TRACE THEOREMS FOR SOME NONLOCAL FUNCTION SPACES WITH HETEROGENEOUS LOCALIZATION

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ABSTRACT. It is a classical result of Sobolev spaces that any H^1 function has a well-defined $H^{1/2}$ trace on the boundary of a sufficient regular domain. In this work, we present stronger and more general versions of such a trace theorem in a new nonlocal function space $\mathcal{S}(\Omega)$ satisfying $H^1(\Omega) \subset \mathcal{S}(\Omega) \subset L^2(\Omega)$. The new space $\mathcal{S}(\Omega)$ is associated with a nonlocal norm characterized by a nonlocal interaction kernel that is defined heterogeneously with a special localization feature on the boundary. Through the heterogeneous localization, we are able to show that the $H^{1/2}$ norm of the trace on the boundary can be controlled by the nonlocal norm that are weaker than the classical H^1 norm. In fact, the trace theorems can be essentially shown without imposing any extra regularity of the function in the interior of the domain other than being square integrable. Implications of the new trace theorems to the coupling of local and nonlocal equations and possible further generalizations are also discussed.

1. Introduction

On a domain $\Omega \in \mathbb{R}^d$ with boundary $\partial\Omega$, a trace operator T on a subset Γ of $\partial\Omega$ is defined as

$$Tu = u|_\Gamma \qquad \forall u \in C^1(\bar{\Omega}),$$

where $\bar{\Omega}$ is the closure of Ω. It is a classical result of Gagliardo [15] that the linear operator T can be extended continuously as a map from $H^1(\Omega)$, the standard Sobolev space of L^2 functions with square integrable derivatives, to $H^{1/2}(\Gamma)$. The purpose of this paper is to show that the same trace map exists and is continuous on a nonlocal function space $\mathcal{S}(\Omega)$ that is the completion of $C^1(\bar{\Omega})$ with respect to the norm

$$\|\cdot\|_{\mathcal{S}(\Omega)} = (\|\cdot\|^2_{L^2(\Omega)} + |\cdot|^2_{\mathcal{S}(\Omega)})^{1/2}, \tag{1}$$

with the associated nonlocal semi-norm $|\cdot|_{\mathcal{S}(\Omega)}$ defined by

$$|u|^2_{\mathcal{S}(\Omega)} = \int_\Omega \int_{\Omega\cap\mathcal{H}(\boldsymbol{x})} \gamma(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 d\boldsymbol{y} d\boldsymbol{x}, \tag{2}$$

corresponding to some nonlocal interaction kernel $\gamma = \gamma(\boldsymbol{x},\boldsymbol{y})$ with its support or the so called effective interaction neighborhood denoted by $\mathcal{H}(\mathbf{x})$ to be heterogeneously localized as $\boldsymbol{x}$ approaches the boundary of Ω. These new results on the trace map can be viewed as generalizations of the classical trace theorem for local Sobolev spaces to a nonlocal setting. Indeed, for the class of nonlocal spaces under consideration, the Sobolev space $H^1(\Omega)$ can be continuously embedded in $\mathcal{S}(\Omega)$, as shown in this work, so that the new trace theorems

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effectively provide new ways to improve classical result in the H^1 space with the latter becoming a consequence.

The nonlocal interaction kernel γ considered here is taken to be spatially heterogeneous, which is different from those represented by typical translate-invariant kernels studied in the literature [4]. We note that variable order and variable growth function spaces have been a popular subject with a rich history and much recent interest, see for instance [7, 28]. The special feature of our work lies in the heterogeneous localization of nonlocal interactions at the boundary. To be more specific, we work with nonlocal kernels $\gamma = \gamma(\boldsymbol{x}, \boldsymbol{y})$ with a finite range of interactions, that is, having their support for $\boldsymbol{y}$ over an effective neighborhood $\mathcal{H}(\boldsymbol{x})$ at every $\boldsymbol{x} \in \Omega$. This choice is inherited from the peridynamic model of continuum mechanics [32]. In previous mathematical analysis of peridynamics and related nonlocal diffusion models, see [10] for instance, $\mathcal{H}(\boldsymbol{x})$ is usually taken as a ball of a fixed radius δ for all $\boldsymbol{x}$ in the domain. The parameter δ is called the peridynamic or influence horizon [32]. We adopt the practice of having δ and $\mathcal{H}(\boldsymbol{x})$ vary with $\boldsymbol{x}$, thus leading to the study of nonlocal spaces and nonlocal operators with a variable horizon. For an earlier study of peridynamics with a variable horizon, we refer to [33], which is primarily concerned with issues relevant to computational modeling. Heuristically we may want $\gamma(\boldsymbol{x}, \boldsymbol{y})|\boldsymbol{x} - \boldsymbol{y}|^2$ behaving more and more like a Dirac delta function while approaching Γ, a part of the boundary $\partial\Omega$. Such a localization leads naturally to a seamless transition from a domain featuring nonlocal interactions governed by nonlocal models to a domain having localized interactions as described by classical PDEs, see section 2 for more discussions. A class of kernels that fits the desired profile is

$$\gamma(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma} \left(\frac{|\boldsymbol{y} - \boldsymbol{x}|}{\delta(\boldsymbol{x})} \right) \tag{3}$$

where $\hat{\gamma} = \hat{\gamma}(s)$ is a non-increasing nonnegative function defined for $s \in (0, 1)$ with a finite $d + 1$ moment. For illustration, we focus on the case that $\hat{\gamma}$ is a constant function for most of the paper and present some discussions for more general kernel functions towards the end. The influence horizon $\delta = \delta(\boldsymbol{x})$ is a function defined on Ω that approaches zero when $\boldsymbol{x}$ approaches the boundary. A simple choice would be

$$\delta(\boldsymbol{x}) = \sigma \operatorname{dist}(\boldsymbol{x}, \Gamma), \quad \boldsymbol{x} \in \Omega \,, \tag{4}$$

for some $\sigma \in (0, 1]$ and $\Gamma \subset \partial\Omega$. The associated nonlocal neighborhood $\mathcal{H}(\boldsymbol{x})$ is defined by

$$\mathcal{H}(\boldsymbol{x}) := \{ \boldsymbol{y} \in \Omega : |\boldsymbol{y} - \boldsymbol{x}| \le \delta(\boldsymbol{x}) \} \,.$$

The main contribution of this work is to show that, by allowing heterogeneous localization as described in the case with the influence horizon $\delta = \delta(\boldsymbol{x})$ and the vanishing effective neighborhood $\mathcal{H}(\boldsymbol{x})$ when $\boldsymbol{x}$ approaches the boundary $\partial\Omega$, we expect to have a well defined continuous trace map from the associated nonlocal space $\mathcal{S}(\Omega)$ to $H^{1/2}(\partial\Omega)$.

As in studies of classical trace theorems for standard Sobolev spaces, we proceed first to a special stripe domain $\Omega = (0, r) \times \mathbb{R}^{d-1}$ with a portion of its boundary $\Gamma = \{0\} \times \mathbb{R}^{d-1}$ where r is any given positive constant. The following result gives a special instance, in terms of choices of the spatial domain and the nonlocal interaction kernel, of the more general trace theorem.

Theorem 1.1 (Special trace theorem)**.** *For $\Omega = (0, r) \times \mathbb{R}^{d-1}$ and $\Gamma = \{0\} \times \mathbb{R}^{d-1}$, there exists a constant C depends only on d such that such that for any $u \in C^1(\bar{\Omega}) \cap \mathcal{S}(\Omega)$,*

$$\|u\|_{L^2(\Gamma)} \le C \Big(r^{-1/2} \|u\|_{L^2(\Omega)} + r^{1/2} |u|_{\mathcal{S}(\Omega)} \Big) \,, \tag{5}$$

and for $d \geq 2$*,*

$$|u|_{H^{1/2}(\Gamma)} \leq C\Big(r^{-1}\|u\|_{L^2(\Omega)} + |u|_{\mathcal{S}(\Omega)}\Big)\,. \tag{6}$$

***Remark* 1.2.** *We are mainly interested in the small* r *dependence of the imbedding coefficients. For large* r*, the results remain true with uniformly bounded coefficients, that is,* $\|u\|_{H^{1/2}(\Gamma)} \leq C\|u\|_{\mathcal{S}(\Omega)}$ *where* C *is a constant as* $r \to \infty$*. We note in addition that the need of the* $L^2(\Omega)$ *norm in* (5) *and* (6) *is similar to that for standard trace inequalities in Sobolev spaces. The dependence on the* $L^2(\Omega)$ *norm may be removed by considering* L^p *type norms for the trace with a suitable choice of* p*, just like the classical counterpart in Sobolev spaces.*

We leave the proof of the theorem 1.1 to section 4. Once the special trace theorem is established, we know that the trace map T admits a continuous extension on $\mathcal{S}(\Omega)$. The continuous mapping from $\mathcal{S}(\Omega)$ to $H^{1/2}(\Gamma)$ allows us to make sense the notion of restriction of a function to the boundary of domain. Thus suitable Dirchlet boundary value problems become well-defined for nonlocal models with solutions in the space $\mathcal{S}(\Omega)$. Furthermore, analogous to the case of classical Sobolev spaces, see for example [9], the previous theorem has a more general version valid for Lipschitz domains Ω in $\mathbb{R}^d$ for $d \geq 2$, given in the following theorem.

Theorem 1.3 (General trace theorem)**.** *Assume that* Ω *is a bounded simply connected Lipschitz domain in* $\mathbb{R}^d$ *(*$d \geq 2$*) and* $\Gamma = \partial\Omega$*, then there exists a constant* C *depending only on* Ω *and* Γ *such that*

$$\|Tu\|_{H^{\frac{1}{2}}(\Gamma)} \leq C\|u\|_{\mathcal{S}(\Omega)}\,, \quad \forall u \in \mathcal{S}(\Omega)\,. \tag{7}$$

The proof of the previous theorem can be found in section 5. For completeness, let us recall the standard notion of Lipschitz domain as a domain whose boundary is locally the graph of a Lipschitz continuous function, or more precisely, we have

Definition 1.4. *A bounded open subset* Ω *of* $\mathbb{R}^d$ *is called a Lipschitz domain, if, for every* $\boldsymbol{p} \in \partial\Omega$*, there exists a pair* $\{B(\boldsymbol{p}, r), \varphi_{\boldsymbol{p}}\}$ *where* $\varphi_{\boldsymbol{p}} : \mathbb{R}^{d-1} \to \mathbb{R}$ *such that*

(1) $|\varphi_{\boldsymbol{p}}(\bar{\boldsymbol{x}}) - \varphi_{\boldsymbol{p}}(\bar{\boldsymbol{y}})| \leq M|\bar{\boldsymbol{x}} - \bar{\boldsymbol{y}}|$ *for* $\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}} \in \mathbb{R}^{d-1}$ *,*
(2) $\Omega \cap B(\boldsymbol{p}, r) = \{(x_1, \bar{\boldsymbol{x}}) \in \mathbb{R}^d | x_1 > \varphi_{\boldsymbol{p}}(\bar{\boldsymbol{x}})\} \cap B(\boldsymbol{p}, r)$ *.*

The remainder of the paper is organized as follows. Section 2 contains some comments on the motivation of our work and discussions on the connections to classical results. Some more detailed studies on nonlocal spaces and the associated norms are given in section 3. A few interesting inequalities needed in the proofs of the trace theorems, such as nonlocal Hardy type inequalities that extend their classical local versions, are also presented there.

We note that in order to avoid notation complication, a special kernel is used in sections 3 to 5, namely, we choose $\hat{\gamma}$ to be the characteristic function on $(0, 1)$, and

$$\gamma(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \chi_{(0,1)}(|\boldsymbol{y} - \boldsymbol{x}|)\,, \quad \text{where } \delta(\boldsymbol{x}) = \sigma \mathrm{dist}(\boldsymbol{x}, \Gamma)\,. \tag{8}$$

Specifically, $\sigma = 1$ is used in sections 3 to 5 but one can easily check that discussions contained in these sections remain valid for $\sigma < 1$ as well. In fact, further discussions on the particular set-up that we choose to study in this work, such as the forms of $\delta = \delta(\boldsymbol{x})$ and settings involving more general kernels, are presented in section 6, along with discussions on the uniformity of the inequalities with respect to $\sigma \to 0$. Given that our focus here is on the trace map as motivated by recent studies on nonlocal models such as peridynamics, other interesting issues concerning the nonlocal space and its various generalizations are not discussed at length in this work. Some brief discussions on more general and open issues are included in section 7 with more in depth studies to be given in upcoming works.

2. Motivation and connection to classical results

A few comments on the motivation of our work are in order. There have been many recent studies on nonlocal spaces and associated nonlocal operators that have appeared naturally in various branches of physical, biological and social sciences, see [3, 4, 5, 10, 11, 14, 16, 21, 24, 26, 27, 30, 34] and references cited therein on related applications and mathematical analysis. Moreover, developing nonsmooth calculus [17] has also been an active subject of mathematical research with strong connections to geometry [13].

Generically, nonlocal equations posed on a domain $\Omega \subset \mathbb{R}^d$ are complemented by nonlocal boundary conditions, or more precisely, constraints on a some domain with nonzero d-dimensional volume, hence leading to so-called constrained value problems [10]. To avoid the use of such nonlocal constraints, the nonlocal operators need to be properly modified near the boundary, which is often the case for fractional differential equations [5]. For a more recent survey on the nonlocal elliptic equations, we refer to [31]. In order to have well-defined nonlocal problems on Ω with Dirichlet type data on part of its boundary $\partial\Omega$ of codimension-1, study of the trace map becomes a necessity. More often the trace map is defined if functions under consideration enjoy suitable interior regularity. A consequence of a well-defined trace map with the trace belonging to a space similar to that for standard Sobolev spaces would allow a seamless coupling between a classical, local PDE (for instance the Poisson equation $-\Delta u = f$) on one side Ω_- of a codimension-1 interface Γ with a nonlocal equation (say the variational equation $-\mathcal{L}u = f$ associated with the nonlocal energy) on the other side Ω_+ of Γ, see Fig. 1 for an illustration (the circular domains depict domains of nonlocal interactions associated with a heterogeneously defined horizon parameter). The study on transmission conditions and the well-posedness of the coupled local and nonlocal models is left to a separate work. Furthermore, having varying horizon allows one to harvest the flexibility in working with nonlocal interactions on a wide range of scales so that more effective numerical simulations can be carried out, along the lines of asymptotically compatible schemes [35].

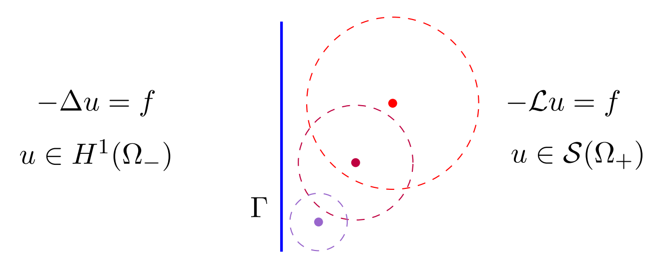

Figure 1. A PDE model (in Ω_-) is coupled with a nonlocal model (in Ω_+) using suitably defined boundary trace and transmission condition on Γ.

We note that the new nonlocal trace theorems can be viewed as extensions and refinement of their classical counterparts. Indeed, this can be appreciated from different perspectives.

First, the approach taken in this work provides one avenue to achieve sufficient regularity for defining the trace map without imposing extra regularity away from the boundary. More specifically, for a proper subdomain Ω' of Ω, with a positive distance away from $\partial\Omega$, one can show that with the kernel given by (8), functions in $\mathcal{S}(\Omega)$ are generally not expected to have regularity better than $L^2(\Omega')$ over the subdomain Ω', or may be significantly less regular away from the boundary than H^1 functions. Yet, as elucidated in the introduction and rigorously established in the theorems, due to the shrinking horizon towards the boundary, there is enough regularity for these functions to have well-defined traces just on the boundary itself. Intuitively, this is a natural consequence of the localization of nonlocal interactions on the boundary.

On the other hand, it is well-known that (see [4]) for a translation invariant and radial kernel $\gamma(\mathbf{x},\mathbf{y}) = \tilde{\gamma}(|\mathbf{x}-\mathbf{y}|)$ with a finite second moment

$$\int_{\mathbb{R}^d} \tilde{\gamma}(|\mathbf{x}|)|\mathbf{x}|^2 d\mathbf{x} < \infty,$$

the nonlocal norm is bounded from above by a suitable multiple of the conventional H^1 norm and the Sobolev space H^1 is continuously imbedded in the corresponding nonlocal function space. Moreover, a result in the paper [4] states that when $\tilde{\gamma} = \tilde{\gamma}(|\mathbf{x}|)$ is taken to be a suitable sequence with $\tilde{\gamma}(|\mathbf{x}|)|\mathbf{x}|^2$ approximating the Dirac delta measure, the limit of the corresponding nonlocal spaces recovers the H^1 space. Such localization, given the translation invariance of the kernel, can be viewed as a uniform localization. This does not serve our purpose of constructing a function space that allow functions having weak regularity in the interior of the domain but still having a well-defined trace on the boundary. Nevertheless, it is natural to expect that the continuous imbedding of H^1 into the nonlocal space remains true for a variable horizon with the localization feature on the boundary, so that the classical trace theorem of H^1 space becomes a direct consequence of the nonlocal counterpart established in this work.

We now present a proposition showing the relation between the classical Sobolev space $H^1(\Omega)$ and the new nonlocal space $\mathcal{S}(\Omega)$ under consideration here.

Proposition 2.1. *For $\delta(\boldsymbol{x}) = \sigma \cdot dist(x, \Gamma)$ with $\sigma \in (0,1)$, the space $H^1(\Omega)$ is continuously imbedded in $\mathcal{S}(\Omega)$ and there exists a constant C depending only on σ and Ω such that*

$$\|u\|_{\mathcal{S}(\Omega)} \le C\|u\|_{H^1(\Omega)}\,, \quad \forall u \in H^1(\Omega)\,. \tag{9}$$

Moreover, C is independent of σ for σ small.

Proof. We begin with a proof of (9) for a smooth function $u \in C^1(\bar{\Omega}) \cap H^1(\Omega)$.

Let the kernel γ be defined as (3). We can have a standard extension of u to $\mathbb{R}^d$ such that

$$\|u\|_{H^1(\mathbb{R}^d)} \le C_1\|u\|_{H^1(\Omega)}\,,$$

where C_1 only depends on Ω. Notice that for any $\boldsymbol{h} \in \mathbb{R}^d$,

$$|u(\boldsymbol{x}+\boldsymbol{h}) - u(\boldsymbol{x})|^2 = \Big|\int_0^1 \nabla u(\boldsymbol{x}+t\boldsymbol{h})\cdot \boldsymbol{h}dt\Big|^2 \le |\boldsymbol{h}|^2 \int_0^1 |\nabla u(\boldsymbol{x}+t\boldsymbol{h})|^2 dt\,.$$

So

$$\begin{aligned}
|u|^2_{\mathcal{S}(\Omega)} &\le \int_{\mathbb{R}^d}\int_{|\boldsymbol{h}|<\delta(\boldsymbol{x})} \gamma(\mathbf{x},\mathbf{x}+\boldsymbol{h})|u(\boldsymbol{x}+\boldsymbol{h}) - u(\boldsymbol{x})|^2 d\boldsymbol{h}d\boldsymbol{x}\\
&\le \int_{\mathbb{R}^d}\int_{|\boldsymbol{h}|<\delta(\boldsymbol{x})} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}}\hat{\gamma}(\frac{|\boldsymbol{h}|}{\delta(\boldsymbol{x})})|u(\boldsymbol{x}+\boldsymbol{h}) - u(\boldsymbol{x})|^2 d\boldsymbol{h}d\boldsymbol{x}\\
&= \int_{\mathbb{R}^d}\int_{|\boldsymbol{h}|<1} \frac{1}{|\delta(\boldsymbol{x})|^{2}}\hat{\gamma}(\boldsymbol{h})|u(\boldsymbol{x}+\delta(\boldsymbol{x})\boldsymbol{h}) - u(\boldsymbol{x})|^2 d\boldsymbol{h}d\boldsymbol{x}\\
&\le \int_{\mathbb{R}^d}\int_{|\boldsymbol{h}|<1} |\boldsymbol{h}|^2\hat{\gamma}(\boldsymbol{h})\int_0^1 |\nabla u(\boldsymbol{x}+t\delta(\boldsymbol{x})\boldsymbol{h})|^2 dt d\boldsymbol{h}d\boldsymbol{x}\,.
\end{aligned}$$

Let $\boldsymbol{y} = \boldsymbol{x} + t\delta(\boldsymbol{x})\boldsymbol{h}$, we see that

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = I + t\nabla\delta(\boldsymbol{x}) \otimes \boldsymbol{h}\,,$$

and its inverse are uniformly bounded everywhere if $\|\nabla\delta\| = \sigma < 1$. Moreover, the bounds are independent of σ if σ is small. Thus, there is a generic constant $C > 0$ such that

$$\begin{aligned} |u|^2_{\mathcal{S}(\Omega)} &\leq C(\int_{|\boldsymbol{h}|<1} |\boldsymbol{h}|^2 \hat{\gamma}(\boldsymbol{h}) d\boldsymbol{h}) |u|^2_{H^1(\mathbb{R}^d)} \\ &\leq C\|u\|^2_{H^1(\Omega)} . \end{aligned}$$

The constant C may depend on Ω but is independent of σ for σ small.

Putting together, we have the inequality (9) for $u \in C^1(\bar{\Omega})$. Now invoking density argument, since $H^1(\Omega)$ is the completion of $C^1(\bar{\Omega})$ under the $\|\cdot\|_{H^1(\Omega)}$ norm, we get (9) valid in $H^1(\Omega)$ and the continuous imbedding $H^1(\Omega)$ in the space $\mathcal{S}(\Omega)$, which is the completion of $C^1(\bar{\Omega})$ under the weaker nonlocal norm. □

The above proposition implies that the new trace theorems for nonlocal spaces are indeed refinement of the the classical $H^{1/2}$ trace theorems that are direct consequences of their nonlocal counterpart due to Proposition 2.1. One may recover the classical results by simply combining (7) and (9):

$$C\|Tu\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq \|u\|_{\mathcal{S}(\Omega)} \leq C\|u\|_{H^1(\Omega)} , \quad \forall u \in H^1(\Omega) .$$

3. More properties of the semi-norm $|\cdot|_{\mathcal{S}(\Omega)}$

In this section, we look more closely at the nonlocal semi-norm $|\cdot|_{\mathcal{S}(\Omega)}$ and derive results needed for the trace theorems. The discussion mainly consists of two parts. The first part focuses on generalizing Hardy's type inequalities to the nonlocal spaces, yielding new results that are of interests on their own. The second part introduces some special quantities mimicking norms of directional derivatives that are particularly matched with our nonlocal setting. The study of such norms, however, are more involved technically than their local analog and require new techniques that, to our knowledge, have not beed used in the literature before.

To begin, we can see intuitively that when $\boldsymbol{x}$ gets closer to Γ, the nonlocal norm around the neighborhood of $\boldsymbol{x}$ behaves more like that of H^1. We then expect properties similar to those classical results associated with H^1 functions hold. For example, a generalization of the Hardy inequality can be shown.

Proposition 3.1 (Nonlocal Hardy-type inequality)**.** *Let $\Omega = (0, r)$ for some $r > 0$ and $u \in C^1(\bar{\Omega})$ with $u(0) = 0$, then we have*

$$\int_\Omega \frac{|u(x)|^2}{|x|^2} dx \leq C_{a,b} \int_\Omega \int_{ax}^{bx} \frac{|u(y) - u(x)|^2}{|x|^3} dy dx , \tag{10}$$

where $C_{a,b} = \frac{4(2+b+a)}{(b-a)(2-b-a)^2}$ with a and b satisfy $0 \leq a < b \leq 1$. In particular, this implies the Hardy-type inequality, for a constant $C > 0$ independent of Ω such that,

$$\int_\Omega \frac{|u(x)|^2}{dist(x,\Gamma)^2} dx \leq C|u|^2_{\mathcal{S}(\Omega)} \tag{11}$$

where $\Gamma = \{0\}$.

Proof. For any $x \in \Omega$ and $y \in \Omega$, let us write

$$u(x) = u(x) - u(y) + u(y) ,$$

from which we get

$$|u(x)|^2 \leq (1 + \frac{1}{\epsilon})|u(y) - u(x)|^2 + (1 + \epsilon)|u(y)|^2 ,$$

where ϵ is a small number to be determined. Now integrating y over the interval (ax, bx), we get for $x \in \Omega$,

$$|u(x)|^2 \leq \frac{1+1/\epsilon}{bx-ax}\int_{ax}^{bx}|u(y)-u(x)|^2dy + \frac{1+\epsilon}{bx-ax}\int_{ax}^{bx}|u(y)|^2dy\,,$$

so that

$$\begin{aligned}\int_\Omega \frac{|u(x)|^2}{|x|^2}dx &\leq \frac{1+1/\epsilon}{b-a}\int_\Omega\int_{ax}^{bx}\frac{|u(y)-u(x)|^2}{|x|^3}dydx + \frac{1+\epsilon}{b-a}\int_\Omega\int_{ax}^{bx}\frac{|u(y)|^2}{|x|^3}dydx\\ &= \mathrm{I}+\mathrm{II}\,.\end{aligned}$$

The term I is our desired bound, and for the term II, since $u \in C^1(\bar{\Omega})$ and $u(0) = 0$, we can use Fubini's theorem to change the order of integration to get

$$\begin{aligned}\mathrm{II} &= \frac{1+\epsilon}{b-a}\int_0^r\int_{ax}^{bx}\frac{|u(y)|^2}{|x|^3}dydx\\ &= \frac{1+\epsilon}{b-a}\int_0^{br}\int_{y/b}^{\max\{y/a,r\}}\frac{|u(y)|^2}{|x|^3}dxdy\,.\end{aligned}$$

Notice that $0 \leq a < b \leq 1$, we get

$$\begin{aligned}\mathrm{II} &\leq \frac{1+\epsilon}{b-a}\int_0^r\int_{y/b}^{y/a}\frac{|u(y)|^2}{|x|^3}dxdy\\ &= \frac{1+\epsilon}{b-a}\cdot\frac{1}{2}(b^2-a^2)\int_\Omega\frac{|u(y)|^2}{|y|^2}dy\\ &= \frac{(1+\epsilon)(b+a)}{2}\int_\Omega\frac{|u(x)|^2}{|x|^2}dx\,.\end{aligned}$$

Now since $b + a < 2$, we can pick $\epsilon = \frac{1}{b+a} - \frac{1}{2}$ to get

$$\begin{aligned}\int_\Omega\frac{|u(x)|^2}{|x|^2}dx &\leq \frac{1+1/\epsilon}{b-a}\cdot\frac{2}{2-(1+\epsilon)(b+a)}\int_\Omega\int_{ax}^{bx}\frac{|u(y)-u(x)|^2}{|x|^3}dydx\\ &= C_{a,b}\int_\Omega\int_{ax}^{bx}\frac{|u(y)-u(x)|^2}{|x|^3}dydx\,.\end{aligned}$$

At last, inequality (11) follows straightforwardly from (10). □

Although the Proposition 3.1 only shows the nonlocal Hardy-type inequality for the one dimensional case, it is not hard to see that the more general cases are also true.

Corollary 3.2 (Nonlocal Hardy's inequality in a multi-dimensional stripe domain)**.** *Let* $\Omega = (0, r) \times \mathbb{R}^{d-1}$ *(*$d \geq 2$*) and* $\Gamma = \{0\} \times \mathbb{R}^{d-1}$*. Assume that* $u \in C^1(\bar{\Omega})$ *and* $u(0, \bar{\boldsymbol{x}}) = 0$ *for* $\bar{\boldsymbol{x}} \in \mathbb{R}^{d-1}$*, then*

$$\int_\Omega\frac{|u(\boldsymbol{x})|^2}{dist(\boldsymbol{x},\Gamma)^2}d\boldsymbol{x} \leq C|u|^2_{\mathcal{S}(\Omega)}\,. \tag{12}$$

Proof. Use Proposition 3.1, we have

$$\begin{aligned}\int_\Omega\frac{|u(\boldsymbol{x})|^2}{(\mathrm{dist}(\boldsymbol{x},\Gamma))^2}d\boldsymbol{x} &= \int_{\mathbb{R}^{d-1}}\int_0^r\frac{|u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2}dx_1d\bar{\boldsymbol{x}}\\ &\leq \int_{\mathbb{R}^{d-1}}\int_0^r\int_{ax_1}^{bx_1}\frac{|u(y_1,\bar{\boldsymbol{x}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^3}dy_1dx_1d\bar{\boldsymbol{x}}\,, \qquad (13)\end{aligned}$$

where the last term is bounded by $C|u|^2_{\mathcal{S}(\Omega)}$ by Lemma 3.6 that is shown later in this section. □

***Remark* 3.3.** *We note first that while Proposition 3.1 and Corollary 3.2 are shown for a specific kernel with influence horizon $\delta(\boldsymbol{x}) = dist(\boldsymbol{x}, \Gamma)$, from the proof, we can see that the nonlocal Hardy's inequality also holds for any $\delta(\boldsymbol{x}) = \sigma\, dist(\boldsymbol{x}, \Gamma)$ with constant C depending continuously on σ. Moreover, we will see in section 6 that C can be made a uniform constant with respect to σ for $\sigma \to 0$.*

Secondly, although the nonlocal Hardy's inequality is presented only for a strip domain here, it is not hard to see that it also holds for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. This will also be further illustrated in section 7.

The last integral in (13) involves weighted variations of the function u in its first component. In the same spirit of norms of directional derivatives in classical, local function spaces, we introduce the following definition as a nonlocal analog that refines our understanding of how the nonlocal norm $\|\cdot\|_{\mathcal{S}(\Omega)}$ provides control on the function variation in different directions. This not only helps proving (13) but also plays important roles in proving the new trace theorems. For brevity of notation, $⨍$ is used to represent the integral average over the respective domain, that is, the integral over the domain divided by the volume of domain.

Definition 3.4. *On the domain $\Omega = (0, r) \times \mathbb{R}^{d-1}$, we define in the following two directional nonlocal semi-norms $|\cdot|_n$ and $|\cdot|_t$, standing for normal and tangential directions respectively with reference to the boundary segment $\Gamma = \{0\} \times \mathbb{R}^{d-1}$,*

$$|u|_n^2 = \int_{\mathbb{R}^{d-1}} \int_0^r ⨍_{ax_1}^{bx_1} \frac{|u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 dx_1 d\bar{\boldsymbol{x}} \tag{14}$$

$$|u|_t^2 = \int_{\mathbb{R}^{d-1}} \int_0^r ⨍_{B_{cx_1}(\bar{\boldsymbol{x}})} \frac{|u(x_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \tag{15}$$

where $0 \le a < b \le 1$ and $0 < c < 1$ are constants.

***Remark* 3.5.** *To offer some insight, we make some heuristic comments. For a smooth function $u = u(\boldsymbol{x})$, we may approximately have, in an informal manner, that*

$$\begin{aligned}
|u|_n^2 &\approx \int_{\mathbb{R}^{d-1}} \int_0^r ⨍_{ax_1}^{bx_1} \frac{|y_1 - x_1|^2}{|x_1|^2} |u_{x_1}(x_1, \bar{\boldsymbol{x}})|^2 dy_1 dx_1 d\bar{\boldsymbol{x}} \\
&= C_n(a, b) \int_{\mathbb{R}^{d-1}} \int_0^r |u_{x_1}(x_1, \bar{\boldsymbol{x}})|^2 dx_1 d\bar{\boldsymbol{x}} \\
|u|_t^2 &\approx \int_{\mathbb{R}^{d-1}} \int_0^r ⨍_{B_{cx_1}(\bar{\boldsymbol{x}})} \frac{|\nabla_{\bar{\boldsymbol{x}}} u(x_1, \bar{\boldsymbol{x}}) \cdot (\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \\
&= C_t(c, d) \int_{\mathbb{R}^{d-1}} \int_0^r |\nabla_{\bar{\boldsymbol{x}}} u(x_1, \bar{\boldsymbol{x}})|^2 dx_1 d\bar{\boldsymbol{x}}
\end{aligned}$$

for some constants $C_n(a, b)$ and $C_t(c, d)$ that can be computed explicitly. This provides a hint that $|\cdot|_n$ and $|\cdot|_t$ may indeed mimic norms of directional directives. We thus see that it is reasonable to call $|\cdot|_n$ and $|\cdot|_t$ directional semi-norms. In comparison, we may also informally expand $|\cdot|^2_{\mathcal{S}(\Omega)}$ as

$$|u|^2_{\mathcal{S}(\Omega)} \sim \int_{\mathbb{R}^{d-1}} \int_0^r ⨍_{B_{x_1}(x_1, \bar{\boldsymbol{x}}) \cap \Omega} \frac{|u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \approx C \|\nabla u\|^2_{L^2(\Omega)} .$$

As the norms of classical (local) directional derivatives are obviously bounded by that of the total gradient, we extend to the nonlocal case by establishing the following lemma saying that $|\cdot|_n$ and $|\cdot|_t$ are controlled by the original semi-norm $|\cdot|_{\mathcal{S}(\Omega)}$.

Lemma 3.6. *Let* $\Omega = (0, r) \times \mathbb{R}^{d-1}$ *for some* $r > 0$, a, b *and* c *satisfy* $0 \le a < b \le 1$, $0 < c < 1$ *and* $(a-1)^2 + c^2 \le 1$. *Then there exists a constant* C *depending only on* a, b *and* c *such that for any* $u \in \mathcal{S}(\Omega)$,

$$|u|_n \le C|u|_{\mathcal{S}(\Omega)}\,, \tag{16}$$

$$|u|_t \le C|u|_{\mathcal{S}(\Omega)}\,. \tag{17}$$

Proof. First, let us briefly describe the idea of the proof. Instead of showing (16) and (17) directly, we show the following two inequalities instead.

$$|u|_n^2 \le c_1|u|_t^2 + C|u|_{\mathcal{S}(\Omega)}^2 \tag{18}$$

$$|u|_t^2 \le c_2|u|_n^2 + C|u|_{\mathcal{S}(\Omega)}^2 \tag{19}$$

where $c_1 c_2 < 1$. We see that they immediately yield both (16) and (17). Moreover, by density argument, we can focus on showing (18) and (19) only for $u \in C^1(\bar{\Omega}) \cap \mathcal{S}(\Omega)$.

For any $(y_1, \bar{\boldsymbol{y}}) \in \Omega$, let us write

$$u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}}) = u(y_1, \bar{\boldsymbol{x}}) - u(y_1, \bar{\boldsymbol{y}}) + u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})\,,$$

and we get the estimate

$$|u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}})|^2 \le (1+\epsilon)|u(y_1, \bar{\boldsymbol{x}}) - u(y_1, \bar{\boldsymbol{y}})|^2 + (1+\frac{1}{\epsilon})|u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2\,,$$

where ϵ is a small number to be determined. The relative positions of those points are depicted in Figure 2. The purple horizontal dotted line shows the range of $(y_1, \bar{\boldsymbol{x}})$, the blue vertical dotted line for $(y_1, \bar{\boldsymbol{y}})$, and the red vertical dashed line for $(x_1, \bar{\boldsymbol{y}})$. The key to choose these positions is to make sure that $(y_1, \bar{\boldsymbol{y}})$ stays in the effective neighborhood of $(x_1, \bar{\boldsymbol{x}})$ bounded by the black dashed circle.

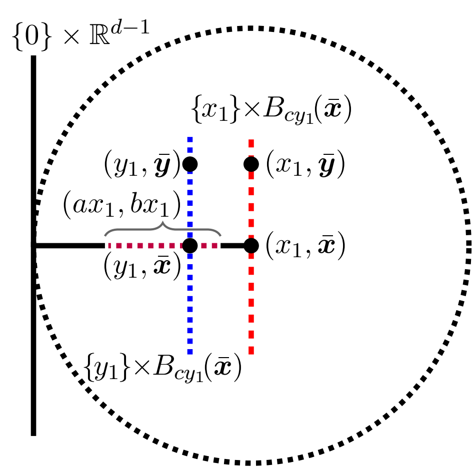

FIGURE 2. Depiction of geometry used in the proof of Lemma 3.6.

Now integrating $\bar{\boldsymbol{y}}$ over the ball $B_{cy_1}(\bar{\boldsymbol{x}})$ we have

$$\begin{aligned}|u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}})|^2 \le &(1+\epsilon) \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} |u(y_1, \bar{\boldsymbol{x}}) - u(y_1, \bar{\boldsymbol{y}})|^2 d\bar{\boldsymbol{y}} \\ &+ (1+1/\epsilon) \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} |u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2 d\bar{\boldsymbol{y}}\,.\end{aligned}$$

So, we have

$$\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{x}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 dx_1 d\bar{\boldsymbol{x}}$$

$$\le (1+\epsilon)\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{ax_1}^{bx_1} \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} \frac{|u(y_1,\bar{\boldsymbol{x}})-u(y_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dy_1 dx_1 d\bar{\boldsymbol{x}} \tag{20}$$

$$+ (1+1/\epsilon)\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{ax_1}^{bx_1} \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dy_1 dx_1 d\bar{\boldsymbol{x}}$$

$$= \mathrm{I} + \mathrm{II}\,.$$

It is easy to see that II is controlled by $|u|^2_{\mathcal{S}(\Omega)}$ since

$$\mathrm{II} = (1+1/\epsilon)\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{ax_1}^{bx_1} \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dy_1 dx_1 d\bar{\boldsymbol{x}}$$

$$\le C(1+1/\epsilon)\int_{\Omega}\fint_{\mathcal{H}(\boldsymbol{x})\cap\Omega} \frac{|u(\boldsymbol{y})-u(\boldsymbol{x})|^2}{|x_1|^2} d\boldsymbol{y} d\boldsymbol{x}\,,$$

where the last inequality is true since $\boldsymbol{y}\in\mathcal{H}(\boldsymbol{x})\cap\Omega$, a result we can see by using the assumption on a, b and c,

$$|\boldsymbol{y}-\boldsymbol{x}|^2 = (y_1-x_1)^2 + |\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2 \le (a-1)^2|x_1|^2 + c^2b^2|x_1|^2 \le |x_1|^2\,, \tag{21}$$

where $\bar{\boldsymbol{y}}\in B_{cy_1}(\bar{\boldsymbol{x}})$ and $ax_1\le y_1\le bx_1$ are used.

Now for I, by using Fubini's thoerem, we have

$$\mathrm{I} = \frac{(1+\epsilon)}{b-a}\int_{\mathbb{R}^{d-1}}\int_0^r \int_{ax_1}^{bx_1} \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{x}})|^2}{|x_1|^3} d\bar{\boldsymbol{y}} dy_1 dx_1 d\bar{\boldsymbol{x}}$$

$$\le \frac{(1+\epsilon)}{b-a}\int_{\mathbb{R}^{d-1}}\int_0^r \Big(\int_{y_1/b}^{y_1/a} \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{x}})|^2}{|x_1|^3} d\bar{\boldsymbol{y}} dx_1\Big) dy_1 d\bar{\boldsymbol{x}}$$

$$= \frac{(1+\epsilon)(b+a)}{2}\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{B_{cy_1}(\bar{\boldsymbol{x}})} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{x}})|^2}{|y_1|^2} d\bar{\boldsymbol{y}} dy_1 d\bar{\boldsymbol{x}}$$

$$= \frac{(1+\epsilon)(b+a)}{2}\int_{\mathbb{R}^{d-1}}\int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \frac{|u(x_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}}$$

Together, (18) is proved for $c_1 = (1+\epsilon)(b+a)/2$.

As for (19), we consider a point $(y_1,\bar{\boldsymbol{y}})\in\Omega$ (as depicted in Figure 2) so that $(y_1,\bar{\boldsymbol{y}})$ is in the effective neighborhood of $(x_1,\bar{\boldsymbol{x}})$. $|u(x_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2$ is estimated in a similar fashion by

$$|u(x_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2 \le (1+\epsilon)|u(x_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{y}})|^2 + (1+\frac{1}{\epsilon})|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2\,.$$

Integrating y_1 over the interval (ax_1,bx_1), we get

$$|u(x_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2 \le (1+\epsilon)\fint_{ax_1}^{bx_1} |u(x_1,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{y}})|^2 dy_1$$

$$+ (1+1/\epsilon)\fint_{ax_1}^{bx_1} |u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2 dy_1\,.$$

This implies

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \frac{|u(x_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \\
&\leq (1+\epsilon) \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(x_1,\bar{\boldsymbol{y}}) - u(y_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \\
&\quad + (1+1/\epsilon) \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \\
&= \mathrm{III} + \mathrm{IV}\,.
\end{aligned}$$

The term IV is clearly controlled by $|u|^2_{\mathcal{S}(\Omega)}$,

$$\begin{aligned}
\mathrm{IV} &= (1+1/\epsilon) \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \\
&\leq C(1+1/\epsilon) \int_{\Omega} \fint_{\mathcal{H}(\boldsymbol{x})\cap\Omega} \frac{|u(\boldsymbol{y}) - u(\boldsymbol{x})|^2}{|x_1|^2} d\boldsymbol{y} d\boldsymbol{x}\,,
\end{aligned}$$

where the last inequality is derived based on the observation that $\boldsymbol{y} \in \mathcal{H}(\boldsymbol{x})$:

$$|\boldsymbol{y} - \boldsymbol{x}|^2 = (y_1 - x_1)^2 + |\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^2 \leq (a-1)^2|x_1|^2 + c^2|x_1|^2 \leq |x_1|^2\,, \tag{22}$$

following assumptions on a, b and c.

For the term III, we use Fubini's theorem to get

$$\begin{aligned}
\mathrm{III} &= (1+\epsilon) \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \\
&= (1+\epsilon) \int_0^r \int_{\mathbb{R}^{d-1}} \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{y}} d\bar{\boldsymbol{x}} dx_1 \\
&= (1+\epsilon) \int_0^r \int_{\mathbb{R}^{d-1}} \fint_{B_{cx_1}(\bar{\boldsymbol{y}})} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dy_1 d\bar{\boldsymbol{x}} d\bar{\boldsymbol{y}} dx_1 \\
&= (1+\epsilon) \int_0^r \int_{\mathbb{R}^{d-1}} \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{y}})|^2}{|x_1|^2} dx_1 d\bar{\boldsymbol{y}} dy_1 \\
&= (1+\epsilon) \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{x}}) - u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 dx_1 d\bar{\boldsymbol{x}}\,.
\end{aligned}$$

This implies (19) with $c_2 = (1+\epsilon)$. The product of c_1 and c_2 is

$$c_1 c_2 = \frac{(1+\epsilon)^2(b+a)}{2}\,.$$

Since $b+a<2$, by choosing ϵ small enough such that $(1+\epsilon)^2(b+a)<2$, we have $c_1c_2<1$, so that (16) and (17) are true, and hence we have the lemma. □

4. Proof of the theorem 1.1

We note first that in this section, theorem 1.1 is only shown for the kernel defined in (8). The discussions for more general kernels are in section 6. Now let us show (5). For any $(x_1, \bar{\boldsymbol{x}}) \in (0,r) \times \mathbb{R}^{d-1}$, write

$$u(0,\bar{\boldsymbol{x}}) = u(x_1,\bar{\boldsymbol{x}}) - (u(x_1,\bar{\boldsymbol{x}}) - u(0,\bar{\boldsymbol{x}}))\,,$$

from which we have

$$u^2(0,\bar{\boldsymbol{x}}) \leq 2u^2(x_1,\bar{\boldsymbol{x}}) + 2(u(x_1,\bar{\boldsymbol{x}}) - u(0,\bar{\boldsymbol{x}}))^2.$$

Now by integrating x_1 over $(0,r)$, we obtain

$$u^2(0,\bar{\boldsymbol{x}}) \leq \frac{2}{r}\int_0^r u^2(x_1,\bar{\boldsymbol{x}})dx_1 + \frac{2}{r}\int_0^r |u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2 dx_1\,.$$

So, we get by Proposition 3.1 and Lemma 3.6 that

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} u^2(0,\bar{\boldsymbol{x}})d\bar{\boldsymbol{x}} &\leq \frac{2}{r}\|u\|^2_{L^2(\Omega)} + 2r\int_{\mathbb{R}^{d-1}}\int_0^r \frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|x_1|^2}dx_1 d\bar{\boldsymbol{x}}\\ &\leq \frac{2}{r}\|u\|^2_{L^2(\Omega)} + 2r\int_{\mathbb{R}^{d-1}}\int_0^r\int_{ax_1}^{bx_1} \frac{|u(y_1,\bar{\boldsymbol{x}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^3}dy_1 dx_1 d\bar{\boldsymbol{x}}\\ &\leq C\Big(\frac{\|u\|^2_{L^2(\Omega)}}{r} + r\|u\|^2_{\mathcal{S}(\Omega)}\Big)\,.\end{aligned}$$

Let us show (6) next. First, for $d\geq 2$, by definition of $|u(0,\cdot)|^2_{H^{1/2}(\mathbb{R}^{d-1})}$,

$$\begin{aligned}|u(0,\cdot)|^2_{H^{1/2}(\mathbb{R}^{d-1})} &= \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}} \frac{|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^d}d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\\ &= \int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})} \frac{|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^d}d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\\ &\quad + \int_{\mathbb{R}^{d-1}}\int_{B^c_{r/2}(\bar{\boldsymbol{x}})} \frac{|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^d}d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\,.\end{aligned}$$

Now the second part can be estimated by

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}}\int_{B^c_{r/2}(\bar{\mathbf{0}})} \frac{|u(0,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{h}}|^d}d\bar{\boldsymbol{h}}d\bar{\boldsymbol{x}}\\ &\leq \int_{B^c_{r/2}(\bar{\mathbf{0}})}\frac{1}{|\bar{\boldsymbol{h}}|^d}\int_{\mathbb{R}^{d-1}}\big(2u^2(0,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})+2u^2(0,\bar{\boldsymbol{x}})\big)d\bar{\boldsymbol{x}}d\bar{\boldsymbol{h}}\\ &\leq \frac{C}{r}\|u(0,\cdot)\|^2_{L^2(\mathbb{R}^{d-1})}\,,\end{aligned}\tag{23}$$

where C is a constant depend only on d. Thus we only have to prove the following inequality to get (6),

$$\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})} \frac{|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^d}d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}} \leq C|u|^2_{\mathcal{S}(\Omega)}\,.\tag{24}$$

The idea is again to split the left-hand side into three parts that can be controlled by the right hand side.

As shown in Figure 3, we choose $(x_1,\bar{\boldsymbol{x}}),(y_1,\bar{\boldsymbol{y}})\in\Omega$ and rewrite

$$\begin{aligned}u(0,\bar{\boldsymbol{y}}) &= u(0,\bar{\boldsymbol{y}}) - u(y_1,\bar{\boldsymbol{y}}) + u(y_1,\bar{\boldsymbol{y}})\\ u(0,\bar{\boldsymbol{x}}) &= u(0,\bar{\boldsymbol{x}}) - u(x_1,\bar{\boldsymbol{x}}) + u(x_1,\bar{\boldsymbol{x}})\,.\end{aligned}$$

Notice that the blue solid horizontal line and the red horizontal dashed line in Figure 3 show the possible positions of $(x_1,\bar{\boldsymbol{x}})$ and $(y_1,\bar{\boldsymbol{y}})$ respectively. The key is to determine the end points of these lines so that any $(y_1,\bar{\boldsymbol{y}})$ over the blue solid line should stand in the effective neighborhood (shown as red solid circle) of any $(x_1,\bar{\boldsymbol{x}})$ on the red horizontal dashed line, in particular, the left-most end point whose effective neighborhood is given by the dashed purple circle.

By splitting terms, we have

$$|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2 \leq 3|u(0,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{y}})|^2 + 3|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2 + 3|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2\,.$$

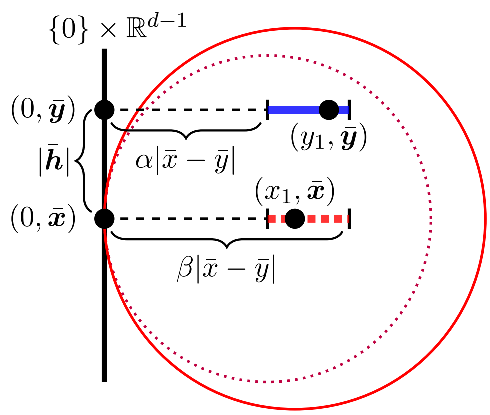

FIGURE 3. Depiction of geometry used in the proof of Theorem 1.1.

Now let α and β be numbers to be determined and satisfy $1<\alpha<\beta\le 2$. Integrating both x_1 and y_1 in the interval $(\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|,\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|)$, we have

$$\begin{aligned}|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2 \le{}& \frac{3}{(\beta-\alpha)|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}|u(0,\bar{\boldsymbol{y}})-u(y_1,\bar{\boldsymbol{y}})|^2dy_1\\ &+\frac{3}{(\beta-\alpha)|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2dx_1\\ &+\frac{3}{(\beta-\alpha)^2|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2dy_1dx_1\,.\end{aligned}$$

So our integral can controlled by,

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})}\frac{|u(0,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^d}d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\\ &\le\frac{3}{\beta-\alpha}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\frac{|u(y_1,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{y}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^{d+1}}dy_1d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\\ &\quad+\frac{3}{\beta-\alpha}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^{d+1}}dx_1d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\\ &\quad+\frac{3}{(\beta-\alpha)^2}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^{d+2}}dy_1dx_1d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\\ &=\frac{6}{\beta-\alpha}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\frac{|u(y_1,\bar{\boldsymbol{y}})-u(0,\bar{\boldsymbol{y}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^{d+1}}dy_1d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\\ &\quad+\frac{3}{(\beta-\alpha)^2}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\bar{\boldsymbol{x}})}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^{d+2}}dy_1dx_1d\bar{\boldsymbol{y}}d\bar{\boldsymbol{x}}\\ &=\mathrm{I}+\mathrm{II}\,.\end{aligned}$$

Let us first check that the term II is bounded by $C|u|^2_{\mathcal{S}(\Omega)}$. We take notice of Fubini's theorem and the fact that $\beta \le 2$ to get

$$\begin{aligned}
\mathrm{II} &= \frac{3}{(\beta-\alpha)^2}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\mathbf{0})}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{h}}|^{d+2}}dy_1dx_1d\bar{\boldsymbol{h}}d\bar{\boldsymbol{x}}\\
&\le \frac{3\beta^{d+2}}{(\beta-\alpha)^2}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\mathbf{0})}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1dx_1d\bar{\boldsymbol{h}}d\bar{\boldsymbol{x}}
\end{aligned}$$

where $B_{r/2}(\mathbf{0})$ denotes the $d-1$ dimensional ball at the origin. The integral can be further estimated by

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\mathbf{0})}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1dx_1d\bar{\boldsymbol{h}}d\bar{\boldsymbol{x}}\\
&=\int_{\mathbb{R}^{d-1}}\int_{S^{d-2}}\int_0^{r/2}\Big(\int_{\alpha h}^{\beta h}\int_{\alpha h}^{\beta h}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1dx_1\Big)|J|dhdS^{d-2}d\bar{\boldsymbol{x}}
\end{aligned}$$

where $|J|=|J(h)|$ is the volume element of $d-1$ dimensional ball and dS^{d-2} is the volume element of the $d-2$ dimensional unit sphere. After a change of order of integration, since $1<\alpha<\beta\le 2$, we end up with

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}}\int_{S^{d-2}}\Big(\int_0^{r/2}\int_{\alpha h}^{\beta h}\int_{\alpha h}^{\beta h}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1dx_1|J|dh\Big)dS^{d-2}d\bar{\boldsymbol{x}}\\
&\le\int_{\mathbb{R}^{d-1}}\int_{S^{d-2}}\Big(\int_0^{r}\int_{\frac{x_1}{\beta}}^{\frac{x_1}{\alpha}}\int_{\alpha h}^{\beta h}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1|J|dhdx_1\Big)dS^{d-2}d\bar{\boldsymbol{x}}\\
&=\int_{\mathbb{R}^{d-1}}\int_0^{r}\Big(\int_{S^{d-2}}\int_{\frac{x_1}{\beta}}^{\frac{x_1}{\alpha}}\int_{\alpha h}^{\beta h}\frac{|u(y_1,\bar{\boldsymbol{x}}+\bar{\boldsymbol{h}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1|J|dhdS^{d-2}\Big)dx_1d\bar{\boldsymbol{x}}\\
&\le\int_{\mathbb{R}^{d-1}}\int_0^{r}\Big(\int_{\frac{x_1}{\beta}\le|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|\le\frac{x_1}{\alpha}}\int_{\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}^{\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|}\frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|x_1|^{d+2}}dy_1d\bar{\boldsymbol{y}}\Big)dx_1d\bar{\boldsymbol{x}}\\
&\le\int_{\Omega}\int_{\mathcal{H}(\boldsymbol{x})}\frac{|u(\boldsymbol{y})-u(\boldsymbol{x})|^2}{|x_1|^{d+2}}d\boldsymbol{y}d\boldsymbol{x}\,.
\end{aligned}$$

Note that the last inequality is true only if $\boldsymbol{y}=(y_1,\bar{\boldsymbol{y}})\in\mathcal{H}(\boldsymbol{x})$. Since $\alpha|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|\le y_1\le\beta|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|$ and $x_1/\beta\le|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|\le x_1/\alpha$ implies that $\alpha x_1/\beta\le y_1\le\beta x_1/\alpha$, we have

$$|\boldsymbol{y}-\boldsymbol{x}|^2=(y_1-x_1)^2+|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2\le\max\{(1-\frac{\beta}{\alpha})^2+\frac{1}{\alpha^2},(1-\frac{\alpha}{\beta})^2+\frac{1}{\alpha^2}\}|x_1|^2\le|x_1|^2\,, \tag{25}$$

if we pick α and β such that

$$\max\left\{(1-\frac{\beta}{\alpha})^2+\frac{1}{\alpha^2},(1-\frac{\alpha}{\beta})^2+\frac{1}{\alpha^2}\right\}\le 1\,. \tag{26}$$

Then this in fact leaves us many chocies of α and β, for example, $\alpha=\frac{3}{2}$ and $\beta=2$ would work.

The term I is bounded by

$$\begin{aligned}
&\frac{6}{\beta-\alpha}\int_{\mathbb{R}^{d-1}}\int_{B_{r/2}(\mathbf{0})}\int_{\alpha|\bar{\boldsymbol{h}}|}^{\beta|\bar{\boldsymbol{h}}|}\frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{h}}|^{d+1}}dx_1 d\bar{\boldsymbol{h}} d\bar{\boldsymbol{x}}\\
&=\frac{6C(d)}{\beta-\alpha}\int_{\mathbb{R}^{d-1}}\int_0^{r/2}\int_{\alpha h}^{\beta h}\frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|h|^{d+1}}h^{d-2}dx_1 dh d\bar{\boldsymbol{x}}\\
&\le\frac{6C(d)}{\beta-\alpha}\int_{\mathbb{R}^{d-1}}\int_0^{r}\Big(\int_{\frac{x_1}{\beta}}^{\frac{x_1}{\alpha}}\frac{1}{h^3}dh\Big)|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2 dx_1 d\bar{\boldsymbol{x}}\\
&\le 3C(d)(\beta+\alpha)\int_{\mathbb{R}^{d-1}}\int_0^{r}\frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|x_1|^2}dx_1 d\bar{\boldsymbol{x}}\,.
\end{aligned}$$

By Proposition 3.1 and Lemma 3.6, we have

$$\int_{\mathbb{R}^{d-1}}\int_0^{r}\frac{|u(x_1,\bar{\boldsymbol{x}})-u(0,\bar{\boldsymbol{x}})|^2}{|x_1|^2}dx_1 d\bar{\boldsymbol{x}}\le C|u|^2_{\mathcal{S}(\Omega)}\,.$$

This completes the proof of theorem 1.1. □

***Remark* 4.1.** *The problem of relating boundary estimates and interior estimates appears often in the study of PDE boundary value problems, such as in Kellogg's theorem for deriving C^α regularity estimates up to the boundary with prescribed C^α data* [20]*, and in deriving interior regularity estimates from the coincidence set for free boundary problems* [23]*. Indeed, the idea of relating boundary points to interior points in order to get an estimate of boundary from those in the interior leads to a popular approach to establish the classical trace theorem, see for example,* [22, chapter 15]. *However, a new challenge in our work here in the nonlocal case, unlike the straightforward constructions in the classical case, is that the interior points need to be carefully chosen to make the nonlocal norm $\|u\|_{\mathcal{S}(\Omega)}$ coming into play. The lemma 3.6 provides us analogies of estimates on tangential and normal derivatives that are important to complete our derivation.*

5. Proof of the theorem 1.3

First, let us show that the theorem 1.3 is true when Ω is a special Lipschitz domain, namely, assume there exists a Lipschitz continuous function $\varphi:\mathbb{R}^{d-1}\to\mathbb{R}$ such that

$$\Omega=\{\boldsymbol{x}\in\mathbb{R}^d | x_1>\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}}\in\mathbb{R}^{d-1}\}\,,$$

and

$$\partial\Omega=\{\boldsymbol{x}\in\mathbb{R}^d | x_1=\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}}\in\mathbb{R}^{d-1}\}\,.$$

Then we can define two linear operators $G_\varphi: L^2(\Omega)\to L^2(\mathbb{R}^d_+)$, where $\mathbb{R}^d_+=(0,\infty)\times\mathbb{R}^{d-1}$ and $D_\varphi: L^2(\partial\Omega)\to L^2(\mathbb{R}^{d-1})$ by: for $\boldsymbol{x}=(x_1,\bar{\boldsymbol{x}})\in(0,\infty)\times\mathbb{R}^{d-1}$,

$$\begin{aligned}
(G_\varphi u)(\boldsymbol{x})&=u(x_1+\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}}),\\
(D_\varphi u)(\bar{\boldsymbol{x}})&=u(\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}}).
\end{aligned}$$

It is known that D_φ is a bounded operator from $H^{\frac12}(\partial\Omega)$ to $H^{\frac12}(\mathbb{R}^{d-1})$, and its inverse D_φ^{-1} on the two spaces is also a bounded operator (see, for instance, Lemma 3 in [9]). The next step is to show that G_φ is a bounded operator from $\mathcal{S}(\Omega)$ to $\mathcal{S}(\mathbb{R}^d_+)$. We note that $\delta(\boldsymbol{x})$ used for the two spaces $\mathcal{S}(\Omega)$ and $\mathcal{S}(\mathbb{R}^d_+)$ may need to have different scalings, though this does not affect the purpose of proving the trace inequality.

$$
\begin{aligned}
\|G_\varphi u\|_{\mathcal{S}(\mathbb{R}^d_+)} &= \int_{\mathbb{R}^{d-1}}\int_0^\infty |u(x_1+\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}})|^2 dx_1 d\bar{\boldsymbol{x}} \\
&\quad + \int_{\mathbb{R}^d_+}\int_{\mathcal{H}(\boldsymbol{x})} \frac{|u(y_1+\varphi(\bar{\boldsymbol{y}}),\bar{\boldsymbol{y}})-u(x_1+\varphi(\bar{\boldsymbol{x}}),\bar{\boldsymbol{x}})|^2}{|\sigma_1\cdot x_1|^{d+2}} d\boldsymbol{y}d\boldsymbol{x} \\
&= \|u\|^2_{L^2(\Omega)} + \int_\Omega\int_{\mathcal{H}'(\boldsymbol{x})} \frac{|u(y_1,\bar{\boldsymbol{y}})-u(x_1,\bar{\boldsymbol{x}})|^2}{|\sigma_1\cdot(x_1-\varphi(\bar{\boldsymbol{x}}))|^{d+2}} d\boldsymbol{y}d\boldsymbol{x}
\end{aligned} \tag{27}
$$

where $\mathcal{H}'(\boldsymbol{x}) = \{\boldsymbol{y}\in\Omega : |y_1-x_1-(\varphi(\bar{\boldsymbol{y}})-\varphi(\bar{\boldsymbol{x}}))|^2+|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2 \le \sigma_1^2\cdot|x_1-\varphi(\bar{\boldsymbol{x}})|^2\}$. Now since $(x_1,\varphi(\bar{\boldsymbol{x}}))\in\partial\Omega$, we know that

$$
\mathrm{dist}(\boldsymbol{x},\partial\Omega) \le |x_1-\varphi(\bar{\boldsymbol{x}})| \le K_1\mathrm{dist}(\boldsymbol{x},\partial\Omega)\,,
$$

for some K_1 independent of $\boldsymbol{x}$. Then for any $\boldsymbol{y}\in\mathcal{H}'(\bar{\boldsymbol{x}})$,

$$
\begin{aligned}
|y_1-x_1|^2+|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2 &\le 2|y_1-x_1-(\varphi(\bar{\boldsymbol{y}})-\varphi(\bar{\boldsymbol{x}}))|^2+2|(\varphi(\bar{\boldsymbol{y}})-\varphi(\bar{\boldsymbol{x}}))|^2+|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2 \\
&\le \max\{2,2M^2+1\}\big(|y_1-x_1-(\varphi(\bar{\boldsymbol{y}})-\varphi(\bar{\boldsymbol{x}}))|^2+|\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}|^2\big) \\
&\le (\sigma_1K_2\cdot\mathrm{dist}(\boldsymbol{x},\partial\Omega))^2 =: (\sigma_2\cdot\mathrm{dist}(\boldsymbol{x},\partial\Omega))^2\,.
\end{aligned}
$$

This together with (27) implies that

$$
\|G_\varphi u\|_{\mathcal{S}(\mathbb{R}^d_+)} \le C\|u\|_{\mathcal{S}(\Omega)}\,,
$$

with $\delta(\boldsymbol{x})$ defined as $\sigma_1\mathrm{dist}(\boldsymbol{x},\partial\Omega)$ and $\sigma_2\mathrm{dist}(\boldsymbol{x},\partial\Omega)$ for $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}(\Omega)$ respectively, and σ_1,σ_2 satisfy $\sigma_2=\sigma_1K_2$. Taking into account the above observations and applying the special nonlocal trace theorem (1.1) already shown for a stripe domain, we have

$$
\begin{aligned}
\|Tu\|_{H^{\frac12}(\partial\Omega)} &= \|D_\varphi^{-1}(D_\varphi Tu)\|_{H^{\frac12}(\partial\Omega)} \\
&\le C_1\|D_\varphi Tu\|_{H^{\frac12}(\mathbb{R}^{d-1})} \\
&= C_1\|(G_\varphi u)(0,\cdot)\|_{H^{\frac12}(\mathbb{R}^{d-1})} \\
&\le C_2\|G_\varphi u\|_{\mathcal{S}(\mathbb{R}^d_+)} \\
&\le C_3\|u\|_{\mathcal{S}(\Omega)}\,.
\end{aligned}
$$

Now for Ω, which is a bounded simply connected Lipschitz domain, there exists a finite number of pairs $\{B(\boldsymbol{x}_i,r_i),\varphi_i\}_{i=1}^N$ such that $\partial\Omega\subset\bigcup_{i=1}^N B(\boldsymbol{x}_i,r)$. Each φ_i is Lipschitz continuous, and we assume they have a uniform Lipschitz constant M. Now let $\{\zeta_i\}_{i=1}^N$ be a partition of unity of $\partial\Omega$, i.e.,

(1) $\zeta_i\in C_c^\infty(B(\boldsymbol{x}_i,r_i))$, $1\le i\le N$,

(2) $0\le\zeta_i\le1$ and $\displaystyle\sum_{i=1}^N\zeta_i(\boldsymbol{x})=1$ for all $\boldsymbol{x}\in\partial\Omega$.

Then for $\boldsymbol{x}\in\partial\Omega$,

$$
Tu(\boldsymbol{x}) = \sum_{i=1}^N T(\zeta_iu)(\boldsymbol{x})\,,
$$

so

$$
\|Tu\|_{H^{\frac12}(\partial\Omega)} \le \sum_{i=1}^N\|T(\zeta_iu)\|_{H^{\frac12}(\partial\Omega)}.
$$

Now since $\zeta_i\in C_c^\infty(B(\boldsymbol{x}_i,r_i))$, we may assume without loss of generality that

$$
\mathrm{dist}(supp(\zeta_i),\partial B(\boldsymbol{x}_i,r_i)) \ge r_i-b,\quad \forall i=1,2,...N
$$

for some $b < r_i$. Then instead of considering the semi-$H^{\frac{1}{2}}(\partial\Omega)$ norm as integral over $\partial\Omega \times \partial\Omega$, we treat it as the integral over $\partial\Omega \times \partial\Omega \cap \{|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}| \leq b\}$, since the other part can be thrown into the $L^2(\partial\Omega)$ norm as we did before. Under this alternative definition of $H^{\frac{1}{2}}(\partial\Omega)$, we have

$$\|T(\zeta_i u)\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|T(\zeta_i u)\|_{H^{\frac{1}{2}}(\partial\Omega\cap B(\boldsymbol{x}_i,r_i))} .$$

Now since

$$\begin{aligned}\Omega \cap B(\boldsymbol{x}_i, r_i) &= \{\boldsymbol{x} \in \mathbb{R}^d | x_1 > \varphi_i(\bar{\boldsymbol{x}})\} \cap B(\boldsymbol{x}_i, r_i),\\ \partial\Omega \cap B(\boldsymbol{x}_i, r_i) &= \{\boldsymbol{x} \in \mathbb{R}^d | x_1 = \varphi_i(\bar{\boldsymbol{x}})\} \cap B(\boldsymbol{x}_i, r_i)\,,\end{aligned}$$

we may apply a zero extension and consider $\zeta_i u$ as a function defined on $\{\boldsymbol{x} \in \mathbb{R}^d | x_1 > \varphi_i(\bar{\boldsymbol{x}})\}$. Hence the estimate in the beginning of this proof can be applied. Therefore

$$\begin{aligned}\|Tu\|_{H^{\frac{1}{2}}(\partial\Omega)} &\leq C_1 \sum_{i=1}^{N} \|T(\zeta_i u)\|_{H^{\frac{1}{2}}(\{\boldsymbol{x}\in\mathbb{R}^d | x_1 = \varphi_i(\bar{\boldsymbol{x}})\})}\\ &\leq C_2 \sum_{i=1}^{N} \|\zeta_i u\|_{\mathcal{S}(\{\boldsymbol{x}\in\mathbb{R}^d | x_1 > \varphi_i(\bar{\boldsymbol{x}})\})}\\ &\leq C_3 \sum_{i=1}^{N} \|\zeta_i u\|_{\mathcal{S}(\Omega\cap B(\boldsymbol{x}_i,r_i))}\\ &\leq C_4 \|u\|_{\mathcal{S}(\Omega)}\,,\end{aligned}$$

where the last inequality is true because

$$\begin{aligned}&\int_{\Omega\cap B(\boldsymbol{x}_i,r_i)} \int_{\Omega\cap B(\boldsymbol{x}_i,r)\cap\mathcal{H}(\boldsymbol{x})} \frac{(\zeta_i(\boldsymbol{y})u(\boldsymbol{y}) - \zeta_i(\boldsymbol{x})u(\boldsymbol{x}))^2}{|\delta(\boldsymbol{x})|^{d+2}} d\boldsymbol{y} d\boldsymbol{x}\\ &\leq 2 \int_{\Omega\cap B(\boldsymbol{x}_i,r_i)} \int_{\Omega\cap B(\boldsymbol{x}_i,r)\cap\mathcal{H}(\boldsymbol{x})} \frac{\zeta_i^2(\boldsymbol{y})(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 + u^2(\boldsymbol{x})(\zeta_i(\boldsymbol{y}) - \zeta_i(\boldsymbol{x}))^2}{|\delta(\boldsymbol{x})|^{d+2}} d\boldsymbol{y} d\boldsymbol{x}\\ &\leq 2 \left(\|\zeta_i\|_{C^0}^2 \|u\|_{\mathcal{S}(\Omega\cap B(\boldsymbol{x}_i,r))}^2 + \|\zeta_i\|_{C^1}^2 \int_{\Omega\cap B(\boldsymbol{x}_i,r_i)} \int_{|\boldsymbol{y}-\boldsymbol{x}|\leq\delta(\boldsymbol{x})} u^2(\boldsymbol{x}) \frac{|\boldsymbol{y} - \boldsymbol{x}|^2}{|\delta(\boldsymbol{x})|^{d+2}} d\boldsymbol{y} d\boldsymbol{x} \right)\\ &\leq C \left(\|\zeta_i\|_{C^0}^2 \|u\|_{\mathcal{S}(\Omega\cap B(\boldsymbol{x}_i,r))}^2 + \|\zeta_i\|_{C^1}^2 \int_{\Omega\cap B(\boldsymbol{x}_i,r_i)} u^2(\boldsymbol{x}) d\boldsymbol{x} \right) .\end{aligned}$$

This completes the proof. □

6. More general kernels

Although much of our discussion so far is focused on the choice that $\hat{\gamma}$ takes on a constant value over its support, the new nonlocal trace theorems can also be established for more general nonlocal interactions that are discussed here. The special choice of $\hat{\gamma}$ avoids technical complication while keeping the essence of the issues to be investigated. More importantly, the nonlocal norm of u corresponding to this special case is among the weakest of nonlocal norms associated with popular kernels that have been used in the literature. For example, for a typical fractional power law kernel $\hat{\gamma}(s) = 1/s^\lambda$, for $\lambda \in [0, d+2)$ [8, 2], we have the fractional type kernels

$$\gamma^\lambda(\boldsymbol{x}, \boldsymbol{y}) = \frac{c_\lambda}{|\delta(\boldsymbol{x})|^{d+2-\lambda}} \cdot \frac{1}{|\boldsymbol{y} - \boldsymbol{x}|^\lambda} \quad \text{for } \boldsymbol{y} \in \mathcal{H}(\boldsymbol{x}), \quad \lambda \in [0, d+2)\,. \tag{28}$$

Notice that λ has to be less than $d+2$ to ensure that $\hat{\gamma}$ has a finite $d+2$ order moment so that all $C^1(\bar{\Omega})$ functions have finite nonlocal norms. For such kernels, it is easy to make the following comparison of norms.

Lemma 6.1. *For γ^λ defined in* (28),

$$\int_\Omega \int_{\Omega\cap\mathcal{H}(\boldsymbol{x})} \gamma^0(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{y})-u(\boldsymbol{x}))^2 d\boldsymbol{y}d\boldsymbol{x} \le C \int_\Omega \int_{\Omega\cap\mathcal{H}(\boldsymbol{x})} \gamma^\lambda(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{y})-u(\boldsymbol{x}))^2 d\boldsymbol{y}d\boldsymbol{x}\,,$$

with $\lambda \in (0, d+2)$.

Proof. This is obvious since for $\boldsymbol{y} \in \mathcal{H}(\boldsymbol{x})$, i.e., $|\boldsymbol{y}-\boldsymbol{x}| \le \delta(\boldsymbol{x})$,

$$\frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \le \frac{1}{|\delta(\boldsymbol{x})|^{d+2-\lambda}} \cdot \frac{1}{|\boldsymbol{y}-\boldsymbol{x}|^\lambda}$$

for $\lambda \in (0, d+2)$. □

The lemma shows that $|u|_{\mathcal{S}(\Omega)}$ defined with $\lambda = 0$ indeed gives the weakest norm among ones corresponding to a large class of kernels either associated with (28) or are bounded below and above by such kernels. It is also possible to consider generalizing the choices of the variable horizon. For example, we are going to show that the nonlocal trace inequalities and Hardy-type inequalities proved previously also hold for $\delta(\boldsymbol{x}) = \sigma \mathrm{dist}(\boldsymbol{x},\Gamma)$ where $\sigma \in (0,\sigma_0]$ for $\sigma_0 > 0$. More importantly, the embedding constants in these inequalities only depend on σ_0. For this matter, we define some notations first.

$$|u|^2_{\delta(\boldsymbol{x}),r} = \int_{\Omega_r} \int_{\Omega_r\cap\mathcal{H}(\boldsymbol{x})} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma}(\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\delta(\boldsymbol{x})})(u(\boldsymbol{y})-u(\boldsymbol{x}))^2 d\boldsymbol{y}d\boldsymbol{x}\,.$$

where $\Omega_r = (0,r)\times\mathbb{R}^{d-1}$. The next lemma shows that the smaller σ is, the larger the nonlocal norm we can get.

Lemma 6.2. *Let $\delta(\boldsymbol{x}) = \sigma\, dist(\boldsymbol{x},\Gamma)$, where $\sigma \in [\frac{1}{2}, 1)$ and $\Gamma = \{0\}\times\mathbb{R}^{d-1}$, then there exists a constant C depending only on d such that the following inequality holds for any $r > 0$ and $\alpha \in (0,1]$,*

$$|u|^2_{\delta(\boldsymbol{x}),r/2} \le C\Big(\frac{1+\sigma}{1-\sigma}\Big)^{d+2} |u|^2_{\alpha\delta(\boldsymbol{x}),r}\,. \tag{29}$$

Proof. First, $|u|_{\delta(\boldsymbol{x}),r/2}$ can be rewrite as

$$|u|^2_{\delta(\boldsymbol{x}),r/2} = \int_{\Omega_{r/2}} \int_{D_{\delta(\boldsymbol{x}),r/2}} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma}(\frac{|\boldsymbol{s}|}{\delta(\boldsymbol{x})})(u(\boldsymbol{x}+\boldsymbol{s})-u(\boldsymbol{x}))^2 d\boldsymbol{s}d\boldsymbol{x}\,.$$

where $D_{\delta(\boldsymbol{x}),r/2} = \{\boldsymbol{s}\in\mathbb{R}^d : |\boldsymbol{s}| \le \delta(\boldsymbol{x}), \boldsymbol{x}+\boldsymbol{s} \in \Omega_{r/2}, \text{ for some } \boldsymbol{x}\in\Omega_{r/2}\}$.

Now for any $n \in \mathbb{N}$, we decompose $u(\boldsymbol{x}+\boldsymbol{s}) - u(\boldsymbol{x})$ into n parts,

$$u(\boldsymbol{x}+\boldsymbol{s}) - u(\boldsymbol{x}) = \Big(u(\boldsymbol{x}+\boldsymbol{s}) - u(\boldsymbol{x}+\frac{n-1}{n}\boldsymbol{s})\Big) + \cdots + \Big(u(\boldsymbol{x}+\frac{1}{n}\boldsymbol{s}) - u(\boldsymbol{x})\Big)\,.$$

By using the inequality $(\sum_{i=1}^n a_i)^2 \le n\sum_{i=1}^n a_i^2$, we get

$$|u|^2_{\delta(\boldsymbol{x}),r/2} \le n\sum_{i=1}^n \int_{\Omega_{r/2}} \int_{D_{\delta(\boldsymbol{x}),r/2}} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma}(\frac{|\boldsymbol{s}|}{\delta(\boldsymbol{x})})(u(\boldsymbol{x}+\frac{i}{n}\boldsymbol{s}) - u(\boldsymbol{x}+\frac{i-1}{n}\boldsymbol{s}))^2 d\boldsymbol{s}d\boldsymbol{x}\,.$$

For each fixed i, let $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \frac{i-1}{n}\boldsymbol{s}$, then $\tilde{\boldsymbol{x}} \in \Omega_{r/2}$ as a result of $\boldsymbol{x}+\boldsymbol{s}\in\Omega_{r/2}$ and $\boldsymbol{x}\in\Omega_{r/2}$. Since $\delta(\boldsymbol{x}) = \sigma x_1$ and $|\boldsymbol{s}| \le \delta(\boldsymbol{x})$, we have

$$(1-\sigma)\delta(\boldsymbol{x}) \le \delta(\tilde{\boldsymbol{x}}) \le (1+\sigma)\delta(\boldsymbol{x})\,.$$

Then by the fact that $\delta(\boldsymbol{x}) \le \delta(\tilde{\boldsymbol{x}})/(1-\sigma)$, $1/\delta(\boldsymbol{x}) \le (1+\sigma)/\delta(\tilde{\boldsymbol{x}})$ and $\hat{\gamma}$ nonincreasing we have

$$\begin{aligned}
|u|^2_{\delta(\boldsymbol{x}),r/2} &\le n^2 \int_{\Omega_{r/2}} \int_{|\boldsymbol{s}|\le \frac{\delta(\tilde{\boldsymbol{x}})}{1-\sigma}} \frac{(1+\sigma)^{d+2}}{|\delta(\tilde{\boldsymbol{x}})|^{d+2}} \hat{\gamma}\Big(\frac{|\boldsymbol{s}|}{\delta(\tilde{\boldsymbol{x}})/(1-\sigma)}\Big)\Big(u(\tilde{\boldsymbol{x}}+\frac{1}{n}\boldsymbol{s})-u(\tilde{\boldsymbol{x}})\Big)^2 d\boldsymbol{s}d\tilde{\boldsymbol{x}} \\
&= n^{d+2} \int_{\Omega_{r/2}} \int_{n|\boldsymbol{s}|\le \frac{\delta(\tilde{\boldsymbol{x}})}{1-\sigma}} \frac{(1+\sigma)^{d+2}}{|\delta(\tilde{\boldsymbol{x}})|^{d+2}} \hat{\gamma}\Big(\frac{n|\boldsymbol{s}|}{\delta(\tilde{\boldsymbol{x}})/(1-\sigma)}\Big)\big(u(\tilde{\boldsymbol{x}}+\boldsymbol{s})-u(\tilde{\boldsymbol{x}})\big)^2 d\boldsymbol{s}d\tilde{\boldsymbol{x}} \\
&= \Big(\frac{1+\sigma}{1-\sigma}\Big)^{d+2} \int_{\Omega_{r/2}} \int_{|\boldsymbol{s}|\le \frac{\delta(\boldsymbol{x})}{n(1-\sigma)}} \frac{1}{\big|\frac{\delta(\boldsymbol{x})}{n(1-\sigma)}\big|^{d+2}} \hat{\gamma}\Big(\frac{|\boldsymbol{s}|}{\frac{\delta(\boldsymbol{x})}{n(1-\sigma)}}\Big)\big(u(\boldsymbol{x}+\boldsymbol{s})-u(\boldsymbol{x})\big)^2 d\boldsymbol{s}d\boldsymbol{x} \\
&\le \Big(\frac{1+\sigma}{1-\sigma}\Big)^{d+2} |u|^2_{\frac{\delta(\boldsymbol{x})}{n(1-\sigma)},r},
\end{aligned}$$

where n is chosen as any number such that $n(1-\sigma)\ge 1$. This shows that (29) is true for any $\alpha = \frac{1}{n(1-\sigma)}$ with $n\in\mathbb{N}$ and $n(1-\sigma)\ge 1$. Now for a general $\alpha\in(0,1]$, we can find a number $n\ge 1$ such that

$$\frac{1}{(n+1)(1-\sigma)} < \alpha \le \frac{1}{n(1-\sigma)}.$$

Then it is easy to see that

$$|u|^2_{\delta(\boldsymbol{x}),r/2} \le \Big(\frac{1+\sigma}{1-\sigma}\Big)^{d+2} |u|^2_{\frac{\delta(\boldsymbol{x})}{(n+1)(1-\sigma)},r} \le \Big(\frac{1+\sigma}{1-\sigma}\Big)^{d+2}\Big(\frac{n+1}{n}\Big)^{d+2} |u|^2_{\alpha\delta(\boldsymbol{x}),r}.$$

So (29) is true with $C = 2^{d+2}$. □

Using this lemma, we arrive at the conclusion that our embedding results can be extend to any $\delta(\boldsymbol{x}) = \sigma \mathrm{dist}(\boldsymbol{x},\Gamma)$.

Proposition 6.3. *The results of Theorem 1.1, Theorem 1.3 and Corollary 3.2 remain valid for influence horizon of the form $\delta(\boldsymbol{x}) = \sigma\, dist(\boldsymbol{x},\Gamma)$ where $\sigma\in(0,\sigma_0]$ for some $\sigma_0>0$. Moreover, the embedding constant C depends only on Ω, Γ and σ_0.*

Proof. First we observe that theorem 1.3 follows completely from the theorem 1.1, so we only need to establish the corresponding results in theorem 1.1 and corollary 3.2, using directly the Proposition 3.1 and Lemma 3.6. It is not hard to see that the result holds for $\sigma\in[\frac{1}{2},\sigma_0]$. Indeed, if we choose $a=\frac{1}{2}$, $b=1$, $c=\frac{1}{2}$ in the proof of Proposition 3.1 (to assure $a\ge 1-\sigma$) and Lemma 3.6 (to satisfy the equivalent versions of (21) and (22) corresponding to $\sigma\ge\frac{1}{2}$), and $\alpha=\frac{3}{2}$, $\beta=\frac{7}{4}$ in section 4 (to satisfy the equivalent versions of (25) and (26) corresponding to $\sigma\ge\frac{1}{2}$), we see that the inequalities in these proofs hold with C depending on σ_0. Then for the other case that $\sigma\in(0,\frac{1}{2})$, the result is obtained from Lemma 6.2. □

Moreover, the proportionality of the horizon on the distance to the boundary is only a specific choice that can be generalized. One instance is that $\delta(\boldsymbol{x})$ is proportional to $\mathrm{dist}(\boldsymbol{x},\Gamma)$ for $\boldsymbol{x}$ only on a boundary layer of finite positive width but remains constant elsewhere. A possible form of such a $\delta(\boldsymbol{x})$ might be

$$\delta(\boldsymbol{x}) = \min\{\sigma \mathrm{dist}(x,\Gamma),\eta\},$$

for some $\eta>0$ to be specified. Another possibility is to have $\delta(\boldsymbol{x})$ vanishes in some other nonlinear ways as $\boldsymbol{x}$ approaches the boundary. Similar results can be shown in these cases and they follow naturally from the fact that it is the nonlocal interaction in the boundary layer, rather than the interior of the domain, that provides the essential control on the $H^{\frac{1}{2}}$ trace.

The discussion on the general form of $\delta(\boldsymbol{x})$ is meaningful since it is important in many applications to note that the imbedding constant in (9) does not depend on σ, just like the constants appearing in the new nonlocal trace inequalities. For example, for the coupled PDE and nonlocal model depicted in Fig. 1, we may recover a coupled PDE models in the local limit as $\sigma \to 0$. This again implies that the nonlocal trace theorems are refinement and improvement of the classical trace theorems in $H^1(\Omega)$.

7. Discussion

We now make some further discussions on the main results given in the paper. As important as the role that Sobolev space plays in the study of partial differential equations, the mathematical theory of nonlocal space provides the essential tool towards rigorous analysis of nonlocal equations. When nonlocality is incorporated in the models, it could lead to more subtle definitions of suitable boundary value problems. Nonlocal equations on domains with boundary are often supplemented not by additional constraints on the codimension-1 boundary, but rather volumetric constraints [10]. We, however, are able to define new nonlocal spaces as presented here, allowing nonlocality to diminish when approaching the domain boundary, so that standard (local) Dirichlet type boundary conditions can be defined on a codimension-1 boundary. We note that boundary value problems with local boundary conditions have been widely studied as well for fractional differential equations that are also instances of nonlocal models [5]. Indeed, classical fractional derivatives may be seen as having a vanishing horizon near the boundary or a diminishing history dependence near the initial time [2, 8], however, their scaling features are completely different from our setting so that the boundary trace or initial value are sensible largely due to the sufficiently strong interior regularity for fractional derivatives, and not through the localization effect described in this work. We also have not found parallel results in the vast literature on generalizations of Sobolev spaces such as Besov and Lizorkin-Triebel spaces [1, 37, 38]. We present next in this section the by-products of our trace theorems and at last conclude with some possible generalizations.

7.1. By-products.

The main focus of this work is on trace theorems, but it is worthwhile to point out that along the way, the results used to establish the main theorems are also of independent interests.

For example, the first step towards having the generalized trace inequality is to prove a nonlocal Hardy type inequality, that itself is also an interesting extension of the classical Hardy's inequality. The classical Hardy's inequality, see for instance [6], involves a bound on a weighted function norm by some norm of first order derivatives over the domain. There are naturally many extensions to spaces associated with variable orders and position dependent weights. Intuitively, our generalizations are derived by saying that the first derivative does not need to be well defined everywhere in the domain, but only at the place where the weighting factor blows up or when the nonlocal interactions are localized. This may not be surprising itself, and our new definition of a variable-horizon based nonlocal interaction and the special heterogeneous localization feature help making such generalizations possible.

As mentioned in remark 3.3, we may also establish a more general version for more general Lipshitz domains.

Proposition 7.1 (Nonlocal Hardy's inequality). *Given a bounded Lipshitz domain Ω, there exists a constant $C > 0$ such that if $Tu = 0$ on $\partial\Omega$, then*

$$\int_\Omega \frac{|u(\boldsymbol{x})|^2}{(dist(\boldsymbol{x}, \partial\Omega))^2} d\boldsymbol{x} \le C|u|^2_{\mathcal{S}(\Omega)} \,. \tag{30}$$

The procedure to establish (30) follows the similar path as in the proof of Theorem 1.3 starting from the Theorem 1.1. We do not repeat the detailed argument here.

7.2. **Trace theorems on portions of the domain boundary.** We first give some comments on the generalization of the trace theorem to portions of the domain boundary. In the classical, local case, we note that the trace inequality on $\partial\Omega$ automatically implies the same result for the trace on a subset Γ of $\partial\Omega$. This is not, however, as straightforward in the case for our nonlocal space whose definition involves the Γ dependent horizon, and thus the Γ dependent nonlocal kernel.

We expect similar results remain valid, as demonstrated by the special case given in Theorem 1.1, but careful investigations are needed for more general domains. A possible route is to consider first a special domain that is a (rectangular) section of the strip domain, for instance, $\Omega = (0, r) \times (a, b) \times \mathbb{R}^{d-2}$ and $\Gamma = \{0\} \times (a, b) \times \mathbb{R}^{d-2}$. By a suitable extension in the second variable from the interval (a, b) to the whole real line, we may first utilize the result in Theorem 1.1 for the whole strip domain to get the desired result on its subsection. One may then employ similar partition of unity techniques and domain transformations to more general domains and more general subset of their boundary.

In terms of further generalizations of the trace theorems, we note that although the results of this paper are only shown for the L^2 or the Hilbert space setting, it is not surprising that they can be generalized to the L^p and other more general Banach spaces. With the choices of more general kernels, one may also consider nonlocal extensions of trace results in fractional $W^{s,p}$ type spaces. Extensions of the notion of trace may also go beyond co-dimensional one manifolds to other more general subdomains or sets. Furthermore, the position-dependent and heterogeneous feature in the nonlocal norms may be related for the study of more general Morrey, Campanato, Besov and Lizorkin-Triebel spaces, possibly of variable order and growth conditions, to obtain new type of spaces and the associated trace maps [19, 29]. In addition, connections with the study of Sobolev and other function spaces on metric measure spaces may also be explored [17, 18]. Mathematically, one may also ask questions concerning optimal constants in the trace inequality, as in the classical case [12]. Moreover, while it is known that the H^1 space gives the smallest Sobolev space with continuous $H^{1/2}$ boundary trace map, we now see much larger spaces can also preserve the same property, even in spaces like what we define here whose functions may only be in L^2 over any compact subset away from the boundary. Thus, the issue of how large such a space can be, as communicated to us by Luis Caffarelli, becomes very interesting to study.

Another direction, motivated naturally by interests in nonlocal mechanics, is to consider analogous results for spaces of vector fields such as those studied in [25, 26]. Likewise, one may investigate high order extensions as well, following the discussions of high order nonlocal spaces likes ones in [36], which were relevant to the studies of beams and shells.

In closing, the main results presented here are indicative of the conceptually simple observation on the improved regularity of functions in nonlocal spaces associated with a vanishing nonlocal horizon, either uniformly across the domain of interest, or when approaching a codimension-1 surface. In the former case we recover the limit of nonlocal spaces being the classical Sobolev space, as in [4, 30], while in the latter case we obtain the analogue and extension of the classical trace theorem. One may further investigate regularity estimates, multiscale analysis and homogenization issues associated with nonlocal problems having a heterogeneous choice of variable horizon and nonlocal interaction kernels.

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