HETEROGENEOUSLY LOCALIZED NONLOCAL OPERATORS, BOUNDARY TRACES AND VARIATIONAL PROBLEMS

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Abstract. It is a classical result of Sobolev spaces that any H^1 function has a well-defined $H^{1/2}$ trace on the boundary of a sufficient regular domain. We discuss its recent extensions given in [45] in some heterogeneously localized nonlocal function spaces. The new trace theorems are stronger and more general than the classical result. They can be established essentially for all functions having only square integrability away from the boundary or in any compact subset of interior domain. Yet, the heterogeneous localization offers the necessary regularity precisely at the boundary to have well-defined traces. A consequence is that we may study associated Dirichlet type boundary value problems, as well as the coupling of local and nonlocal equations through co-dimension-1 interfaces.

1. Introduction

To study boundary value problems and interface problems, it is often necessary to study traces of associated function spaces. For standard second order elliptic problems defined on a spatial domain Ω, the associated function space is typically given by $H^1(\Omega)$, the standard Sobolev space of L^2 functions with square integrable derivatives. The following mathematical result is well-known: on a domain $\Omega \in \mathbb{R}^d$ with boundary $\partial\Omega$, a trace operator T on a subset Γ of $\partial\Omega$ can be extended continuously as a map from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$. In other words, we have the classical trace inequality of Gagliardo [17]:

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}, \quad \forall\, u \in H^1(\Omega)$$

for a positive constant C depending only on Ω. Moreover, it is known that for suitably smooth domains, $H^{1/2}(\Gamma)$ functions can have $H^1(\Omega)$ interior extensions, which is a key element in the formulation of boundary value problems of second order elliptic operators with Dirichlet type data. For generic $H^{1/2}(\Gamma)$ data, the $H^1(\Omega)$ regularity of the interior extension is the best possible.

A mathematical question underlying the study in [45] is to characterize some subspaces, denote by of $\mathcal{S}(\Omega)$, that are significantly larger than $H^1(\Omega)$, to have a continuous trace map into $H^{1/2}(\Gamma)$ and thus a stronger and more general trace inequality

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|u\|_{\mathcal{S}(\Omega)}\,, \ \ \forall u \in \mathcal{S}(\Omega)\,.$$

In this work, we present the main findings of [45] concerning some recent extensions of the classical trace theorem in the Sobolev space to some suitably defined nonlocal function spaces. To provide the background and motivation of such investigations, we first recall some elements of nonlocal operators with finite range of interactions and their connections with classical second order elliptic operators. Related works on the associated nonlocal and

2010 *Mathematics Subject Classification.* 46E35, 35A23, 47G10, 35Q74, 49N60.

Key words and phrases. Nonlocal operator, nonlocal function space, trace inequality, boundary value problem, heterogeneous localization, vanishing horizon.

Supported in part by the U.S. NSF grants DMS-1558744, AFOSR MURI Center for Material Failure Prediction Through Peridynamics, and OSD/ARO/MURI W911NF-15-1-0562 on Fractional PDEs for Conservation Laws and Beyond: Theory, Numerics and Applications.

local (classical Sobolev) function spaces are also briefly discussed. Then the key technical results in [45] are reviewed, finally, we point out some implications on the well-posedness of boundary value problems associated with nonlocal elliptic operators and the coupling of local and nonlocal variational problems,

2. Background and motivation

In recent studies of nonlocal mechanical models and nonlocal diffusion, we have become interested in the study of nonlocal operators of the following form: for a given parameter $\delta > 0$ and a given density function (radially symmetric kernel) $\underline{\omega}_\delta = \underline{\omega}_\delta(s) = \underline{\omega}_\delta(|s|)$ compactly defined on $B_\delta(0) \subset \mathbb{R}^d$ the ball of radius δ at the origin),

$$\mathcal{L}_\delta u\,(\boldsymbol{x}) = d \int_{B_\delta(0)} \frac{u(\boldsymbol{x}+s) - 2u(\boldsymbol{x}) + u(\boldsymbol{x}-s)}{|s|^2} \underline{\omega}_\delta(s) ds \quad \forall u = u(\boldsymbol{x}) : \mathbb{R}^d \to \mathbb{R}. \tag{1}$$

The operator $\mathcal{L}_\delta$ is said to be nonlocal since $\mathcal{L}_\delta u\,(\boldsymbol{x})$ depends not only on the properties of u at $\boldsymbol{x}$ but also its δ neighborhood. The positive parameter δ is commonly called the nonlocal horizon parameter that characterizes the range of nonlocal interactions associated with $\mathcal{L}_\delta$.

The nonlocal operator can be seen as a continuum average of a finite difference operator, where the weight is specified by the kernel $\underline{\omega}_\delta$. Meanwhile, formally by Taylor expansion, we have

$$\mathcal{L}_\delta u\,(\boldsymbol{x}) = \Delta u(\boldsymbol{x}) \int_{B_\delta(0)} \underline{\omega}_\delta(|s|) ds + O(\delta^2) \approx \Delta u(\boldsymbol{x}) = \mathcal{L}_0 u(\boldsymbol{x}), \quad \text{as } \delta \to 0,$$

Thus we may get the local differential operator $\mathcal{L}_0 = \Delta$ being the local limit. This can be made rigorous for kernels taking the form $\underline{\omega}_\delta(s) = \delta^{-d}\omega(s/\delta)$.

For a given spatial domain Ω, we may define a nonlocal space $\mathcal{S}(\Omega)$ that is given by the completion of $C^1(\Omega)$ with respect to the nonlocal norm

$$\|\cdot\|_{\mathcal{S}(\Omega)} = (\|\cdot\|^2_{L^2(\Omega)} + |\cdot|^2_{\mathcal{S}(\Omega)})^{1/2}, \tag{2}$$

where the associated nonlocal semi-norm $|\cdot|_{\mathcal{S}(\Omega)}$ is defined by

$$|u|^2_{\mathcal{S}(\Omega)} = \int_\Omega \int_{\Omega\cap B_\delta(\boldsymbol{x})} \gamma(\boldsymbol{x},\boldsymbol{y}) \frac{(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2}{|\boldsymbol{y}-\boldsymbol{x}|^2} d\boldsymbol{y} d\boldsymbol{x} \tag{3}$$

for some nonlocal interaction kernel $\gamma(\boldsymbol{x},\boldsymbol{y})$.

For the nonlocal operator and nonlocal considered here with an associated nonlocal horizon parameter δ, a natural and popular choice of the kernel is in fact

$$\gamma(\boldsymbol{x},\boldsymbol{y}) = \underline{\omega}_\delta(|\boldsymbol{x}-\boldsymbol{y}|) = \delta^{-d}\omega(|\boldsymbol{x}-\boldsymbol{y}|/\delta),$$

for a density $\omega = \omega(|\boldsymbol{x}|)$, though other forms may prove to also be useful, as shown later.

A nonlocal model $-\mathcal{L}_\delta u_\delta = f$ defines in $\mathbb{R}^d$ or Ω with a given data f and the unknown solution u can be seen as a more general model than differential/difference models or a bridge connecting them.

In recent years, there has been significant progress towards a systematic (or even axiomatic) development of a rigorous mathematical framework for nonlocal operators and nonlocal models. In [12], attempts have been made to develop a nonlocal vector calculus as the basic mathematical building blocks for nonlocal continuum models, mimicking the classical Newton's vector calculus for local models in the form of partial differential equations. A comparison of a few key ingredients and concepts is presented in the Table 1.

For further development and applications of the framework to the analysis of various nonlocal models such as linear peridynamic bond-based and state-based models, nonlinear hyperelastic peridynamic models, and nonlocal in time dynamic models, we refer to [11,

Newton's vector calculus	$\Leftrightarrow$	Nonlocal vector calculus
Local balance (PDE)	$\Leftrightarrow$	Nonlocal balance (PD)
Differential gradient/divergence	$\Leftrightarrow$	Nonlocal gradient/divergence
$-\nabla\cdot(K\nabla u)=f$	$\Leftrightarrow$	$-\mathscr{D}\cdot(\omega_\delta\mathscr{D}^*u)=f$
Boundary conditions	$\Leftrightarrow$	Volumetric constraints
Integral identities/integration by parts	$\Leftrightarrow$	Nonlocal integration by parts
$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\partial_n v - v\partial_n u$	$\Leftrightarrow$	$\iint u\mathscr{D}(\mathscr{D}^*v) - v\mathscr{D}(\mathscr{D}^*u) = 0$

TABLE 1. Nonlocal vector calculus vesus Newton's vector calculus

12, 29, 30, 31, 32, 13]. Some of the main distinctions of these series of works include the theory developed for systems of equations involving vectors and tensor fields, the finite range of nonlocal interactions and the precise dependence of the mathematical estimates on the horizon δ, and the minimal regularity assumptions associated with the underlying function spaces.

Generically, nonlocal equations posed on a domain $\Omega\subset\mathbb{R}^d$ are complemented by nonlocal boundary conditions, or more precisely, constraints on a some domain with nonzero d-dimensional volume, hence leading to the so-called constrained value problems [11]. A pictorial depiction of nonlocal problems with a nonlocal horizon $\delta>0$ and volumetric constraints is given in Figure.1, together with its possible differential equation limit in the form of a standard Dirichlet boundary value problem. We note that some key ingredients in the development of the well-posedness theory for linear nonlocal models are the nonlocal Kohn's inequality and nonlocal Poincare inequalities, which serve as the nonlocal analog of their classical, local counterparts for linear PDEs.

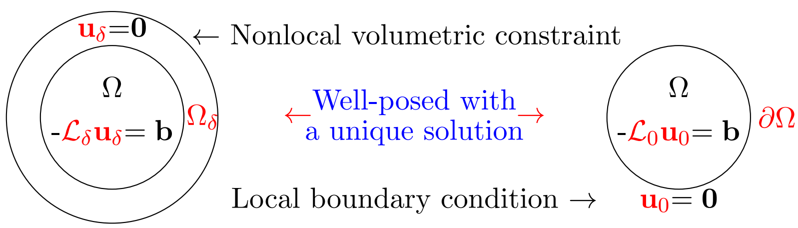

FIGURE 1. A nonlocal constrained value problem and a local boundary value problem.

In general, nonlocal models may allow more singular solutions in a function space $\mathcal{S}_\delta$ larger than H^1. The ability to produce more singular solutions is a major motivation for studyingf nonlocal models in subjects such as fracture mechanics and anomalous diffusion. For more studies on nonlocal spaces and associated nonlocal operators, we refer to [3, 5, 6, 11, 12, 16, 18, 24, 28, 31, 33, 37, 42] and references cited therein on related applications and mathematical analysis. Moreover, developing nonsmooth calculus [20] has also been an active subject of mathematical research with strong connections to geometry [15].

To avoid the use of such nonlocal constraints, the corresponding nonlocal operators need to be properly modified near the boundary, which is often the case for fractional differential equations [6]. For a more recent survey on the nonlocal elliptic equations, we refer to [38].

In order to have well-defined nonlocal problems on Ω with Dirichlet type data on part of its boundary $\partial\Omega$ of codimension-1, study of the trace map in $\mathcal{S}(\Omega)$ becomes natural and at the same time a necessity. However, it is known that in general $\mathcal{S}(\Omega)$ contains $H^1(\Omega)$

as a subspace and it is sufficient large so that it may contain more singular functions as desired by nonlocal models. Meanwhile, we know from localization results that as $\delta \to 0$, we may recover the classical H^1 space, that means that in the local limit, the trace property is attained together with the improvement in regularity homogeneously over the whole domain. This of course does not bode well with our goal, but it leads to the natural idea of heterogeneous localization. That is, we desire to define a nonlocal space which maintains the nonlocality inside the domain (thus allowing more singular functions to be included), but gets localized at the boundary (giving rise to the well-defined trace map). This is the path followed in [45].

3. New trace theorem in heterogeneously localized nonlocal spaces

The main findings of [45] are that the trace map exists and is continuous on a nonlocal function space $\mathcal{S}(\Omega)$ if the support of the kernel is heterogeneously localized as $\boldsymbol{x}$ approaches the boundary of Ω. By considering such a class of kernels, the study departs from many existing works corresponding to typical translate-invariant kernels (such as the celebrated work [5]), though in other settings, variable order and variable growth function spaces have been a popular subject with a rich history and much recent interest, see for instance [8, 34].

In [45], the class of kernels under consideration is given by

$$\gamma(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{|\delta(\boldsymbol{x})|^{d+2}}\hat{\gamma}\left(\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\delta(\boldsymbol{x})}\right) \tag{4}$$

where $\hat{\gamma} = \hat{\gamma}(s)$ is a non-increasing nonnegative function defined for $s \in (0,1)$ with a finite $d+1$ moment. The influence horizon $\delta = \delta(\boldsymbol{x})$ is a function defined on Ω that approaches zero when $\boldsymbol{x}$ approaches the boundary. A simple choice taken in [45] is

$$\delta(\boldsymbol{x}) = \sigma\,\mathrm{dist}(\boldsymbol{x},\Gamma), \quad \boldsymbol{x}\in\Omega\,, \tag{5}$$

for some $\sigma \in (0,1]$ and $\Gamma \subset \partial\Omega$. The associated nonlocal neighborhood $\mathcal{H}(\boldsymbol{x})$ is defined by

$$\mathcal{H}(\boldsymbol{x}) := \{\boldsymbol{y}\in\Omega : |\boldsymbol{y}-\boldsymbol{x}| \le \delta(\boldsymbol{x})\} = B_{\delta(\boldsymbol{x})}(\boldsymbol{x})\,.$$

A key observation of [45] is that, by allowing heterogeneous localization with vanishing effective neighborhood $\mathcal{H}(\boldsymbol{x})$ when $\boldsymbol{x}$ approaches the boundary $\partial\Omega$, we expect to have a well defined continuous trace map from the associated nonlocal space $\mathcal{S}(\Omega)$ to $H^{1/2}(\partial\Omega)$. This leads to the generalized trace theorem of [45] that is stated below.

Theorem 3.1 (General trace theorem)**.** *Assume that Ω is a bounded simply connected Lipschitz domain in $\mathbb{R}^d$ $(d\ge 2)$ and $\Gamma = \partial\Omega$, then there exists a constant C depending only on Ω and Γ such that the trace map T for Γ satisfies*

$$\|Tu\|_{H^{\frac{1}{2}}(\Gamma)} \le C\|u\|_{\mathcal{S}(\Omega)}\,, \quad \forall u\in\mathcal{S}(\Omega)\,. \tag{6}$$

The following proposition has been established in [45], which is of independent interests on its own while showing the relation between the classical Sobolev space $H^1(\Omega)$ and the new heterogeneously localized nonlocal space $\mathcal{S}(\Omega)$. The result generalizes a similar conclusion for nonlocal spaces defined with a constant horizon and translation invariant kernel known in the literature, see [5].

Proposition 3.2. *For $\delta(\boldsymbol{x}) = \sigma\mathrm{dist}(x,\Gamma)$ with $\sigma\in(0,1)$, the space $H^1(\Omega)$ is continuously imbedded in $\mathcal{S}(\Omega)$ and there exists a constant C depending only on σ and Ω such that*

$$\|u\|_{\mathcal{S}(\Omega)} \le C\|u\|_{H^1(\Omega)}\,, \quad \forall u\in H^1(\Omega)\,. \tag{7}$$

Moreover, C is independent of σ for σ small.

The proof of (7) can be done for a smooth function $u \in C^1(\bar{\Omega}) \cap H^1(\Omega)$ first. One key observation is that

$$\begin{aligned}
|u|^2_{\mathcal{S}(\Omega)} &\le \int_{\mathbb{R}^d} \int_{|\boldsymbol{h}|<\delta(\boldsymbol{x})} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma}(\frac{|\boldsymbol{h}|}{\delta(\boldsymbol{x})}) |u(\boldsymbol{x}+\boldsymbol{h}) - u(\boldsymbol{x})|^2 d\boldsymbol{h} d\boldsymbol{x} \\
&= \int_{\mathbb{R}^d} \int_{|\boldsymbol{h}|<1} \frac{1}{|\delta(\boldsymbol{x})|^2} \hat{\gamma}(\boldsymbol{h}) |u(\boldsymbol{x}+\delta(\boldsymbol{x})\boldsymbol{h}) - u(\boldsymbol{x})|^2 d\boldsymbol{h} d\boldsymbol{x} \\
&\le \int_{\mathbb{R}^d} \int_{|\boldsymbol{h}|<1} |\boldsymbol{h}|^2 \hat{\gamma}(\boldsymbol{h}) \int_0^1 |\nabla u(\boldsymbol{x}+t\delta(\boldsymbol{x})\boldsymbol{h})|^2 dt d\boldsymbol{h} d\boldsymbol{x}\,.
\end{aligned}$$

For $\boldsymbol{y} = \boldsymbol{x} + t\delta(\boldsymbol{x})\boldsymbol{h}$, by noting that

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = I + t\nabla\delta(\boldsymbol{x}) \otimes \boldsymbol{h}\,,$$

and its inverse are uniformly bounded everywhere if $\|\nabla\delta\| = \sigma < 1$, with the bounds independent of σ if σ is small, we get the existence of a generic constant $C > 0$ such that

$$\begin{aligned}
|u|^2_{\mathcal{S}(\Omega)} &\le C(\int_{|\boldsymbol{h}|<1} |\boldsymbol{h}|^2 \hat{\gamma}(\boldsymbol{h}) d\boldsymbol{h}) |u|^2_{H^1(\mathbb{R}^d)} \\
&\le C\|u\|^2_{H^1(\Omega)}\,.
\end{aligned}$$

The constant C may depend on Ω but is independent of σ for σ small. For a complete proof, we refer to [45].

By the Proposition 3.2, we have

$$\|Tu\|_{H^{\frac{1}{2}}(\Gamma)} \le C\|u\|_{\mathcal{S}(\Omega)} \le C\|u\|_{H^1(\Omega)}\,, \quad \forall u \in H^1(\Omega)\,,$$

so we can see that the trace inequality (6) is indeed a refinement of the classical trace inequality in the conventional Sobolev space $H^1(\Omega)$. The latter follows as a simple consequence.

Corollary 3.3 (Sobolev trace theorem)**.** *Assume that Ω is a bounded simply connected Lipschitz domain in $\mathbb{R}^d$ $(d \ge 2)$ and $\Gamma = \partial\Omega$, then there exists a constant C depending only on Ω and Γ such that the trace map T for Γ satisfies*

$$\|Tu\|_{H^{\frac{1}{2}}(\Gamma)} \le C\|u\|_{H^1(\Omega)}\,, \quad \forall u \in H^1(\Omega)\,. \tag{8}$$

4. A special trace theorem as an illustrative example

The full proof of the trace theorem 3.1 is presented in [45]. To help understanding what the result conveys and how it compares with other relevant works, it is suggestive to consider a special case. That is, in terms of Ω and Γ , we take a special stripe domain $\Omega = (0, r) \times \mathbb{R}^{d-1}$ and a portion of its boundary $\Gamma = \{0\} \times \mathbb{R}^{d-1}$ where r is any given positive constant, while for the kernel, we adopt a special kernel with $\hat{\gamma}$ to be the characteristic function $\chi_{(0,1)}$ for $|\boldsymbol{y} - \boldsymbol{x}| \in (0, 1)$ with the horizon $\delta(\boldsymbol{x})$ given by the first component of $\boldsymbol{x}$, that is,

$$\gamma(\boldsymbol{x}, \boldsymbol{y}) = \frac{\chi_{(0,1)}(|\boldsymbol{y} - \boldsymbol{x}|)}{|\delta(\boldsymbol{x})|^{d+2}}\,, \quad \delta(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \Gamma) = x_1\,, \quad \text{for } \boldsymbol{x} = (x_1, \tilde{\boldsymbol{x}}), \quad \tilde{\boldsymbol{x}} \in \mathbb{R}^{d-1}. \tag{9}$$

The geometric set up is depicted in Figure 2.

As shown in [45] , this special case serves as not only a helpful step towards proving the more general result but also an illustrative example.

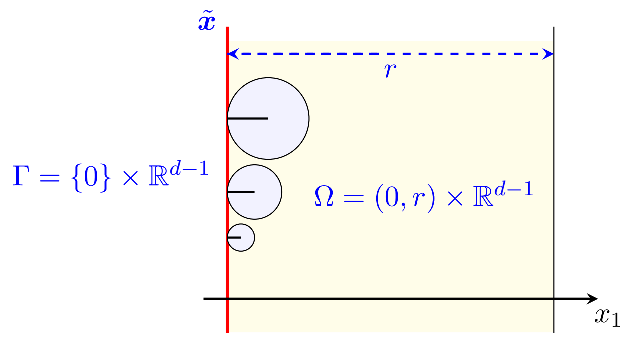

FIGURE 2. Depiction of the stripe geometry.

Let us now take a look at the special form of the nonlocal norms in this case. By definition, we get

$$|u|^2_{\mathcal{S}(\Omega)} = \int_\Omega \int_{\Omega \cap \{|\boldsymbol{y}-\boldsymbol{x}|<|x_1|\}} \frac{(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2}{|x_1|^{2+d}} d\boldsymbol{y} d\boldsymbol{x} \,.$$

Clearly, the denominator x_1 penalizes the spatial variation only at $x_1 = 0$, thus $\mathcal{S}(\Omega)$ contains all functions that are in $L^2(\tilde{\Omega})$ for any domain $\tilde{\Omega}$ with its closure being a compact subset of Ω. Hence, functions in $\mathcal{S}(\Omega)$ are generally not expected to have regularity better than $L^2(\Omega')$ over the subdomain Ω', or may be significantly less regular away from the boundary than H^1 functions. Yet, as elucidated in [45], due to the shrinking horizon towards the boundary, the penalization of spatial variations provides enough regularity for the functions in $\mathcal{S}(\Omega)$ to have well-defined traces just on the boundary itself. Intuitively, this is a natural consequence of the localization of nonlocal interactions on the boundary.

In contrast, we may recall standard norms associated with fractional Sobolev spaces defined on Ω.

$$|u|^2_{H^\alpha(\Omega)} = \int_\Omega \int_\Omega \frac{(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2}{|\boldsymbol{y} - \boldsymbol{x}|^{2\alpha+d}} d\boldsymbol{y} d\boldsymbol{x} \,.$$

The regularity of the functions is effected by the denominator which vanishes at $\boldsymbol{x} = \boldsymbol{y}$.

We now state the trace theorem in the special form.

Theorem 4.1 (Special trace theorem)**.** *For $\Omega = (0, r) \times \mathbb{R}^{d-1}$ and $\Gamma = \{0\} \times \mathbb{R}^{d-1}$, there exists a constant C depends only on d such that such that for any $u \in C^1(\bar{\Omega}) \cap \mathcal{S}(\Omega)$,*

$$\|u\|_{L^2(\Gamma)} \le C\Big(r^{-1/2}\|u\|_{L^2(\Omega)} + r^{1/2}|u|_{\mathcal{S}(\Omega)}\Big) \,, \tag{10}$$

and for $d \ge 2$,

$$|u|_{H^{1/2}(\Gamma)} \le C\Big(r^{-1}\|u\|_{L^2(\Omega)} + |u|_{\mathcal{S}(\Omega)}\Big) \,. \tag{11}$$

The special form allows us to give more quantitative estimates on the constants appearing in the trace inequality. Note that we are mainly interested in the small r dependence of the imbedding coefficients. For large r, the results remain true with uniformly bounded coefficients, that is, $\|u\|_{H^{1/2}(\Gamma)} \le C\|u\|_{\mathcal{S}(\Omega)}$ where C is a constant as $r \to \infty$.

In [45], it has also been noted that the need of the $L^2(\Omega)$ norm in (10) and (11) is similar to that for standard trace inequalities in Sobolev spaces. The dependence on the $L^2(\Omega)$ norm may be removed by considering L^p type norms for the trace with a suitable choice of p, just like the classical counterpart in Sobolev spaces.

5. More on the nonlocal spaces and norms

In the process of establishing the trace inequality, there are a number of results presented in [45] that are of independent interests. For example, a generalization of the classical Hardy's inequality has been shown there, with the one dimensional version given below.

Proposition 5.1 (Nonlocal Hardy-type inequality in one-dimension)**.** *Let $\Omega = (0, r)$ for some $r > 0$, $\Gamma = \{0\}$, and a and b satisfy $0 \le a < b \le 1$. Then we have for any $u \in C^1(\bar{\Omega})$ with $u\,|_\Gamma = u(0) = 0$,*

$$\int_\Omega \frac{|u(x)|^2}{|x|^2} dx \le C_{a,b} \int_\Omega \int_{ax}^{bx} \frac{|u(y) - u(x)|^2}{|x|^3} dy dx\,, \quad \text{where } C_{a,b} = \frac{4(2+b+a)}{(b-a)(2-b-a)^2}. \tag{12}$$

In particular, this implies a Hardy-type inequality: there is a constant $C > 0$ independent of Ω such that,

$$\int_\Omega \frac{|u(x)|^2}{dist(x,\Gamma)^2} dx \le C|u|^2_{\mathcal{S}(\Omega)}\,. \tag{13}$$

We recall the classical Hardy's inequality is of the form:

$$\int_0^r \frac{|u(x)|^2}{\mathrm{dist}(x,\Gamma)^2} dx \le C \int_0^r |u'(x)|^2 dx\,. \tag{14}$$

Obviously, in the above classiical Hardy's inequality, the requirement of $u' \in L^2$ is redundant away from $x = 0$ since a simple observation shows that the singular weight on the left hand side of (14) only becomes effective right on the boundary $x = 0$. Thus, we need that though widely useful, the inequality itself is not sharp. On the other hand, the singular weight for the L^2 norm is in perfect synchrony with the effects of similar singular weight used in the nonlocal norm. In this sense, the nonlocal Hardy's inequality is in fact much more natural.

One can expect a higher dimensional nonlocal Hardy's inequality as well, which is also given in [45] and stated below.

Proposition 5.2 (Nonlocal Hardy's inequality)**.** *Given a bounded Lipshitz domain Ω, there exists a constant $C > 0$ such that if $Tu = 0$ on $\partial\Omega$, then*

$$\int_\Omega \frac{|u(\boldsymbol{x})|^2}{(dist(\boldsymbol{x}, \partial\Omega))^2} d\boldsymbol{x} \le C|u|^2_{\mathcal{S}(\Omega)}\,. \tag{15}$$

The one-dimensional version of the nonlocal Hardy's inequality (13) can be proved via elementary means, but the proof for the more general multidimensional case is somewhat more involved and it is shown with the help of another estimate on the variations of u along the normal direction. discussed later. Such an estimate requires careful studies. Hence, in the same spirit of norms of directional derivatives in classical, local function spaces, the following definition is introduced in [45] as a nonlocal analog that refines our understanding of how the nonlocal norm $\|\cdot\|_{\mathcal{S}(\Omega)}$ provides control on the function variation in different directions. This not only helps proving the nonlocal Hardy's inequality, but also plays important roles in proving the new trace theorems.

Definition 5.3. *On the domain $\Omega = (0, r) \times \mathbb{R}^{d-1}$, we define in the following two directional nonlocal semi-norms $|\cdot|_n$ and $|\cdot|_t$, standing for normal and tangential directions respectively* *with reference to the boundary segment* $\Gamma = \{0\} \times \mathbb{R}^{d-1}$,

$$|u|_n^2 = \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{ax_1}^{bx_1} \frac{|u(y_1, \bar{\boldsymbol{x}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} dy_1 dx_1 d\bar{\boldsymbol{x}} \tag{16}$$

$$|u|_t^2 = \int_{\mathbb{R}^{d-1}} \int_0^r \fint_{B_{cx_1}(\bar{\boldsymbol{x}})} \frac{|u(x_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|^2}{|x_1|^2} d\bar{\boldsymbol{y}} dx_1 d\bar{\boldsymbol{x}} \tag{17}$$

where $0 \le a < b \le 1$ *and* $0 < c < 1$ *are constants, and* $\fint$ *represents the integral average over a domain, that is, the integral over the domain divided by the domain volume.*

Some heuristic comments are provided in [45] to offer insight into the above definitions. For a smooth function $u = u(\boldsymbol{x})$, we may approximately have, in an informal manner, that t $|\cdot|_n$ and $|\cdot|_t$ mimic norms of directional directives, and they are thus named nonlocal normal and tangential directional semi-norms. In comparison, we may also informally see that $|\cdot|^2_{\mathcal{S}(\Omega)}$ involves nonlocal variations in all directions like a total gradient semi-norm.

While the norms of classical (local) directional derivatives are obviously bounded by that of the total gradient, a challenging step in [45] is to establish a nonlocal analog, stated in the the following lemma, that shows that $|\cdot|_n$ and $|\cdot|_t$ are indeed controlled by the original semi-norm $|\cdot|_{\mathcal{S}(\Omega)}$.

Lemma 5.4. *Let* $\Omega = (0, r) \times \mathbb{R}^{d-1}$ *for some* $r > 0$, a, b *and* c *satisfy* $0 \le a < b \le 1$, $0 < c < 1$ *and* $(a-1)^2 + c^2 \le 1$. *Then there exists a constant* C *depending only on* a, b *and* c *such that for any* $u \in \mathcal{S}(\Omega)$,

$$|u|_n \le C|u|_{\mathcal{S}(\Omega)}\,, \tag{18}$$

$$|u|_t \le C|u|_{\mathcal{S}(\Omega)}\,. \tag{19}$$

The above inequalities are established as consequences of the following inequalities

$$|u|_n^2 \le c_1 |u|_t^2 + C|u|^2_{\mathcal{S}(\Omega)} \tag{20}$$

$$|u|_t^2 \le c_2 |u|_n^2 + C|u|^2_{\mathcal{S}(\Omega)} \tag{21}$$

where $c_1 c_2 < 1$.

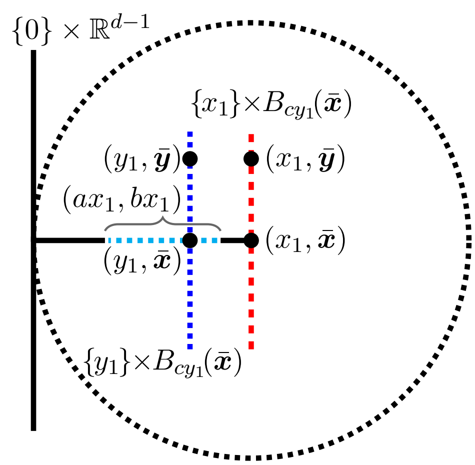

FIGURE 3. Depiction of geometry used in the proof of Lemma 5.4 in [45].

The Figure 3 can help explaining how the above is achieved with the suitable choices of a, b and c. Geometrically, the three constants determines the geometric picture in Figure 3. The cyan horizontal dotted line shows the range of $(y_1, \bar{\boldsymbol{x}})$, the blue vertical dotted line for $(y_1, \bar{\boldsymbol{y}})$, and the red vertical dashed line for $(x_1, \bar{\boldsymbol{y}})$. The key to choose these positions is to make sure that $(y_1, \bar{\boldsymbol{y}})$ stays in the effective neighborhood of $(x_1, \bar{\boldsymbol{x}})$, which is bounded by the black dashed circle. This essentially becomes the issue of how to pick the constants a, b and c.

6. More about the proof of theorem 4.1 on the stripe domain

For showing the bound on the $H^{1/2}$ semi-norm in the inequality (11), we essentially need to prove the following inequality

$$\int_{\mathbb{R}^{d-1}} \int_{B_{r/2}(\bar{\boldsymbol{x}})} \frac{|u(0,\bar{\boldsymbol{y}}) - u(0,\bar{\boldsymbol{x}})|^2}{|\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}|^d} d\bar{\boldsymbol{y}} d\bar{\boldsymbol{x}} \le C|u|^2_{\mathcal{S}(\Omega)} \,. \tag{22}$$

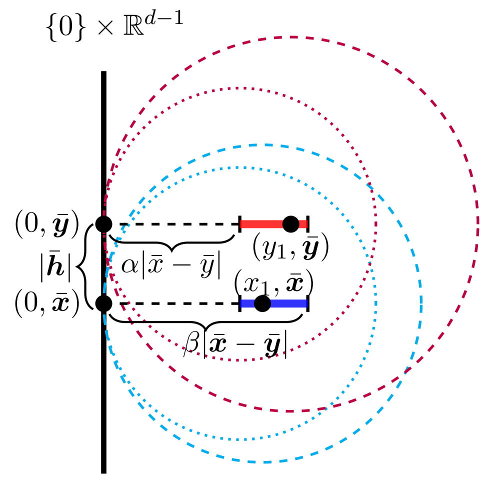

Figure 4. Depiction of geometry used in the proof of Theorem 4.1.

The idea is to make the interior extension using

$$|u(0,\bar{\boldsymbol{y}}) - u(0,\bar{\boldsymbol{x}})| \le |u(0,\bar{\boldsymbol{y}}) - u(y_1,\bar{\boldsymbol{y}})| + |u(y_1,\bar{\boldsymbol{y}}) - u(x_1,\bar{\boldsymbol{x}})| + |u(x_1,\bar{\boldsymbol{x}}) - u(0,\bar{\boldsymbol{x}})|$$

and to control all the terms associated with the right hand side by the nonlocal norm. The blue and red solid horizontal line segments in Figure 4 show the possible positions of $(x_1, \bar{\boldsymbol{x}})$ and $(y_1, \bar{\boldsymbol{y}})$ respectively. The key is to determine the end points of these line segments so that any $(y_1, \bar{\boldsymbol{y}})$ on the red solid line should remain in the effective neighborhood (shown as balls bounded by dotted and dashed spheres, with the dotted ones giving the smallest neighborhoods) of any $(x_1, \bar{\boldsymbol{x}})$ on the other blue solid line and vice versa. The term associated with $|u(y_1, \bar{\boldsymbol{y}}) - u(x_1, \bar{\boldsymbol{x}})|$ is most similar to the nonlocal norm, while the other two terms involves essentially variations in the normal direction that can be estimated with the help of Lemma 5.4.

Concerning the process of interior extension, it is worthy noting that the problem of relating boundary estimates and interior estimates appears often in the study of PDE boundary value problems, such as in Kellogg's theorem for deriving C^α regularity estimates up to the boundary with prescribed C^α data [23], and in deriving interior regularity estimates from the coincidence set for free boundary problems [26].

Indeed, the idea of relating boundary points to interior points in order to get an estimate of boundary from those in the interior leads to a popular approach to establish the classical trace theorem, see for example, [25, chapter 15]. However, a new challenge in our work here in the nonlocal case, unlike the straightforward constructions in the classical case, is that the interior points need to be carefully chosen to make the nonlocal norm $\|u\|_{\mathcal{S}(\Omega)}$ coming into play. The lemma 5.4 provides us analogies of estimates on tangential and normal derivatives that are important for the derivation. The details are very involved and can be found in [45].

7. Generalizations

7.1. **More general kernels.** Although much of our discussion so far is focused on the choice that $\hat{\gamma}$ takes on a constant value over its support, the new nonlocal trace theorem is valid for more general nonlocal interactions. The special choice of $\hat{\gamma}$ not only avoids technical complication but also corresponds to some of the weakest nonlocal norms among those associated with popular kernels used in the literature. For example, for a typical fractional power law kernel $\hat{\gamma}(s) = 1/s^\lambda$ [2, 9], we have the rescaled fractional type kernel

$$\gamma^\lambda(\boldsymbol{x}, \boldsymbol{y}) = \frac{c_\lambda}{|\delta(\boldsymbol{x})|^{d+2-\lambda}} \cdot \frac{1}{|\boldsymbol{y} - \boldsymbol{x}|^\lambda} \quad \text{for } \boldsymbol{y} \in \mathcal{H}(\boldsymbol{x}), \quad \lambda \in [0, d+2)\,. \tag{23}$$

For such kernels, it is easy to make the following comparison of norms.

Lemma 7.1. *For γ^λ defined in (23) with $\lambda \in (0, d+2)$, there is a constant $C > 0$ such that the following inequality holds for any function u,*

$$\int_\Omega \int_{\Omega \cap \mathcal{H}(\boldsymbol{x})} \gamma^0(\boldsymbol{x}, \boldsymbol{y})(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 d\boldsymbol{y} d\boldsymbol{x} \leq C \int_\Omega \int_{\Omega \cap \mathcal{H}(\boldsymbol{x})} \gamma^\lambda(\boldsymbol{x}, \boldsymbol{y})(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 d\boldsymbol{y} d\boldsymbol{x}\,.$$

The lemma shows that the nonlocal norm defined with the constant kernel associated with $\lambda = 0$ indeed gives the weakest norm among ones corresponding to a large class of kernels either associated with (23) or are bounded below and above by such kernels when $\boldsymbol{x}$ and $\boldsymbol{y}$ are close by.

7.2. **More general choices of horizon.** To generalize the choices of the variable horizon, we may consider, for example, the case $\delta(\boldsymbol{x}) = \sigma \text{dist}(\boldsymbol{x}, \Gamma)$ where $\sigma \in (0, \sigma_0]$ for $\sigma_0 > 0$. On this matter, we define some notations first.

$$|u|^2_{\delta(\boldsymbol{x}), r} = \int_{\Omega_r} \int_{\Omega_r \cap \mathcal{H}(\boldsymbol{x})} \frac{1}{|\delta(\boldsymbol{x})|^{d+2}} \hat{\gamma}\Big(\frac{|\boldsymbol{y} - \boldsymbol{x}|}{\delta(\boldsymbol{x})}\Big)(u(\boldsymbol{y}) - u(\boldsymbol{x}))^2 d\boldsymbol{y} d\boldsymbol{x}\,.$$

where $\Omega_r = (0, r) \times \mathbb{R}^{d-1}$. The next lemma shows that the smaller σ is, the larger the nonlocal norm we can get.

Lemma 7.2. *Let $\delta(\boldsymbol{x}) = \sigma\, dist(\boldsymbol{x}, \Gamma)$, where $\sigma \in [\frac{1}{2}, 1)$ and $\Gamma = \{0\} \times \mathbb{R}^{d-1}$, then there exists a constant C depending only on d such that the following inequality holds for any $r > 0$ and $\alpha \in (0, 1]$,*

$$|u|^2_{\delta(\boldsymbol{x}), r/2} \leq C \Big(\frac{1+\sigma}{1-\sigma}\Big)^{d+2} |u|^2_{\alpha\delta(\boldsymbol{x}), r}\,. \tag{24}$$

Using this lemma, we may have the extension to more general horizons given by $\delta(\boldsymbol{x}) = \sigma \text{dist}(\boldsymbol{x}, \Gamma)$.

Proposition 7.3. *The results of Theorem 3.1 and Theorem 4.1 remain valid for influence horizon of the form $\delta(\boldsymbol{x}) = \sigma\, dist(\boldsymbol{x}, \Gamma)$ where $\sigma \in (0, \sigma_0]$ for some $\sigma_0 > 0$. Moreover, the embedding constant C depends only on Ω, Γ and σ_0.*

A horizon proportional to the distance to the boundary is a specific choice that can be further generalized. One instance is that $\delta(\boldsymbol{x})$ is proportional to $\text{dist}(\boldsymbol{x}, \Gamma)$ for $\boldsymbol{x}$ only on a boundary layer of finite positive width but remains constant elsewhere. A possible form of such a $\delta(\boldsymbol{x})$ could be

$$\delta(\boldsymbol{x}) = \min\{\sigma \mathrm{dist}(x, \Gamma), \eta\},$$

for some $\eta > 0$ to be specified. Another possibility is to have $\delta(\boldsymbol{x})$ vanishes in other nonlinear ways as $\boldsymbol{x}$ approaches the boundary. Similar results can be shown in these cases and they follow naturally from the fact that it is the nonlocal interaction in the boundary layer, rather than the interior of the domain, that provides the essential control on the $H^{\frac{1}{2}}$ trace.

The discussion on the general form of $\delta(\boldsymbol{x})$ is meaningful since it is important in many applications to note that the imbedding constant in (7) does not depend on σ, just like the constants appearing in the new nonlocal trace inequalities. For example, for the coupled PDE and nonlocal model depicted in Fig. 6, we may recover a coupled PDE models in the local limit as $\sigma \to 0$. This again implies that the nonlocal trace theorems are refinement and improvement of the classical trace theorems in $H^1(\Omega)$.

7.3. **More general domains and trace theorems on portions of the domain boundary.** Concerning extending the trace theorem from the special stripe domain to more general Liphshitz domain, one can establish theorem 3.1 using partition of unity techniques.

As for generalization of the trace theorem to portions of the domain boundary. we note that the classical, local trace inequality on $\partial\Omega$ automatically implies the same result for the trace on a subset Γ of $\partial\Omega$. This is not, however, as straightforward for the nonlocal space whose definition involves the Γ dependent horizon, and thus the Γ dependent nonlocal kernel.

To demonstrate that similar results remain valid, a possible route is pointed out in [45]. That is, one can consider first a special domain in the form of a (rectangular) section of the stripe domain, for instance, $\Omega = (0, r) \times (a, b) \times \mathbb{R}^{d-2}$ and $\Gamma = \{0\} \times (a, b) \times \mathbb{R}^{d-2}$. By a suitable extension in the second variable from the interval (a, b) to the whole real line, the result in Theorem 4.1 can be utilized for the whole stripe domain to get the desired result on its subsection. One may then employ similar partition of unity techniques and domain transformations to more general domains and more general subset of their boundary.

7.4. **Other generalizations.** In terms of further generalizations of the trace theorems, we note that although the results of this paper are only shown for the L^2 or the Hilbert space setting, it is not surprising that they can be generalized to the L^p and other more general Banach spaces. With the choices of more general kernels, one may also consider nonlocal extensions of trace results in fractional $W^{s,p}$ type spaces. Extensions of the notion of trace may also go beyond co-dimensional one manifolds to other more general subdomains or sets. Furthermore, the position-dependent and heterogeneous feature in the nonlocal norms may be related for the study of more general Morrey, Campanato, Besov and Lizorkin-Triebel spaces, possibly of variable order and growth conditions, to obtain new type of spaces and the associated trace maps [22, 35]. In addition, connections with the study of Sobolev and other function spaces on metric measure spaces may also be explored [20, 21]. Mathematically, one may also ask questions concerning optimal constants in the trace inequality, as in the classical case [14]. Moreover, while it is known that the H^1 space gives the smallest Sobolev space with continuous $H^{1/2}$ boundary trace map, we now see much larger spaces can also preserve the same property, even in spaces like what we define here whose functions may only be in L^2 over any compact subset away from the boundary. Thus, the issue of how large such a space can be, as communicated to us by Luis Caffarelli, becomes very interesting to study.

Another direction, motivated naturally by interests in nonlocal mechanics, is to consider analogous results for spaces of vector fields such as those studied in [30, 31]. Likewise, one may investigate high order extensions as well, following the discussions of high order nonlocal spaces likes ones in [44], which were relevant to the studies of beams and shells.

In closing, the main results presented here are indicative of the conceptually simple observation on the improved regularity of functions in nonlocal spaces associated with a vanishing nonlocal horizon, either uniformly across the domain of interest, or when approaching a codimension-1 surface. In the former case we recover the limit of nonlocal spaces being the classical Sobolev space, as in [5, 37], while in the latter case we obtain the analogue and extension of the classical trace theorem. One may further investigate regularity estimates, multiscale analysis and homogenization issues associated with nonlocal problems having a heterogeneous choice of variable horizon and nonlocal interaction kernels.

8. Variational problems associated with heterogeneously localized nonlocal operators

8.1. **Boundary value problems for heterogeneously localized nonlocal models.** Generically, as alluded earlier, nonlocal equations posed on a domain $\Omega \subset \mathbb{R}^d$ are complemented by nonlocal boundary conditions. More precisely, constraints on a some domain with nonzero d-dimensional volume, hence leading to so-called constrained value problems [11], as illustrated in Figure 1. To avoid the use of such nonlocal constraints, the nonlocal operators need to be properly modified near the boundary, which is often the case for fractional differential equations [6]. For a more recent survey on the nonlocal elliptic equations, we refer to [38].

With heterogeneous localization of the nonlocal interactions, we can have well-posed boundary value problem with Dirichlet data on the boundary only, thanks to the trace theorem and nonlocal Poincare's inequality. This is depicted in Figure 5.

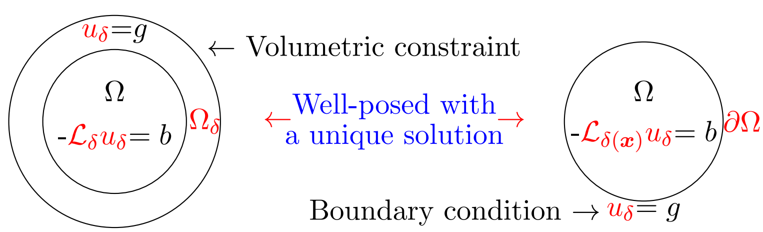

Figure 5. Nonlocal problems with either volumetric constraints or local boundary conditions.

One may further establish the convergence of numerical methods, for example, conforming finite element methods as long as the finite element spaces are dense subspaces of the nonlocal energy space. The latter is assured if we have the density in the classical Sobolev space H^1, a result that is reminiscent and consistent to the findings on asymptotically compatible schemes for the numerical solution of nonlocal problems involving a constant horizon [43].

8.2. **Coupled local and nonlocal models.** Another consequence of a well-defined trace map with the trace belonging to a space similar to that for standard Sobolev spaces would allow a seamless coupling between a classical, local PDE (for instance the Poisson equation $-\Delta u = f$) on one side Ω_- of a codimension-1 interface Γ with a nonlocal equation (say the variational equation $-\mathcal{L}u = f$ associated with the nonlocal energy) on the other side Ω_+ of Γ, see Fig. 6 for an illustration (the circular domains depict domains of nonlocal interactions associated with a heterogeneously defined horizon parameter). The study on transmission conditions and the well-posedness of the coupled local and nonlocal models can be readily obtained from a variational formulation

$$\min_{u_-=u_+ \text{ on } \Gamma} \quad E_{\text{local}}(u_-) + E_{\text{nonlocal}}(u_+) = |\nabla u_-|^2_{L^2(\Omega_-)} + |u|^2_{\mathcal{S}(\Omega_+)} \,.$$

Furthermore, having varying horizon allows one to harvest the flexibility in working with nonlocal interactions on a wide range of scales so that more effective numerical simulations can be carried out, along the lines of asymptotically compatible schemes [43].

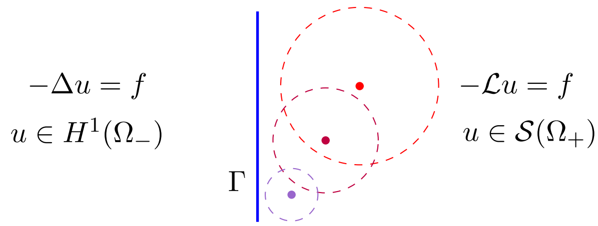

FIGURE 6. A PDE model (in Ω_-) is coupled with a nonlocal model (in Ω_+) using suitably defined boundary trace and transmission condition on Γ.

9. Conclusion

Nonlocality is intrinsic and ubiquitous in nature. In theory and modeling, it is also a generic feature of multiscale analysis and model reduction as it has often been (knowingly or implicitly) encoded in many earlier works (by the names of Mori-Zwanzig formalism, Dyson formula, Duhamel principle, ...). Nonlocal continuum models are arguably more general than local continuum models and discrete models and can also serve as a bridge connecting different models. Meanwhile, developing a systematic/axiomatic mathematical framework for nonlocal models is challenging and important in many important applications (that often involve defects/anomalies). Moreover, the study of nonlocal continuum models can be linked to many other concepts in mathematics, data and computational sciences as well as applications, like the combinatorial hodge theory, graph Laplacian, diffusion map, Levy flights, SPH, RKPM, etc [4, 19, 27, 36, 41, 48]. Hence, there are many good reasons to get better mathematical understanding of nonlocal models and nonlocality.

The main focus of the work presented here (based on [45]) is the heterogeneous localization of nonlocal models. This study leads to interesting mathematical extensions of classical results in functional analysis, and it could also be of practical significance for developing coupled local and nonlocal models that can retain nonlocal features wherever necessary and utilize effective local models wherever feasible, hence achieving effectiveness while maintaining generality. As a guidance for practical mathematical modeling, our findings can be summarized into a simple slogan, that is, think nonlocal, act local.

Acknowledgement

The first author would like to thank the organizing committee of ICCM 2016 for the invitation to speak at the conference.

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