamma(\mathbf{y}, \mathbf{x}) \geq 0$. It is shown in [16] that if $\gamma(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\Delta_{\mathbf{y}}\delta(\mathbf{y} - \mathbf{x})$, with $\delta(\cdot)$ denoting the Dirac delta measure, then $\mathcal{L} = -\Delta$ in the distributional sense. We note that the operator $\mathcal{L}$ can be written as $\mathcal{L} = -\mathcal{D}\gamma\mathcal{D}^*$, where $\mathcal{D}$ and $\mathcal{D}^*$ are the basic nonlocal divergence operator and its dual defined in a nonlocal vector calculus [16]. Such a formulation draws a natural analogy between the nonlocal operator and local second order elliptic differential operator $-\nabla \cdot (\mathbf{C} \cdot \nabla)$, where $\mathbf{C}$ is a second order tensor.

In this section, we only consider kernels of radial type, i.e., $\gamma(\mathbf{x}, \mathbf{y}) = \gamma(|\mathbf{x} - \mathbf{y}|)$. Also we assume that $\mathrm{supp}(\gamma) \subset B_1(\mathbf{0})$ (the unit ball centered at the origin), and there exists a constant $\lambda > 0$ such that $B_\lambda(\mathbf{0}) \subset \mathrm{supp}(\gamma)$. Moreover, $\gamma$ is a nonnegative and nonincreasing function with a bounded second order moment. That is,

$$\hat{\gamma}(|\boldsymbol{\xi}|) = |\boldsymbol{\xi}|^2 \gamma(|\boldsymbol{\xi}|) \in L^1_{loc}(\mathbb{R}^d) \quad \text{and} \quad \int_{B(0,1)} \hat{\gamma}(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = d. \tag{3.3}$$

And we denote the rescaling kernels

$$\hat{\gamma}_\delta(|\boldsymbol{\xi}|) = \frac{1}{\delta^d}\hat{\gamma}\left(\frac{|\boldsymbol{\xi}|}{\delta}\right), \quad \gamma_\delta(|\boldsymbol{\xi}|) = \frac{1}{\delta^{d+2}}\gamma\left(\frac{|\boldsymbol{\xi}|}{\delta}\right) \tag{3.4}$$

for $\delta \in (0, 1]$ and let $\mathcal{L}_\delta$ denote the nonlocal operator corresponding to the kernel $\gamma_\delta$. Note that more general forms of the kernels can be considered as well; the essential features are that they in some sense are approximations of the distributional Laplacian of the Dirac-delta measure near the origin. The assumptions of the above form are made to simplify the presentation.

The model equation to be studied is the *nonlocal volume-constrained problem*:

$$\begin{cases} \mathcal{L}_\delta u = f & \text{on } \Omega, \\ \quad u = 0 & \text{on } \Omega_{\mathcal{I}_\delta} \end{cases} \tag{3.5}$$

where $\Omega_{\mathcal{I}_\delta} = \{\mathbf{y} \in \mathbb{R}^d \backslash \Omega \ \operatorname{dist}(\mathbf{y}, \partial\Omega) \le \delta\}$ and $\Omega_\delta = \Omega \cup \Omega_{\mathcal{I}_\delta}$. The second equation in (3.5) is a constraint imposed on a domain $\Omega_{\mathcal{I}_\delta}$ with a nonzero measure. It is a natural extension of *Dirichelet* boundary condition for classical PDEs [14]. With $\mathcal{L}_0 = -\Delta$, the nonlocal equation (3.5) is an analogue of the classical problem

$$\begin{cases} \mathcal{L}_0 u = f & \text{on } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

The natural energy space associated with (3.5) is [14, 25]

$$\mathcal{S}_\delta = \left\{ u \in L^2(\Omega_\delta) : \int_\Omega \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mathbf{x} d\mathbf{y} < \infty, \ u|_{\Omega_{\mathcal{I}_\delta}} = 0 \right\}$$

for $\delta \in (0, 1]$. It is clear that $\mathcal{S}_\delta$ is a subspace of $L^2(\Omega_\delta)$ with an inner product $(\cdot, \cdot)_{\mathcal{S}_\delta}$ defined as

$$(u, v)_{\mathcal{S}_\delta} = \int_\Omega \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y})) d\mathbf{x} d\mathbf{y}$$

and $\|\cdot\|_{\mathcal{S}_\delta}$ the associated norm. Let $\mathcal{S}_0 = H_0^1(\Omega)$ with an inner product and norm

$$(u, v)_{\mathcal{S}_0} = \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}, \quad \|u\|_{\mathcal{S}_0} = \left( \int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2}.$$

We note that $\|\cdot\|_{\mathcal{S}_\delta}$ are usually seminorms, but for $\{\mathcal{S}_\delta\}$ they are equivalent to full norms, as demonstrated by the Poincaré inequality given later (see also [25]), just as on $H_0^1(\Omega)$, $|u|_{H^1(\Omega)}$ and $\|u\|_{H^1(\Omega)}$ are equivalent. It can be shown that $\mathcal{S}_\delta$ is also the completion of $C_0^\infty(\Omega)$ in $L^2(\Omega_\delta)$ under the norm $\|\cdot\|_{\mathcal{S}_\delta}$ [25].

In order to apply the framework given earlier, it is convenient to have functions in the different spaces $\{\mathcal{S}_\delta, \delta \in [0, 1]\}$ be specified in a common spatial domain, say, $\Omega_w = \Omega \cup \Omega_{\mathcal{I}}$; we thus make all functions in $\mathcal{S}_\delta$ equivalent to themselves with zero extension outside $\Omega$ and norms defined by

$$\left\{ \int_\Omega \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mathbf{x} d\mathbf{y} \right\}^{1/2}$$

and

$$\left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \gamma_\delta(|\mathbf{x}-\mathbf{y}|)(u(\mathbf{x})-u(\mathbf{y}))^2 d\mathbf{x}d\mathbf{y} \right\}^{1/2}$$

are also equivalent for such functions, independently of $\delta$. These equivalence properties will be implicitly used throughout the manuscript unless otherwise noted.

Now we present weak formulations for the nonlocal (and local) diffusion models. Define a family of bilinear forms $\{b_\delta\}$ by

$$b_\delta(u,v) = \begin{cases} \int_\Omega \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{y}-\mathbf{x}|)(u(\mathbf{y})-u(\mathbf{x}))(v(\mathbf{y})-v(\mathbf{x}))d\mathbf{y}d\mathbf{x} \quad (\delta > 0), \\ \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x}) \quad (\delta = 0) \end{cases} \tag{3.7}$$

for $u$, $v \in \mathcal{S}_\delta$. Then the weak formulations of (3.5) and (3.6) are as follows:

$$\text{Find } \; u_\delta \in \mathcal{S}_\delta \text{ such that } b_\delta(u_\delta, v) = (f,v)_{L^2} \quad \forall v \in \mathcal{S}_\delta. \tag{3.8}$$

Now for each $\delta$, we introduce the finite element spaces $\{V_{\delta,h}\} \subset \mathcal{S}_\delta$ associated with the triangulation $\tau_h = \{K\}$ of the domain $\Omega_\delta$ (or $\Omega_w$). We set

$$V_{\delta,h} := \{v \in \mathcal{S}_\delta : v|_K \in P(K) \quad \forall K \in \tau_h\},$$

where $P(K) = \mathcal{P}_p(K)$ is the space of polynomials on $K \in \tau_h$ of degree less than or equal to $p$. Again, for different $\delta$, in order to have the finite element functions defined on a common spatial domain, we also assume, as in the case for $\mathcal{S}_\delta$, that any element in $V_{\delta,h}$ automatically vanishes outside $\Omega$. As $h \to 0$, $\{V_{\delta,h}\}$ is dense in $\mathcal{S}_\delta$, i.e., for any $v \in \mathcal{S}_\delta$, there exists a sequence $(v_h \in V_{\delta,h})$ such that

$$\|v_h - v\|_{\mathcal{S}_\delta} \to 0 \quad \text{as} \quad h \to 0. \tag{3.9}$$

These properties are easily satisfied by standard finite element spaces.

The Galerkin approximation is to replace $\mathcal{S}_\delta$ by $V_{\delta,h}$ in (3.8):

$$\text{Find } \; u_{\delta,h} \in V_{\delta,h} \text{ such that } b_\delta(u_{\delta,h}, v) = (f,v)_{L^2} \quad \forall v \in V_{\delta,h}. \tag{3.10}$$

**3.2. Asymptotically compatible schemes.** To apply the abstract framework to the ND model, we define $\mathcal{T}_\sigma$ in the context of section 2 as

$$\mathcal{T}_\sigma = \begin{cases} \mathcal{S}_{1/\sigma} \quad \text{for } \sigma \in [1,\infty], \\ L^2_0(\Omega) \quad \text{for } \sigma = 0, \\ \mathcal{T}_1 \quad \text{for } \sigma \in (0,1), \end{cases} \tag{3.11}$$

where $L^2_0(\Omega)$ contains all elements in $L^2(\Omega)$ that vanish outside $\Omega$. We define $\mathcal{T}_\sigma$ for $\sigma \in (0,1)$ the same as $\mathcal{T}_1$, since this would not affect the limiting behavior as $\sigma \to \infty$, or equivalently, $\delta \to 0$. Indeed, we are interested in approximations of both nonlocal problems with a finite horizon parameter and their local limits.

For the family of spaces, we need to verify the assumptions made in the earlier section. First, let us state a simple lemma below where fractional Sobolev spaces are used. We use $H_0^{\alpha/2}(\Omega)$ to denote the closure of $C_c^\infty(\Omega)$ in $H^{\alpha/2}(\Omega)$ for $\alpha \in (0,2]$. More discussions on these spaces can be found in [7, 13].

LEMMA 3.1. *For $\alpha \in (0,2]$ and a kernel $\gamma_\delta$ satisfying $|\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) \in L^1(\mathbb{R}^d)$, we have a constant $C$ depending only on $\Omega$ such that*

$$\|u\|^2_{\mathcal{S}_\delta} \le C \left( \int |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} \right) \|u\|^2_{H^{\alpha/2}(\Omega)} \quad \forall\, u \in H_0^{\alpha/2}(\Omega) \cap L_0^2(\Omega). \tag{3.12}$$

*Proof.* We consider the zero extension of functions to $\mathbb{R}^d$, so that there exists a constant $C = C(\Omega)$, independent of $\alpha$ and $\delta$, such that $\|u\|_{H^{\alpha/2}(\mathbb{R}^d)} \le C\|u\|_{H^{\alpha/2}(\Omega)}$ $\forall u \in H_0^{\alpha/2}(\Omega) \cap L_0^2(\Omega)$. Here we denote the extension of $u$ by the same notation. The lemma is then a consequence of the following:

$$\int_{\mathbb{R}^d} |u(\mathbf{x} + \boldsymbol{\xi}) - u(\mathbf{x})|^2 d\mathbf{x} \le C|\boldsymbol{\xi}|^\alpha \|u\|^2_{H^{\alpha/2}(\mathbb{R}^d)}.$$

To see the above, we have by the Fourier transform that

$$|u|^2_{H^{\alpha/2}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\mathbf{k}|^\alpha \hat{u}^2(\mathbf{k}) d\mathbf{k}, \quad \int_{\mathbb{R}^d} |u(\mathbf{x} + \boldsymbol{\xi}) - u(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^d} |e^{i\mathbf{k}\cdot\boldsymbol{\xi}} - 1|^2 \hat{u}^2(\mathbf{k}) d\mathbf{k}.$$

So the desired inequality follows from an elementary inequality $|e^{i\mathbf{k}\cdot\boldsymbol{\xi}} - 1|^2 \le 2|\boldsymbol{\xi}\cdot\mathbf{k}|^\alpha$ for $\alpha \in (0,2]$. Hence, we get

$$\begin{aligned}\int_\Omega \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{y} - \mathbf{x}|)(u(\mathbf{y}) - u(\mathbf{x}))^2 d\mathbf{y} d\mathbf{x} &\le \int_{\mathbb{R}^d} \gamma_\delta(|\boldsymbol{\xi}|) \int_{\mathbb{R}^d} (u(\mathbf{x} + \boldsymbol{\xi}) - u(\mathbf{x}))^2 d\mathbf{x} d\boldsymbol{\xi} \\ &\le C\|u\|_{H^{\alpha/2}(\Omega)} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi},\end{aligned}$$

which leads to the lemma. ∎

By applying the above to functions in $\mathcal{S}_0 = \mathcal{T}_\infty$ for the case of $\alpha = 2$, we have the uniform embedding of $\mathcal{T}_\infty$ in $\mathcal{T}_\sigma$ since for the kernel $\gamma_\delta$, we have

$$\int |\boldsymbol{\xi}|^2 \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = 1.$$

To verify Assumption 2.1 for $\{\mathcal{T}_\sigma\}$, it remains to apply a uniform Poincaré-type inequality for the uniform embedding of $\mathcal{T}_\sigma$ in $\mathcal{T}_0$.

LEMMA 3.2 (uniform Poincaré inequality). *There exists $C > 0$ independent of $\delta$ such that $\forall \delta \in (0,1]$,*

$$\|u\|^2_{L^2(\Omega_\delta)} \le C\|u\|^2_{\mathcal{S}_\delta} \quad \forall u \in \mathcal{S}_\delta. \tag{3.13}$$

The above is a special case of [26, Proposition 5.3] for scalar valued functions (see [26] for the proof). Also from [25], we know that $\mathcal{S}_\delta$ is complete, thus a Hilbert space.

To check Assumption 2.1(ii), we need a compactness lemma that can be found in [7, Theorem 4] and [29, Theorems 1.2, 1.3].

LEMMA 3.3. *Suppose $u_n \in \mathcal{S}_{\delta_n}$ with $\delta_n \to 0$. If*

$$\sup_n \int_\Omega \int_{B_{\delta_n}(\mathbf{x})} \gamma_{\delta_n}(|\mathbf{x} - \mathbf{y}|)(u_n(\mathbf{x}) - u_n(\mathbf{y}))^2 d\mathbf{x} d\mathbf{y} \le \infty,$$

*then $u_n$ is precompact in $L_0^2(\Omega)$. Moreover, any limit point $u \in \mathcal{S}_0$.*

We note that in establishing Lemmas 3.2 and 3.3, an argument of [7] often can be used, which requires that $\hat{\gamma}$ is nonincreasing. It has been noted that by techniques introduced in [2], the results remain true under a less restrictive condition where $\gamma$ is assumed to be nonincreasing. Moreover, [29] provided an even more general argument that works for $d \geq 2$ without the assumption on $\gamma$ being nonincreasing. Related discussions on these issues can be found in [25, 26].

We next move to the bilinear forms. Note that $b_\delta$ is exactly the inner product defined on $\mathcal{S}_\delta$, so Assumption 2.2 is naturally satisfied with $C_1 = C_2 = 1$.

Assumption 2.3 is about the convergence of the operator $\mathcal{L}_\delta$, a result that has been shown in many works, such as [16, 26]. We state it here as a proposition without proof. It is a pointwise convergence property of a smooth function under the action of the nonlocal integral operator $\mathcal{L}_\delta$ (generally interpreted in the principal value sense [26]) to that under the negative Laplace operator.

PROPOSITION 3.4. *For all $v \in C_c^\infty(\Omega)$, and all $\mathbf{x} \in \Omega$, we have*

$$\mathcal{L}_\delta v(\mathbf{x}) \longrightarrow -\Delta v(\mathbf{x}) \quad \text{as } \delta \to 0. \tag{3.14}$$

*Moreover, there exists a constant $C = C(d, v)$ such that*

$$\sup_{\delta \in (0,1)} \sup_{\mathbf{x} \in \Omega} |\mathcal{L}_\delta v(\mathbf{x})| \leq C. \tag{3.15}$$

With pointwise convergence and uniform boundedness estimate of $\mathcal{L}_\delta v$, Assumption 2.3 is obviously true by the bounded convergence theorem. This is stated in the following lemma, which is a stronger result than what Assumption 2.3(ii) requires.

LEMMA 3.5. *For $\mathcal{L}_\delta$ and $\mathcal{L}_0$ defined in section 3.1, $\forall v \in C_c^\infty(\Omega)$,*

$$\|\mathcal{L}_\delta v - \mathcal{L}_0 v\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as } \delta \to 0.$$

As for Assumption 2.4, (3.9) ensures that $V_{\delta,h}$ satisfies (i). To check (ii), for convenience, we define a special family of spaces $\hat{V}_{\delta,h}$.

DEFINITION 3.6. *Let $\hat{V}_{\delta,h} \subset V_{0,h} \subset \mathcal{S}_0$ be the continuous piecewise linear finite element space that corresponds to the same mesh $\tau_h$ with $V_{\delta,h}$.*

The following lemma is simply a restatement of a simple fact that the continuous piecewise linear subspace of $H_0^1$ approximates the whole space as mesh size goes to zero.

LEMMA 3.7. *The family $\{\hat{V}_{\delta,h}\}$ is asymptotically dense in $\mathcal{S}_0$, that is, it satisfies Assumption 2.4(ii).*

Now we see that if $\hat{V}_{\delta,h} \subset V_{\delta,h}$, then $V_{\delta,h}$ also satisfies Assumption 2.4(ii).

With all assumptions 2.1–2.4 verified, the following theorem offers a remedy for developing asymptotically compatible schemes when one wants to solve ND equations.

THEOREM 3.8. *Let $u_\delta$ and $u_{\delta,h}$ be solutions of (3.8) and (3.10), respectively, and $\hat{V}_{\delta,h}$ is defined in Definition 3.6. If $\hat{V}_{\delta,h} \subset V_{\delta,h}$, then $\|u_{\delta,h} - u_0\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$, $h \to 0$.*

*Proof.* Taking $\mathcal{T}_\sigma := \mathcal{S}_{1/\sigma}$, $a_\sigma := b_{1/\sigma}$, $\mathcal{A}_\sigma := \mathcal{L}_{1/\sigma}$, and $W_{\sigma,h} := V_{1/\sigma,h}$, we see that the above theorem follows from Theorem 2.9, since in the above discussions we have verified all assumptions 2.1–2.4 for this case. ∎

In short, we see that if the finite element spaces contain a continuous finite element subspace that have desired approximation properties in $\mathcal{S}_0$, then the corresponding discretization is asymptotically compatible. This is particularly true for any continuous or discontinuous finite element spaces containing at least the subspace of continuous piecewise linear elements.

We now examine the local limit of discrete solutions on a fixed mesh, following the discussions in Theorem 2.7. By condition (2.11), we see that some compatibility of the discrete spaces is needed. We choose to work with a fixed mesh as a simplification, although there is still ample freedom to choose different finite element spaces for nonlocal and local problems. To verify all the additional assumptions required for Theorem 2.7, we state a couple of technical results.

For a given triangulation $\tau_h$, we define a space of continuous and piecewise smooth functions given by

$$\mathcal{V}_h := \{v \in C(\overline{\Omega_\delta}) : \ v|_K \in C^\infty(\overline{K}),\ K \in \tau_h,\ v|_{\Omega_{\mathcal{I}_\delta}} = 0\}.$$

Note that functions in $\mathcal{V}_h$ are again set to vanish outside $\Omega$. Then, we have the convergence of the bilinear forms on the subspace $\mathcal{V}_h$.

Lemma 3.9. *For any $u,\ v \in \mathcal{V}_h$, as $\delta \to 0$, we have*

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\nabla u, \nabla v)_{L^2(\Omega)} \to 0.$$

*Consequently, for any $u_h, v_h \in V_{0,h}$,*

$$\lim_{\delta\to 0} b_\delta(u_h, v_h) = b_0(u_h, v_h).$$

*Proof.* First, we note that

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\nabla u, \nabla v)_{L^2(\Omega)} = \sum_{K\in\tau_h} \int_K u\mathcal{L}_\delta v - \sum_{K\in\tau_h} \int_K \nabla u \cdot \nabla v.$$

Now, for any mesh element $K \in \tau_h$, integration by parts on each $K$ gives

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\nabla u, \nabla v)_{L^2(\Omega)} = \sum_{K\in\tau_h} \int_K u(\mathcal{L}_\delta v + \Delta v) - \sum_{e\in\mathcal{E}_h^0} \int_e u[\![\nabla v]\!]_e,$$

where $\mathcal{E}_h^0$ is the set of internal edges of $\tau_h$ and $[\![\nabla v]\!]_e$ is the jump of the vector on the edge $e$. For the first term, by [17, Theorem 3.7], which remains valid for the kernel under consideration here, we have

$$\int_K u(\Delta v + \mathcal{L}_\delta v) \to \frac{1}{2} \sum_{e\in edge(K)} \int_e u[\![\nabla v]\!]_e \quad \text{as } \delta \to 0.$$

Summing over $K \in \tau_h$, we get

$$\sum_{K\in\tau_h} \int_K u(\Delta v + \mathcal{L}_\delta v) \to \sum_{e\in\mathcal{E}_h^0} \int_e u[\![\nabla v]\!]_e.$$

Thus we have $(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} \to (\nabla u, \nabla v)_{L^2(\Omega)}$ and the lemma follows. ∎

We first consider a simple case when $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$, which means that all functions in $V_{\delta,h}$ are in $H^1(\Omega)$, are continuous over $\Omega$, and vanish outside $\Omega$. In this case, we state an inverse inequality.

Lemma 3.10. *For $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$, there exists a constant $C > 0$, independent of $\delta$ such that for any $u_h \in V_{\delta,h}$,*

$$\|u_h\|_{\mathcal{S}_\delta} \le C\|u_h\|_{L^2(\Omega)}.$$

*Proof.* Since $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$, we first invoke the standard inverse inequality for finite element functions in $H^1$ to get

$$\|u_h\|_{\mathcal{S}_0} \le C\|u_h\|_{L^2(\Omega)} \quad \forall u_h \in V_{\delta,h}$$

for some generic constant $C > 0$ that only depends on the triangulation and the finite element basis. The lemma then follows from the uniform embedding of $\mathcal{S}_\delta$ in $\mathcal{S}_0$. ☐

THEOREM 3.11. *Suppose $u_{\delta,h}$ and $u_{0,h}$ are discrete solutions as defined in* (3.10) *with $\delta > 0$ and $\delta = 0$, respectively. If $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$, then for each fixed $h$ and $\tau_h$,*

$$\|u_{\delta,h} - u_{0,h}\|_{\mathcal{S}_0} \to 0 \text{ as } \delta \to 0.$$

*Proof.* We first check the additional conditions assumed in Theorem 2.7. Obviously condition (2.11) holds since $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$. As for condition (2.12), that is, the convergence of the approximate bilinear forms on the finite element spaces, we get it from Lemma 3.9, Next, combining the continuity of the bilinear form $b_\delta(\cdot,\cdot)$ and the uniform bound of $\|\cdot\|_{\mathcal{S}_\delta}$ by $\|\cdot\|_{\mathcal{S}_0}$ with the uniform inverse inequality given in Lemma 3.10, we get

$$|b_\delta(w_{\delta,h}, v_h)| \le \|w_{\delta,h}\|_{\mathcal{S}_\delta}\|v_h\|_{\mathcal{S}_\delta} \le C\|w_{\delta,h}\|_{L^2(\Omega)}\|v_h\|_{\mathcal{S}_0}$$

for some constant $C > 0$, independent of $\delta$. Condition (2.13), that is, the strengthened continuity of the approximate bilinear form, thus follows. Theorem 2.7 then implies that $\|u_{\delta,h} - u_{0,h}\|_{L^2(\Omega)} \to 0$ and the result of the above theorem follows again from the standard inverse inequality. ☐

The above theorem shows that if the finite element spaces are taken to be the same conforming finite elements for nonlocal problems and their local limit, then the discrete nonlocal solution also converges to the local discrete solution in the same space as $\delta \to 0$. The following results give an extension: it only requires that all continuous piecewise linear functions form a subspace of the finite element space.

THEOREM 3.12. *Let $u_{\delta,h}$ and $u_{0,h}$ be discrete solutions as defined in* (3.10) *with $\delta > 0$ and $\delta = 0$, respectively. Assume further that $V_{\delta,h} \subset \mathcal{S}_\delta$ is a finite element space that contains all continuous piecewise linear functions. Moreover, $V_{0,h} = \mathcal{S}_0 \cap (\bigcap_{\delta>0} V_{\delta,h})$. Then, for fixed $h$ and $\tau_h$, we have $\|u_{\delta,h} - u_{0,h}\|_{L^2} \to 0$ as $\delta \to 0$.*

*Proof.* We only need to show that $\forall v_h \in V_{0,h}$,

$$b_0(u_{*,h}, v_h) = (f, v_h),$$

where $u_{*,h}$ is a limit point of $u_{\delta,h}$, i.e., $\|u_{\delta,h} - u_{*,h}\|_{L^2} \to 0$. Consider

$$\begin{aligned}(f, v_h) - b_0(u_{*,h}, v_h) &= b_\delta(u_{\delta,h}, v_h) - b_0(u_{*,h}, v_h)\\ &= b_\delta(u_{\delta,h} - u_{*,h}, v_h) + \big[b_\delta(u_{*,h}, v_h) - b_0(u_{*,h}, v_h)\big] = \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

First, $\mathrm{I}_2 \to 0$ comes from Lemma 3.9. As for $\mathrm{I}_1$, let $w_{\delta,h} := u_{\delta,h} - u_{*,h}$; we now prove that $\mathrm{I}_1 = b_\delta(w_{\delta,h}, v_h) \to 0$. (Notice that $w_{\delta,h} \notin \mathcal{S}_0$, we cannot apply the technique used in the proof of Theorem 3.11.)

Since $w_{\delta,h}$ and $v_h$ are smooth on each element $K \subset \tau_h$, we will prove the result on each $K \subset \tau_h$. Also, we define $\Gamma_K$ for each $K \subset \tau_h$ by

$$\Gamma_K := \{\mathbf{x} \notin K \mid \mathrm{dist}(\mathbf{x}, K) \le \delta\}.$$

Then

$$b_\delta(w_{\delta,h}, v_h) = \sum_{K\in\tau_h} \int_K \int_{K\cup\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x}.$$

By [7, Theorem 1], for smooth $w_{\delta,h}$ and $v_h$ restricted on $K$,

$$\begin{aligned}&\int_K \int_K \gamma_\delta(\mathbf{x}'-\mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x}\\ &\quad\le \|w_{\delta,h}\|_{\mathcal{S}_\delta(K)}\|v_h\|_{\mathcal{S}_\delta(K)} \le C\|w_{\delta,h}\|_{H^1(K)}\|v_h\|_{H^1(K)}.\end{aligned}$$

Now by the norm equivalence of finite dimensional spaces,

$$\|w_{\delta,h}\|_{H^1(K)} \le C\|w_{\delta,h}\|_{L^2(K)} \to 0 \quad \text{as } \delta\to 0,$$

so

$$\int_K \int_K \gamma_\delta(\mathbf{x}'-\mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x} \to 0.$$

For the second term,

$$\begin{aligned}&\int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x}))d\mathbf{x}'d\mathbf{x}\\ &\quad\le 2\|w_{\delta,h}\|_{L^\infty} \int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|v_h(\mathbf{x}') - v_h(\mathbf{x})|d\mathbf{x}'d\mathbf{x}.\end{aligned}$$

Now by the norm equivalence of finite dimensional spaces,

$$\|w_{\delta,h}\|_{L^\infty} \le C\|w_{\delta,h}\|_{L^2} \to 0,$$

it remains to prove that for any $v_h \in V_{0,h}$,

$$\int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|v_h(\mathbf{x}') - v_h(\mathbf{x})|d\mathbf{x}'d\mathbf{x}$$

is bounded uniformly in $\delta$.

Since $v_h$ is piecewise smooth for $\mathbf{x}\in K$ and $\mathbf{x}'\in\Gamma_K$, respectively, we use $\mathbf{s}$ to denote the intersection of $\partial K$ and the line between $\mathbf{x}'$ and $\mathbf{x}$. By Taylor expansion, we have

$$v_h(\mathbf{x}') = v_h(\mathbf{x}) + \nabla v_h(\mathbf{x})\cdot(\mathbf{s}-\mathbf{x}) + \nabla v_{\Gamma_K}(\mathbf{s})\cdot(\mathbf{x}'-\mathbf{s}) + o(\delta).$$

Denote $K_{out} := K\cap B_\delta(\partial K)$ (the latter being a $\delta$ neighborhood of $\partial K$); then

$$\begin{aligned}&\int_K \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|v_h(\mathbf{x}') - v_h(\mathbf{x})|d\mathbf{x}'d\mathbf{x}\\ &\quad= \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|\nabla v_h(\mathbf{x})\cdot(\mathbf{s}-\mathbf{x}) + \nabla v_{\Gamma_K}(\mathbf{s})\cdot(\mathbf{x}'-\mathbf{s})|d\mathbf{x}'d\mathbf{x}\\ &\qquad+ \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta\cdot o(\delta)d\mathbf{x}'d\mathbf{x}\\ &\quad\le 2\|\nabla v\|_{L^\infty} \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|d\mathbf{x}'d\mathbf{x} + \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta\cdot o(\delta)d\mathbf{x}'d\mathbf{x}.\end{aligned}$$

Now it is easy to see that the second term on the above right-hand side tends to zero as $\delta \to 0$. For the first term, following the proof of [17, Theorem 3.7] and in particular [17, (3.35)], we have

$$2 \int_{K_{out}} \int_{\Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x}) |\mathbf{x}' - \mathbf{x}| d\mathbf{x}' d\mathbf{x} \le \int_{B_\delta(0)} \gamma_\delta(\mathbf{z}) |\mathbf{z}|^2 d\mathbf{z} \left( \sum_{e \in K} |e| \right),$$

which is bounded uniformly in $\delta$ under the assumption on the kernel $\gamma_\delta$.

In summary, we now have proved that for each $K \in \tau_h$,

$$\int_K \int_{K \cup \Gamma_K} \gamma_\delta(\mathbf{x}' - \mathbf{x})(w_{\delta,h}(\mathbf{x}') - w_{\delta,h}(\mathbf{x}))(v_h(\mathbf{x}') - v_h(\mathbf{x})) d\mathbf{x}' d\mathbf{x} \to 0,$$

which leads to $\mathrm{I}_1 \to 0$ and thus completes the proof. ∎

We note that the above theorem implies that as long as all piecewise continuous linear elements are included, the finite element spaces for nonlocal problems may not be conforming subspaces of the local limit problem but can still have solutions that converge to the conforming local finite element solution.

**3.3. A case of conditional asymptotic stability.** It is known that [38] some finite element approximations to the ND models are not asymptotically capable. In particular, the counterexamples given in [38] demonstrate that if $\delta$ is taken to be proportional to $h$, then as $h \to 0$, the discrete solutions may be convergent, but to the wrong limit. It is interesting from a practical point of view to provide some constructive remedies to avoid such undesirable effects. This is the purpose of the discussion here. We show that as long as the condition $h = o(\delta)$ is met as $\delta \to 0$, then we are able to obtain the correct local limit even for discontinuous piecewise constant finite element approximations when they are of conforming type.

THEOREM 3.13. *Let $u_\delta$, $u_{\delta,h}$ be solutions of* (3.8) *and* (3.10). *If $V_{\delta,h}$ is the piecewise constant space, then $\|u_{\delta,h} - u_0\|_{L^2} \to 0$ if $h = o(\delta)$ as $\delta \to 0$.*

*Proof.* We revisit the proof of Theorem 2.9. Recall that $a_\infty(u, v) - (f, v)$ is split into four parts. Without Assumption 2.4(ii), three of the four terms are not affected. We need to prove that $\mathrm{III} \to 0$ if $\sigma_n \cdot h_n \to 0$ as $n \to \infty$. In fact, by Lemma 3.1,

$$\mathrm{III} \le C\|v - v_n\|_{\mathcal{S}_{\sigma_n}} \le C\|v - v_n\|_{H^{\alpha/2}(\Omega)} \left( \int |\boldsymbol{\xi}|^\alpha \gamma_{\delta_n}(|\boldsymbol{\xi}|) d\boldsymbol{\xi} \right)^{1/2},$$

where $v_n \in V_{\delta_n,h_n} = W_{\sigma_n,h_n}$. A direct calculation shows

$$\int |\boldsymbol{\xi}|^\alpha \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = \delta^{\alpha-2} \int_{B(0,1)} |\boldsymbol{\xi}|^\alpha \gamma(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = C\delta^{\alpha-2}.$$

So, $\mathrm{III} \le C\sigma_n^{1-\alpha/2} \|v - v_n\|_{H^{\alpha/2}(\Omega)}$ for $\alpha \in [0, 1]$. Now, by taking $v_n$ as the piecewise constant $L^2$-orthogonal projection of $v \in \mathcal{S}_0$ onto $V_{\delta_n,h_n}$, we have [5, (1.3)]

$$\|v - v_n\|_{H^{\alpha/2}(\Omega)} \le Ch_n^{1-\alpha/2} \|v\|_{H^1(\Omega)}.$$

Thus, $\mathrm{III} \le C(\sigma_n \cdot h_n)^{1-\alpha/2} \|v\|_{H^1(\Omega)} \to 0$ as $n \to \infty$, which completes the proof. ∎

**4. Applications to the state-based PD system.** State-based PD models were presented in [35, 33] as a generalization of bond-based PD models. We refer to [27] for the mathematical analysis. Given the similarity between linear state-based models and ND models in applying the abstract framework introduced in this work, we omit most of the technical details here but emphasize filling in the necessary ingredients (and references) for verifying all the needed assumptions.

**4.1. Linear PD solids.** Using the same notation as for the ND model, we present a PD model [33], using the terms similar to [15, 27], for a constitutively linear, isotropic solid undergoing deformation. For simplicity, we omit mechanical descriptions and define directly the corresponding bilinear form:

$$\begin{aligned} B_\delta(\mathbf{u},\mathbf{v}) := \int_\Omega \Bigg( \left( k(\mathbf{x}) - \frac{\alpha(\mathbf{x})\, m(\mathbf{x})}{d^2} \right) \mathrm{Tr}(\mathcal{D}^*_\omega \mathbf{u})(\mathbf{x}) \mathrm{Tr}(\mathcal{D}^*_\omega \mathbf{v})(\mathbf{x}) \\ + \alpha(\mathbf{x}) \int_\Omega \gamma_\delta(|\mathbf{x}'-\mathbf{x}|) \mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x}) \mathrm{Tr}(\mathcal{D}^*\mathbf{v})(\mathbf{x}',\mathbf{x}) d\mathbf{x}' \Bigg) d\mathbf{x}, \end{aligned} \tag{4.1}$$

where $k(\mathbf{x})$ and $\alpha(\mathbf{x})$ are scalar functions that are closely related to the bulk and shear modulus of the material, respectively, and $\gamma_\delta$ is a kernel as defined for the ND model given earlier. The function $m(\mathbf{x})$ is defined as

$$m(\mathbf{x}) = \int_\Omega \gamma_\delta(|\mathbf{x}'-\mathbf{x}|)|\mathbf{x}'-\mathbf{x}|^2 d\mathbf{x}'.$$

$\mathrm{Tr}(\mathcal{D}^*)$ is the trace of the nonlocal gradient operator $\mathcal{D}^*$ defined in [16]:

$$\mathcal{D}^*\mathbf{u}(\mathbf{x},\mathbf{y}) := (\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \otimes \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}.$$

$\mathrm{Tr}(\mathcal{D}^*_\omega)$ is the trace of the nonlocal weighted gradient $\mathcal{D}^*_\omega$ defined in [16]:

$$\mathcal{D}^*_\omega(\mathbf{u})(\mathbf{x}) := \int_{\boldsymbol{\Omega}} \mathcal{D}^*(\mathbf{u})(\mathbf{x},\mathbf{y})\omega(\mathbf{y},\mathbf{x})\mathbf{dy}, \quad \text{where } \ \omega(\mathbf{x}',\mathbf{x}) = \frac{\mathbf{d}}{\mathbf{m}(\mathbf{x})}\gamma_\delta(|\mathbf{x}'-\mathbf{x}|)|\mathbf{x}'-\mathbf{x}|.$$

Let $\mathcal{S}^* = L^2(\Omega_\delta;\mathbb{R}^d)$. Using the same notation for vector-valued function spaces as for the scalar ND model, the energy spaces are given by

$$\mathcal{S}_\delta = \left\{ \mathbf{u} \in \mathcal{S}^* : \int_\Omega \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{x}'-\mathbf{x}|) \left( \mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x}) \right)^2 d\mathbf{x}' d\mathbf{x} < \infty, \mathbf{u} = 0 \text{ on } \Omega_{\mathcal{I}_\delta} \right\}$$

with an inner product

$$(\mathbf{u},\mathbf{v})_{\mathcal{S}_\delta} = \int_\Omega \int_{B_\delta(\mathbf{x})} \gamma_\delta(|\mathbf{x}'-\mathbf{x}|) \mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x}) \mathrm{Tr}(\mathcal{D}^*\mathbf{v})(\mathbf{x}',\mathbf{x})\, d\mathbf{x}' d\mathbf{x}$$

and an induced norm $\|\cdot\|_{\mathcal{S}_\delta}$, where the $\delta$-dependence is due to the kernel $\gamma_\delta$. Zero extensions to functions in $\mathcal{S}_\delta$ are again assumed as in the scalar case.

By the following uniform Poincaré-type inequality proved in [27, Proposition 3], we know that $(\cdot,\cdot)_{\mathcal{S}_\delta}$ is indeed a well-defined inner product that also induces a well-defined norm $\|\cdot\|_{\mathcal{S}_\delta}$.

LEMMA 4.1 (uniform Poincaré inequality). *There exists a constant $C > 0$ independent of $\delta$ such that* $\forall \delta \in (0, 1]$,

$$\|\mathbf{u}\|^2_{L^2(\Omega_\delta)} \le C\|\mathbf{u}\|^2_{\mathcal{S}_\delta} \quad \forall \mathbf{u} \in \mathcal{S}_\delta.$$

Furthermore, by [27, Lemma 3], $B_\delta$ is a bounded and coercive bilinear operator on $\mathcal{S}_\delta$, i.e., there exist positive constants $C_1$ and $C_2$ independent of $\delta$ such that, $\forall \mathbf{u}, \mathbf{v} \in \mathcal{S}_\delta$,

$$B_\delta(\mathbf{u}, \mathbf{v}) \le C_2 \|\mathbf{u}\|_{\mathcal{S}_\delta} \|\mathbf{v}\|_{\mathcal{S}_\delta} \quad \text{and} \quad B_\delta(\mathbf{u}, \mathbf{u}) \ge C_1 \|\mathbf{u}\|^2_{\mathcal{S}_\delta}.$$

Thus $B_\delta$ induces the nonlocal PD Navier operator $\mathcal{L}_\delta : \mathcal{S}_\delta \to \mathcal{S}_\delta^*$, which is a linear operator bounded uniformly in $\delta$ and is defined by

$$B_\delta(\mathbf{u}, \mathbf{v}) = \langle \mathcal{L}_\delta(\mathbf{u}), \mathbf{v} \rangle \ \ \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{S}_\delta. \tag{4.2}$$

We also denote the space $\mathcal{S}_0$ to be

$$\mathcal{S}_0 = \left\{ \mathbf{u} \in L^2(\Omega; \mathbb{R}^d) : \int_\Omega |(\nabla \mathbf{u} + \nabla \mathbf{u}^T)(\mathbf{x})|^2 d\mathbf{x} < \infty, \mathbf{u}|_{\partial\Omega} = 0 \right\} \tag{4.3}$$

equipped with a norm equivalent to $\|\cdot\|_{H_0^1(\Omega)}$.

Concerning the local limit of $\mathcal{L}_\delta$, we quote the following result [27, Thoerem 3].

LEMMA 4.2. *Assume that $k(\mathbf{x})$ and $\alpha(\mathbf{x})$ are smooth functions (say, of the class $C^1$). Then for $\mathbf{w} \in C_c^\infty(\Omega; \mathbb{R}^d)$, $\mathcal{L}_\delta \mathbf{w}$ is uniformly bounded in $L^\infty(\Omega; \mathbb{R}^d)$, and*

$$\mathcal{L}_\delta \mathbf{w}(\mathbf{x}) \longrightarrow \mathcal{L}_0 \mathbf{w}(\mathbf{x}) \quad \text{as} \ \ \delta \to 0 \quad \forall\, \mathbf{x} \in \Omega,$$

*where $\mathcal{L}_0$ is defined by $\mathcal{L}_0 \mathbf{w}(\mathbf{x}) = -\mathrm{div}(\mu(\mathbf{x}) \nabla \mathbf{w}(\mathbf{x})) - \nabla((\mu(\mathbf{x}) + \lambda(\mathbf{x}))\,\mathrm{div}\,\mathbf{w}(\mathbf{x}))$ with $\mu(\mathbf{x}) = \alpha(\mathbf{x})/[d(d+2)]$ and $\lambda(\mathbf{x}) = k(\mathbf{x}) - 2\alpha(\mathbf{x})/[d^2(d+2)]$.*

For the given $\mathbf{w}$, combining the above pointwise convergence of $\mathcal{L}_\delta \mathbf{w}$ to $\mathcal{L}_0 \mathbf{w}$ with the uniform boundedness of $\mathcal{L}_\delta \mathbf{w}$, we get $\|\mathcal{L}_\delta \mathbf{w} - \mathcal{L}_0 \mathbf{w}\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$, a result stronger than what is needed later.

**4.2. Asymptotically compatible scheme.** As before, we define the spaces $\mathcal{T}_\sigma$ in Assumption 2.1 the same way as in (3.11) except with $\{\mathcal{S}_\delta\}$ denoting vector-valued function spaces associated with the state-based PD model. We then define $\mathbf{u}_\delta$.

$$\text{Find} \ \ \mathbf{u}_\delta \in \mathcal{S}_\delta \text{ such that } B_\delta(\mathbf{u}_\delta, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall v \in \mathcal{S}_\delta. \tag{4.4}$$

Similarly, we define the limiting bilinear form on $\mathcal{S}_0$:

$$B_0(\mathbf{u}, \mathbf{v}) := \langle \mathcal{L}_0 \mathbf{u}, \mathbf{v} \rangle \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{S}_0.$$

It is well known that $B_0$ is bounded and coercive on $\mathcal{S}_0$ [11, 19]. We set $a_\sigma := B_{1/\sigma}$ and $\mathcal{A}_\sigma := \mathcal{L}_{1/\sigma}$ for $\sigma \in [1, \infty]$. Then one part of Assumption 2.1(i) is given by Lemma 4.1, while the other is precisely [26, Lemma 2.2] restated as the lemma below.

LEMMA 4.3. *There exists a constant $C > 0$ only depending on $\Omega$ such that*

$$\|\mathbf{u}\|^2_{\mathcal{S}_\delta} \le C \left( \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^2 \gamma_\delta(|\boldsymbol{\xi}|) d\boldsymbol{\xi} \right) \|\mathbf{u}\|^2_{H^1(\Omega)} \quad \forall \mathbf{u} \in H^1(\Omega) \cap \mathcal{S}_\delta.$$

Assumption 2.1(ii) is just [27, Lemma 7] that is restated below without proof.

LEMMA 4.4. *Let $\mathbf{u}_\delta \in \mathcal{S}_\delta$ for $\delta > 0$. If $\sup_{\delta>0} \|\mathbf{u}_\delta\|_{\mathcal{S}_\delta} < \infty$, then the sequence $(\mathbf{u}_\delta)$ is precompact in $L^2(\Omega; \mathbb{R}^d)$. Moreover, any limit point $\mathbf{u} \in \mathcal{S}_0$.*

Meanwhile, the discussions in the previous subsection easily lead to Assumptions 2.2 and 2.3 in the present context.

For discrete approximations, as in the ND case, let $\{V_{\delta,h}\} \subset \mathcal{S}_\delta = \mathcal{T}_\sigma$ denote a family of finite element subspaces where $h$ characterizes the mesh size and for any

$\mathbf{v} \in \mathcal{S}_\delta$, we have a family of elements $\{\mathbf{v}_h \in V_{\delta,h}\}$ such that $\|\mathbf{v}_h - \mathbf{v}\|_{\mathcal{S}_\delta} \to 0$ as $h \to 0$. Then, the Galerkin approximation is to replace $\mathcal{S}_\delta$ by $V_{\delta,h}$ in (4.4):

$$\text{Find } \mathbf{u}_{\delta,h} \in V_{\delta,h} \text{ such that } B_\delta(\mathbf{u}_{\delta,h}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2} \quad \forall \mathbf{v} \in V_{\delta,h}. \tag{4.5}$$

Clearly, $W_{\sigma,h} := V_{1/\sigma,h}$ satisfies Assumption 2.4(i). The Assumption 2.4(ii) is also satisfied with $\hat{V}_{\delta,h} \subset V_{\delta,h}$ and $\hat{V}_{\delta,h}$ being a vector-valued version of the continuous piecewise linear element subspace that approximates $\mathcal{S}_0 = \mathcal{T}_\infty$ as $h \to 0$.

Now we are ready to state the convergence theorem on the finite element approximations of the linear state-based PD model, as a direct consequence of Theorem 2.9. We skip the detailed proof.

THEOREM 4.5. *Let* $\mathbf{u}_\delta$*,* $\mathbf{u}_{\delta,h}$ *be the solutions of* (4.4) *and* (4.5)*, and* $\hat{V}_{\delta,h} \subset \mathcal{S}_\delta$ *is described as above. If* $\hat{V}_{\delta,h} \subset V_{\delta,h}$*, then* $\|\mathbf{u}_{\delta,h} - \mathbf{u}_0\|_{L^2} \to 0$ *as* $\delta \to 0$*,* $h \to 0$*.*

Consequently, we see also that for the state-based PD models, the asymptotic compatibility is preserved for conforming finite element approximations that contain continuous piecewise linear finite element subspaces.

By extending the convergence of the discrete linear forms from the ND models to the state-based PD models, we can also get similar results on the convergence of the discrete solutions between the PD models and the local Navier equations as $\delta \to 0$ on a fixed mesh.

**5. Numerical experiments.** Here, we report numerical results that validate our analysis and provide results on the order of convergence that cannot be seen from our convergence theorems. We use a discontinuous piecewise linear finite element to solve a one-dimensional nonlocal problem $-\mathcal{L}_\delta u = f$ on $(0,1)$ with the nonlocal constraint $u = 0$ outside $(0,1)$ and the nonlocal operator given by

$$\mathcal{L}_\delta u = 2\int_{-\delta}^{\delta} \gamma_\delta(s)(u(x+s) - u(x))ds.$$

A special kernel is chosen to be $\gamma_\delta(s) = \delta^{-2}s^{-1}$ in our numerical examples.

We choose a relatively smooth function as $u_0$ given by a fourth order B-spline:

$$u_0(x) = \frac{1}{h^5}\begin{cases} 0, & x < 0, \\ \dfrac{x^4}{120}, & 0 \le x < 0.2, \\ -\dfrac{x^4}{30} + \dfrac{x^3}{30} - \dfrac{x^2}{100} + \dfrac{x}{750} - \dfrac{1}{15000}, & 0.2 \le x < 0.4, \\ \dfrac{x^4}{20} - \dfrac{x^3}{10} + \dfrac{7x^2}{100} - \dfrac{x}{50} + \dfrac{31}{15000}, & 0.4 \le x < 0.5, \end{cases}$$

with $u_0$ symmetric (even) with respect to $x = 0.5$. Its graph shown in Figure 2.

We then calculate analytically $f := -u_0''$ and solve nonlocal problems on a uniform mesh using a discontinuous piecewise linear finite element space. The corresponding pointwise errors $e(x) = u_{\delta,h}(x) - u_0(x)$ are plotted in Figures 3–5 for the three cases, respectively. Note that the red dots are highlighted to show errors at nodal points. Qualitatively, one may observe some common features in these plots: first, while the errors are generally discontinuous at the nodal points given the use of discontinuous finite element functions, the magnitude of discontinuity diminishes as $\delta \to 0$, leading to a continuous (and conforming) approximation to the local limit solution as predicted by the theory; second, the error profiles, in particular, the maximum and minimum

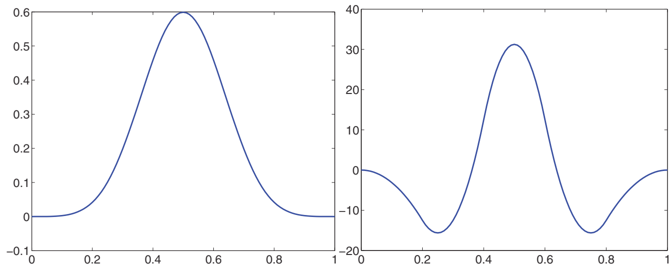

FIG. 2. *Graph of* $u_0(x)$ *and its second order derivative.*

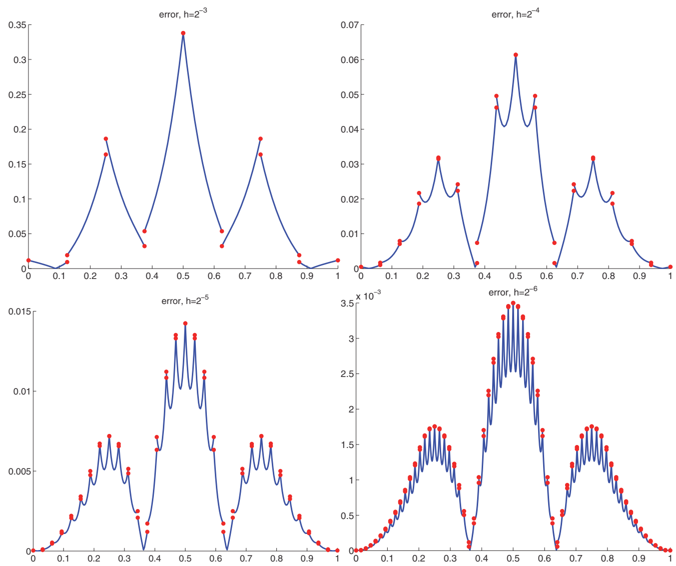

FIG. 3. *Pointwise error* $u_{\delta,h}(x) - u_0(x)$ *with* $r = \frac{\delta}{h} = 3$ *and* $h = 2^{-k}$, $k = 3, 4, 5, 6$.

envelopes of the errors, are all nicely correlated with the second derivatives of $u_0$ shown in Figure 2. While this does not follow from our analytical framework here, this is consistent with the errors of typical piecewise linear interpolations and may not also tie this with the more detailed truncation error analysis given in [38]. Meanwhile, the error plots also show different oscillation patterns of the errors inside the mesh intervals in comparison with those at nodal points for the three cases. A possible explanation is that oscillations are related to discretization errors that become more pronounced with smaller $\delta$ due to the reduction of modeling errors (between nonlocal and local equations).

To be more quantitative, the $L^2$ error of the function values $\|e\|_0$ and the piecewise first order derivatives $\|e\|_1$ are computed along with the mesh weighted discrete $\ell^2$ errors of functions values and the first order derivatives at midpoints of mesh intervals (denoted by $\|\bar{e}\|_0$ and $\|\bar{e}\|_1$, respectively). Tables 1–3 provide errors and convergence orders (given inside parentheses) in different norms and with different relations between $\delta$ and $h$.

Table 1 shows the errors and convergence rates when $\delta/h$ is fixed as a constant as the mesh is refined with a decreasing $h$. The errors are measured against the exact solution $u_0$, the local limit. The $L^2$ convergence rate for function values is of second

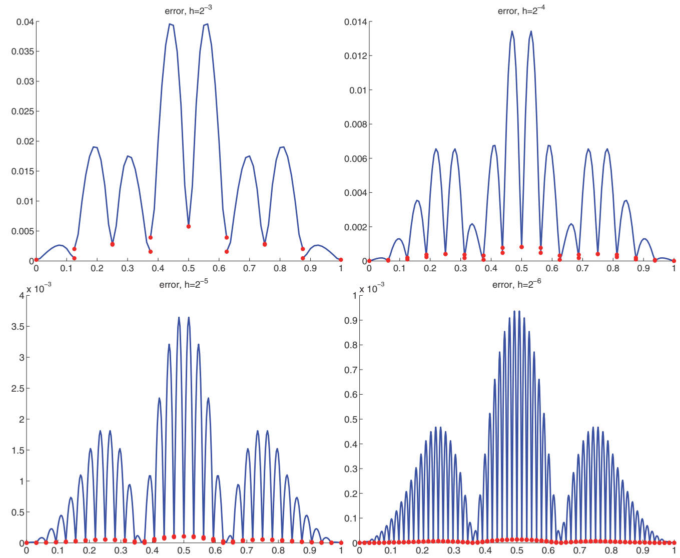


FIG. 4. *Pointwise error* $u_{\delta,h}(x) - u_0(x)$ *with* $\delta = h^2$ *and* $h = 2^{-k}$*,* $k = 3, 4, 5, 6$.

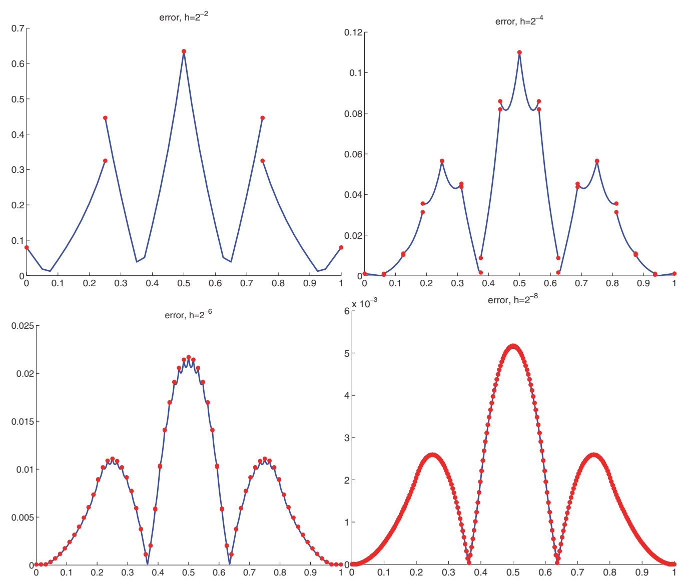


FIG. 5. *Pointwise error* $u_{\delta,h}(x) - u_0(x)$ *with* $\delta = \sqrt{h}$ *and* $h = 2^{-k}$*,* $k = 3, 4, 6, 8$.

order and that for piecewise first order derivatives is of first order. Given that we expect the modeling error (that is, the difference between solutions of the nonlocal and local models) is of the order $\delta^2$ [38, 40], we see that the orders of the numerical approximation errors are consistent with the optimal orders predicted by standard approximation theory.

For Table 2, we let $\delta = h^2$ when refining the mesh. This is the case where $\delta$ decreases faster than $h$ while they both go to zero. We find that the $L^2$ convergence

Table 1
*Errors and convergence rates for fixed $r := \delta/h = 3$.*

| $h$ | $\|e\|_0$ | $\|e\|_1$ | $\|\bar{e}\|_0$ | $\|\bar{e}\|_1$ |
|---|---|---|---|---|
| $2^{-3}$ | $1.15 \times 10^{-1}(-)$ | $1.50 \times 10^{0}(-)$ | $1.50 \times 10^{-1}(-)$ | $1.32 \times 10^{0}(-)$ |
| $2^{-4}$ | $2.25 \times 10^{-2}(2.4)$ | $3.79 \times 10^{-1}(2.0)$ | $2.79 \times 10^{-2}(2.4)$ | $2.58 \times 10^{-1}(2.3)$ |
| $2^{-5}$ | $5.26 \times 10^{-3}(2.1)$ | $1.42 \times 10^{-1}(1.4)$ | $6.43 \times 10^{-3}(2.1)$ | $5.81 \times 10^{-2}(2.2)$ |
| $2^{-6}$ | $1.29 \times 10^{-3}(2.0)$ | $6.54 \times 10^{-2}(1.1)$ | $1.58 \times 10^{-3}(2.0)$ | $1.42 \times 10^{-2}(2.0)$ |
| $2^{-7}$ | $3.22 \times 10^{-4}(2.0)$ | $3.20 \times 10^{-2}(1.0)$ | $3.92 \times 10^{-4}(2.0)$ | $3.51 \times 10^{-3}(2.0)$ |

Table 2
*Errors and convergence rates for $\delta = h^2$.*

| $h$ | $\|e\|_0$ | $\|e\|_1$ | $\|\bar{e}\|_0$ | $\|\bar{e}\|_1$ |
|---|---|---|---|---|
| $2^{-3}$ | $1.67 \times 10^{-2}(-)$ | $4.94 \times 10^{-1}(-)$ | $2.79 \times 10^{-3}(-)$ | $8.19 \times 10^{-2}(-)$ |
| $2^{-4}$ | $4.62 \times 10^{-3}(1.9)$ | $2.52 \times 10^{-1}(1.0)$ | $3.79 \times 10^{-4}(3.0)$ | $2.53 \times 10^{-2}(1.8)$ |
| $2^{-5}$ | $1.21 \times 10^{-3}(1.9)$ | $1.27 \times 10^{-1}(1.0)$ | $4.76 \times 10^{-5}(3.0)$ | $6.40 \times 10^{-3}(2.0)$ |
| $2^{-6}$ | $3.08 \times 10^{-4}(2.0)$ | $6.34 \times 10^{-2}(1.0)$ | $5.96 \times 10^{-6}(3.0)$ | $1.63 \times 10^{-3}(2.0)$ |

Table 3
*Errors and convergence rates for $\delta = \sqrt{h}$.*

| $h$ | $\|e\|_0$ | $\|e\|_1$ | $\|\bar{e}\|_0$ | $\|\bar{e}\|_1$ |
|---|---|---|---|---|
| $2^{-2}$ | $2.35 \times 10^{-1}(-)$ | $3.32 \times 10^{0}(-)$ | $4.22 \times 10^{-1}(-)$ | $2.74 \times 10^{0}(-)$ |
| $2^{-4}$ | $4.31 \times 10^{-2}(1.2)$ | $5.94 \times 10^{-1}(1.2)$ | $4.94 \times 10^{-2}(1.5)$ | $5.12 \times 10^{-1}(1.2)$ |
| $2^{-6}$ | $9.54 \times 10^{-3}(1.1)$ | $1.29 \times 10^{-1}(1.1)$ | $9.84 \times 10^{-3}(1.2)$ | $1.11 \times 10^{-1}(1.1)$ |
| $2^{-8}$ | $2.31 \times 10^{-3}(1.0)$ | $3.11 \times 10^{-2}(1.0)$ | $2.31 \times 10^{-3}(1.0)$ | $2.66 \times 10^{-2}(1.0)$ |

orders stay the same as in the above case, though the errors are smaller than the other cases, for a given mesh of the same size.

On the other hand, in the results for $\delta = \sqrt{h}$ listed in Table 3, since $\delta$ decreases more slowly than $h$, although the correct local limit is obtained as predicted by our theory, the $L^2$ convergence order for function values drops to first order. A possible explanation is that the modeling error dominates and it is of the order $O(\delta^2) = O(h)$.

Data in these tables on the discrete $\ell^2$ norms show similar patterns in convergence order as the continuous error norms in Tables 1 and 3, but some superconvergence order can be observed in Table 2 for discrete norms. For analysis of super convergence properties for nonlocal equations, we refer to some related findings in [38]. In addition, we note that with the same mesh spacing, say, $h = 2^{-6}$, the errors decrease as $\delta$ changes from $O(\sqrt{h})$ to $O(h)$ and $O(h^2)$, a reasonable and desirable behavior showing the efficiency of localization (small horizon) if the objective is to capture the local limit when the latter is well defined.

**6. Conclusion.** In this work, we established an abstract mathematical framework to the analysis of a class of asymptotically compatible schemes for the approximations of parametrized problems. The motivation was to develop a robust discretization of nonlocal models for multiscale problems where the nonlocal models can be seen as parametrized by the horizon that measures the range of nonlocal interactions. Yet, the abstract framework allows us to put the discussion in a broader context. Not only does it reveal the true essence of the nonlocal problems, but it may also be applicable to other parametrized problems. The analysis is valid with minimal assumptions on the underlying problems, the solution regularity, and the approximation spaces. Among various studies of numerical methods and their asymptotic behavior with a parameter approaching to a limit (ranging from uniformly convergent schemes for singularly perturbed problems [30], numerical viscosity solutions of conservation laws [9] to asymptotically preserving schemes for kinetic equations [22]), perhaps the analysis in [21] offers the closest resemblance to the work here in spirit.[1] In [21], the approximations to the zero mean free path $\epsilon \to 0$ limit or diffusive limit of radiative transport models have been studied. The models studied there share similar features as the nonlocal models considered here in that the parametrized problems may have singular solutions but they approach a more regular solution of the diffusive limit. It has been concluded in [21] that piecewise constant approximations would only lead to a uniform constant solution in such a limit but that finite elements containing enough continuous elements can recover the correct limit as both mesh size and mean free path go to zero, a phenomenon that is reminiscent to our finding here for the local limit of nonlocal problems. This provides additional motivation to present the more abstract framework developed here so it may be applied to problems that arise from different applications.

Meanwhile, the illustrative applications of the framework discussed here offered some new results on the numerical analysis of nonlocal problems as well. For a homogeneous Dirichlet type nonlocal constrained value problems associated with a scalar ND equation, we showed that any finite element discretization that contains piecewise linear functions provides an asymptotically compatible scheme and thus is a robust discretization to both the nonlocal problems and the local limit. The convergence of approximations to the correct solutions and models is ensured independent of the relations between the horizon parameter $\delta$ and the discretization parameter $h$ as shown in the diagram 1. Moreover, we showed that such discrete schemes of the nonlocal problem converge to the conforming finite element scheme of the local differential problem as the horizon goes to zero for fixed $h$. We further extended similar results to a nonlocal state-based PD system to show the generality of the abstract framework.

We note also that while our earlier works exposed possible risks in using piecewise constant finite element for nonlocal problem when the horizon is proportional to the mesh size [38], the present study provided new remedy to deal with the issue by showing that piecewise constant finite element for the ND problem, when conforming, would be a conditionally asymptotically compatible discretization, under the natural condition that $h = o(\delta)$, which has been pointed out in some simulation-based studies [6, 10].[2]

In addition, to compensate for the lack of analysis on the order of convergence, we carried out numerical experiments of a one-dimensional ND equation discretized with conforming discontinuous piecewise linear finite elements. The discontinuous linear finite element solutions of the nonlocal problem converge to the solution of the correct local differential problem as predicted no matter how $\delta$ varies with $h$, but the

[1]We thank Chi-Wang Shu and Cory Hauck for bringing [21] to our attention when we presented our results at recent conferences.

[2]This has also been discussed in personal communications with M. Parks and R. Lehoucq.

convergence rates show dependence on the choices of $\delta$ and $h$. The convergence and superconvergence orders observed lead to interesting theoretical issues to be studied further along with the development of possible postprocessing techniques [12] to improve the order of convergence especially for derivatives and stress variables when singular behaviors are likely to be present in practice.

Finally, we note that our study is restricted to conforming approximations and linear problems. More studies are underway to extend them to other varieties of approximation methods including particle-based or meshfree methods and also to nonlinear and multiscale settings.

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