LEAST ACTION PRINCIPLES FOR INCOMPRESSIBLE FLOWS AND OPTIMAL TRANSPORT BETWEEN SHAPES

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ABSTRACT. As V. I. Arnold observed in the 1960s, the Euler equations of incompressible fluid flow correspond formally to geodesic equations in a group of volume-preserving diffeomorphisms. Working in an Eulerian framework, we study incompressible flows of shapes as critical paths for action (kinetic energy) along transport paths constrained to be shape densities (characteristic functions). The formal geodesic equations for this problem are Euler equations for incompressible, inviscid potential flow of fluid with zero pressure and surface tension on the free boundary. The problem of minimizing this action exhibits an instability associated with microdroplet formation, with the following outcomes: Any two shapes of equal volume can be approximately connected by an Euler spray—a countable superposition of ellipsoidal geodesics. The infimum of the action is the Wasserstein distance squared, and is almost never attained except in dimension 1. Every Wasserstein geodesic between bounded densities of compact support provides a solution of the (compressible) pressureless Euler system that is a weak limit of (incompressible) Euler sprays. Each such Wasserstein geodesic is also the unique minimizer of a relaxed least-action principle for a two-fluid mixture theory corresponding to incompressible fluid mixed with vacuum.

1. INTRODUCTION

1.1. **Overview.** In this paper we develop several points of connection between least-action principles for incompressible fluids with free boundary and Wasserstein distance between shapes (as represented by characteristic-function densities). In particular, we show how Wasserstein distance between shapes (and more generally, compactly supported measures with bounded densities) arises naturally as a completion or relaxation of the problem of determining geodesic distance along a 'manifold' of equal-volume fluid domains.

The geometric interpretation of solutions of the Euler equations of incompressible inviscid fluid flow as geodesic paths in the group of volume-preserving diffeomorphisms was famously

pioneered by V. I. Arnold [4]. If we consider an Eulerian description for an incompressible body of constant-density fluid moving freely in space, such geodesic paths correspond to critical paths for the action

$$\mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho |v|^2 \, dx \, dt \,, \tag{1.1}$$

where $\rho = (\rho_t)_{t \in [0,1]}$ is a path of *shape densities* transported by a velocity field $v \in L^2(\rho \, dx \, dt)$ according to the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \,. \tag{1.2}$$

Here, saying that ρ_t is a shape density means that ρ_t is constrained to be a characteristic function for a fluid domain Ω_t:

$$\rho_t = \mathbb{1}_{\Omega_t} \,, \quad t \in [0,1]. \tag{1.3}$$

Naturally, then, the velocity field must be divergence free in the interior of Ω_t, satisfying $\nabla \cdot v = 0$ there. Equation (1.2) holds in the sense of distributions in $\mathbb{R}^d \times [0,1]$, interpreting ρv as 0 wherever $\rho = 0$.

In this Eulerian framework, it is natural to study the action in (1.1) subject to given endpoint conditions of the form

$$\rho_0 = \mathbb{1}_{\Omega_0}, \quad \rho_1 = \mathbb{1}_{\Omega_1} \,. \tag{1.4}$$

These conditions differ from Arnold-style conditions that fix the flow-induced volume-preserving diffeomorphism between Ω_0 and Ω_1, and correspond instead to fixing only the *image* of this diffeomorphism. As we show in section 3 below, it turns out that the geodesic equations that result are precisely the Euler equations for *potential flow* of an incompressible, inviscid fluid occupying domain Ω_t, with *zero pressure and zero surface tension* on the free boundary $\partial \Omega_t$. In short, the geodesic equations are classic water wave equations with zero gravity and surface tension. The initial-value problem for these equations has recently been studied in detail—the works [36, 18, 19] extend the breakthrough works of Wu [56, 57] to deal with nonzero vorticity and zero gravity, and establish short-time existence and uniqueness for sufficiently smooth initial data in certain bounded domains.

The problem of *minimizing* the action in (1.1) subject to the constraints above turns out to be ill-posed if the dimension $d > 1$, as we will show in this paper. By this we mean that action-minimizing paths that satisfy all the constraints (1.2), (1.3) and (1.4) do not exist in general, even locally. Nevertheless, the infimum of the action defines a distance between equal-volume sets which we will call *shape distance*, determined by

$$d_s(\Omega_0, \Omega_1)^2 = \inf \mathcal{A} \,, \tag{1.5}$$

where the infimum is taken subject to the constraints (1.2), (1.3), (1.4) above. By the well-known result of Benamou and Brenier [6], it is clear that

$$d_s(\Omega_0, \Omega_1) \geq d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1}), \tag{1.6}$$

where $d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})$ denotes the usual Wasserstein distance (Monge-Kanotorvich distance with quadratic cost) between the measures with densities $\mathbb{1}_{\Omega_0}$ and $\mathbb{1}_{\Omega_1}$. This is so because the result of [6] characterizes the squared Wasserstein distance $d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2$ as the infimum in (1.5) subject to the same transport and endpoint constraints as in (1.2) and (1.4), but *without* the constraint (1.3) that makes ρ a characteristic function.

Our objective in this paper is to develop several results that precisely relate the infimum in (1.5) and corresponding geodesics (critical paths for action) on the one hand, to Wasserstein distance and corresponding length-minimizing Wasserstein geodesics—also known as displacement interpolants—on the other hand. Wasserstein geodesic paths typically do not

have characteristic-function densities, and thus do not correspond to geodesics for shape distance. A common theme in our results is the observation that the least-action problem in (1.5) is subject to an instability associated with *microdroplet* formation.

The idea that Arnold's least action principle for incompressible flows may suffer analytically from instability or non-attainment appears to have led Brenier and others starting in the late 1980s to investigate various forms of relaxed least-action problems for incompresible flows [8, 10, 48, 11, 12, 13, 1, 37]. Such relaxed problems involve generalizing the notion of flows of diffeomorphisms to formulate a framework in which existence of minimizers can be proved, using convex analysis or variational methods. Our microdroplet constructions also provide a precise connection between Wasserstein geodesic paths (which correspond to pressureless, compressible fluid flows) and relaxed least-action problems for flows of incompressible-fluid–vacuum mixtures.

1.2. **Main results.** Broadly speaking, our aim is to investigate the geometry of the space of shapes (corresponding to characteristic-function densities), focusing on the geodesics for shape distance and the corresponding distance induced by (1.5). Studies of this type have been carried out by many other authors, as will be discussed in subsection 1.3. One issue about which we have little to say is that of geodesic completeness, which corresponds here to global existence in time for weak solutions of the free-boundary Euler equations. In addition to other well-known difficulties for Euler equations, here there arise further thorny problems such as collisions of fluid droplets, for example.

Geodesic connections. Our first results instead address the question of determining which targets and sources are connected by geodesics for shape distance, and how these relate to (1.5). The general question of determining all exact connections is an interesting one that seems difficult to answer. In regard to a related question in a space of smooth enough volume-preserving diffeomorphisms of a fixed manifold, Ebin and Marsden in [24, 15.2(vii)] established a covering theorem showing that the geodesic flow starting from the identity diffeomorphism covers a full neighborhood. By contrast, what our first result will show essentially is that for an arbitrary bounded open source domain Ω_0, targets for shape-distance geodesics are globally dense in the 'manifold' of bounded open sets of the same volume. The idea is that it is possible to construct geodesics comprised of disjoint microdroplets (which we call *Euler sprays*) that approximately reach an arbitrarily specified Ω_1 as closely as desired in terms of an optimal-transport distance.

Below, it is convenient to denote the distance between two bounded measurable sets Ω_0, Ω_1 that is induced by Wasserstein distance by the overloaded notation

$$d_W(\Omega_0, \Omega_1) = d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1}), \tag{1.7}$$

and similarly with L^p-Wasserstein distance d_p for any value of $p \in [1, \infty]$.

Theorem 1.1. *Let Ω_0, Ω_1 be any pair of bounded open sets in $\mathbb{R}^d$ with equal volume. Then for any $\varepsilon > 0$, there is an Euler spray which transports the source Ω_0 (up to a null set) to a target Ω_1^ε satisfying $d_\infty(\Omega_1, \Omega_1^\varepsilon) < \varepsilon$. The action $\mathcal{A}^\varepsilon$ of the spray satisfies*

$$d_s(\Omega_0, \Omega_1^\varepsilon)^2 \leq \mathcal{A}^\varepsilon \leq d_W(\Omega_0, \Omega_1)^2 + \varepsilon \,.$$

The precise definition of an Euler spray and the proof of this result will be provided in section 4. A particular, simple geodesic for shape distance will play a special role in our analysis. Namely, we observe in Proposition 3.4 that a path $t \mapsto \Omega_t$ of ellipsoids determines a critical path for the action (1.1) constrained by (1.2)–(1.4) if and only if the d-dimensional vector $a(t) = (a_1(t), \ldots, a_d(t))$, formed by the principal axis lengths, follows a geodesic curve on

the hyperboloid-like surface in $\mathbb{R}^d$ determined by the constraint that corresponds to constant volume,
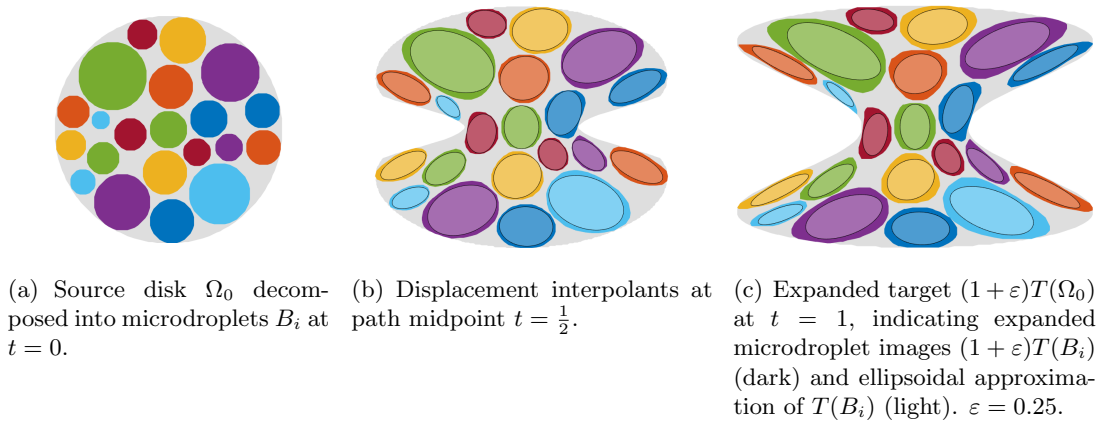
$$a_1 a_2 \cdots a_d = \text{const.} \tag{1.8}$$

(a) Source disk Ω_0 decomposed into microdroplets B_i at $t = 0$.

(b) Displacement interpolants at path midpoint $t = \frac{1}{2}$.

(c) Expanded target $(1 + \varepsilon)T(\Omega_0)$ at $t = 1$, indicating expanded microdroplet images $(1 + \varepsilon)T(B_i)$ (dark) and ellipsoidal approximation of $T(B_i)$ (light). $\varepsilon = 0.25$.

FIGURE 1. Illustration of Wasserstein geodesic flow from Ω_0 to $\Omega_1 = T(\Omega_0)$, where T is the Brenier map. Source Ω_0 is decomposed into countably many small balls, few shown. Matching shades indicate corresponding droplets transported by displacement interpolation. Euler spray droplets are nested inside Wasserstein ellipsoids and remain disjoint.

To prove Theorem 1.1, we decompose the source domain Ω_0, up to a set of measure zero, as a countable union of tiny disjoint open balls using a Vitali covering lemma. These 'microdroplets' are transported by ellipsoidal geodesics that approximate a local linearization of the Wasserstein geodesic (displacement interpolant) which produces straight-line transport of points from the source Ω_0 to the target Ω_1. Crucially, the droplets remain disjoint, and the total action or cost along the resulting path of 'spray' shape densities is then shown to be close to that attained by the Wasserstein geodesic.

The ideas behind the construction of the Euler sprays are illustrated in Figure 1. The shaded background in panel (c) indicates the target $\Omega_1 = T(\Omega_0)$, expanded by a factor $(1 + \varepsilon)$, where $T: \Omega_0 \to \Omega_1$ is a computed approximation to the Brenier (optimal transport) map. The expanded images $(1 + \varepsilon)T(B_i)$ of balls B_i in the source are shown in dark shades, and (nested inside) ellipsoidal approximations to $T(B_i)$ in corresponding light shades. We show that along Wasserstein geodesics (displacement interpolants), nested images remain nested, and that the ellipsoidal Euler geodesics (not shown) remain nested inside the Wasserstein-transported ellipses indicated in light shades.

The result of Theorem 1.1 directly implies that a natural relaxation of the shape distance d_s—the lower semicontinuous envelope with respect to Wasserstein distance—agrees with the induced Wasserstein distance d_W. (See [7, section 1.7.2] regarding the general notion of relaxation of variational problems.) In fact, by a rather straightforward completion argument we can identify the shape distance in (1.5) as follows.

Theorem 1.2. *For every pair of bounded measurable sets in $\mathbb{R}^d$ of equal volume,*

$$d_s(\Omega_0, \Omega_1) = d_W(\Omega_0, \Omega_1).$$

As is well known, Wasserstein distance between measures of a given mass that are supported inside a fixed compact set induces the topology of weak-$\star$ convergence. In this topology, the closure of the set of such measures with characteristic-function densities is the set of measurable functions $\rho\colon \mathbb{R}^d \to [0,1]$ with compact support. The result above is a corollary of the following more general result that indicates how Euler-spray geodesic paths approximately connect arbitrary endpoints in this set.

Theorem 1.3. *Let ρ_0, $\rho_1\colon \mathbb{R}^d \to [0,1]$ be measurable functions of compact support that satisfy*

$$\int_{\mathbb{R}^d} \rho_0 = \int_{\mathbb{R}^d} \rho_1\,.$$

Then

(a) *For any $\varepsilon > 0$ there are open sets Ω_0, Ω_1 which satisfy*

$$d_\infty(\rho_0, \mathbb{1}_{\Omega_0}) + d_\infty(\rho_1, \mathbb{1}_{\Omega_1}) < \varepsilon,$$

and are connected by an Euler spray whose total action $\mathcal{A}^\varepsilon$ satisfies

$$\mathcal{A}^\varepsilon \le d_W(\rho_0, \rho_1)^2 + \varepsilon.$$

(b) *For any $\varepsilon > 0$ there is a path $\rho^\varepsilon = (\rho_t^\varepsilon)_{t \in (0,1)}$ on $(0,1)$ consisting of a countable concatenation of Euler sprays, such that*

$$\rho_t^\varepsilon \overset{\star}{\rightharpoonup} \rho_0 \quad as\ t \to 0^+\,, \qquad \rho_t^\varepsilon \overset{\star}{\rightharpoonup} \rho_1 \quad as\ t \to 1^-\,,$$

and the total action $\mathcal{A}^\varepsilon$ of the path satisfies

$$\mathcal{A}^\varepsilon = \int_0^1 \int_{\mathbb{R}^d} \rho_t^\varepsilon |v^\varepsilon|^2\,dx\,dt \le d_W(\rho_0, \rho_1)^2 + \varepsilon.$$

The results of Theorems 1.1 and 1.3 concern geodesics for shape distance that only approximately connect arbitrary sources Ω_0 and targets Ω_1. A uniqueness property of Wasserstein geodesics allows us to establish the following sharp criterion for existence and non-existence of *length-minimizing* shape geodesics that exactly connect source to target.

Theorem 1.4. *Let Ω_0, Ω_1 be bounded open sets in $\mathbb{R}^d$ with equal volume, and let $\rho = (\rho_t)_{t \in [0,1]}$ be the density along the Wasserstein geodesic path that connects $\mathbb{1}_{\Omega_0}$ and $\mathbb{1}_{\Omega_1}$. Then the infimum for shape distance in (1.5) is achieved by some path satisfying the constraints (1.2),(1.3),(1.4) if and only if ρ is a characteristic function.*

For dimension $d = 1$ the Wasserstein density is always a characteristic function. For dimension $d > 1$ however, this property of being a characteristic function requires that the Wasserstein geodesic is given piecewise by rigid body motion. (See Remarks 2.2–2.3.)

Limits of Euler sprays. For the Euler sprays constructed in the proof of Theorem 1.1, the fluid domains Ω_t do not typically have smooth boundary, due to the presence of cluster points of the countable set of microdroplets. The geodesic equations that they satisfy, then, are not quite classical free-boundary water-wave equations. Rather, our Euler sprays provide a family of weak solutions $(\rho^\varepsilon, v^\varepsilon, p^\varepsilon)$ to the following system of Euler equations:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \tag{1.9}$$

$$\partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla p = 0, \tag{1.10}$$

with the "incompressibility" constraint that ρ^ε is a shape density, meaning it is a characteristic function as in (1.3). Both of these equations hold in the sense of distributions on $\mathbb{R}^d \times [0,1]$,

which means the following: For any smooth test functions $q \in C_c^\infty(\mathbb{R}^d \times [0,1], \mathbb{R})$ and $\tilde{v} \in C_c^\infty(\mathbb{R}^d \times [0,1], \mathbb{R}^d)$,

$$\int_0^1 \int_{\mathbb{R}^d} \rho(\partial_t q + v \cdot \nabla q) \, dx \, dt = \int_{\mathbb{R}^d} \rho q \, dx \bigg|_{t=0}^{t=1}, \qquad (1.11)$$

$$\int_0^1 \int_{\mathbb{R}^d} \rho v \cdot (\partial_t \tilde{v} + v \cdot \nabla \tilde{v}) + p \nabla \cdot \tilde{v} \, dx \, dt = \int_{\mathbb{R}^d} \rho v \cdot \tilde{v} \, dx \bigg|_{t=0}^{t=1}. \qquad (1.12)$$

Now, limits as $\varepsilon \to 0$ of these Euler-spray geodesics can be considered. By dealing with a sequence of initial and final data $\rho_0^k = \mathbb{1}_{\Omega_0^k}$, $\rho_1^k = \mathbb{1}_{\Omega_1^k}$ that converge weak-$\star$, we find that it is possible to approximate a general family of Wasserstein geodesic paths, in the following sense.

Theorem 1.5. *Let $\rho_0, \rho_1 \colon \mathbb{R}^d \to [0,1]$ be measurable functions of compact support that satisfy*

$$\int_{\mathbb{R}^d} \rho_0 = \int_{\mathbb{R}^d} \rho_1 \,.$$

Let (ρ, v) be the density and transport velocity determined by the unique Wasserstein geodesic that connects the measures with densities ρ_0 and ρ_1 as described in section 2.

Then there is a sequence of weak solutions (ρ^k, v^k, p^k) to (1.11)–(1.12), associated to Euler sprays as provided by Theorem 1.1, that converge to $(\rho, v, 0)$, and (ρ, v) is a weak solution of the pressureless Euler system

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \qquad (1.13)$$

$$\partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) = 0. \qquad (1.14)$$

The convergence holds in the the following sense: $p^k \to 0$ uniformly, and

$$\rho^k \overset{\star}{\rightharpoonup} \rho, \qquad \rho^k v^k \overset{\star}{\rightharpoonup} \rho v, \qquad \rho^k v^k \otimes v^k \overset{\star}{\rightharpoonup} \rho v \otimes v, \qquad (1.15)$$

weak-$\star$ in L^∞ on $\mathbb{R}^d \times [0,1]$.

This result shows that one can approximate a large family of solutions of pressureless Euler equations, ones coming from Wasserstein geodesics having bounded densities of compact support, by solutions of incompressible Euler equations with vacuum. (For densities taking values in $[0, R]$ instead of $[0, 1]$, one can simply scale the densities coming from the Euler sprays, by multiplying by R.)

The convergence in (1.15) can be strengthened in terms of the TL^p topology that was introduced in [29] to compare two functions that are absolutely continuous with respect to different probability measures—see subsection 6.2. The result of Theorem 6.7 essentially shows that while oscillations exist in space and time for the densities ρ^k and velocities v^k in Theorem 1.5, there are no oscillations following the flow lines. Our analysis of convergence in the TL^p topology is based upon an improved stability result regarding the stability of transport maps. We describe and establish this stability result separately in an Appendix, due to its potential for independent interest.

Relaxed least-action principles. Our next result establishes a precise connection between Wasserstein geodesics and a relaxed least-action principle for incompressible flow of two-fluid mixtures. In particular this relates to work of Brenier on relaxations of Arnold's least-action principle for incompressible flow [8, 10, 11, 12, 13, 14]. The mixture model is a variant of Brenier's model for homogenized vortex sheets [11], and is related to the variable-density model studied by Lopes et al. [37]. Our model, however, also allows one fluid to have zero density, corresponding to a fluid-vacuum mixture. In this degenerate case, we show that the

Wasserstein geodesic provides the unique minimizer of the relaxed least-action principle—see Theorem 7.2.

An important point of constrast between our results and those of Brenier [12] and Lopes et al. [37] concerns the issue of consistency of the relaxed theory with classical solutions. The results of [12] and [37] establish that classical smooth solutions of the incompressible fluid equations do provide action minimizers locally, for sufficiently short time. However, the result of Theorem 1.4 above shows that for any smooth free-boundary fluid motion (corresponding to a shape geodesic) that is not given locally by rigid motion, the solution *never* achieves minimum action, over any positive interval of time.

Shape distance without volume constraint. Our investigations in this paper were motivated in part by an expanded notion of shape distance that was introduced and examined by Schmitzer and Schnörr in [46]. These authors considered a shape distance determined by restricting the Wasserstein metric to smooth paths of 'shape measures' consisting of uniform distributions on bounded open sets in $\mathbb{R}^2$ with connected smooth boundary. This allows one to naturally compare shapes of different volume. In our present investigation, the only smoothness properties of shapes and paths that we require are those intrinsically associated with Wasserstein distance. Thus, we investigate the geometry of a 'submanifold' of the Wasserstein space consisting of uniform distributions on shapes regarded as arbitrary bounded measurable sets in $\mathbb{R}^d$. As we will see in Section 8 below, geodesics for this extended shape distance correspond to a modified water-wave system with spatially uniform compressibility and zero *average* pressure. In Theorem 8.1 below we extend the result of Theorem 1.2, for volume-constrained paths of shapes, to deal with paths of uniform measures connecting two arbitrary bounded measurable sets. We show that the extended shape distance again agrees with the Wasserstein distance between the endpoints. The proof follows directly from the construction of concatenated Euler sprays used to prove Theorem 1.3(b).

1.3. **Related work on the geometry of image and shape space.** The shape distance that we defined in (1.5) is related to a large body of work in imaging science and signal processing.

The general problem of finding good ways to compare two signals (such as time series, images, or shapes) is important in a number of application areas, including computer vision, machine learning, and computational anatomy. The idea to use deformations as a means of comparing images goes back to pioneering work of D'Arcy Thompson [49].

Distances derived from optimal transport theory (Monge-Kantorovich, Wasserstein, or earth-mover's distance) have been found useful in analyzing images by a number of workers [28, 32, 43, 47, 53, 54]. The transport distance with quadratic cost (Wasserstein distance) is special as it provides a (formal) Riemannian structure on spaces of measures with fixed total mass [3, 42, 51].

Methods which endow the space of signals with the metric structure of a Riemannian manifold are of particular interest, as they facilitate a variety of image processing tasks. This geometric viewpoint, pioneered by Dupuis, Grenander & Miller [23, 31], Trouvé [50], Younes [58] and collaborators, has motivated the study of a variety of metrics on spaces of images over a number of years—see [23, 30, 33, 46, 59] for a small selection.

The main thrust of these works is to study Riemannian metrics and the resulting distances in the space of image deformations (diffeomorphisms). Connections with the Arnold viewpoint of fluid mechanics were noted from the outset [58], and have been further explored by Holm, Trouvé, Younes and others [30, 33, 59]. This work has led to the *Euler-Poincaré theory of*

metamorphosis [33], which sets up a formalism for analyzing least-action principles based on Lie-group symmetries generated by diffeomorphism groups.

A different way to consider shapes is to study them only via their boundary, and consider Riemannian metrics defined in terms of normal velocity of the boundary. Such a point of view has been taken by Michor, Mumford and collaborators [16, 39, 40, 60]. As they show in [39], a metric given by only the L^2 norm of normal velocity does not lead to a viable geometry, as any two states can be connected by an arbitrarily short curve. On the other hand it is shown in [16] that if two or more derivatives of the normal velocity are penalized, then the resulting metric on the shape space is geodesically complete.

In this context, we note that what our work shows is that if the metric is determined by the L^2 norm of the transport velocity in the bulk, then the global metric distance is not zero, but that it is still degenerate in the sense that a length-minimizing geodesic typically may not exist in the shape space. While our results do not directly involve smooth deformations of smooth shapes, it is arguably interesting to consider shape spaces which permit 'pixelated' approximations, and our results apply in that context.

We speculate that to create a shape distance that (even locally) admits length-minimizing paths in the space of shapes, one needs to prevent the creation a large perimeter at negligible cost. This is somewhat analogous to the motivation for the metrics on the space of curves considered by Michor and Mumford [39]. Possibilities include introducing a term in the metric which penalizes deforming the boundary, or a term which enforces greater regularity for the vector fields considered.

A number of existing works obtain regularity of geodesic paths and resulting diffeomorphisms by considering Riemannian metrics given in terms of second-order derivatives of velocities, as in the Large Deformation Diffeomorphic Metric Mapping (LDDMM) approach of [5]. Metrics based on conservative transport which penalize only one derivative of the velocity field are connected with viscous dissipation in fluids and have been considered by Fuchs et al. [27], Rumpf, Wirth and collaborators [44, 55], as well as by Brenier, Otto, and Seis [15], who established a connection to optimal transport.

1.4. **Plan.** The plan of this paper is as follows. In section 2 we collect some basic facts and estimates that concern geodesics for Monge-Kantorovich/Wasserstein distance. In section 3 we derive formally the geodesic equations for paths of shape densities and describe the special class of ellipsoidal solutions. The construction of Euler sprays and the proof of Theorem 1.1 is carried out in section 4. Theorem 1.2 is proved in section 5. The connection between Wasserstein geodesics and a relaxed least-action priniciple motivated by Brenier's work is developed in section 7. The paper concludes in section 8 with a treatment of the extended notion of shape distance related to that examined by Schmitzer and Schnörr in [46].

2. Preliminaries: Wasserstein geodesics between open shapes

In this section we recall some basic properties of the standard minimizing geodesic paths (displacement interpolants) for the Wasserstein or Monge-Kantorovich distance between shape densities on open sets, and establish some basic estimates. Two properties that are key in the sequel are that the density ρ is (i) smooth on an open subset of full measure, and (ii) it is *convex* along the corresponding particle paths, see Lemma 2.1.

2.1. **Standard Wasserstein geodesics.** Let Ω_0 and Ω_1 be two bounded open sets in $\mathbb{R}^d$ with equal volume. Let μ_0 and μ_1 be measures with respective densities

$$\rho_0 = \mathbb{1}_{\Omega_0}, \qquad \rho_1 = \mathbb{1}_{\Omega_1}.$$

As is well known [9, 35], there exists a convex function ψ such that $T = \nabla\psi$ (called the *Brenier map* in [51]) is the optimal transportation map between Ω_0 and Ω_1 corresponding to the quadratic cost. Moreover, this map is unique a.e. in Ω_0; see [9] or [51, Thm. 2.32].

McCann [38] later introduced a natural curve $t \mapsto \mu_t$ that interpolates between μ_0 and μ_1, called the *displacement interpolant*, which can be described as the push-forward of the measure μ_0 by the interpolation map T_t given by

$$T_t(z) = (1-t)z + t\nabla\psi(z), \qquad 0 \leq t \leq 1. \tag{2.1}$$

Because ψ is convex we have $\langle \nabla\psi(z) - \nabla\psi(\hat{z}), z - \hat{z} \rangle \geq 0$ for all $z, \hat{z}$, hence the interpolating maps T_t are injective for $t \in [0,1)$, satisfying

$$|T_t(z) - T_t(\hat{z})| \geq (1-t)|z - \hat{z}|. \tag{2.2}$$

Note that particle paths $z \mapsto T_t(z)$ follow straight lines with constant velocity:

$$v(T_t(z), t) = \nabla\psi(z) - z. \tag{2.3}$$

Furthermore μ_t has density ρ_t that satisfies the continuity equation

$$\partial_t\rho + \operatorname{div}(\rho v) = 0. \tag{2.4}$$

In terms of these quantities, the Wasserstein distance satisfies

$$d_W(\mu_0, \mu_1)^2 = \int_{\Omega_0} |\nabla\psi(z) - z|^2 \, dz = \int_0^1 \int_{\Omega_t} \rho|v|^2 \, dx \, dt, \tag{2.5}$$

and the L^∞ transport distance may be defined as a minimum over maps S that push forward the measure μ_0 to μ_1 [45, Thm. 3.24], satsifying

$$\begin{aligned} d_\infty(\mu_0, \mu_1) &= \min\{\|S - \operatorname{id}\|_{L^\infty(\mu_0)} : S_\sharp\mu_0 = \mu_1\} \\ &\geq |\Omega_0|^{-1/2} \min\{\|S - \operatorname{id}\|_{L^2(\mu_0)} : S_\sharp\mu_0 = \mu_1\} \\ &= |\Omega_0|^{-1/2} d_W(\Omega_0, \Omega_1). \end{aligned} \tag{2.6}$$

The displacement interpolant has the property that

$$d_W(\mu_s, \mu_t) = (t - s)d_W(\mu_0, \mu_1), \quad 0 \leq s \leq t \leq 1. \tag{2.7}$$

The property (2.7) implies that the displacement interpolant is a *constant-speed geodesic* (length-minimizing path) with respect to Wasserstein distance. The displacement interpolant $t \mapsto \mu_t$ is the *unique* constant-speed geodesic connecting μ_0 and μ_1, due to the uniqueness of the Brenier map and Proposition 5.32 of [45] (or see [2, Thm. 3.10]). For brevity the path $t \mapsto \mu_t$ is called the *Wasserstein geodesic* from μ_0 to μ_1.

At this point it is convenient to mention that the result of Theorem 1.4, providing a sharp criterion for the existence of a minimizer for the shape distance in (1.5), will be derived by combining the uniqueness property of Wasserstein geodesics with the result of Theorem 1.2— see the end of section 5 below.

Extending the regularity theory of Caffarelli [17], Figalli [25] and Figalli & Kim [26] have shown (see Theorem 3.4 in [20] and also [21]) that the optimal transportation potential ψ is smooth away form a set of measure zero. More precisely, there exist relatively closed sets of measure zero, $\Sigma_i \subset \Omega_i$ for $i = 0, 1$ such that $T : \Omega_0 \backslash \Sigma_0 \to \Omega_1 \backslash \Sigma_1$ is a smooth diffeomorphism between two open sets.

Let $\lambda_1(z), \ldots, \lambda_d(z)$ be the eigenvalues of $\operatorname{Hess}\psi(z)$ for $z \in \Omega_0 \backslash \Sigma_0$. Due to convexity and regularity of ψ, $\lambda_i > 0$ for all $i = 1, \ldots, n$. Furthermore, because $\nabla\psi$ is a map that pushes forward the Lebesgue measure on Ω_0 to that on Ω_1, it follows that the Jacobian of T has value 1 and thus $\lambda_1 \cdots \lambda_d = 1$.

Along the particle paths of displacement interpolation starting from any $z \in \Omega_0 \setminus \Sigma_0$, the mass density satisfies

$$\rho(T_t(z),t)^{-1} = \det \frac{\partial T_t}{\partial z} = \det((1-t)I + t\nabla^2 \psi(z)) = \prod_{j=1}^{d}(1 - t + t\lambda_j(z)). \qquad (2.8)$$

We now show that the density ρ is convex along these paths. The stronger fact that $\rho^{-1/d}$ is concave along particle paths follows from more general classical results stated in [38] and related to a well-known proof of the Brunn-Minkowski inequality by Hadwiger and Ohmann. Since a simple proof is available for our case, we present it here for completeness.

Lemma 2.1. *Along the particle paths $t \mapsto T_t(z)$ of displacement interpolation between the measures μ_0 and μ_1 with respective densities $\mathbb{1}_{\Omega_0}$ and Ω_1 as above, the map $t \mapsto \rho(T_t(z),t)^{-1/d}$ is concave. Further, the map $t \mapsto \rho(T_t(z),t)$ is convex. Moreover, $\rho \leq 1$.*

Proof. Fix z and let $g(t) = \rho(T_t(z),t)^{-1/d}$. We compute

$$\frac{g'}{g} = \frac{1}{d}\sum_{j=1}^{d} \frac{\lambda_j - 1}{1 - t + t\lambda_j},$$

$$\frac{g''}{g} = \left(\frac{1}{d}\sum_{j=1}^{d} \frac{\lambda_j - 1}{1 - t + t\lambda_j}\right)^2 - \frac{1}{d}\sum_{j=1}^{d}\left(\frac{\lambda_j - 1}{1 - t + t\lambda_j}\right)^2 \leq 0 \qquad (2.9)$$

due to the Cauchy-Schwartz (or Jensen's) inequality. This shows g is concave. That $t \mapsto \rho(T_t(z),t)$ is convex follows directly. Because ρ equals 1 when $t = 0$ and $t = 1$, we infer $\rho \leq 1$ along particle paths. $\qquad \square$

We also note that computations above and continuity equation (2.3) imply

$$\text{div } v = -\frac{1}{\rho}\left(\frac{d\rho}{dt}\right) = -\frac{d}{dt}\log\rho = \sum_{j=1}^{d} \frac{\lambda_j - 1}{1 - t + t\lambda_j}. \qquad (2.10)$$

Remark 2.2. We remark that according to the result of Theorem 1.4, a minimizer for (1.5) will exist if and only if $\rho(T_t(z),t) \equiv 1$ for all z in the non-singular set $\Omega_0 \setminus \Sigma_0$. For this, clearly it is a necessary consequence of (2.9) that $\lambda_j \equiv 1$ everywhere in $\Omega_0 \setminus \Sigma_0$. This means T is a rigid translation on each component of $\Omega_0 \setminus \Sigma_0$. Thus Ω_1 represents some kind of decomposition of Ω_0 by fracturing into pieces that can separate without overlapping.

As a nontrivial example in the case of one dimension ($d = 1$), let $\mathcal{C} \subset [0,1]$ be the standard Cantor set, and let $\Omega_0 = (0,1)$. Define the Brenier map $T(x) = x + c(x)$ with c given by the Cantor function, increasing and continuous on $[0,1]$ with $c(0) = 0$, $c(1) = 1$ and $c' = 0$ on $(0,1) \setminus \mathcal{C}$. Then $T(\Omega_0) = (0,2)$, but the pushforward of uniform measure on Ω_0 is the uniform measure on the set $\Omega_1 = T(\Omega_0 \setminus \mathcal{C})$, which has countably many components, and total length $|\Omega_1| = 1$.

Remark 2.3. Actually, in the case $d = 1$ it is always the case that $\rho(T_t(z),t) \equiv 1$ for all z in the non-singular set. This is so because the diffeomorphism $T : \Omega_0 \setminus \Sigma_0 \to \Omega_1 \setminus \Sigma_1$ must be a rigid translation on each component, as it pushes forward Lebesgue measure to Lebesgue measure.

2.2. **Local linear approximation and estimates.** Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of Hess $\psi(x)$, as before. Recall that $\lambda_i > 0$ for all $i = 1, \ldots, n$ and $\lambda_1 \cdots \lambda_d = 1$. Let $\underline{\lambda}(x)$ and $\overline{\lambda}(x)$ be the minimal and maximal eigenvalues of Hess $\psi(x) = DT(x)$ respectively. We define, for any $U \subset \Omega_0 \backslash \Sigma_0$,

$$\underline{\lambda}_U = \inf\{\underline{\lambda}(x) : x \in U\}, \qquad \overline{\lambda}_U = \sup\{\overline{\lambda}(x) : x \in U\}, \tag{2.11}$$

and note that for any $x \in U$ and $\hat{x} \in \mathbb{R}^d$,

$$\underline{\lambda}_U |\hat{x}| \leq |DT(x)\hat{x}| \leq \overline{\lambda}_U |\hat{x}|. \tag{2.12}$$

For $U \in \Omega_0 \backslash \Sigma_0$ we also let

$$\|D^3\psi\|_U := \sup_{x \in U} \max_{|u|=|v|=|w|=1} \left| \sum_{i,j,k=1}^{d} \frac{\partial^3 \psi(x)}{\partial x_i \partial x_j \partial x_k} u_i v_j w_k \right|. \tag{2.13}$$

Taylor expansion provides a basic estimate on the difference between the optimal transport map and its linearization: Whenever $B(x_0, r) \subset \Omega_0 \setminus \Sigma_0$ and $x \in B(x_0, r)$,

$$|T(x) - T(x_0) - DT(x_0)(x - x_0)| < \frac{1}{2}\|D^3\psi\|_{B(x_0,r)} r^2. \tag{2.14}$$

3. Geodesics and incompressible fluid flow

3.1. **Incompressible Euler equations for smooth critical paths.** In this subsection, for completeness we derive the Euler fluid equations that formally describe smooth geodesics (paths with stationary action) for the shape distance in (1.5). To cope with the problem of moving domains we work in a Lagrangian framework, computing variations with respect to flow maps that preserve density and the endpoint shapes Ω_0 and Ω_1.

Toward this end, suppose that

$$Q = \bigcup_{t \in [0,1]} \Omega_t \times \{t\} \quad \subset \mathbb{R}^d \times [0,1] \tag{3.1}$$

is a space-time domain generated by smooth deformation of Ω_0 due to a smooth velocity field $v \colon \bar{Q} \to \mathbb{R}^d$. That is, the t-cross section of Q is given by

$$\boxed{\Omega_t = X(\Omega_0, t),} \tag{3.2}$$

where X is the Lagrangian flow map associated to v, satisfying

$$\dot{X}(z,t) = v(X(z,t), t), \qquad X(z,0) = z, \tag{3.3}$$

for all $(z,t) \in \Omega_0 \times [0,1]$.

For any (smooth) extension of v to $\mathbb{R}^d \times [0,1]$, the solution of the mass-transport equation in (1.2) with given initial density ρ_0 supported in $\bar{\Omega}_0$ is

$$\rho(x,t) = \rho_0(z) \det\left(\frac{\partial X}{\partial z}(z,t)\right)^{-1}, \quad x = X(z,t) \in \Omega_t,$$

with $\rho = 0$ outside Q.

Considering a smooth family $X = X_\varepsilon$ of flow maps defined for all small values of a variational parameter ε, the variation $\delta X = (\partial X/\partial \varepsilon)|_{\varepsilon=0}$ induces a variation in density satisfying

$$-\frac{\delta\rho}{\rho} = \delta \log \det\left(\frac{\partial X}{\partial z}(z,t)\right) = \operatorname{tr}\left(\frac{\partial \delta X}{\partial z}\left(\frac{\partial X}{\partial z}\right)^{-1}\right) \tag{3.4}$$

Introducing $\tilde{v}(x,t) = \delta X(z,t)$, $x = X(z,t)$, it follows

$$-\frac{\delta\rho}{\rho} = \nabla \cdot \tilde{v}. \tag{3.5}$$

For variations that leave the density invariant, necessarily $\nabla \cdot \tilde{v} = 0$.

We now turn to consider the variation of the action or transport cost as expressed in terms of the flow map:

$$\mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho|v|^2 dx\,dt = \int_0^1 \int_{\Omega_0} |\dot{X}(z,t)|^2 dz\,dt. \tag{3.6}$$

For flows preserving $\rho = 1$ in Q, of course $\nabla \cdot v = 0$. Computing the first variation of $\mathcal{A}$ about such a flow, after an integration by parts in t and changing to Eulerian variables, we find

$$\begin{aligned}
\frac{\delta\mathcal{A}}{2} &= \int_0^1 \int_{\Omega_0} \dot{X} \cdot \delta\dot{X}\,dz\,dt \\
&= \int_{\Omega_0} \dot{X} \cdot \delta X\,dz\bigg|_{t=1} - \int_0^1 \int_{\Omega_0} \ddot{X} \cdot \delta X\,dz\,dt \\
&= \int_{\Omega_t} v \cdot \tilde{v}\,dx\bigg|_{t=1} - \int_0^1 \int_{\Omega_t} (\partial_t v + v \cdot \nabla v) \cdot \tilde{v}\,dx\,dt.
\end{aligned} \tag{3.7}$$

Recall that any L^2 vector field u on Ω_t has a unique Helmholtz decomposition as the sum of a gradient and a field L^2-orthogonal to all gradients, which is divergence-free with zero normal component at the boundary:

$$u = \nabla p + w, \qquad \nabla \cdot w = 0 \ \text{ in } \Omega_t, \quad w \cdot n = 0 \ \text{ on } \partial\Omega_t. \tag{3.8}$$

If we loosen the requirement that $w \cdot n = 0$ on the boundary, it is still the case that

$$\int_{\partial\Omega_t} w \cdot n\,dS = \int_{\Omega_t} \nabla \cdot w\,dx = 0,$$

It follows that the space orthogonal to all divergence-free fields on Ω_t is the space of gradients ∇p such that p is constant on the boundary, and we may take this constant to be zero:

$$\boxed{p = 0 \quad \text{on } \partial\Omega_t.} \tag{3.9}$$

Requiring $\delta\mathcal{A} = 0$ for arbitrary virtual displacements having $\nabla \cdot \tilde{v} = 0$ (and $\tilde{v} = 0$ at $t = 1$ at first), we find that necessarily $u = -(\partial_t v + v \cdot \nabla v)$ is such a gradient. Thus the incompressible Euler equations hold in Q:

$$\partial_t v + v \cdot \nabla v + \nabla p = 0, \quad \nabla \cdot v = 0 \quad \text{in } Q, \tag{3.10}$$

where $p : \bar{Q} \to \mathbb{R}$ is smooth and satisfies (3.9).

Finally, we may consider variations $\tilde{v}$ that do not vanish at $t = 1$. However, we require $\tilde{v} \cdot n = 0$ on $\partial\Omega_1$ in this case because the target domain Ω_1 should be fixed. That is, the allowed variations in the flow map X must fix the image at $t = 1$:

$$\Omega_1 = X(\Omega_0, 1). \tag{3.11}$$

The vanishing of the integral term at $t = 1$ in (3.7) then leads to the requirement that v is a gradient at $t = 1$. For $t = 1$ we must have

$$v = \nabla\phi \quad \text{in } \Omega_t. \tag{3.12}$$

We claim this gradient representation actually must hold for all $t \in [0,1]$. Let $v = \nabla\phi + w$ be the Helmholtz decomposition of v, and for small ε consider the family of flow maps generated by

$$\dot{X}(z,t) = (v + \varepsilon w)(X(z,t),t) \qquad X(z,0) = z. \tag{3.13}$$

Corresponding to this family, the action from (3.6) takes the form

$$\mathcal{A} = \int_0^1 \int_{\Omega_0} |\dot{X}(z,t)|^2 dz\, dt = \int_0^1 \int_{\Omega_t} |\nabla\phi|^2 + |(1+\varepsilon)w|^2\, dx\, dt \tag{3.14}$$

Because $w \cdot n = 0$ on $\partial\Omega_t$, the domains Ω_t do not depend on ε, and the same is true of $\nabla\phi$ and w, so this expression is a simple quadratic polynomial in ε. Thus

$$\frac{1}{2} \frac{d\mathcal{A}}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_0^1 \int_{\Omega_t} |w|^2\, dx\, dt \tag{3.15}$$

and consequently it is necessary that $w = 0$ if $\delta\mathcal{A} = 0$. This proves the claim.

The Euler equation in (3.10) is now a spatial gradient, and one can add a function of t alone to ϕ to ensure that

$$\boxed{\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + p = 0, \quad \Delta\phi = 0 \quad \text{in } \Omega_t.} \tag{3.16}$$

The equations boxed above, including (3.16) together with the zero-pressure boundary condition (3.9) and the kinematic condition that the boundary of Ω_t moves with normal velocity $v \cdot n$ (coming from (3.2)-(3.3)), comprise what we shall call the *Euler droplet* equations, for incompressible, inviscid, potential flow of fluid with zero surface tension and zero pressure at the boundary.

Definition 3.1. *A smooth solution of the Euler droplet equations is a triple (Q, ϕ, p) such that $\phi, p \colon \bar{Q} \to \mathbb{R}$ are smooth and the equations (3.1), (3.2), (3.3), (3.12), (3.16), (3.9) all hold.*

Proposition 3.2. *For smooth flows X that deform Ω_0 as above, that respect the density constraint $\rho = 1$ and fix $\Omega_1 = X(\Omega_0, 1)$, the action $\mathcal{A}$ in (3.6) is critical with respect to smooth variations if and only if X corresponds to a smooth solution of the Euler droplet equations.*

3.2. **Weak solutions and Galilean boost.** Here we record a couple of simple basic properties of solutions of the Euler droplet equations.

Proposition 3.3. *Let (Q, ϕ, p) be a smooth solution of the Euler droplet equations. Let $\rho = \mathbb{1}_Q$ and $v = \mathbb{1}_Q \nabla\phi$, and extend p as zero outside Q.*
(a) The Euler equations (1.9)-(1.10) hold in the sense of distributions on $\mathbb{R}^d \times [0,1]$.
(b) The mean velocity

$$\bar{v} = \frac{1}{|\Omega_t|} \int_{\Omega_t} v(x,t)\, dx \tag{3.17}$$

is constant in time, and the action decomposes as

$$\mathcal{A} = \int_0^1 \int_{\Omega_t} |v - \bar{v}|^2 dx\, dt + |\Omega_0||\bar{v}|^2. \tag{3.18}$$

(c) *Given any constant vector $b \in \mathbb{R}^d$, another smooth solution $(\hat{Q}, \hat{\phi}, \hat{p})$ of the Euler droplet equations is given by a Galilean boost, via*

$$\hat{Q} = \bigcup_{t \in [0,1]} (\Omega_t + bt) \times \{t\}, \tag{3.19}$$

$$\hat{\phi}(x + bt, t) = \phi(x, t) + b \cdot x + \frac{1}{2}|b|^2 t, \quad \hat{p}(x + bt, t) = p(x, t). \tag{3.20}$$

Proof. To prove (a), what we must show is the following: For any smooth test functions $q \in C_c^\infty(\mathbb{R}^d \times [0, 1], \mathbb{R})$ and $\tilde{v} \in C_c^\infty(\mathbb{R}^d \times [0, 1], \mathbb{R}^d)$,

$$\int_Q (\partial_t q + v \cdot \nabla q)\, dx\, dt = \int_{\Omega_t} q\, dx \Big|_{t=0}^{t=1} \tag{3.21}$$

$$\int_Q v \cdot (\partial_t \tilde{v} + v \cdot \nabla \tilde{v}) + p\nabla \cdot \tilde{v}\, dx\, dt = \int_{\Omega_t} \tilde{v} \cdot v\, dx \Big|_{t=0}^{t=1} \tag{3.22}$$

Changing to Lagrangian variables via $x = X(z, t)$, writing $\hat{q}(z, t) = q(x, t)$, and using incompressibility, equation (3.21) is equivalent to

$$\int_0^1 \int_{\Omega_0} \frac{d}{dt}\hat{q}(z, t)\, dz\, dt = \int_{\Omega_0} \hat{q}(z, t)\, dz \Big|_{t=0}^{t=1}. \tag{3.23}$$

Evidently this holds. In (3.22), we integrate the pressure term by parts, and treat the rest as in (3.7) to find that (3.22) is equivalent to

$$\int_Q (\partial_t v + v \cdot \nabla v + \nabla p) \cdot \tilde{v}\, dx\, dt = 0. \tag{3.24}$$

Then (a) follows. The proof of parts (b) and (c) is straightforward. $\square$

3.3. **Ellipsoidal Euler droplets.** The intial-value problem for the Euler droplet equations is a difficult fluid free boundary problem, one that may be treated by the methods developed by Wu [56, 57]. For flows with vorticity and smooth enough intial data, smooth solutions for short time have been shown to exist in [36, 18, 19].

In this section, we describe simple, particular Euler droplet solutions for which the fluid domain Ω_t remains ellipsoidal for all t. Our main result is the following.

Proposition 3.4. *Given a constant $r > 0$, let $a(t) = (a_1(t), \dots, a_d(t))$ be any constant-speed geodesic on the surface in $\mathbb{R}_+^d$ determined by the relation*

$$a_1 \cdots a_d = r^d. \tag{3.25}$$

Then this determines an Euler droplet solution (Q, ϕ, p) with Ω_t equal to the ellipsoid $E_{a(t)}$ given by

$$E_a = \Big\{ x \in \mathbb{R}^d : \sum_j (x_j/a_j)^2 < 1 \Big\}, \tag{3.26}$$

and potential and pressure given by

$$\phi(x, t) = \frac{1}{2}\sum_j \frac{\dot{a}_j x_j^2}{a_j} - \beta(t), \qquad p(x, t) = \dot{\beta}\left(1 - \sum_j \frac{x_j^2}{a_j^2}\right), \tag{3.27}$$

with

$$\dot{\beta}(t) = \frac{1}{2}\frac{\sum_j \dot{a}_j^2/a_j^2}{\sum_j 1/a_j^2}. \tag{3.28}$$

For clarity, we first derive the result in the planar case, then treat the case of general dimension $d \geq 2$.

3.3.1. *Droplets in dimension $d = 2$.* We seek incompressible flows inside a time-dependent elliptical domain where

$$\frac{x^2}{a(t)^2} + \frac{y^2}{b(t)^2} < 1, \tag{3.29}$$

with the geometric mean $r = (ab)^{1/2}$ constant in time for volume conservation. We will find such flows as time-stretched straining flows (X, Y), satisfying

$$(\dot{X}, \dot{Y}) = v(X, Y, t) = \alpha(t)(X, -Y).$$

Such flows have velocity potential satisfying $v = \nabla\phi$, with

$$\phi(x, y, t) = \frac{1}{2}\alpha(t)(x^2 - y^2) - \beta(t), \tag{3.30}$$

$$\partial_t\phi = \frac{1}{2}\dot{\alpha}(x^2 - y^2) - \dot{\beta}, \quad \frac{1}{2}|\nabla\phi|^2 = \frac{1}{2}\alpha^2(x^2 + y^2).$$

To satisfy the Bernoulli equation we require $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = 0$ on the boundary of the ellipse $(x, y) = (a\cos\theta, b\sin\theta)$, or

$$(\dot{\alpha} + \alpha^2)a^2\cos^2\theta + (-\dot{\alpha} + \alpha^2)b^2\sin^2\theta = 2\dot{\beta}$$

In order for this to hold independent of θ, we require

$$(\dot{\alpha} + \alpha^2)a^2 = -(\dot{\alpha} - \alpha^2)b^2 = 2\dot{\beta}.$$

Due to the motion of the boundary points $(a, 0)$, $(0, b)$ we need

$$\dot{a} = \alpha a, \quad \dot{b} = -\alpha b,$$

whence

$$2\dot{\beta} = a\ddot{a} = \frac{2b^2\dot{a}^2}{(a^2 + b^2)} = \frac{2r^4\dot{a}^2}{(a^4 + r^4)}$$

because $r^2 = ab$ is constant. Notice $\ddot{a} > 0$ in all cases. There is a first integral (because kinetic energy is conserved) which we can find by writing

$$\frac{\ddot{a}}{\dot{a}} = 2\dot{a}\left(\frac{1}{a} - \frac{a^3}{r^4 + a^4}\right),$$

whence we find that $a(t)$ and $b(t)$ are determined by solving

$$\frac{\dot{a}}{a} = \frac{c}{\sqrt{a^2 + b^2}} = -\frac{\dot{b}}{b} = \alpha(t). \tag{3.31}$$

for some real constant c. From the derivation of the Bernoulli equation, inside the ellipse the pressure is

$$p = -\partial_t\phi - \frac{1}{2}|\nabla\phi|^2 = \dot{\beta}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{3.32}$$

where $\dot{\beta}$ is recovered from the equation

$$\dot{\beta}(t) = \left(\frac{cab}{a^2 + b^2}\right)^2. \tag{3.33}$$

To summarize, an elliptical Euler droplet solution (Q, ϕ, p) is determined in terms of any solution $(a(t), b(t))$ of (3.31) (with any real c) by (3.29), (3.30), (3.32), and (3.33). We note that the speed of motion of the point (a, b) on the hyperbola $ab = r^2$ is constant: by (3.31),

$$\dot{a}^2 + \dot{b}^2 = c^2. \tag{3.34}$$

In the context of the fixed-endpoint problem, then, $|c|$ is the distance along the hyperbola betweeen $(a(0), b(0))$ and $(a(1), b(1))$.

3.3.2. *Droplets in dimension $d \geq 2$.* Let us now derive the result stated in Proposition 3.4. The flow X associated with a velocity potential of the form in (3.27) must satisfy

$$\dot{X}_j = \alpha_j(t) X_j, \qquad \alpha_j = \frac{\dot{a}_j}{a_j}, \quad j = 1, \ldots, d. \tag{3.35}$$

Then $(X_j/a_j)^{\cdot} = 0$ for all j, so the flow is purely dilational along each axis and consequently ellipsoids are deformed to ellipsoids as claimed. Note that incompressibility corresponds to the relation

$$\Delta \phi = \sum_j \alpha_j = \sum_j \frac{\dot{a}_j}{a_j} = \frac{d}{dt} \log(a_1 \cdots a_d) = 0.$$

From (3.27) we next compute

$$\partial_t \phi_t + \frac{1}{2} |\nabla \phi|^2 = -\dot{\beta} + \frac{1}{2} \sum_j (\dot{\alpha}_j + \alpha_j^2) x_j^2 = -\dot{\beta} + \frac{1}{2} \sum_j \frac{\ddot{a}_j x_j^2}{a_j}.$$

This must equal zero on the boundary where $x_j = a_j s_j$ with $s \in S_{d-1}$ arbitrary. We infer that for all j,

$$a_j \ddot{a}_j = 2\dot{\beta}. \tag{3.36}$$

The expression for pressure in (3.27) in terms of $\dot{\beta}$ then follows from (3.16), and $p = 0$ on $\partial \Omega_t$.

We recover $\dot{\beta}$ by differentiating the constraint twice in time. We find

$$\begin{aligned}
0 &= \sum_j \left(\sum_k a_1 \cdots a_d \frac{\dot{a}_k}{a_k} \frac{\dot{a}_j}{a_j} + a_1 \ldots a_d \frac{a_j \ddot{a}_j - \dot{a}_j^2}{a_j^2} \right) \\
&= 0 + \sum_j \frac{2\dot{\beta} - \dot{a}_j^2}{a_j^2}
\end{aligned}$$

whence (3.28) holds.

To get the first integral that corresponds to kinetic energy, multiply (3.36) by $2\dot{a}_j/a_j$ and sum to find

$$0 = \sum_j \dot{a}_j \ddot{a}_j, \quad \text{whence} \quad \sum_j \dot{a}_j^2 = c^2$$

and we see that $c = |\dot{a}(t)|$ is the constant speed of motion.

It remains to see that (3.36) are the geodesic equations on the constraint surface. To see this, recall that geodesic flow on the constraint surface corresponds to a stationary point for the augmented action

$$\int_0^1 \frac{1}{2} |\dot{a}|^2 + \lambda(t) \left(\prod_j a_j - r^d \right) dt$$

which leads to the Euler-Lagrange equations

$$-\ddot{a}_j + \frac{\lambda(t)r^d}{a_j} = 0.$$

Correspondingly, $\lambda r^d = 2\dot{\beta}$. This finishes the demonstration of Proposition 3.4.

Remark 3.5. For later reference, we note that $\ddot{a}_j > 0$ for all t, due to (3.36) and (3.28).

Remark 3.6. Given any two points on the surface described by the constraint (3.25), there exists a constant-speed geodesic connecting them. This fact is a straightforward consequence of the Hopf-Rinow theorem on geodesic completeness [34, Theorem 1.7.1], because all closed and bounded subsets on the surface are compact.

Remark 3.7. The Euclidean metric on the hyperboloid-like surface arises, in fact, as the metric induced by the Wasserstein distance [52, Chap. 15], because, given any dilational flow satisfying (3.35) with $a_1 \cdots a_d = r^d$,

$$\int_{\Omega_t} |v|^2 \, dx = \int_{\Omega_t} \sum_j \alpha_j^2 x_j^2 \, dx = \sum_j \dot{a}_j^2 \int_{|z| \leq 1} z_j^2 \, dz \, r^d = \frac{\omega_d r^d}{d+2} \sum_j \dot{a}_j^2 \, ,$$

where $\omega_d = |B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^d$. For a geodesic, this expression is constant for $t \in [0,1]$ and equals the action $\mathcal{A}_a$ in (3.6) for the ellipsoidal Euler droplet.

3.4. Ellipsoidal Wasserstein droplets.
Let (Q, ϕ, p) be an ellipsoidal Euler droplet solution as given by Proposition 3.4, so that $\Omega_0 = E_{a(0)}$ and $\Omega_1 = E_{a(1)}$ are co-axial ellipsoids. We will call the optimal transport map T between these co-axial ellipsoids an *ellipsoidal Wasserstein droplet*. This is described and related to the Euler droplet as follows.

Given $A \in \mathbb{R}^d$, let $D_A = \operatorname{diag}(A_1, \ldots, A_d)$ denote the diagonal matrix with diagonal A. Then, given $\Omega_0 = E_{a(0)}$, $\Omega_1 = E_{a(1)}$ as above, the particle paths for the Wasserstein geodesic between the corresponding shape densities are given by linear interpolation via

$$T_t(z) = D_{A(t)} D_{A(0)}^{-1} z \, , \qquad A(t) = (1-t)a(0) + ta(1) \, . \tag{3.37}$$

Note that a point $z \in E_A$ if and only if $D_A^{-1} z$ lies in the unit ball $B(0,1)$ in $\mathbb{R}^d$. Thus the Wasserstein geodesic flow takes ellipsoids to ellipsoids:

$$T_t(\Omega_0) = E_{A(t)} \, , \quad t \in [0,1].$$

Let $a(t)$, $t \in [0,1]$, be the geodesic on the hyperboloid-like surface that corresponds to the Euler droplet that we started with. Recall that $\Omega_t = E_{a(t)}$ from Proposition 3.4. Because each component $t \mapsto a_j(t)$ is convex by Remark 3.5, it follows that for each $j = 1, \ldots, d$,

$$a_j(t) \leq A_j(t), \quad t \in [0,1]. \tag{3.38}$$

Because $E_A = D_A B(0,1)$, we deduce from this the following important nesting property, which is illustrated in Figure 2 (where for visibility the ellipses at times $t = \frac{1}{2}$ and $t = 1$ are offset horizontally by $\frac{b}{2}$ and b respectively).

Proposition 3.8. *Given any corresponding elliptical Euler droplet and Wasserstein droplet that deform one ellipsoid $\Omega_0 = E_{a(0)}$ to another $\Omega_1 = E_{a(1)}$, the Euler domains remain nested inside their Wasserstein counterparts, with*

$$X(\Omega_0, t) = \Omega_t \subset T_t(\Omega_0), \quad t \in [0,1]. \tag{3.39}$$
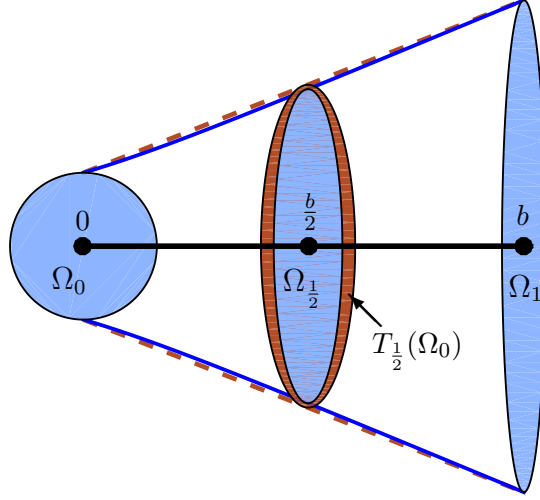
FIGURE 2. Euler droplet (light blue) deforming a circle to an ellipse, nested inside a Wasserstein droplet (dark orange). Tracks of the center and endpoints of vertical major axis are indicated for both droplets.

Remark 3.9. In terms of the notation of this subsection, the straining flow X associated with the Euler droplet is given by $X(z,t) = D_{a(t)}D_{a(0)}^{-1}z$ in terms of the constant-speed geodesic $a(t)$ of Proposition 3.4. Due to (3.38), this flow satisfies, for each $j = 1, \ldots, d$ and $z \in \mathbb{R}^d$,

$$|X_j(z,t)| = \frac{a_j(t)}{a_j(0)}|z_j| \leq \frac{A_j(t)}{A_j(0)}|z_j| = |T_t(z)_j|.$$

For the nesting property $X(\hat{\Omega}, t) \subset T_t(\hat{\Omega})$ to hold, convexity of $\hat{\Omega}$ is not sufficient in general. However, a sufficient condition is that whenever $\alpha_j \in [0,1]$ for $j = 1, \ldots, d$,

$$x = (x_1, \ldots, x_d) \in \hat{\Omega} \quad \text{implies} \quad D_\alpha x = (\alpha_1 x_1, \ldots, \alpha_n x_n) \in \hat{\Omega}.$$

For later use below, we describe how to bound the action for a boosted elliptical Euler droplet in terms of action for the corresponding boosted elliptical Wasserstein droplet, in the case when the source and target domains are respectively a ball and translated ellipse:

Lemma 3.10. *Given $r > 0$, $\hat{a} \in \mathbb{R}_+^d$ with $\hat{a}_1 \cdots \hat{a}_d = r^d$, and $b \in \mathbb{R}^d$, let*

$$\Omega_0 = B(0, r), \qquad \Omega_1 = E_{\hat{a}} + b.$$

Let $a(t)$, $t \in [0,1]$, be the minimizing geodesic on the surface (3.25) with

$$a(0) = \hat{r} = (r, \ldots, r), \qquad a(1) = \hat{a} = (\hat{a}_1, \ldots, \hat{a}_d).$$

Let (Q, ϕ, p) be the elliptical Euler droplet solution corresponding to the geodesic a, and let $\mathcal{A}_a$ denote the corresponding action. Then

$$d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 \leq \mathcal{A}_a \leq d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \frac{\overline{\lambda}^4}{\underline{\lambda}^2}\,\omega_d r^{d+2}, \tag{3.40}$$

where

$$\underline{\lambda} = \min \frac{\hat{a}_i}{r}, \qquad \overline{\lambda} = \max \frac{\hat{a}_i}{r}. \tag{3.41}$$

Proof. First, consider the transport cost for mapping Ω_0 to Ω_1. The (constant) velocity of particle paths starting at $x \in B(0, r)$ is

$$u(x) = (r^{-1}D_{\hat{a}} - I)x + b,$$

and the squared transport cost or action is (substituting $x = rz$)

$$d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 = \int_{B(0,r)} |u(x)|^2 \, dx = \sum_j \int_{B(0,r)} \left(\frac{\hat{a}_j}{r} - 1\right)^2 z_j^2 + b_j^2 \, dz$$

$$= \omega_d r^d \left(|b|^2 + \frac{|\dot{A}|^2}{d+2}\right), \tag{3.42}$$

where $A(t) = (1-t)\hat{r} + t\hat{a}$ is the straight-line path from $\hat{r}$ to $\hat{a}$.

The mass density inside the transported ellipsoid $T_t(\Omega_0)$ is constant in space, given by

$$\rho(t) = \det DT_t^{-1} = \prod_i \frac{r}{A_i(t)} = \prod_i \left(1 - t + t\frac{\hat{a}_i}{r}\right)^{-1}.$$

Due to Remark 3.7, the corresponding action for the Euler droplet is bounded by that of the constant-volume path found by dilating the elliptical Wasserstein droplet: Let

$$\gamma_j(t) = \rho(t)^{1/d} A_j(t).$$

Then the flow $S_t(z) = r^{-1}D_{\gamma(t)}z$ is dilational and volume-preserving (with $\prod_j \gamma_j(t) \equiv r^d$) and has zero mean velocity. The flow $z \mapsto S_t(z) + tb$ takes Ω_0 to Ω_1, as on Figure 2, with action

$$\mathcal{A}_\gamma = \int_0^1 \int_{B(0,r)} \sum_j \left(b_j + \frac{\dot{\gamma}_j z_j}{r}\right)^2 dz \, dt$$

$$= \omega_d r^d \left(|b|^2 + \frac{1}{d+2} \int_0^1 |\dot{\gamma}|^2 \, dt\right). \tag{3.43}$$

Note that $\sum_j (\dot{\gamma}_j/\gamma_j)^2 \le \sum_j (\dot{A}_j/A_j)^2$, because

$$\frac{\dot{\gamma}_j}{\gamma_j} = \frac{\dot{A}_j}{A_j} + \frac{\dot{\rho}}{d\rho} = \frac{\dot{A}_j}{A_j} - \frac{1}{d}\sum_i \frac{\dot{A}_i}{A_i}.$$

Because ρ is convex we have $\rho \le 1$, hence $\gamma_j^2 \le \max A_i^2$. Thus

$$|\dot{\gamma}|^2 \le (\max A_i^2) \sum_j \frac{\dot{A}_j^2}{A_j^2} \le \left(\frac{\max A_i^2}{\min A_i^2}\right)|\dot{A}|^2 \le \left(\frac{\max \hat{a}_i^2}{\min \hat{a}_i^2}\right)|\hat{a} - \hat{r}|^2. \tag{3.44}$$

Plugging this back into (3.43) and using (3.42), we deduce that

$$\mathcal{A}_\gamma \le d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \frac{\omega_d r^d}{d+2}\left(\frac{\max \hat{a}_i^2}{\min \hat{a}_i^2}\right)|\hat{a} - \hat{r}|^2. \tag{3.45}$$

With the notation in (3.41), $\overline{\lambda}$ and $\underline{\lambda}$ respectively are the maximum and minimum eigenvalues of DT_t, and because $|1 - \hat{a}_i/r| \le \max(1, \hat{a}_i/r) \le \overline{\lambda}$ for all $i = 1, \ldots, d$, this estimate implies

$$\mathcal{A}_a \le \mathcal{A}_\gamma \le d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \frac{d}{d+2}\frac{\overline{\lambda}^4}{\underline{\lambda}^2}\omega_d r^{d+2}. \tag{3.46}$$

$\square$

3.5. **Velocity and pressure estimates.** Lastly in this section we provide bounds on the velocity $v = \nabla\phi$ and pressure p for the ellipsoidal Euler droplet solutions. Note that because $1/a_j^2 \leq \sum_i (1/a_i^2)$,

$$0 \leq p \leq \dot{\beta} \leq \frac{1}{2} \sum_j \dot{a}_j^2 \leq \frac{1}{2} \int_0^1 |\dot{\gamma}|^2 \, dt$$

Using (3.44) and the notation in (3.41), it follows

$$0 \leq p \leq \frac{\overline{\lambda}^4}{\underline{\lambda}^2} \, r^2 d. \tag{3.47}$$

For the velocity, it suffices to note that in (3.35), $|X_j/a_j| \leq 1$ hence $|\dot{X}|^2 \leq \sum_j \dot{a}_j^2$. Thus the same bounds as above apply and we find

$$|\nabla\phi|^2 \leq \frac{\overline{\lambda}^4}{\underline{\lambda}^2} \, r^2 d. \tag{3.48}$$

Finally, for a boosted elliptical Euler droplet, with velocity boosted as in (3.20) by a constant vector $b \in \mathbb{R}^d$, the same pressure bound as above in (3.47) applies, and the same bound on velocity becomes

$$|\nabla\hat{\phi} - b|^2 \leq \frac{\overline{\lambda}^4}{\underline{\lambda}^2} \, r^2 d. \tag{3.49}$$

4. EULER SPRAYS

Heuristically, an Euler spray is a countable disjoint superposition of solutions of the Euler droplet equations. Recall that the notation $\sqcup_n \Omega_n$ means the union of disjoint sets Ω_n.

Definition 4.1. An **Euler spray** is a triple (Q, ϕ, p), with Q a bounded open subset of $\mathbb{R}^d \times [0,1]$ and $\phi, p : Q \to \mathbb{R}$, such that there is a sequence $\{(Q_n, \phi_n, p_n)\}_{n \in \mathbb{N}}$ of smooth solutions of the Euler droplet equations, such that $Q = \sqcup_{n=1}^\infty Q_n$ is a disjoint union of the sets Q_n, and for each $n \in \mathbb{N}$, $\phi_n = \phi|_{\Omega_n}$ and $p_n = p|_{\Omega_n}$.

With each Euler spray that satisfies appropriate bounds we may associate a weak solution (ρ, v, p) of the Euler system (1.9)-(1.10). The following result is a simple consequence of the weak formulation in (1.11)-(1.12) together with Proposition 3.3(a) and the dominated convergence theorem.

Proposition 4.2. *Suppose (Q, ϕ, p) is an Euler spray such that $|\nabla\phi|^2$ and p are integrable on Q. Then with $\rho = \mathbb{1}_Q$ and $v = \mathbb{1}_Q \nabla\phi$ and with p extended as zero outside Q, the triple (ρ, v, p) satisfies the Euler system (1.9)-(1.10) in the sense of distributions on $\mathbb{R}^d \times [0,1]$.*

Our main goal in this section is to prove Theorem 1.1. The strategy of the proof is simple to outline: We will approximate the optimal transport map $T \colon \Omega_0 \to \Omega_1$ for the Monge-Kantorovich distance, up to a null set, by an 'ellipsoidal transport spray' built from a countable collection of ellipsoidal Wasserstein droplets. The spray maps Ω_0 to a target Ω_1^ε whose shape distance from Ω_1 is as small as desired. Then from the corresponding ellipsoidal Euler droplets nested inside, we construct the desired Euler spray (Q, ϕ, p) that connects Ω_0 to Ω_1^ε by a critical path for the action in (1.1).

Remark 4.3. In general, for the Euler sprays that we construct, the domain $Q = \sqcup_{n=1}^\infty Q_n$ has an irregular boundary ∂Q strictly larger than the infinite union $\sqcup_{n=1}^\infty \partial Q_n$ of smooth boundaries of individual ellipsoidal Euler droplets, since ∂Q contains limit points of sequences belonging to infinitely many Q_n.

4.1. **Approximating optimal transport by an ellipsoidal transport spray.** Heuristically, an ellipsoidal transport spray is a countable disjoint superposition of transport maps on ellipsoids, whose particle trajectories do not intersect.

Definition 4.4. *An **ellipsoidal transport spray** on Ω_0 is a map $S \colon \Omega_0 \to \mathbb{R}^d$, such that*

$$\Omega_0 = \bigsqcup_{n \in \mathbb{N}} \Omega_0^n$$

is a disjoint union of ellipsoids, the restriction of S to Ω_0^n is an ellipsoidal Wasserstein droplet, and the linear interpolants S_t defined by

$$S_t(z) = (1-t)z + tS(z), \qquad z \in \Omega_0,$$

remain injections for each $t \in [0,1]$.

Proposition 4.5. *Let Ω_0, Ω_1 be a pair of shapes in $\mathbb{R}^d$ of equal volume, and let $T \colon \Omega_0 \to \Omega_1$ be the optimal transport map for the Monge-Kantorovich distance with quadratic cost. For any $\varepsilon > 0$, there is an ellipsoidal transport spray $S^\varepsilon \colon \Omega_0^\varepsilon \to \mathbb{R}^d$ such that*

(i) *Ω_0^ε is a countable union of balls in the non-singular set $\Omega_0 \setminus \Sigma_0$ with $|\Omega_0 \setminus \Omega_0^\varepsilon| = 0$, and*

(ii) *$\displaystyle \sup_{z \in \Omega_0^\varepsilon} |T(z) - S^\varepsilon(z)| < \varepsilon \operatorname{diam} \Omega_1$.*

(iii) *The L^∞ transportation distance between the uniform distributions on Ω_1^ε and Ω_1 satisfies $d_\infty(\Omega_1^\varepsilon, \Omega_1) < \varepsilon \operatorname{diam} \Omega_1$.*

The proof of this result will comprise the remainder of this subsection. The strategy is as follows. The set Ω_0^ε is chosen to be the union of a suitable Vitali covering of Ω_0 a.e. by balls B_i. We expand the Brenier map T by a factor of $1 + \varepsilon$ and consider the displacement interpolation map between Ω_0 and $(1 + \varepsilon)\Omega_1$ given by

$$T_t^\varepsilon(x) = (1-t)x + t(1+\varepsilon)T(x). \tag{4.1}$$

Next, on each ball B_i we approximate T by an affine map which takes the ball center x_i to $(1 + \varepsilon)T(x_i)$. Namely, this approximation will take the form

$$S^\varepsilon(x) = (1+\varepsilon)T(x_i) + DT(x_i)(x - x_i), \qquad x \in B_i. \tag{4.2}$$

The corresponding displacement interpolation map has three key properties: (i) it is locally affine so maps balls to ellipsoids, (ii) it is volume-preserving, and (iii) the dilation by $1 + \varepsilon$ grants each ellipsoidal image sufficient 'personal space' to ensure the injectivity of the piecewise affine approximation.

4.1.1. *Vitali covering.* We suppose $0 < \varepsilon < 1$. The first step in the proof of Proposition 4.5 is to produce a suitable Vitali covering of Ω_0, up to a null set, by a countable disjoint union of balls. By a simple translation of target and source so that the origin is the midpoint of two points in $\bar{\Omega}_1$ separated by distance $\operatorname{diam} \Omega_1$, because the distance from any point in Ω_1 to each of the two points is also no more than $\operatorname{diam} \Omega_1$ we may assume that

$$\sup_{x \in \Omega_0} |T(x)| \leq \frac{\sqrt{3}}{2} \operatorname{diam} \Omega_1. \tag{4.3}$$

Recall that there is a relatively closed null set $\Sigma_0 \subset \Omega_0$ such that $T = \nabla \psi$ is a smooth diffeomorphism from $\Omega_0 \setminus \Sigma_0$ to its image. Then for every $x \in \Omega_0 \setminus \Sigma_0$, there exists $\bar{r}(x, \varepsilon) \in (0, \operatorname{diam} \Omega_1)$ such that whenever $0 < r < \bar{r}$, then $B(x, r) \subset \Omega_0 \setminus \Sigma_0$ and both

$$\frac{\varepsilon}{4} > \frac{r \|D^3 \psi\|_{B(x,r)}}{\underline{\lambda}_{B(x,r)}^2}, \qquad \varepsilon > \left(\frac{\bar{\lambda}_{B(x,r)}^2}{\underline{\lambda}_{B(x,r)}} \frac{r}{\operatorname{diam} \Omega_1} \right)^2, \tag{4.4}$$

where $\underline{\lambda}_U$ and $\overline{\lambda}_U$ are defined by (2.11) and $\|D^3\psi\|_U$ is defined by (2.13). This follows by noting that the right-hand sides are continuous functions of r with value 0 when $r = 0$. The family of balls

$$\{B(x,r) : x \in \Omega_0 \backslash \Sigma_0, \ 0 < r < \overline{r}(x,\varepsilon)\}$$

forms a Vitali cover of $\Omega_0 \backslash \Sigma_0$. Therefore, by Vitali's covering theorem [22, Theorem III.12.3], there is a countable family of mutually disjoint balls $B(x_i, r_i)$, with $x_i \in \Omega_0 \backslash \Sigma_0$ and $0 < r_i < \overline{r}(x_i, \varepsilon)$, such that

$$|(\Omega_0 \backslash \Sigma_0) \backslash \cup_{i \in \mathbb{N}} B(x_i, r_i)| = 0 \ .$$

We let

$$\Omega_0^\varepsilon = \bigcup_{i \in \mathbb{N}} B_i \,, \qquad B_i = B(x_i, r_i). \tag{4.5}$$

For further use below, we note that $\underline{\lambda}_i \le 1 \le \overline{\lambda}_i$ for all i, where

$$\underline{\lambda}_i = \underline{\lambda}_{B(x_i,r_i)} \,, \qquad \overline{\lambda}_i = \overline{\lambda}_{B(x_i,r_i)} \,, \quad i \in \mathbb{N}. \tag{4.6}$$

We observe that from the first constraint in (4.4) follows

$$\|D^3\psi\|_{B_i} r_i < \frac{\varepsilon}{4}\underline{\lambda}_i^2 \le \frac{\varepsilon}{4} \ . \tag{4.7}$$

4.1.2. *An approximating ellipsoidal transport spray.* We shall approximate the optimal transport map T on Ω_0^ε through linear approximation on each ball B_i, combined with a homothetic expansion of the ball centers to maintain injectivity.

For each $i \in \mathbb{N}$, we denote the linear approximation to T on B_i by

$$A^i(x) = T(x_i) + DT(x_i)(x - x_i). \tag{4.8}$$

Then we define $S^\varepsilon \colon \Omega_0^\varepsilon \to \mathbb{R}^d$ by (4.2) whenever $x \in B_i$, so that

$$S^\varepsilon(x) = A^i(x) + \varepsilon T(x_i) \,. \tag{4.9}$$

Because each B_i is a ball and $DT(x_i) = \operatorname{Hess}\psi(x_i)$ whose determinant is 1, the affine map A^i is an ellipsoidal Wasserstein droplet, so the same is true for the restriction of S^ε to B_i.

For every $x \in B_i$, note that we have the estimate by Taylor's theorem

$$\begin{aligned} |T(x) - S^\varepsilon(x)| &\le |T(x) - A^i(x)| + \varepsilon|T(x_i)| \\ &\le \frac{1}{2}\|D^3\psi\|_{B_i} r_i^2 + \frac{\varepsilon}{2}\operatorname{diam}\Omega_1 \\ &\le \frac{1}{8}\varepsilon r_i + \frac{\sqrt{3}}{2}\varepsilon\operatorname{diam}\Omega_1 \,. \end{aligned} \tag{4.10}$$

Because $r_i \le \operatorname{diam}\Omega_1$, the estimate in part (ii) holds. In order to show that S^ε is an ellipsoidal transport spray and complete the proof of Proposition 4.5, it remains to show that the interpolants S_t^ε defined as in Definition 4.4 are injections for each $t \in [0,1]$.

Lemma 4.6 (Injectivity of interpolants). *For each $t \in [0,1]$, the interpolant*

$$S_t^\varepsilon = (1-t)I + tS^\varepsilon$$

is an injection. Its image is a union of the disjoint ellipsoids $S_t^\varepsilon(B_i)$, $i \in \mathbb{N}$, separated according to

$$\operatorname{dist}(S_t^\varepsilon(B_i), S_t^\varepsilon(B_j)) \ge \frac{\varepsilon t}{2}(\underline{\lambda}_i^2 r_i + \underline{\lambda}_j^2 r_j), \qquad i \ne j. \tag{4.11}$$

Proof. Define T_t^ε by (4.1). Because $T = \nabla\psi$ with ψ convex, as for (2.2), for all $x, \hat{x} \in \Omega_0$ we have

$$|T_t^\varepsilon(x) - T_t^\varepsilon(\hat{x})| \geq (1-t)|x - \hat{x}|\,. \tag{4.12}$$

Hence the images $T_t^\varepsilon(B_i)$ and $T_t^\varepsilon(B_j)$ are disjoint whenever $i \neq j$ and $t \in [0, 1)$. In order to prove the lemma it suffices to show that for each k, $S_t^\varepsilon(B_k) \subset T_t^\varepsilon(B_k)$ and that

$$\mathrm{dist}(S_t^\varepsilon(B_k), T_t^\varepsilon(\partial B_k)) \geq \frac{\varepsilon t}{2}\underline{\lambda}_k^2 r_k\,.$$

Fix $k \in \mathbb{N}$ and for $x \in B_k$ define an affine approximation to T_t^ε as

$$U_t^\varepsilon(x) = T_t^\varepsilon(x_k) + DT_t^\varepsilon(x_k)(x - x_k)\,. \tag{4.13}$$

Then by Taylor's theorem, for all $x \in B_k$ we have the estimate

$$|T_t^\varepsilon(x) - U_t^\varepsilon(x)| \leq \frac{1}{2}(1+\varepsilon)t\delta_k < t\delta_k\,, \tag{4.14}$$

where $\delta_k := \|D^3\psi\|_{B_k} r_k^2 < \frac{1}{4}\varepsilon\underline{\lambda}_k^2 r_k$ due to (4.7). Now, because

$$S_t^\varepsilon(x) = T_t^\varepsilon(x_k) + DT_t(x_k)(x - x_k)$$

we see that the ellipsoids given by

$$E = S_t^\varepsilon(B_k) - T_t^\varepsilon(x_k)\,, \qquad \hat{E} = U_t^\varepsilon(B_k) - T_t^\varepsilon(x_k)$$

are concentric with identical principal axes having respective principal stretches

$$1 - t + t\lambda\,, \qquad 1 - t + t(1+\varepsilon)\lambda\,,$$

for each eigenvalue λ of $DT(x_k) = \mathrm{Hess}\,\psi(x_k)$. Due to the definition of $\underline{\lambda}_k$ from (4.6), each such eigenvalue satisfies $\lambda \geq \underline{\lambda}_k$. Because $\underline{\lambda}_k \leq 1$, the ratio of the principal stretches has a common lower bound:

$$\frac{1 - t + t(1+\varepsilon)\lambda}{1 - t + t\lambda} \geq 1 + \frac{\varepsilon t\underline{\lambda}_k}{1 - t + t\underline{\lambda}_k} > 1 + \varepsilon t\underline{\lambda}_k =: \alpha\,. \tag{4.15}$$

Thus the uniform dilation αE is contained in $\hat{E}$ and hence the distance from E to $\partial\hat{E}$ is greater than the distance from E to $\partial(\alpha E)$, which is easily seen to be $(\alpha - 1)\underline{\lambda}_k r_k$. This means

$$\mathrm{dist}(S_t^\varepsilon(B_k), U_t^\varepsilon(\partial B_k)) = \mathrm{dist}(E, \hat{E}) \geq \mathrm{dist}(E, \partial(\alpha E)) = (\alpha - 1)\underline{\lambda}_k r_k = \varepsilon t\underline{\lambda}_k^2 r_k\,.$$

Combining this with (4.14) we deduce that for $\varepsilon < 1$,

$$\mathrm{dist}(S_t^\varepsilon(B_k), T_t^\varepsilon(\partial B_k)) \geq \varepsilon t\underline{\lambda}_k^2 r_k - t\delta_k \geq \frac{\varepsilon t}{2}\underline{\lambda}_k^2 r_k\,. \tag{4.16}$$

The inclusion $S_t^\varepsilon(B_k) \subset T_t^\varepsilon(B_k)$ now follows by continuation from the common point $T_t^\varepsilon(x_k)$. $\qquad\square$

This completes the proof of parts (i) and (ii) of Proposition 4.5. For part (iii), we note that the set $\Omega_0^\varepsilon = (S^\varepsilon)^{-1}(\Omega_1^\varepsilon)$ has full measure in $\Omega_0 \setminus \Sigma_0$, and T is a smooth diffeomorphism from this set to $\Omega_1 \setminus \Sigma_1$ so maps null sets to null sets. It follows $T \circ (S^\varepsilon)^{-1}$ maps Ω_0^ε to a set of full measure in Ω_1, satisfies

$$\sup_{x \in \Omega_1^\varepsilon} |T \circ (S^\varepsilon)^{-1}(x) - x| < \varepsilon\,\mathrm{diam}\,\Omega_1,$$

and pushes forward uniform measure to uniform measure. The result claimed in part (iii) follows, due to (2.6).

4.2. **Action estimate for Euler spray.** Each of the ellipsoidal Wasserstein droplets that make up the ellipsoidal transport spray S^ε is associated with a boosted ellipsoidal Euler droplet nested inside, due to the nesting property in Proposition 3.8. The disjoint superposition of these Euler droplets make up an Euler spray that deforms Ω_0^ε to the same set Ω_1^ε.

In order to complete the proof of Theorem 1.1, it remains to bound the action of this Euler spray in terms of the Wasserstein distance between the uniform measures on Ω_0 and Ω_1. Toward this goal, we first note that because the maps T and S^ε are volume-preserving, due to the estimate in part (ii) of Proposition 4.5 we have

$$d_W(T(B_i), S^\varepsilon(B_i))^2 \le (\varepsilon K_1)^2 |B_i|, \qquad K_1 = \operatorname{diam} \Omega_1.$$

(One obtains this by bounding the transport cost of straight-line motion from $T(z)$ to $S^\varepsilon(z)$ using the Lagrangian form of the action in (3.6).) Now by the triangle inequality,

$$
\begin{aligned}
d_W(B_i, S^\varepsilon(B_i))^2 &\le \left(d_W(B_i, T(B_i)) + \varepsilon K_1 |B_i|^{1/2} \right)^2 \\
&\le d_W(B_i, T(B_i))^2 (1+\varepsilon) + (\varepsilon + \varepsilon^2) K_1^2 |B_i|
\end{aligned}
\tag{4.17}
$$

Recall that by inequality (3.40) of Lemma 3.10, the action of the i-th ellipsoidal Euler droplet, denoted by $\mathcal{A}_i$, satisfies

$$
\begin{aligned}
\mathcal{A}_i &\le d_W(B_i, S^\varepsilon(B_i))^2 + \frac{\overline{\lambda}_i^4}{\underline{\lambda}_i^2} r_i^2 |B_i| \\
&\le d_W(B_i, T(B_i))^2 (1+\varepsilon) + 2\varepsilon K_1^2 |B_i|,
\end{aligned}
\tag{4.18}
$$

where we make use of the second constraint in (4.4).

By summing over all i, we obtain the required bound,

$$\mathcal{A}^\varepsilon = \sum_i \mathcal{A}_i \le d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + K\varepsilon$$

where

$$K = d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + 2|\Omega_0|(\operatorname{diam}\Omega_1)^2.$$

This concludes the proof of Theorem 1.1.

5. Shape distance equals Wasserstein distance

Our main goal in this section is to prove Theorem 1.3, which establishes the existence of paths of shape densities (as countable concatenations of Euler sprays) that exactly connect any two compactly supported measures having densities with values in $[0,1]$ and have action as close as desired to the Wasserstein distance squared between the measures. Theorem 1.2 follows as an immediate corollary, showing that shape distance between arbitrary bounded measurable sets with positive, equal volume is the Wasserstein distance between the corresponding characteristic functions.

Theorem 1.3 will be deduced from Theorem 1.1 by essentially 'soft' arguments. Theorem 1.1 shows that the relaxation of shape distance, in the sense of lower-semicontinuous envelope with respect to the topology of weak-$\star$ convergence of characteristic functions, is Wasserstein distance. Essentially, here we use this result to compute the completion of the shape distance in the space of bounded measurable sets.

Lemma 5.1. *Let $\rho\colon \mathbb{R}^d \to [0,1]$ be a measurable function of compact support. Then for any $\varepsilon > 0$ there is an open set Ω such that its volume is the total mass of ρ and the L^∞ transport distance from ρ to its characteristic function is less than ε:*

$$|\Omega| = \int_{\mathbb{R}^d} \rho\, dx \quad and \quad d_\infty(\rho, \mathbb{1}_\Omega) < \varepsilon\,.$$

Proof. We recall that weak-$\star$ convergence of probability measures supported in a fixed compact set is equivalent to convergence in (either L^2 or L^∞) Wasserstein distance. Given $k \in \mathbb{N}$, cover the support of ρ a.e. by a grid of disjoint open rectangles of diameter less than $\varepsilon_k = 1/k$. For each rectangle R in the grid, shrink the rectangle homothetically from any point inside to obtain a sub-rectangle $\hat{R} \subset R$ such that $|\hat{R}| = \int_R \rho\, dx$. Let Ω_k be the disjoint union of the non-empty rectangles $\hat{R}$ so obtained. Then the sequence of characteristic functions $\mathbb{1}_{\Omega_k}$ evidently converges weak-$\star$ to ρ in the space of fixed-mass measures on a fixed compact set: for any continuous test function f on $\mathbb{R}^d$, as $k \to \infty$ we have

$$\int_{\Omega_k} f(x)\, dx \to \int_{\mathbb{R}^d} f(x)\rho(x)\, dx\,.$$

Choosing $\Omega = \Omega_k$ for some sufficiently large k yields the desired result. $\square$

Proof of Theorem 1.3 part (a). Let ρ_0, ρ_1 have the properties stated, and suppose $D := d_W(\rho_0, \rho_1) > 0$. (The other case is trivial.) Let $\varepsilon > 0$. By Lemma 5.1 we may choose open sets Ω_0 and $\hat{\Omega}_1$ whose volume is $\int_{\mathbb{R}^d} \rho_0$ and such that

$$d_\infty(\rho_0, \mathbb{1}_{\Omega_0}) + d_\infty(\rho_1, \mathbb{1}_{\hat{\Omega}_1}) < \frac{\varepsilon}{2}, \qquad d_W(\Omega_0, \hat{\Omega}_1)^2 \le d_W(\rho_0, \rho_1)^2 + \frac{\varepsilon}{2}. \tag{5.1}$$

Then we can apply Theorem 1.1 to find an Euler spray that connects Ω_0 to a set $\hat{\Omega}_1^\varepsilon =: \Omega_1$ close to $\hat{\Omega}_1$ with the properties

$$d_\infty(\Omega_1, \hat{\Omega}_1) < \frac{\varepsilon}{3}, \qquad \mathcal{A}^\varepsilon \le d_W(\Omega_0, \hat{\Omega}_1)^2 + \frac{\varepsilon}{3}, \tag{5.2}$$

where $\mathcal{A}^\varepsilon$ is the action of this Euler spray. By combining the inequalities in (5.1) and (5.2) we find that the sets Ω_0, Ω_1 have the properties required. $\square$

Before we establish part (b), we separately discuss the concatenation of transport paths. Let $\rho^k = (\rho_t^k)_{t\in[0,1]}$ be a path of shape densities for each $k = 1, 2, \ldots, K$, with associated transport velocity field $v^k \in L^2(\rho^k)$ and action

$$\mathcal{A}_k = \int_0^1 \int_{\mathbb{R}^d} \rho_t^k(x) |v^k(x,t)|^2\, dx\, dt.$$

We say this set of paths forms a *chain* if $\rho_1^k = \rho_0^{k+1}$ for $k = 1, \ldots, K-1$. Given such a chain, and numbers $\tau_k > 0$ such that $\sum_{k=1}^K \tau_k = 1$, we define the *concatenation of the chain of paths ρ^k compressed by τ_k* to be the path $\rho = (\rho_t)_{t\in[0,1]}$ given by

$$\rho_t = \rho_s^k \quad \text{for} \quad t = \tau_k s + \sum_{j<k} \tau_k\,, \quad s \in [0,1]. \tag{5.3}$$

The transport velocity associated with the concatenation is

$$v(\cdot, t) = \tau_k^{-1} v^k(\cdot, s) \quad \text{for} \quad t = \tau_k s + \sum_{j<k} \tau_k\,, \quad s \in [0,1], \tag{5.4}$$

and the action is

$$\mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho_t |v|^2 \, dx \, dt = \sum_{k=1}^K \tau_k^{-1} \int_0^1 \int_{\mathbb{R}^d} \rho_s^k(x) |v(x,s)|^2 \, dx \, ds = \sum_{k=1}^K \tau_k^{-1} \mathcal{A}_k. \tag{5.5}$$

Remark 5.2. We mention here how the triangle inequality for the shape distance defined in (1.5) follows directly from this concatenation procedure. Given the chain ρ^k as above with actions $\mathcal{A}_k$, let $\delta_k = \sqrt{\mathcal{A}_k}$ and set

$$\tau_k = \frac{\delta_k}{\sum_j \delta_j}, \quad k = 1, \dots, K.$$

Let $\mathcal{A}$ be the action of the concatenation of paths ρ^k compressed by τ_k, and let $\delta = \sqrt{\mathcal{A}}$. Then

$$\mathcal{A} = \delta^2 = \sum_k \tau_k^{-1} \delta_k^2 = \left(\sum_k \delta_k \right)^2.$$

From this the triangle inequality follows.

Proof of Theorem 1.3 part (b). Next we establish part (b). The idea is to construct a path of shape densities $\rho = (\rho_t)_{t \in [0,1]}$ connecting ρ_0 to ρ_1 by concatenating the Euler spray from part (a) together with two paths of small action that themselves are concatenated chains of Euler sprays that respectively connect Ω_0 to sets that approximate ρ_0, and connect Ω_1 to sets that approximate ρ_1.

Let $\varepsilon > 0$, and let ρ^ε be a shape density determined by an Euler spray as from part (a) that connects bounded open sets Ω_0 and Ω_1 of volume $\int_{\mathbb{R}^d} \rho_0$, but with the (perhaps tighter) conditions

$$d_W(\mathbb{1}_{\Omega_0}, \rho_0) + d_W(\mathbb{1}_{\Omega_1}, \rho_1) < \frac{1}{4}\varepsilon 2^{-1}, \qquad \mathcal{A}^\varepsilon < d_W(\rho_0, \rho_1)^2 + \varepsilon,$$

where $\mathcal{A}^\varepsilon$ is the action of this spray.

Next we construct a chain of Euler sprays with shape densities ρ^k, $k = 1, 2, \dots$, with action $\mathcal{A}_k$ that connect Ω_1 with a chain of sets Ω_k such that $\mathbb{1}_{\Omega_k} \xrightarrow{\star} \rho_1$ as $k \to \infty$ and

$$d_W(\mathbb{1}_{\Omega_k}, \rho_1) < \frac{1}{4}\varepsilon 2^{-k}, \qquad \mathcal{A}_k < (\varepsilon 2^{-k})^2. \tag{5.6}$$

We proceed by recursion by applying Theorem 1.1 like in the proof of part (a). Given $k \geq 1$, suppose Ω_k is defined and ρ^j are defined for $j < k$. Using Lemma 5.1 we can find a bounded open set $\hat{\Omega}_{k+1}$ such that

$$|\hat{\Omega}_{k+1}| = \int_{\mathbb{R}^d} \rho_0 \qquad \text{and} \qquad d_W(\mathbb{1}_{\hat{\Omega}_{k+1}}, \rho_1) < \frac{1}{8}\varepsilon 2^{-k-1}.$$

Then by invoking Theorem 1.1 and the triangle inequality for d_W, we obtain an Euler spray with action $\mathcal{A}_k$ that connects Ω_k to a bounded open set Ω_{k+1}, such that

$$d_W(\Omega_{k+1}, \hat{\Omega}_{k+1}) < \frac{1}{8}\varepsilon 2^{-k-1} \quad \text{and} \quad \mathcal{A}_k < d_W(\Omega_k, \hat{\Omega}_{k+1})^2 + \frac{1}{2}(\varepsilon 2^{-k})^2 < (\varepsilon 2^{-k})^2.$$

We let $\rho^k = (\rho_t^k)_{t \in [0,1]}$ be the path of shape densities for this spray, so that $\rho_0^k = \mathbb{1}_{\Omega_k}$ and $\rho_1^k = \mathbb{1}_{\Omega_{k+1}}$. This completes the construction of the chain of paths ρ^k satisfying (5.6).

It is straightforward to see that $d_W(\rho_t^k, \rho_1) \to 0$ as $k \to \infty$ uniformly for $t \in [0,1]$. Now we let $\rho^+ = (\rho_t^+)_{t \in [0,1]}$ be the countable concatenation of this chain of paths ρ^k compressed by

$\tau_k = 2^{-k}$ according to the formulas (5.3)–(5.5) above taken with $K \to \infty$, and with $\rho_1^+ = \rho_1$. The action $\mathcal{A}^+$ of this concatenation then satisfies

$$\mathcal{A}^+ = \sum_{k=1}^{\infty} 2^k \mathcal{A}_k < \varepsilon^2. \tag{5.7}$$

In exactly analogous fashion we can construct a countable concatenation $\hat{\rho}^-$ of a chain of paths coming from Euler sprays, that connects $\hat{\rho}_0^- = \mathbb{1}_{\Omega_0}$ with $\hat{\rho}_1^- = \rho_0$ and having action $\mathcal{A}^- < \varepsilon^2$. Then define ρ^- be the *reversal* of $\hat{\rho}^-$, given by

$$\rho_t^- = \hat{\rho}_{1-t}^- .$$

This path ρ^- has the same action $\mathcal{A}^-$.

Finally, define the path ρ by concatenating ρ^-, ρ^{ε}, ρ^+ compressed by ε, $1 - 2\varepsilon$ and ε respectively. This path satisfies the desired endpoint conditions and has action $\mathcal{A}$ that satisfies

$$\mathcal{A} = \varepsilon^{-1}\mathcal{A}^- + (1 - 2\varepsilon)^{-1}\mathcal{A}^{\varepsilon} + \varepsilon^{-1}\mathcal{A}^+ < d_W(\rho_0, \rho_1)^2 + K\varepsilon,$$

for some constant K independent of ε small. The result of part (b) follows. $\qquad\square$

As we indicated in section 2, the result of Theorem 1.4, providing a sharp criterion for the existence of a minimizer for the shape distance in (1.5), follows by combining the uniqueness property of Wasserstein geodesics with the result of Theorem 1.2.

Proof of Theorem 1.4. Clearly, if the Wasserstein geodesic density is a characteristic function, then the Wasserstein geodesic provides a minimizing path for shape distance. On the other hand, if a minimizer for (1.5) exists, it must have constant speed by a standard reparametrization argument. Then by Theorem 1.2 it provides a constant-speed minimizing path for Wasserstein distance as well, hence corresponds to the unique Wasserstein geodesic. Thus the Wasserstein geodesic density is a characteristic function. $\qquad\square$

6. Displacement interpolants as weak limits

6.1. **Proof of Theorem 1.5.** Next we supply the proof of Theorem 1.5. We will accomplish this in two steps, first dealing with the case that the endpoint densities ρ_0, ρ_1 are characteristic functions of bounded open sets. To extend this result to the general case, we will make use of fundamental results on stability of optimal transport plans from [3] and [52].

Proposition 6.1. *Let Ω_0, Ω_1 be bounded open sets of equal volume. Let (ρ, v) be the density and transport velocity determined by the unique Wasserstein geodesic (displacement interpolant) that connects the uniform measures on Ω_0 and Ω_1 as described in section 2.*

Then, as $\varepsilon \to 0$, the weak solutions $(\rho^{\varepsilon}, v^{\varepsilon}, p^{\varepsilon})$ associated to the Euler sprays of Theorem 1.1 converge to $(\rho, v, 0)$, and (ρ, v) is a weak solution to the pressureless Euler system (1.13)–(1.14). The convergence holds in the following sense: $p^{\varepsilon} \to 0$ uniformly, and

$$\rho^{\varepsilon} \overset{\star}{\rightharpoonup} \rho, \qquad \rho^{\varepsilon} v^{\varepsilon} \overset{\star}{\rightharpoonup} \rho v, \qquad \rho^{\varepsilon} v^{\varepsilon} \otimes v^{\varepsilon} \overset{\star}{\rightharpoonup} \rho v \otimes v, \tag{6.1}$$

weak-$\star$ in L^{∞} on $\mathbb{R}^d \times [0, 1]$.

As our first step toward proving this result, we describe the bounds on pressure and velocity that come from the construction of the Euler sprays constructed above, for any given $\varepsilon \in (0, 1)$.

Lemma 6.2. *Let $(Q^{\varepsilon}, \phi^{\varepsilon}, p^{\varepsilon})$, $0 < \varepsilon < 1$, denote the Euler sprays constructed in the proof of Theorem 1.1, and let $X^{\varepsilon} \colon \Omega_0^{\varepsilon} \times [0, 1] \to \mathbb{R}^d$ denote the associated flow maps, which satisfy*

$$\dot{X}^{\varepsilon}(z, t) = \nabla\phi^{\varepsilon}(X^{\varepsilon}(z, t), t), \quad (z, t) \in \Omega_0^{\varepsilon} \times [0, 1],$$

with $X^\varepsilon(z,0) = z$. *Then for some $\hat{K} > 0$ independent of ε, we have*

$$0 \le p^\varepsilon(x,t) \le \hat{K}\varepsilon \tag{6.2}$$

for all $(x,t) \in Q^\varepsilon$, and

$$|X^\varepsilon(z,t) - T_t(z)| + |\dot{X}^\varepsilon(z,t) - \dot{T}_t(z)| \le \hat{K}\sqrt{\varepsilon} \tag{6.3}$$

for all $(z,t) \in \Omega_0^\varepsilon \times [0,1]$, where $(z,t) \mapsto T_t(z)$ is the flow map from (2.1) for the Wasserstein geodesic.

Proof. By the pressure bound for individual droplets in (3.47) together with the second condition in (4.4), we have the pointwise bound

$$0 \le p^\varepsilon \le K_0\varepsilon, \qquad K_0 = K_1^2 d. \tag{6.4}$$

Next consider the velocity. The boosted elliptical Euler droplet that transports B_i to $S^\varepsilon(B_i)$ is translated by x_i, and boosted by the vector

$$b_i := (1 + \varepsilon)T(x_i) - x_i = \dot{T}_t(x_i) + \varepsilon T(x_i). \tag{6.5}$$

In this "i-th droplet," the velocity satisfies, by the estimate (3.49),

$$|\nabla\phi^\varepsilon - b_i| = |v^\varepsilon - b_i| \le K_0\varepsilon. \tag{6.6}$$

Now the particle velocity for the Euler spray compares to that of the Wasserstein geodesic according to

$$
\begin{aligned}
|\dot{X}^\varepsilon(z,t) - \dot{T}_t(z)| &\le |\dot{X}^\varepsilon - b_i| + |b_i - \dot{T}_t(z)| \\
&\le K_0\varepsilon + \varepsilon|T(x_i)| + r_i \max_j |\lambda_j(z) - 1| \\
&\le K_0\varepsilon + K_1\varepsilon + \sqrt{K_0\varepsilon} \le K_2\sqrt{\varepsilon}.
\end{aligned}
\tag{6.7}
$$

(Here $\lambda_j(z)$ denote the eigenvalues of $DT(z) = \nabla\psi(z)$, and we use the fact that $|\lambda_j(z)-1| \le \overline{\lambda}_i$ together with (4.4).) Upon integration in time we obtain both bounds in (6.3). $\qquad\square$

Proof of Proposition 6.1. Now, let (ρ, v) be the density and velocity of the particle paths for the Wasserstein geodesic, from (2.8) and (2.4). To prove $\rho^\varepsilon \overset{\star}{\rightharpoonup} \rho$ weak-$\star$ in L^∞, it suffices to show that as $\varepsilon \to 0$,

$$\int_0^1 \int_{\mathbb{R}^d} (\rho^\varepsilon - \rho)q \, dx \, dt \to 0, \tag{6.8}$$

for every smooth test function $q \in C_c^\infty(\mathbb{R}^d \times [0,1], \mathbb{R})$. Changing to Lagrangian variables using X^ε for the term with $\rho^\varepsilon = \mathbb{1}_{Q^\varepsilon}$ and T_t for the term with ρ, the left-hand side becomes

$$\int_0^1 \int_{\Omega_0} (q(X^\varepsilon(z,t),t) - q(T_t(z),t)) \, dz \, dt. \tag{6.9}$$

Evidently this does approach zero as $\varepsilon \to 0$, due to (6.3).

Next, we claim $\rho^\varepsilon v^\varepsilon \overset{\star}{\rightharpoonup} \rho v$ weak-$\star$ in L^∞. Because these quantities are uniformly bounded, it suffices to show that as $\varepsilon \to 0$,

$$\int_0^1 \int_{\mathbb{R}^d} (\rho^\varepsilon v^\varepsilon - \rho v) \cdot \tilde{v} \, dx \, dt \to 0 \tag{6.10}$$

for each $\tilde{v} \in C_c^\infty(\mathbb{R}^d \times [0,1], \mathbb{R}^d)$. Changing variables in the same way, the left-hand side becomes

$$\int_0^1 \int_{\Omega_0} \left(\dot{X}^\varepsilon(z,t) \cdot \tilde{v}(X^\varepsilon(z,t),t) - \dot{T}_t(z) \cdot \tilde{v}(T_t(z),t) \right) dz \, dt. \tag{6.11}$$

But because $\tilde{v}$ is smooth and due to the bounds in (6.3), this also tends to zero as $\varepsilon \to 0$.

It remains to prove $\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon \overset{\star}{\rightharpoonup} \rho v \otimes v$ weak-$\star$ in L^∞. Considering the terms componentwise, the proof is extremely similar to the previous steps. This finishes the proof of Theorem 1.5. $\quad\square$

To generalize Proposition 6.1 to handle general densities $\rho_0, \rho_1 \colon \mathbb{R}^d \to [0, 1]$, we will use a double approximation argument, comparing Euler sprays to optimal Wasserstein geodesics for open sets whose characteristic functions approximate ρ_0, ρ_1 in the sense of Lemma 5.1, then comparing these to the Wasserstein geodesic that connects ρ_0 to ρ_1. We prove weak-star convergence for the second comparison by extending the results from [3] and [52] on weak-$\star$ stability of transport plans to establish weak-$\star$ stability of Wasserstein geodesic flows (in the Eulerian framework).

Proposition 6.3. *Let (ρ, v) be the density and transport velocity determined by the Wasserstein geodesic that connects the measures with given densities $\rho_0, \rho_1 \colon \mathbb{R}^d \to [0, 1]$, measurable with compact support such that*

$$\int_{\mathbb{R}^d} \rho_0 = \int_{\mathbb{R}^d} \rho_1 \,.$$

Let $(\bar{\rho}^k, \bar{v}^k)$ be the density and transport velocity determined by the Wasserstein geodesic that connects the measures with densities $\mathbb{1}_{\Omega_0^k}$ and $\mathbb{1}_{\Omega_1^k}$, where Ω_0^k, Ω_1^k, $k = 1, 2, \ldots$, are bounded open sets such that $|\Omega_0^k| = |\Omega_1^k| = \int_{\mathbb{R}^d} \rho_0$ and

$$d_\infty(\rho_0, \mathbb{1}_{\Omega_0^k}) + d_\infty(\rho_1, \mathbb{1}_{\Omega_1^k}) \to 0 \quad \text{as } k \to \infty.$$

Then

$$\bar{\rho}^k \overset{\star}{\rightharpoonup} \rho, \qquad \bar{\rho}^k \bar{v}^k \overset{\star}{\rightharpoonup} \rho v, \qquad \bar{\rho}^k \bar{v}^k \otimes \bar{v}^k \overset{\star}{\rightharpoonup} \rho v \otimes v, \tag{6.12}$$

weak-$\star$ in L^∞ on $\mathbb{R}^d \times [0, 1]$. Consequently $0 \le \rho \le 1$ a.e. in $\mathbb{R}^d \times [0, 1]$.

Proof. Let π (resp. π^k) be the optimal transport plan connecting ρ_0 to ρ_1 (resp. $\mathbb{1}_{\Omega_0^k}$ to $\mathbb{1}_{\Omega_0^k}$). These plans take the form $\pi = (\mathrm{id} \times T)_\sharp \rho_0$ (resp. $\pi^k = (\mathrm{id} \times T^k)_\sharp \mathbb{1}_{\Omega_0^k}$) where T (resp. T^k) is the Brenier map. Then by [52, Theorem 5.20] or [3, Proposition 7.1.3], we know that π^k converges weak-$\star$ to π in the space of Radon measures on $\mathbb{R}^d \times \mathbb{R}^d$.

We will prove that $\bar{\rho}^k \bar{v}^k \overset{\star}{\rightharpoonup} \rho v$; it will be clear that the remaining results in (6.12) are similar. Let $\varphi \colon \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$ be smooth with compact support. We claim that

$$\int_0^1 \int_{\mathbb{R}^d} \bar{\rho}^k \bar{v}^k \varphi(x, t) \, dx \, dt \to \int_0^1 \int_{\mathbb{R}^d} \rho v \varphi(x, t) \, dx \, dt. \tag{6.13}$$

Recall from (2.3) that the geodesic velocities $\bar{v}^k(x, t)$ satisfy

$$\bar{v}^k((1 - t)z + tT^k(z), t) = T^k(z) - z.$$

Hence the left-hand side of (6.13) can be written in the form

$$\int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} (y - z)\varphi((1 - t)z + ty, t) \, d\pi^k(z, y) \, dt = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(z, y) \, d\pi^k(z, y) \,,$$

where

$$\psi(z, y) = \int_0^1 (y - z)\varphi((1 - t)z + ty, t) \, dt.$$

Due to the fact that $\pi^k \overset{\star}{\rightharpoonup} \pi$ and all these measures are supported in a fixed compact set, as $k \to \infty$ we obtain the limit

$$
\begin{aligned}
\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(z, y)\, d\pi(z, y) &= \int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} (y - z)\varphi((1 - t)z + ty, t)\, d\pi(z, y)\, dt \\
&= \int_0^1 \int_{\mathbb{R}^d} (T(z) - z)\varphi(T_t(z), t)\, \rho_0(z)\, dz\, dt\,,
\end{aligned}
\tag{6.14}
$$

where $T_t(z) = (1 - t)z + tT(z)$. To conclude the proof, we need to recall how ρ and v are determined by displacement interpolation, in a precise technical sense for the present case when ρ_0 and ρ_1 lack smoothness. Indeed, from the results in Lemma 5.29 and Proposition 5.30 of [45] (also see Proposition 8.1.8 of [3]), we find that with the notation

$$
x_t(z, y) = (1 - t)z + ty\,,
$$

the measure μ_t with density ρ_t is given by the pushforward

$$
\mu_t = (x_t)_\sharp \pi = (x_t)_\sharp (\mathrm{id} \times T)_\sharp \mu_0 = (T_t)_\sharp (\rho_0\, dz)\,,
\tag{6.15}
$$

and the transport velocity is given by

$$
v(x, t) = (T - \mathrm{id}) \circ (T_t)^{-1}(x).
\tag{6.16}
$$

Thus we may use T_t to push forward the measure $\rho_0(z)\, dz = d\mu_0(z)$ in (6.14) to write, for each $t \in [0, 1]$,

$$
\int_{\mathbb{R}^d} (T(z) - z)\varphi(T_t(z), t)\, \rho_0(z)\, dz = \int_{\mathbb{R}^d} v(x, t)\varphi(x, t)\, \rho_t(x)\, dx.
\tag{6.17}
$$

It then follows that (6.13) holds, as desired. $\qquad\square$

Remark 6.4. The validity of the continuity equation (1.13) for (ρ, v) is well known and established in several sources, e.g., see [51, Theorem 5.51] or [45, Chapter 5]. The step above going from (6.14) to (6.17) provides an answer to the related exercise 5.52 in [51]. We are not aware, however, of any source where the momentum equation (1.14) for (ρ, v) is explicitly and rigorously justified.

Proof of Theorem 1.5. Let us now finish the proof of Theorem 1.5. As any ball in $L^\infty(\mathbb{R}^d \times [0, 1])$ is metrizable [22, Theorem V.5.1], we may fix a metric d in a large enough ball, and select $\varepsilon_k > 0$ for each $k \in \mathbb{N}$ such that for the quantities $(\rho^k, v^k, p^k) := (\rho^{\varepsilon_k}, v^{\varepsilon_k}, p^{\varepsilon_k})$ coming from the Euler sprays of Proposition 6.1, the components of ρ^k, $\rho^k v^k$ and $\rho^k v^k \otimes v^k$ approximate the corresponding quantities $\bar\rho^k$, $\bar\rho^k \bar v^k$ and $\bar\rho^k \bar v^k \otimes \bar v^k$ that appear in Proposition 6.3, within distance $1/k$. That is,

$$
\max\left(d(\rho^k, \bar\rho^k),\ d(\rho^k v_i^k, \bar\rho^k \bar v_i^k),\ d(\rho^k v_i^k v_j^k, \bar\rho^k \bar v_i^k \bar v_j^k) \right) < \frac{1}{k}.
$$

Then the convergences asserted in (1.15) evidently hold. $\qquad\square$

6.2. Convergence in the stronger TL^p sense.

The convergences described in Propositions 6.1 and 6.3 and Theorem 1.5 actually hold in a stronger sense related to the TL^p metric that was introduced in [29] to measure differences between functions defined with respect to different measures. We recall the definition of the TP^p metric and a number of its properties in appendix A.

Our first result strengthens the conclusions drawn in Proposition 6.1.

Proposition 6.5. *Under the same hypotheses as Proposition 6.1 and Lemma 6.2, the map that associates $T_t(x)$ with $X_t^\varepsilon(x) = X^\varepsilon(x,t)$, defined by $Y_t^\varepsilon = X_t^\varepsilon \circ T_t^{-1}$, pushes forward ρ_t to ρ_t^ε and we have the estimate*

$$|x - Y_t^\varepsilon(x)| + |v_t(x) - v_t^\varepsilon(Y_t^\varepsilon(x))| \leq \hat{K}\sqrt{\varepsilon} \tag{6.18}$$

for all $t \in [0,1]$ and ρ_t-a.e. x. By consequence, for all $t \in [0,1]$ we have

$$d_{TL^\infty}((\rho_t, v_t), (\rho_t^\varepsilon, v_t^\varepsilon)) \leq \hat{K}\sqrt{\varepsilon}.$$

This result follows immediately from estimate (6.3) of Lemma 6.2. Expressed in terms of couplings, using the transport plan that associates $X^\varepsilon(z,t)$ with $T_t(z)$ given by the pushforward

$$\pi^\varepsilon = (X^\varepsilon(\,\cdot\,,t) \times T_t)_\sharp \rho_0,$$

the estimate (6.3) implies that for π^ε-a.e. (x,y), for all $t \in [0,1]$ we have

$$|x - y| + |v_\varepsilon(x,t) - v(y,t)| \leq \hat{K}\sqrt{\varepsilon}.$$

Next we improve the conclusions of Proposition 6.3 by invoking the results of Proposition A.5 in the Appendix.

Proposition 6.6. *Under the assumptions of Proposition 6.3, there exist transport maps $\bar{S}^k$ that push forward ρ_0 to $\bar{\rho}_0^k = \mathbb{1}_{\Omega_0^k}$, such that*

$$\| \mathrm{id} - \bar{S}^k \|_{L^\infty(\rho_0)} \to 0 \quad as\ k \to \infty, \tag{6.19}$$

and for any such sequence of transport maps, the maps given by

$$\bar{S}_t^k = T_t^k \circ \bar{S}^k \circ T_t^{-1}$$

push forward ρ_t to $\bar{\rho}_t^k$ and satisfy, as $k \to \infty$,

$$\sup_{t \in [0,1]} \int |x - \bar{S}_t^k(x)|^2 \rho_t(x)\, dx \to 0, \tag{6.20}$$

$$\sup_{t \in [0,1]} \int |v_t(x) - \bar{v}_t^k(\bar{S}_t^k(x))|^2 \rho_t(x)\, dx \to 0, \tag{6.21}$$

$$\sup_{t \in [0,1]} \int |(v_t \otimes v_t)(x) - (\bar{v}_t^k \otimes \bar{v}_t^k)(\bar{S}_t^k(x))| \rho_t(x)\, dx \to 0. \tag{6.22}$$

Proof. The existence of the maps $\bar{S}^k$ follow from the fact that $d_\infty(\rho_0, \bar{\rho}_0^k) \to 0$ as $k \to \infty$, and existence of optimal transport maps for these distances, see Theorem 3.24 of [45]. The remaining statements follow from Proposition A.5 in the Appendix. $\square$

By combining the last two results, we obtain the following improvement of the conclusions of Theorem 1.5.

Theorem 6.7. *Under the same hypotheses as Theorem 1.5, we have the following. Let ρ_0^k, $\bar{S}^k$ be as in Proposition 6.6, and let (ρ^k, v^k, p^k) be solutions of the Euler system (1.13)–(1.14) coming from the Euler sprays of Theorem 1.1, chosen as in the proof of Theorem 1.5. Define*

$$S_t^k = X_t^k \circ S^k \circ T_t^{-1}. \tag{6.23}$$

Then $(S_t^k)_\sharp(\rho_t \, dx) = \rho_t^k \, dx$, *and*

$$\sup_{t\in[0,1]} \int |x - S_t^k(x)|^2 \, \rho_t(x) \, dx \to 0 \,, \tag{6.24}$$

$$\sup_{t\in[0,1]} \int |v_t(x) - v_t^k(S_t^k(x))|^2 \, \rho_t(x) \, dx \to 0 \,, \tag{6.25}$$

$$\sup_{t\in[0,1]} \int |(v_t \otimes v_t)(x) - (v_t^k \otimes v_t^k)(S_t^k(x))| \, \rho_t(x) \, dx \to 0. \tag{6.26}$$

Proof. Using Proposition 6.5, we can deduce that

$$\sup_{t\in[0,1]} \int |S_t^k(x) - \bar{S}_t^k(x)|^2 \, \rho_t(x) \, dx \to 0, \tag{6.27}$$

$$\sup_{t\in[0,1]} \int |v_t^k(S_t^k(x)) - \bar{v}_t^k(\bar{S}_t^k(x))|^2 \, \rho_t(x) \, dx \to 0 \,, \tag{6.28}$$

$$\sup_{t\in[0,1]} \int |(v_t^k \otimes v_t^k)(S_t^k(x)) - (\bar{v}_t^k \otimes \bar{v}_t^k)(\bar{S}_t^k(x))| \, \rho_t(x) \, dx \to 0 \,. \tag{6.29}$$

Combining these with the results of Proposition 6.6 finishes the proof. □

7. Relaxed least-action principles for two-fluid incompressible flow and displacement interpolation

In a series of papers that includes [8, 10, 11, 12, 13, 14], Brenier studied Arnold's least-action principles for incompressible Euler flows by introducing relaxed versions that involve convex minimization problems, for which duality principles yield information about minimizers and/or minimizing sequences.

In this section, we describe a simple variant of Brenier's theories that provides a relaxed least-action principle for a two-fluid incompressible flow in which one fluid can be taken as vacuum. For this degenerate case we show that the displacement interpolant (Wasserstein geodesic) provides the unique minimizer. Moreover, the concatenated Euler sprays that we constructed to prove Theorem 1.3 provide a minimizing sequence for the relaxed problem.

We remark that Lopes Filho et al. [37] studied a variant of Brenier's relaxed least-action principles for variable density incompressible flows. As we indicate below, their formulation is closely related to ours, but it requires fluid density to be positive everywhere.

7.1. **Kinetic energy and least-action principle for two fluids.** We recall that a key idea behind Brenier's work is that kinetic energy can be reformulated in terms of convex duality, based on the idea that kinetic energy is a jointly convex function of density and momentum. In order to handle possible vacuum, we extend this idea in the following way. Let $\hat{\varrho} \geq 0$ be a constant (representing the density of one fluid). We define $\hat{K}_{\hat{\varrho}}$ as the Legendre transform of the indicator function of the paraboloid

$$P_{\hat{\varrho}} = \left\{ (a, b) \in \mathbb{R} \times \mathbb{R}^d : a + \tfrac{1}{2}\hat{\varrho}|b|^2 \leq 0 \right\}, \tag{7.1}$$

given for $(x, y) \in \mathbb{R} \times \mathbb{R}^d$ by

$$\hat{K}_{\hat{\varrho}}(x, y) = \sup_{(a,b)\in P_{\hat{\varrho}}} ax + b \cdot y. \tag{7.2}$$

We find the following.

Lemma 7.1. *Let $\hat{\varrho} \geq 0$ and define $\hat{K}$ by (7.2). Then $\hat{K}_{\hat{\varrho}}$ is convex, and*

$$\hat{K}_{\hat{\varrho}}(x,y) = \begin{cases} \dfrac{1}{2}\dfrac{|y|^2}{\hat{\varrho}x} & \text{if } y \neq 0 \text{ and } \hat{\varrho}x > 0, \\ 0 & \text{if } y = 0 \text{ and } x \geq 0, \\ +\infty & \text{else.} \end{cases} \tag{7.3}$$

In case $\hat{\varrho} > 0$, we have the scaling property

$$\hat{K}_{\hat{\varrho}}(\hat{\varrho}x, \hat{\varrho}y) = \hat{K}_1(x,y). \tag{7.4}$$

The proof of this lemma is a straightforward calculation based on cases that we leave to the reader. We emphasize that $\hat{\varrho} = 0$ is allowed. Indeed, for $\hat{\varrho} = 0$, $\hat{K}_0$ reduces to the indicator function for the closed half-line

$$\{(x,y) : y = 0, \; x \geq 0\}.$$

Suppose $c \in \mathbb{R}$ represents the 'concentration' of one fluid and $m \in \mathbb{R}^d$ represents the 'momentum' of this fluid, at some point in the flow. If $\hat{K}_{\hat{\varrho}}(c,m) < +\infty$, then $c \geq 0$ and $m = \hat{\varrho}cv$ for some 'velocity' $v \in \mathbb{R}^d$ which satisfies

$$\hat{K}_{\hat{\varrho}}(c,m) = \frac{1}{2}\hat{\varrho}c|v|^2. \tag{7.5}$$

Next we begin to describe our relaxed least-action principle for two-fluid incompressible flow. Consider fluid flow inside a large box for unit time, with

$$\Omega = [-L, L]^d, \qquad Q = \Omega \times [0,1].$$

Let $\hat{\varrho}_i$, $i = 0, 1$, be constants representing the densities of two fluids, with $\hat{\varrho}_1 > \hat{\varrho}_0 \geq 0$. (More fluids could be considered, but we have no reason to do so at this point.) Next we let $c_i(x,t)$, $i = 0, 1$, represent the concentration of fluid i at the point $(x,t) \in Q$. For classical flows, the fluids should occupy non-overlapping regions of space-time, meaning that the concentrations are characteristic functions $c_i = \mathbb{1}_{Q_i}$ with

$$Q_i = \bigcup_{t \in [0,1]} \Omega_{i,t} \times \{t\}, \qquad Q = \bigsqcup_i Q_i. \tag{7.6}$$

The requirement $c_i(x,t) \in \{0,1\}$ will be relaxed, however, to the requirement $c_i(x,t) \in [0,1]$. This provides a convex restriction that heuristically allows 'mixtures' to form (by taking weak limits, say).

Writing $m_i(x,t)$ for the momentum of fluid i at $(x,t) \in Q$, the action to be minimized is the total kinetic energy

$$K(c,m) = \sum_i \int_Q \hat{K}_{\hat{\varrho}_i}(c_i, m_i)\, dx\, dt, \tag{7.7}$$

subject to three types of constraints—incompressibility, transport that conserves the total mass of each fluid, and endpoint conditions. We require

$$\sum_i c_i = 1 \qquad \text{a.e. in } Q, \tag{7.8}$$

$$\hat{\varrho}_i \partial_t c_i + \nabla \cdot m_i = 0 \qquad \text{in } Q \text{ for all } i, \tag{7.9}$$

$$\frac{d}{dt}\int_\Omega \hat{\varrho}_i c_i = 0 \qquad \text{for } t \in [0,1] \text{ for all } i, \tag{7.10}$$

and fixed endpoint conditions at $t = 0, 1$:

$$c_i(x, 0) = c_{i0}(x), \qquad c_i(x, 1) = c_{i1}(x), \tag{7.11}$$

where $c_{i0}, c_{i1} \in L^\infty(\Omega, [0, 1])$ are prescribed for each i in a fashion compatible with the constraints (7.8) and (7.10). For unmixed, classical flows, these data take the form of characteristic functions:

$$c_{i0}(x) = \mathbb{1}_{\Omega_{i,0}}, \qquad c_{i1}(x) = \mathbb{1}_{\Omega_{i,1}}. \tag{7.12}$$

The constraints above are more properly written and collected in the following weak form, required to hold for all test functions p, ϕ_i in the space $C^0(Q)$ of continuous functions on Q, having $\partial_t \phi_i$, $\nabla_x \phi_i$ also continuous on Q, for $i = 0, 1$:

$$0 = \int_Q p - \sum_i \int_Q \left((p + \hat{\varrho}_i \partial_t \phi_i) c_i + \nabla_x \phi_i \cdot m_i \right)$$
$$- \sum_i \hat{\varrho}_i \left(\int_\Omega c_{i1}(x) \phi_i(x, 1) \, dx - \int_\Omega c_{i0}(x) \phi_i(x, 0) \, dx \right). \tag{7.13}$$

Let us now describe precisely the set $\mathcal{A}_K$ of functions (c, m) that we take as admissible for the relaxed least-action principle. We require $c_i \in L^\infty(Q, [0, 1])$. As we shall see below, it is natural to require that the path

$$t \mapsto c_i(\cdot, t) \, dx$$

is weak-$\star$ continuous into the space of signed Radon measures on Ω, and that $m_i = \hat{\varrho}_i c_i v_i$ with $v_i \in L^2(Q, c_i)$ if $\hat{\varrho}_i > 0$. Then the action in (7.7) becomes

$$K(c, m) = \sum_i \int_Q \frac{1}{2} \hat{\varrho}_i c_i |v_i|^2. \tag{7.14}$$

When $\hat{\varrho}_0 = 0$, we require $m_0 = 0$ a.e., since this condition is necessary to have $K(c, m) < \infty$ in (7.7). In this case we have

$$K(c, m) = \int_Q \frac{1}{2} \hat{\varrho}_1 c_1 |v_1|^2, \tag{7.15}$$

and the constraints on c_0 from (7.13) reduce simply to the requirement that $c_0 = 1 - c_1$.

We let $\mathcal{A}_K$ denote the set of functions (c, m) that have the properties required in the previous paragraph and satisfy the weak-form constraints (7.13). Our *relaxed least-action two-fluid problem* is to find $(\bar{c}, \bar{m}) \in \mathcal{A}_K$ with

$$K(\bar{c}, \bar{m}) = \inf_{(c,m) \in \mathcal{A}_K} K(c, m). \tag{7.16}$$

A formal description of classical critical points of the action in (7.16), subject to the constraints in (7.13), and with each c_i a characteristic function of smoothly deforming sets as in (7.6), will lead to classical Euler equations for two-fluid incompressible flow, along the lines of our calculation in section 3, which applies in the case $\hat{\varrho}_0 = 0$.

We will discuss in subsection 7.3 below how the least-action problem (7.16) is equivalent to a weaker formulation in which (c_i, m_i) are only taken to be signed Radon measures on Q. When $\hat{\varrho}_0 > 0$, this weaker formulation may be compared directly to the variant of Brenier's least-action principle for variable-density flows, as treated by Lopes Filho et al. [37], in the two-fluid special case.

7.2. **Wasserstein geodesics are minimizers of relaxed action.** We focus now on the case $\hat{\varrho}_0 = 0$, and take $\hat{\varrho}_1 = 1$ for convenience.

Theorem 7.2. *Suppose $\hat{\varrho}_0 = 0$, and ρ_0, $\rho_1 \colon \mathbb{R}^d \to [0,1]$ are measurable with compact support and equal integrals in $(-L, L)^d$. Then the relaxed least-action problem in* (7.16), *with endpoint data determined by $c_{10} = \rho_0$, $c_{11} = \rho_1$, has a unique solution $(\bar{c}, \bar{m})$ given inside Q by*

$$\bar{c}_1 = \rho, \quad \bar{m}_1 = \rho v, \qquad \bar{c}_0 = 1 - \rho, \quad \bar{m}_0 = 0, \tag{7.17}$$

in terms of the displacement interpolant (ρ, v) (described in section 2 and (6.15),(6.16)*) between the measures μ_0 and μ_1 with densities ρ_0 and ρ_1.*

Proof. It is clear from the pushforward description of (6.15)–(6.16) that $(\bar{c}, \bar{m})$ as defined in (7.17) belongs to the admissible set $\mathcal{A}_K$, due to the facts that (i) $0 \le \rho \le 1$ by the last assertion of Proposition 6.3 and (ii) the support of (ρ, v) is compactly contained in Ω due to (2.1) and (6.16). We then have, since $\hat{\varrho}_0 = 0$,

$$K(\bar{c}, \bar{m}) = \int_Q \frac{1}{2} \rho |v|^2 = \int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} \rho |v|^2 \, dx \, dt$$

because the pair (ρ, v) is defined on $\mathbb{R}^d \times [0,1]$ and is zero outside Q. But similarly, for *any* admissible pair $(c, m) \in \mathcal{A}_K$, if we extend (c_1, v_1) by zero outside Q, we have

$$K(c, m) = \int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} c_1 |v_1|^2 \, dx \, dt$$

and (c_1, v_1) determines a narrowly continuous path of measures $t \mapsto \mu_t = c_1 \, dx$ on $\mathbb{R}^d$ with $v \in L^2(\mu)$ that satisfies the continuity equation. It is known that (ρ, v) minimizes this expression over this wider class of paths of measures, due to the characterization of Wasserstein distance by Benamou and Brenier [6], see [51, Thm. 8.1]. By consequence we obtain that $(\bar{c}, \bar{m})$ as defined by (7.17) is indeed a minimizer of the relaxed least-action problem (7.16).

Because the Wasserstein minimizing path is unique (as discussed in section 2), it follows that any minimizer in (7.16) must be given as in (7.17). $\qquad \square$

Proposition 7.3. *The family of incompressible flows (concatenated Euler sprays) given for all small $\varepsilon > 0$ by Theorem 1.3(b) determine a minimizing family $(c^\varepsilon, m^\varepsilon)$ for the relaxed least-action principle* (7.16) *according to*

$$c_1^\varepsilon = \rho^\varepsilon, \quad m_1^\varepsilon = \rho^\varepsilon v^\varepsilon, \qquad c_0^\varepsilon = 1 - \rho^\varepsilon, \quad m_0^\varepsilon = 0.$$

That is, $(c^\varepsilon, m^\varepsilon) \in \mathcal{A}_K$ and $\lim_{\varepsilon \to 0} K(c^\varepsilon, m^\varepsilon) = \inf_{\mathcal{A}_K} K(c, m)$.

We remark that in case the endpoint data are classical (unmixed) as in (7.12), we are not able to use the individual Euler sprays that we construct for the proof of Theorem 1.1 to obtain a similar result. The reason is that the target set $\Omega_{1,1} = \Omega_1$ is not hit exactly by our Euler sprays, and this means that the corresponding concentration-momentum pair $(c^\varepsilon, m^\varepsilon) \notin \mathcal{A}_K$ because it would not satisfy the constraint (7.13) as required. We conjecture, however, that for small enough $\varepsilon > 0$, Euler sprays can be constructed that hit an arbitrary target shape Ω_1 (up to a set of measure zero). If that is the case, these Euler sprays would similarly provide a minimizing family for the relaxed least-action principle (7.16).

7.3. **Extended relaxed least-action principle.** In this subsection we discuss an extenstion of the least-action principle (7.16) which facilitates comparison with previous works. Our extension involves expanding the class of admissible concentration-momentum pairs, and is a kind of hybrid of Brenier's 'homogenized vortex sheet' formulation in [11] and the variable-density formulation in [37] for geodesic flow in the diffeomorphism group. The extended formulation reduces, however, to the formulation in (7.16) whenever the action is finite—see Proposition 7.5 below.

The formulations of [11, 12, 37] were designed to make it possible to establish existence of minimizers through convex analysis. The key is to express kinetic energy through duality. We start with the space $C^0(Q)$ of continuous functions on $Q = [-L, L]^d \times [0, 1]$, whose dual is the space $\mathcal{M}(Q)$ of signed Radon measures. The duality pairing is

$$\langle F, c \rangle = \int_Q F \, dc \qquad \text{for } F \in C^0(Q), \, c \in \mathcal{M}(Q).$$

Similarly we write $\langle G, m \rangle = \int_Q G \cdot dm$ for $G \in C^0(Q)^d$ and $m \in \mathcal{M}(Q)^d$.

Next, let $\hat{\varrho} \geq 0$ be a constant representing fluid density. We let

$$\hat{E} = C^0(Q) \times C^0(Q)^d, \quad \hat{E}^* = \mathcal{M}(Q) \times \mathcal{M}(Q)^d,$$

and define $\hat{\mathcal{K}}_{\hat{\varrho}} : \hat{E}^* \to \mathbb{R}$ as the Legendre transform of the indicator function of the parabolic set

$$\mathcal{P}_{\hat{\varrho}} = \{(F, G) \in E : F + \frac{1}{2}\hat{\rho}|G|^2 \leq 0 \text{ in } Q\}, \tag{7.18}$$

given for $(c, m) \in \hat{E}^*$ by

$$\hat{\mathcal{K}}_{\hat{\varrho}}(c, m) = \sup_{(F,G) \in \mathcal{P}_{\hat{\varrho}}} \langle F, c \rangle + \langle G, m \rangle. \tag{7.19}$$

(To compare with [37, eq. (3.8)] it may help to note $\hat{\varrho}\mathcal{P}_{\hat{\varrho}} = \mathcal{P}_1$ when $\hat{\varrho} > 0$.)

The following result follows from [11, Proposition 3.4] in the case $\hat{\varrho} > 0$, and is straightforward to show in the case $\hat{\varrho} = 0$, when the conclusion entails $m = 0$.

Proposition 7.4. *Let $\hat{\varrho} \geq 0$, and let $(c, m) \in \hat{E}^*$. If $\hat{K}_{\hat{\varrho}}(c, m) < \infty$, then c is a nonnegative measure and m is absolutely continuous with respect to c, with Radon-Nikodým derivative $\hat{\varrho}v$ where $v \in L^2(Q, c)$, and*

$$\hat{\mathcal{K}}_{\hat{\varrho}}(c, m) = \int_Q \frac{1}{2}\hat{\varrho}|v|^2 \, dc.$$

Our reformulated least-action problem may now be specified, as follows. Let $\hat{\varrho}_1 > \hat{\varrho}_0 \geq 0$. For $(c, m) \in E^* = \hat{E}^* \times \hat{E}^*$ we write

$$c = (c_0, c_1), \quad m = (m_0, m_1),$$

and we define

$$\mathcal{K}(c, m) = \sum_i \mathcal{K}_{\hat{\varrho}_i}(c_i, m_i). \tag{7.20}$$

We introduce the class $\hat{\mathcal{A}}_{\mathcal{K}}$ of admissible pairs $(c, m) \in E^*$ that satisfy the same weak-form constraints (7.13) as before (with c_i, m_i replaced respectively by dc_i, dm_i). The extended relaxed least-action problem is to find $(\hat{c}, \hat{m}) \in \hat{\mathcal{A}}_{\mathcal{K}}$ such that

$$\mathcal{K}(\hat{c}, \hat{m}) = \inf_{(c,m) \in \hat{\mathcal{A}}_{\mathcal{K}}} \mathcal{K}(c, m). \tag{7.21}$$

This form of the relaxed least-action problem may be compared rather directly with the homogenized vortex sheet model of Brenier [11] and with the variable-density model of Lopes

Filho et al. [37]. Both of these models deal with the endpoint problem for diffeomorphisms rather than mass distributions as is done here. Brenier's model involves a sum over 'phases' as in our model (7.20), but the fluid density in each phase is the same. The variable-density model of [37] allows for mixture density (called c, corresponding to $\hat{\varrho}c$ here) to depend upon both Eulerian and Lagrangian spatial coordinates (called x and a respectively), similar to the formulation in [12].

In both [11] and [37] as well as related works for relaxed least-action principles formulated in a space of measures, the existence of minimizers is established by using the Fenchel-Rockafellar theorem from convex analysis. One expresses the objective function corresponding to $\mathcal{K}(c, m)$ as a sum of Legendre transforms of indicator functions of two sets, corresponding here to the set $\mathcal{P}_{\hat{\varrho}}$ in (7.18) and to another set that accounts for the constraints in (7.13). We do not pursue this analysis as it is outside the scope of this paper. In any case, for the degenerate case $\hat{\varrho}_0 = 0$ that is most relevant to the rest of this paper, existence of a unique minimizer follows from Theorem 7.2 above and Proposition 7.5 below.

We claim that the relaxed least-action problem (7.21) always reduces to the previous problem (7.16), due to the following fact.

Proposition 7.5. *Suppose $(c, m) \in \hat{\mathcal{A}}_{\mathcal{K}}$ and $\mathcal{K}(c, m) < \infty$. Then for some $(\bar{c}, \bar{m}) \in \mathcal{A}_K$ we have $\mathcal{K}(c, m) = K(\bar{c}, \bar{m})$ and*

$$dc_i = \bar{c}_i \, dx \, dt, \quad dm_i = \bar{m}_i \, dx \, dt, \quad i = 0, 1. \tag{7.22}$$

Consequently, the infimum in (7.21) is the same as that in (7.16).

Proof. To prove this result, we first invoke Proposition 7.4 to infer that c_i is a nonnegative measure and m_i is absolutely continuous with respect to c_i for $i = 0, 1$. Next we note that $\sum_i c_i = 1$ by taking $\phi_i = 0$ and p arbitrary in (7.13). Hence the representation in (7.22) holds with $\bar{c}_i \in L^\infty(Q, [0, 1])$ and $m_i = \hat{\varrho}_i \bar{c}_i v_i$ with $v_i \in L^2(Q, \bar{c}_i)$.

Finally, we claim $t \mapsto \bar{c}_i(\cdot, t)$ is weak-$\star$ continuous into $\mathcal{M}(Q)$. By choosing $p = 0$ and ϕ_i to depend only on t in (7.13) we infer that $\int_\Omega \bar{c}_i(x, t) \, dx$ is independent of t. Thus, because Ω is compact, we can invoke Lemma 8.1.2 of [3] to conclude that $t \mapsto \bar{c}_i(\cdot, t)$ is narrowly, hence weak-$\star$, continuous.

It is clear that the infimum in (7.16) is greater or equal to that in (7.21), because the admissible set $\mathcal{A}_K$ is naturally embedded in $\hat{\mathcal{A}}_{\mathcal{K}}$, and the two are equal if either is finite. Recalling that $\inf \emptyset = +\infty$, equality follows in general. $\square$

Remark 7.6. As a last comment, we note that for variable-density flow with strictly positive density, the relaxed least-action problem studied by Lopes et al. [37] was shown to be *consistent* with the classical Euler equations, in the sense that classical solutions of the Euler system induce weak solutions of relaxed Euler equations, and for sufficiently short time the induced solution is the unique minimizer of the relaxed problem. In the case that we consider with $\hat{\rho}_0 = 0$, however, this consistency property cannot hold in general when the space dimension $d > 1$, for the reason that in general we can expect the Wasserstein density $\rho < 1$ in Theorem 7.2 (see Theorem 1.4), while necessarily $\rho \in \{0, 1\}$ for any classical solution of the incompressible Euler equations.

8. A Schmitzer-Schnörr-type shape distance without volume constraint

Theorem 1.2 establishes that restricting the Wasserstein metric to paths of shapes of fixed volume does not provide a new notion of distance on the space of such shapes. Namely it shows that for paths in the space of shapes of fixed volume, the infimum of the length of paths between two given shapes is the Wasserstein distance.

Volume change. It is of interest to consider a more general space of shapes in order to compare shapes of different volumes. In particular, the Schmitzer and Schnörr [46] considered a space that corresponds to the set of bounded, simply connected domains in $\mathbb{R}^2$ with smooth boundary and arbitrary positive area. To each such shape Ω one associates as its corresponding *shape measure* the probability measure having uniform density on Ω, denoted by

$$\mathcal{U}_\Omega = \frac{1}{|\Omega|}\mathbb{1}_\Omega. \tag{8.1}$$

We consider here this same association between sets and shape measures, but allow for more general shapes. Namely for fixed dimension d, let us consider shapes as bounded measurable subsets of $\mathbb{R}^d$ with positive volume. Let $\mathcal{C}$ be the set of all shape measures corresponding to such shapes. Thus $\mathcal{C}$ is the set of all uniform probability distributions of bounded support.

One can formally consider $\mathcal{C}$ as a submanifold of the space of probability measures of finite second moment, endowed with Wasserstein distance. Then we define a distance between shapes as we did in (1.5), requiring

$$d_\mathcal{C}(\Omega_0, \Omega_1)^2 = \inf \mathcal{A}, \qquad \mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho|v|^2 \, dx \, dt, \tag{8.2}$$

where $\rho = (\rho_t)$ is now required to be a path of shape measures in $\mathcal{C}$, with endpoints

$$\rho_0 = \mathcal{U}_{\Omega_0}, \qquad \rho_1 = \mathcal{U}_{\Omega_1}, \tag{8.3}$$

and transported according to the continuity equation (1.2) with a velocity field $v \in L^2(\rho \, dx \, dt)$.

Because the characteristic-function restriction (1.3) is replaced by the weaker requirement that ρ_t has a uniform density, for any two shapes of equal volume scaled to unity for convenience, it is clear that

$$d_s(\Omega_0, \Omega_1) \geq d_\mathcal{C}(\Omega_0, \Omega_1) \geq d_W(\Omega_0, \Omega_1). \tag{8.4}$$

Then as a direct consequence of Theorem 1.2, we have

$$d_\mathcal{C}(\Omega_0, \Omega_1) = d_W(\Omega_0, \Omega_1). \tag{8.5}$$

By a minor modification of the arguments of section 5, in general we have the following.

Theorem 8.1. *Let Ω_0 and Ω_1 be any two shapes of positive volume. Then*

$$d_C(\Omega_0, \Omega_1) = d_W(\mathcal{U}_{\Omega_0}, \mathcal{U}_{\Omega_1}).$$

Proof. By a simple scaling argument, we may assume $\min\{|\Omega_0|, |\Omega_1|\} \geq 1$ without loss of generality, so that both $\rho_0, \rho_1 \leq 1$. Then the concatenated Euler sprays provided by Theorem 1.3(b) supply a path of shape measures in $\mathcal{C}$ (actually shape densities), with action converging to $d_W(\mathcal{U}_{\Omega_0}, \mathcal{U}_{\Omega_1})^2$. $\qquad \square$

Smoothness. For dimension $d = 2$, Theorem 8.1 does not apply to describe distance in the space of shapes considered by Schmitzer and Schnörr in [46], however, for as we have mentioned, they consider shapes to be bounded simply connected domains with smooth boundary.

One point of view on this issue is that it is nowadays reasonable for many purposes to consider 'pixelated' images and shapes, made up of fine-grained discrete elements, to be valid approximations to smooth ones. Thus the microdroplet constructions considered in this paper,

which fit with the mathematically natural regularity conditions inherent in the definition of Wasserstein distance, need not be thought unnatural from the point of view of applications.

Nevertheless one may ask whether the infimum of path length in the space of smooth simply connected shapes is still the Wasserstein distance, as in Theorem 8.1. Our proof of Theorem 1.2 in Section 5 does not provide paths in this space because the union of droplets is disconnected. However, the main mechanism by which we efficiently transport mass, namely by "dividing" the domain into small pieces (droplets) which almost follow the Wasserstein geodesics, is still available. In particular, by creating many deep creases in the domain it might be effectively 'divided' into such pieces while still remaining connected and smooth. Thus we conjecture that even in the class of smooth sets considered in [46], the geodesic distance is the Wasserstein distance between uniform distributions as in Theorem 8.1.

Geodesic equations. It is also interesting to compare our Euler droplet equations from subsection 3.1 with the formal geodesic equations for smooth critical paths of the action $\mathcal{A}$ in the space $\mathcal{C}$ of uniform distributions. These equations correspond to equation (4.12) of Schmitzer and Schnörr in [46].

These geodesic equations may be derived in a manner almost identical to the treatment in subsection 3.1 above. The principal difference is that due to (3.4), the divergence of the Eulerian velocity may be a nonzero function of time, constant in space:

$$\nabla \cdot v = c(t),$$

and the same is true of virtual displacements $\tilde{v}$. The variation of action now satisfies

$$\frac{\delta \mathcal{A}}{2} = \int_{\Omega_t} v \cdot \tilde{v} \, \rho \, dx \bigg|_{t=1} - \int_0^1 \int_{\Omega_t} (\partial_t v + v \cdot \nabla v) \cdot \tilde{v} \, \rho \, dx \, dt. \tag{8.6}$$

Now, the space of vector fields orthogonal to all constant-divergence fields on Ω_t is the space of gradients ∇p such that p vanishes on the boundary and has average zero in Ω_t, satisfying

$$p = 0 \quad \text{on } \partial\Omega_t, \qquad \int_{\Omega_t} p \, dx = 0. \tag{8.7}$$

Because ρ is spatially constant and $\tilde{v}$ can be (locally in time) arbitrary with spatially constant divergence, necessarily $u = -(\partial_t v + v \cdot \nabla v)$ is such a gradient. The remaining considerations in section 3.1 apply almost without change, and we conclude that $v = \nabla\phi$ where

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + p = 0, \qquad \Delta\phi = c(t), \tag{8.8}$$

where $c(t)$ is spatially constant in Ω_t.

These fluid equations differ from those in section 3.1 in that ϕ gains one degree of freedom (a multiple of the solution of $\Delta\phi = 1$ in Ω_t with Dirichlet boundary condition) while the pressure p loses one degree of freedom (as its integral is constrained).

They have elliptical droplet solutions given by displacement interpolation of elliptical Wasserstein droplets as in subsection 3.4, because pressure vanishes and density is indeed spatially constant for these interpolants. Because they are Wasserstein geodesics, these particular solutions are also length-minimizing geodesics in the shape space $\mathcal{C}$.

We remark that unlike in the case of Euler sprays, disjoint superposition will not yield a geodesic in general. This is because the requirement of spatially uniform density leads to a global coupling between all shape components. It seems likely that length-minimizing paths in $\mathcal{C}$ will not generally exist even locally, but we have no proof at present.

APPENDIX A. TL^p CONVERGENCE AND STABILITY OF WASSERSTEIN GEODESICS

Here we recall the notion of TL^p convergence as introduced in [29], which provides a more precise comparison between Wasserstein geodesics than the notion of weak convergence does alone. We recall some of the basic properties, establish new ones and use them to prove the convergence of Wasserstein geodesics considered as weak solutions to pressureless Euler equation.

The TL^p metric provides a natural setting for comparing optimal transport maps between different probability measures. Let $\mathcal{P}_p(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ with finite p-th moments. On the space $TL^p(\mathbb{R}^d)$, consisting of all ordered pairs (μ, g) where $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ and $g \in L^p(\mu)$, the metric is given as follows: For $1 \le p < \infty$,

$$d_{TL^p}((\mu_0, g_0), (\mu_1, g_1)) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} \left(\int |x - y|^p + |g_0(x) - g_1(y)|^p \, d\pi(x, y) \right)^{1/p},$$

and

$$d_{TL^\infty}((\mu_0, g_0), (\mu_1, g_1)) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} \operatorname*{ess\,sup}_\pi (|x - y| + |g_0(x) - g_0(y)|),$$

where $\Pi(\mu_0, \mu_1)$ is the set of transportation plans (couplings) between μ_0 and μ_1.

The following result estabilishes a stability property for optimal transport maps, as a consequence of a known general stability property for optimal plans.

Proposition A.1. *Let $\mu, \mu_k \in \mathcal{P}_p(\mathbb{R}^d)$ be probability measures absolutely continuous with respect to Lebesgue measure, and let $\nu, \nu_k \in \mathcal{P}_p(\mathbb{R}^d)$, for each $k \in \mathbb{N}$. Assume that*

$$d_p(\mu_k, \mu) \to 0 \quad and \quad d_p(\nu_k, \nu) \to 0 \quad as \ k \to \infty.$$

Let T_k and T be the optimal transportation maps between μ_k and ν_k, and μ and ν, respectively. Then

$$(\mu_k, T_k) \xrightarrow{TL^p} (\mu, T) \quad as \ k \to \infty.$$

Proof. The measures $\pi_k = (\mathrm{id} \times T_k)_\sharp \mu_k$ and $\pi = (\mathrm{id} \times T)_\sharp \mu$ are the optimal transportation plans between μ_k and ν_k, and μ and ν, respectively. By stability of optimal transport plans (Proposition 7.1.3 of [3] or Theorem 5.20 in [52]) the sequence π_k is precompact with respect to weak convergence and each of its subsequential limits is an optimal transport plan between μ and ν. Since π is the unique optimal transportation plan between μ and ν the sequence π_k converges to π. Furthermore, by Theorem 5.11 of [45] or Remark 7.1.11 of [3],

$$\lim_{k \to \infty} \int |x|^p + |y|^p \, d\pi_k(x, y) = \lim_{k \to \infty} \int |x|^p \, d\mu_k + \int |y|^p \, d\nu_k$$

$$= \int |x|^p \, d\mu + \int |y|^p \, d\nu = \int |x|^p + |y|^p \, d\pi(x, y).$$

By Lemma 5.1.7 of [3], it follows the π_k have uniformly integrable p-th moments, therefore

$$d_p(\pi_k, \pi) \to 0 \quad as \ k \to \infty,$$

by Proposition 7.1.5 in [3]. Hence there exists (optimal) $\gamma_k \in \Pi(\pi, \pi_k)$ such that

$$\int |x - \tilde{x}|^p + |y - \tilde{y}|^p \, d\gamma_k(x, y, \tilde{x}, \tilde{y}) \to 0 \quad as \ k \to \infty. \tag{A.1}$$

Since π-almost everywhere $y = T(x)$ and π_k-almost everywhere $\tilde{y} = T_k(\tilde{x})$ and the support $\operatorname{supp} \gamma_k$ of γ_k is contained in $\operatorname{supp} \pi \times \operatorname{supp} \pi_k$, we conclude that γ_k-almost everywhere

$(x, y, \tilde{x}, \tilde{y}) = (x, T(x), \tilde{x}, T_k(\tilde{x}))$. Therefore

$$\int |x - \tilde{x}|^p + |T(x) - T_k(\tilde{x})|^p \, d\gamma_k(x, y, \tilde{x}, \tilde{y}) \to 0 \quad \text{as } k \to \infty.$$

Finally let θ_k be the projection of γ_k to $(x, \tilde{x})$ variables. By its definition $\theta_k \in \Pi(\mu, \mu_k)$ and by above

$$\int |x - \tilde{x}|^p + |T(x) - T_k(\tilde{x})|^p \, d\theta_k(x, \tilde{x}) \to 0 \quad \text{as } k \to \infty. \tag{A.2}$$

Thus $(\mu_k, T_k) \xrightarrow{TL^p} (\mu, T)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now consider the convergence of Wasserstein geodesics between measures μ_k and ν_k as in the Lemma A.1, treating only the case $p = 2$. We recall that particle paths along these geodesics are given by

$$T_{k,t}(x) = (1 - t)x + tT_k(x).$$

The displacement interpolation between μ_k and ν_k, and particle velocities (in Eulerian variables) along the geodesics, are given by (cf. (6.15)–(6.16))

$$\mu_{k,t} = T_{k,t\sharp}\mu_k, \quad v_{k,t} = (T_k - \text{id}) \circ T_{k,t}^{-1}, \quad t \in [0, 1).$$

If ν_k is absolutely continuous with respect to Lebesgue measure, then $t = 1$ is allowed. We also recall that

$$\int |v_{k,t}(z)|^2 d\mu_{k,t}(z) = \int |v_{k,0}(x)|^2 d\mu_k(x) = d_2^2(\mu_k, \nu_k).$$

Furthermore it is straightforward to check that $t \mapsto (\mu_{k,t}, v_{k,t})$ is Lipschitz continuous into $TL^2(\mathbb{R}^d)$, satisfying for $0 \le s < t < 1$

$$(t - s)d_2(\mu_k, \nu_k) = d_2(\mu_{k,t}, \mu_{k,s}) \le d_{TL^2}((\mu_{k,t}, v_{k,t}), (\mu_{k,s}, v_{k,s})) \le (t - s)d_2(\mu_k, \nu_k). \tag{A.3}$$

Proposition A.2. *Under the assumptions of Proposition A.1 for the case $p = 2$, as $k \to \infty$ we have*

$$\sup_{t\in[0,1]} d_2(\mu_{k,t}, \mu_t) \to 0, \tag{A.4}$$

$$\sup_{t\in[0,1)} d_{TL^2}((\mu_{k,t}, v_{k,t}), (\mu_t, v_t)) \to 0, \tag{A.5}$$

$$\sup_{t\in[0,1)} d_{TL^1}((\mu_{k,t}, v_{k,t} \otimes v_{k,t}), (\mu_t, v_t \otimes v_t)) \to 0. \tag{A.6}$$

If the measures ν_k and ν are absolutely continuous with respect to Lebesgue measure then the convergence in (A.5) and (A.6) hold also for $t \in [0, 1]$.

Proof. Let $\pi \in \Pi(\mu, \nu)$, $\pi_k \in \Pi(\mu_k, \nu_k)$, and $\gamma_k \in \Pi(\pi, \pi_k)$ be as in the proof of Proposition A.1. Similarly to θ_k, we define $\theta_{k,t} = (z_t \times z_t)_\sharp \gamma_k$ where

$$z_t(x, y) = (1 - t)x + ty \quad \text{and} \quad (z_t \times z_t)(x, y, \tilde{x}, \tilde{y}) = (z_t(x, y), z_t(\tilde{x}, \tilde{y})).$$

We note that $\theta_{k,t} \in \Pi(\mu_t, \mu_{k,t})$ and hence, for all $t \in [0, 1]$,

$$d_2(\mu_t, \mu_{k,t})^2 \le \int |z - \tilde{z}|^2 d\theta_{k,t}(z, \tilde{z})$$

$$= \int |(1 - t)(x - \tilde{x}) + t(y - \tilde{y})|^2 d\gamma_k(x, y, \tilde{x}, \tilde{y})$$

$$\le 2 \int |x - \tilde{x}|^2 + |y - \tilde{y}|^2 d\gamma_k(x, y, \tilde{x}, \tilde{y}),$$

which by (A.1) converges to 0 as $k \to \infty$.

We use the same coupling $\theta_{k,t}$ to compare the velocities. Using that γ_k-almost everywhere $(x, y, \tilde{x}, \tilde{y}) = (x, T(x), \tilde{x}, T_k(\tilde{x}))$, for any $t \in [0, 1)$ we obtain

$$
\int |v_t(z) - v_{k,t}(\tilde{z})|^2 \, d\theta_{k,t}(z, \tilde{z})
$$

$$
= \int |v_t((1-t)x + ty) - v_{k,t}((1-t)\tilde{x} + t\tilde{y})|^2 \, d\gamma_k(x, y, \tilde{x}, \tilde{y})
$$

$$
= \int |v(T_t(x)) - v_{k,t}(T_{k,t}(\tilde{x}))|^2 \, d\theta_k(x, \tilde{x})
$$

$$
= \int |v_0(x) - v_{k,0}(\tilde{x})|^2 \, d\theta_k(x, \tilde{x})
$$

$$
\leq 2 \int |x - \tilde{x}|^2 + |T(x) - T_k(\tilde{x})|^2 \, d\theta_k(x, \tilde{x}),
$$

which converges to 0 as $k \to \infty$, as in (A.2).

The convergence in (A.6) is a straightforward consequence through use of Cauchy-Schwarz inequalities. $\qquad\square$

Remark A.3. If the target measure ν_k is not absolutely continuous with respect to Lebesgue measure, then T_k may fail to be invertible on the support of ν_k and $(\mu_{k,t}, v_{k,t})$ may fail to converge as $t \to 1$ to some point in $TL^2(\mathbb{R}^d)$ due to oscillations in velocity. However, if ν_k and ν are absolutely continuous with respect to Lebesgue measure, then the curves $t \mapsto (\mu_{k,t}, v_{k,t})$, $t \mapsto (\mu_t, v_t)$ extend as continuous maps into TL^2 for all $t \in [0, 1]$, and the uniform convergences in (A.5)–(A.6) hold on $[0, 1]$.

A number of properties of the TL^p metric are established in Section 3 of [29] for measures supported in a fixed bounded set One useful characterization of TL^p-convergence in this case is stated in Proposition 3.12 of [29], which implies the following.

Proposition A.4. *Let $D \subset \mathbb{R}^d$ be open and bounded, and let μ and μ_k $(k \in \mathbb{N})$ be probability measures on D, and suppose μ is absolutely continuous with respect to Lebesgue measure. Call a sequence of transport maps (S_k) that push forward μ to μ_k (satisfying $S_{k\sharp}\mu = \mu_k$) stagnating if*

$$
\lim_{n \to \infty} \int_D |x - S_k(x)| \, d\mu(x) = 0. \tag{A.7}
$$

Then the following are equivalent, for $1 \leq p < \infty$.

(i) $(\mu_k, f_k) \xrightarrow{TL^p} (\mu, f)$ *as $k \to \infty$.*

(ii) *μ_k converges weakly to μ and for every stagnating sequence (S_k) we have*

$$
\int_D |f(x) - f_k(S_k(x))|^p \, d\mu(x) \to 0 \quad \text{as } k \to \infty. \tag{A.8}
$$

Moreover, for (ii) to hold it suffices that (A.8) holds for any single stagnating sequence (S_k).

This result together with Proposition A.2 yields the following.

Proposition A.5. *Make the same assumptions as in Proposition A.2, and assume all measures μ_k, μ, ν_k, ν are absolutely continuous with respect to Lebesgue measure and have support in a bounded open set D. Then for any stagnating sequence of transport maps (S_k) that push forward μ to μ_k, with the notation*

$$
S_{k,t} = T_{k,t} \circ S_k \circ T_t^{-1}
$$

the sequence $(S_{k,t})$ pushes forward μ_t to $\mu_{k,t}$ and is stagnating, and as $k \to \infty$,

$$\sup_{t \in [0,1]} \int |x - S_{k,t}(x)|^2 \, d\mu_t(x) \to 0, \tag{A.9}$$

$$\sup_{t \in [0,1]} \int |v_t(x) - v_{k,t}(S_{k,t}(x))|^2 \, d\mu_t(x) \to 0, \tag{A.10}$$

$$\sup_{t \in [0,1]} \int |(v_t \otimes v_t)(x) - (v_{k,t} \otimes v_{k,t})(S_{k,t}(x))| \, d\mu_t(x) \to 0. \tag{A.11}$$

Proof. First we note that indeed

$$\mu_{k,t} = (T_{k,t})_\sharp \mu_k = (T_{k,t} \circ S_k)_\sharp \mu = (S_{k,t})_\sharp \mu_t.$$

Next, fix any $t \in [0,1]$. Because $d_2(\mu_{k,t}, \mu_t) \to 0$ by (A.4) and $T_{k,t}$ is the optimal transport map pushing forward μ_k to $\mu_{k,t}$, by Proposition A.1 we have $d_2((\mu_k, T_{k,t}), (\mu, T_t)) \to 0$. Now by Proposition A.4, because $(T_t)_\sharp \mu = \mu_t$ we have

$$\int |x - S_{k,t}(x)|^2 \, d\mu_t(x) = \int |T_t(z) - T_{k,t}(S_k(z))|^2 \, d\mu(z) \to 0. \tag{A.12}$$

We infer that $(S_{k,t})$ is stagnating and the convergence in (A.9) holds pointwise in t. But now, the middle quantity in (A.12) is a quadratic function of t, so the uniform convergence in (A.9) holds.

Next, we note that the quantity in (A.10) is actually independent of t. We have

$$\int |v_t(x) - v_{k,t}(S_{k,t}(x))|^2 \, d\mu_t(x) = \int |v_0(z) - v_{k,0}(S_k(z))|^2 \, d\mu(z) \to 0,$$

due to Proposition A.4. The proof of (A.11) is similar. $\qquad \square$

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