A generalized Sz. Nagy inequality in higher dimensions and the critical thin film equation

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Received 15 December 2015, revised 3 October 2016
Accepted for publication 28 October 2016
Published 17 November 2016

Recommended by Professor Andrea L Bertozzi

Abstract

In this paper, we provide an alternative proof for the classical Sz. Nagy inequality in one dimension by a variational method and generalize it to higher dimensions $d \geqslant 1$

$$\mathcal{J}(h) := \frac{\left(\int_{\mathbb{R}^d} |h| \, \mathrm{d}x\right)^{a-1} \int_{\mathbb{R}^d} |\nabla h|^2 \, \mathrm{d}x}{\left(\int_{\mathbb{R}^d} |h|^{m+1} \, \mathrm{d}x\right)^{\frac{a+1}{m+1}}} \geqslant \beta_0,$$

where $m > 0$ for $d = 1, 2$, $0 < m < \frac{d+2}{d-2}$ for $d \geqslant 3$, and $a = \frac{d+2(m+1)}{md}$. The Euler–Lagrange equation for critical points of $\mathcal{J}(h)$ in the non-negative radial decreasing function space is given by a free boundary problem for a generalized Lane–Emden equation, which has a unique solution (denoted by h_c) and the solution determines the best constant for the above generalized Sz. Nagy inequality. The connection between the critical mass $M_c = \int_{\mathbb{R}} h_c \, \mathrm{d}x = \frac{2\sqrt{2}\pi}{3}$ for the thin-film equation and the best constant of the Sz. Nagy inequality in one dimension was first noted by Witelski *et al* (2004 *Eur. J. Appl. Math.* **15** 223–56). For the following critical thin film equation in multi-dimension $d \geqslant 2$

$$h_t + \nabla \cdot (h \, \nabla \Delta h) + \nabla \cdot (h \, \nabla h^m) = 0, \quad x \in \mathbb{R}^d,$$

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where $m = 1 + 2/d$, the critical mass is also given by $M_c := \int_{\mathbb{R}^d} h_c \, dx$. A finite time blow-up occurs for solutions with the initial mass larger than M_c. On the other hand, if the initial mass is less than M_c and a global non-negative entropy weak solution exists, then the second moment goes to infinity as $t \to \infty$ or $h(\cdot, t_k) \rightharpoonup 0$ in $L^1(\mathbb{R}^d)$ for some subsequence $t_k \to \infty$. This shows that a part of the mass spreads to infinity.

Keywords: long-wave instability, free-surface evolution, critical mass, free boundary problem
Mathematics Subject Classification numbers: 35K65, 35K25, 39B62

1. Introduction

In many models of physical and biological systems, coherent system states are formed and maintained by a balance of competing influences. There are processes that disperse, defocus, fragment, or spread things out in some ways, while aggregation, focusing, or concentration effects are generated by nonlinear mechanisms. For physical systems that can be described by a gradient flow driven by a free energy, these competing effects are usually represented by terms with different signs in the free energy. Some functional inequalities have been extensively investigated to determine the domination among these competing effects in the free energy (see [5, 10, 12, 13, 15, 16, 18, 33, 34]).

There are rich phenomena when competition is dynamically balanced in some invariant scalings such as the mass invariant scaling which leads to a critical exponent. We refer a physical system with such critical exponents to as a critical system. Sometimes, equilibrium solutions in a critical system are also solutions to the Euler–Lagrange equation for an associated functional inequality. In other words, equilibrium solutions achieve the equality in the functional inequality and determine the best constant of the functional inequality. Consequently, the best constant provides sharp conditions on initial data to distinguish between global existence and finite time blow-up.

This paper focuses on the following critical-case long-wave unstable thin film equation with $m = 1 + \frac{2}{d}$

$$h_t + \nabla \cdot (h^n \nabla \Delta h) + \nabla \cdot (h^n \nabla (h^m)) = 0, \quad x \in \mathbb{R}^d, \; t > 0, \tag{1.1}$$

which has been derived from a lubrication approximation to model the surface tension dominated motion of viscous liquid films and spreading droplets over a solid substrate. The unknown function $h(x, t)$ represents the height of the evolving free-surface. The parameter n is usually referred to as a mobility exponent. The case $n = 3$ corresponds to films with constant interfacial shear stress and constant surface tension [29], and the case $n = 1$ corresponds to the lubrication approximation of the Hele-Shaw flow [21]. The details can be found in a review [29].

We rewrite (1.1) in a variational form

$$h_t - \nabla \cdot (h^n \nabla \mu) = 0, \quad \mu = \frac{\delta \mathcal{F}}{\delta h}, \tag{1.2}$$

where μ is a chemical potential. It is given by the variation of a free energy functional:

$$\mathcal{F}(h) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla h|^2 \, dx - \frac{1}{m+1} \int_{\mathbb{R}^d} h^{m+1} \, dx. \tag{1.3}$$

In the thin film equation (1.2), the negative chemical potential is referred to as the dynamic pressure, $p = -\mu = \Delta h + h^m$. From the variational form in (1.2), we have the following entropy-dissipation relation for $h \geqslant 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(h(\cdot,t)) = -\int_{\mathbb{R}^d} h^n |\nabla(h^m + \Delta h)|^2 \, \mathrm{d}x \leqslant 0. \tag{1.4}$$

Notice that the second term in (1.1) involves the fourth order derivative and is a stabilizing term, while the third term is a destabilizing second order derivative term. For short-wave solutions, the stabilizing term dominates the destabilizing one so that the linearized equation of (1.1) is well-possed. However, for long-wave solutions, the destabilizing term sometimes dominates the stabilizing term such that the long-wave instability may occur. The competition between the stabilizing term and the destabilizing term is represented by different signs for the corresponding terms in the free energy (1.3).

If $h(x, t)$ is a solution to (1.1), then the mass invariant re-scaled profile $\alpha^d h(\alpha x, \alpha^{d+4} t)$ is also a solution to (1.1). This scaling invariant property indicates a balance between the stabilizing and destabilizing terms in the mass invariant scaling, and hence $m = 1 + 2/d$ is a critical exponent. This is the reason why we call the equation (1.1) a critical case model. Notice that $m = 1 + 2/d$ coincides with the Fujita exponent for the associated Allen–Cahn equation $u_t = -\frac{\delta \mathcal{F}}{\delta u} = \Delta u + u^m$.

For some critical models, there is a critical mass M_c that can be used to distinguish between global existence and finite time blow-up, and the critical mass is usually given by the mass of equilibrium solutions. For an equilibrium solution h_{eq}, the dissipation term is zero, and the equilibrium chemical potential is given by

$$\begin{cases} \mu_{\mathrm{eq}}(x) = \bar{C}, & x \in \mathrm{supp} h_{\mathrm{eq}}, \\ \mu_{\mathrm{eq}}(x) \geqslant \bar{C}, & \text{otherwise} \end{cases} \tag{1.5}$$

for some constant $\bar{C}$. In other words, the equilibrium solution is a Nash equilibrium [9].

Witelski *et al* in [34] found that the best constant of the Sz. Nagy inequality [27] is closely connected to the critical mass M_c in the one-dimensional thin film equation. In this paper, we provide an alternative proof for the classic Sz. Nagy inequality (1941) based on a variational method and extend this inequality to any dimension $d \geqslant 1$.

Define a space

$$X := \{h \in L^1(\mathbb{R}^d), \quad \nabla h \in L^2(\mathbb{R}^d)\},$$

and a functional

$$\mathcal{J}(h) = \frac{\left(\int_{\mathbb{R}^d} |h| \, \mathrm{d}x\right)^{a-1} \int_{\mathbb{R}^d} |\nabla h|^2 \, \mathrm{d}x}{\left(\int_{\mathbb{R}^d} |h|^{m+1} \, \mathrm{d}x\right)^{\frac{a+1}{m+1}}}, \tag{1.6}$$

where $m > 0$ for $d = 1, 2$, $0 < m < \frac{d+2}{d-2}$ for $d \geqslant 3$, $a = \frac{d+2(m+1)}{md}$. A generalized Sz. Nagy inequality can be formulated as the following minimizing problem

$$\beta_0 = \inf_{h \in X} \mathcal{J}(h). \tag{1.7}$$

Thanks to the rearrangement technique, the infimum β_0 can be achieved in the following non-negative radial symmetric decreasing function space

$$X^*_{\rm rad} := \{h \geqslant 0 \big| \; h(x) = h(|x|),\;\; h'(r) \leqslant 0,\;\; h \in L^1(\mathbb{R}^d),\;\; \nabla h \in L^2(\mathbb{R}^d)\}. \tag{1.8}$$

In section 3, we will show that the Euler–Lagrange equation for critical points of $\mathcal{J}(h)$ in $X^*_{\rm rad}$ is given by the following free boundary problem for a generalized Lane–Emden equation up to a re-scaling

$$h'' + \frac{d-1}{r}h' + h^m = 1 \quad \text{for } 0 < r < R, \tag{1.9}$$

$$h'(0) = 0,\;\; h(R) = h'(R) = 0, \tag{1.10}$$

and $\alpha := h(0) > 1$. We will show that this free boundary problem (1.9) and (1.10) has a unique solution h_c which gives the best constant

$$\beta_0 = 2(a-1)^{\frac{1}{m}(1+\frac{2}{d})-1}(a+1)^{-\frac{1}{m}(1+\frac{2}{d})}M_c^{\frac{2}{d}}, \tag{1.11}$$

where $M_c = \int_{\mathbb{R}^d} h_c \,{\rm d}x$. In one dimension, we have an explicit value

$$\beta_0 = 4\cdot 3^{-\frac{3}{m}}m^{-3}(m+3)^{1+\frac{3}{m}}\left(\mathcal{B}\left(\frac{3}{2},\frac{3}{2m}\right)\right)^2.$$

Theorem 1.1. *Suppose $f \in L^1(\mathbb{R}^d)$, $\nabla f \in L^2(\mathbb{R}^d)$. Then $f \in L^{m+1}(\mathbb{R}^d)$ and satisfies the following generalized Sz. Nagy inequality*

$$\left(\frac{1}{a+1}\int_{\mathbb{R}^d}|f|^{m+1}\,{\rm d}x\right)^{\frac{a+1}{m+1}} \leqslant 2^{-1}(a-1)^{1-\frac{d+2}{md}}M_c^{-\frac{2}{d}}\left(\int_{\mathbb{R}^d}|f|\,{\rm d}x\right)^{a-1}\int_{\mathbb{R}^d}|\nabla f|^2\,{\rm d}x, \tag{1.12}$$

where $m > 0$ for $d = 1, 2$, $0 < m < \frac{d+2}{d-2}$ for $d \geqslant 3$, $a = \frac{d+2(m+1)}{md}$, and $M_c = \int_{\mathbb{R}^d} h_c\,{\rm d}x$, h_c is the unique solution to the free boundary problem (1.9) and (1.10) in $X^*_{\rm rad}$ and satisfies that

(i) *the equality holds in (1.12) if $f = A\,h_c(\lambda|x-x_0|)$ for any real numbers $A > 0$, $\lambda > 0$, $x_0 \in \mathbb{R}^d$.*
(ii) *$h_c(0) > 1$, $h_c(r) > 0$, $h_c'(r) < 0$ for $0 < r < R$, and $h_c(r) \equiv 0$ for $r \geqslant R$.*

In particular, for $d = 1$ there is a unique closed form solution to (1.9) and (1.10)

$$h_c(r) = \left((m+1)B^{-1}\left(\sqrt{2}\,m(m+1)^{-\frac{1}{2m}}(R-r);\,\frac{1}{2},\frac{1}{2m}\right)\right)^{\frac{1}{m}}$$

satisfying

$$h_c(0) = \alpha_c = (m+1)^{1/m}, \quad R = \frac{(m+1)^{\frac{1}{2m}}}{2^{1/2}m}\mathcal{B}\left(\frac{1}{2},\frac{1}{2m}\right),$$
$$\|h_c\|_{L^1} = 2^{\frac{1}{2}}m^{-2}(m+1)^{\frac{3}{2m}}(m+3)\mathcal{B}\left(\frac{3}{2},\frac{3}{2m}\right),$$

where $\mathcal{B}(a, b)$ is the Beta function and B^{-1} is the inverse function of the incomplete Beta function $B(x;\, a, b)$.

Remark 1.1. The derivation of $h_c'(R) = 0$ in the free boundary problem (1.9) and (1.10) comes from (i) a Pohozaev type identity between the free energy and the contact angle $\bar{\mathcal{F}}(h_c) = Ch_c'(R)^2$ in lemma 3.1; (ii) $\bar{\mathcal{F}}(h_c) = 0$ in proposition 3.1.

The idea of using a functional inequality to determine a sharp condition on initial data to distinguish global existence from finite time blow-up goes back to the work of Weinstein [33], where a sharp condition for a critical focusing nonlinear Schrödinger equation is established by using the best constant of a Gagliardo–Nirenberg inequality. Although the system studied there is a Hamiltonian system instead of a gradient flow system, the mathematical tools involved are very similar.

More recently it has been shown that the best constant in the logarithmic Hardy–Littlewood–Sobolev inequality [1] determines the critical mass 8π for the two-dimensional (2D) parabolic–elliptic Keller–Segel model [10, 12]. While using the Onofri inequality, Calvez and Corrias [15] showed that 8π is also the critical mass for the 2D parabolic–parabolic Keller–Segel model.

The Hardy–Littlewood–Sobolev inequality was used to study the degenerate Keller–Segel model in higher dimensions $d \geqslant 3$ under the critical exponent $m_e = \frac{2d}{d+2}$, see [16]. The equilibrium equation is given by the Lane–Emden equation $-\Delta u = u^p$ (see [9]) with the critical exponent $p = \frac{d+2}{d-2}$, and its solution achieves the equality for the Hardy–Littlewood–Sobolev inequality. Notice that the critical exponent for the mass invariant scaling is given by $m_c = 2 - \frac{2}{d}$, see [11]. For the diffusion exponent between these two critical exponents, $m_e < m < m_c$, there is a constant s^* depending only on the initial mass and the best constant of the Hardy–Littlewood–Sobolev inequality [17], such that for $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there is a unique global weak solution if $\|\rho_0\|_{L^{\frac{2d}{d+2}}}$ is less than s^*, and a finite time blow-up occurs if $\|\rho_0\|_{L^{\frac{2d}{d+2}}}$ is larger than s^*. We refer to Dolbeault *et al* [18] for recent developments on functional inequalities and applications to global existence for nonlinear partial differential equations.

For the one-dimensional thin film equation, from the pioneering work of Bernis and Friedman [4], global existence of weak solutions, non-negativity, Hölder regularities, finite speed of propagation of the solution support have been elaborated by Bernis [3], Beretta *et al* [2], Bertozzi and Pugh [5–8], Witelski *et al* [34], Giacomelli *et al* [19–21] etc. In 2014, Taranets and King [32] proved local existence of nonnegative weak and strong solutions to the following equation in a bounded domain Ω with smooth boundary in $\mathbb{R}^d$

$$h_t + \nabla \cdot (h^n \nabla \Delta h) + \nabla \cdot (h^m \nabla h) = 0, \quad x \in \Omega,\ t > 0$$

subjecting to the boundary conditions

$$\nabla h \cdot \vec{n} = \nabla \Delta h \cdot \vec{n} = 0, \text{ on } \partial\Omega \times (0, T), \tag{1.13}$$

where $\vec{n}$ is the unit outward normal vector, and the initial condition

$$h(x, 0) = h_0(x) \geqslant 0,\ h_0 \in H^1(\Omega).$$

Moreover, they also obtained global existence of solutions to the above problem under a more restrictive threshold $m_0 < \bar{M}_d$. For $d = 1$, their threshold is $\bar{M}_1 = 1/\sqrt{12} < M_c = \frac{2\sqrt{2}\pi}{3}$. Uniqueness of solutions to the thin film equation with the unstable term has not been extensively studied in multi-dimension. The only result known to our knowledge is that Taranets and King [32, theorem 3] proved uniqueness of initially constant solutions in a special subcritical case.

The organization of the paper is given as follows. In section 2, existence of a minimizer for $\mathcal{J}(h)$ is proved by using the Strauss inequality. In section 3, we show that any critical points of $\mathcal{J}(h)$ in X^*_{rad} satisfy the free boundary problem (1.9) and (1.10) up to a re-scaling. And we prove existence and uniqueness of solutions to (1.9) and (1.10). In section 4, we prove the main theorem. In section 5, we show an important application of the generalized Sz. Nagy inequality (1.12) to the critical thin film equation (1.1), $d \geqslant 2$: (i) a finite time blow-up occurs for solutions with the initial mass larger than the critical mass M_c; (ii) if the initial mass is less than the critical mass M_c and a global non-negative entropy weak solution exists, then the second moment goes to infinity as $t \to \infty$ or $h(\cdot, t_k) \rightharpoonup 0$ in $L^1(\mathbb{R}^d)$ for some subsequence $t_k \to \infty$.

2. Existence of a minimizer for $\mathcal{J}$(h)

In this section we prove a slight more general result on existence of a minimizer for a generalized functional (see (2.2) below). The case of $q = 0$ and $p = 2$ is what we need for the generalized Sz. Nagy inequality (1.12). First, we consider the following generalized minimizing problem

$$\beta = \inf_{h \in X} J(h), \tag{2.1}$$

$$J(h) := \frac{\left(\int_{\mathbb{R}^d} |h|^{q+1}\,\mathrm{d}x\right)^{\frac{a-p/2}{q+1}} \int_{\mathbb{R}^d} |\nabla h|^p\,\mathrm{d}x}{\left(\int_{\mathbb{R}^d} |h|^{m+1}\,\mathrm{d}x\right)^{\frac{a+p/2}{m+1}}}, \tag{2.2}$$

$$X := \{\, h \in L^{q+1}(\mathbb{R}^d), \quad \nabla h \in L^p(\mathbb{R}^d)\}.$$

Here $d \geqslant 1$ and the parameters p, q, m, a are given by the following ranges:

$$p > \max\left\{1, \frac{2d}{d+2}\right\}, \quad \sigma = \begin{cases} \dfrac{(p-1)d+p}{d-p} & \text{if } p < d, \\ \infty & \text{if } p \geqslant d, \end{cases} \tag{2.3}$$

$$0 \leqslant q < \min\{p-1, \sigma-1\}, \quad q < m < \sigma, \tag{2.4}$$

$$a = \frac{p(m+1)(q+1) + (p/2-1)d(m+q) - dqm + (p-1)d}{d(m-q)}. \tag{2.5}$$

Remark 2.1. When $q = 0, p = 2$, the minimizing problem (2.1) and (2.2) becomes the minimizing problem (1.6) and (1.7), and $\beta = \beta_0$.

Existence of the positive lower bound of $J(h)$ can be directly obtained from the Gagliardo–Nirenberg–Sobolev inequality [26, p 176, formula (2.3.50)]. In other words, the minimizing problem (2.1) is well-defined, i.e. there exists $\beta > 0$ such that (2.1) holds.

Denote X^*_{rad} as a function space of non-negative radial symmetric decreasing functions

$$X^*_{\text{rad}} := \{h \geqslant 0 \,\big|\, h(x) = h(|x|),\ h'(r) \leqslant 0,\ h \in L^{q+1}(\mathbb{R}^d),\ \nabla h \in L^p(\mathbb{R}^d)\}. \tag{2.6}$$

Lemma 2.1. *The minimizing problem (2.1) is equivalent to the following minimizing problem*

$$\beta = \inf_{h \in X^*_{\rm rad}} J(h), \tag{2.7}$$

where $X^*_{\rm rad}$ *is given by* (2.6).

Proof. First from the book [22, lemma 7.6], we know that if $\nabla h \in L^p(\mathbb{R}^d)$, then $\nabla |h|$ is also in $L^p(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} |\nabla h|^p \,\mathrm{d}x = \int_{\mathbb{R}^d} |\nabla |h||^p \,\mathrm{d}x, \text{ for any } p > 1. \tag{2.8}$$

Hence $J(|h|) = J(h)$. As a result we can find a minimizer in $X^+ = X \cap \{h | h(x,t) \geqslant 0\}$.

Next, we use the classical rearrangement technique to further reduce the range for finding a minimizer to $X^*_{\rm rad}$.

Let $h^* : \mathbb{R}^d \to [0, +\infty)$ be the radial decreasing rearrangement of $h \in X^+$ (see [23, chapter 3]). Then the rearrangement function satisfies

$$\int_{\mathbb{R}^d} h^*(x)^p \,\mathrm{d}x = \int_{\mathbb{R}^d} h(x)^p \,\mathrm{d}x, \quad 0 < p < \infty. \tag{2.9}$$

Moreover, from the classical Pólya–Szegő inequality [14], we know

$$\int_{\mathbb{R}^d} |\nabla h^*|^p \,\mathrm{d}x \leqslant \int_{\mathbb{R}^d} |\nabla h|^p \,\mathrm{d}x, \quad 1 \leqslant p \leqslant \infty. \tag{2.10}$$

Using (2.9) and (2.10), it holds that

$$J(h^*) = \frac{\left(\int_{\mathbb{R}^d} (h^*)^{q+1} \mathrm{d}x\right)^{\frac{a-p/2}{q+1}} \int_{\mathbb{R}^d} |\nabla h^*|^p \,\mathrm{d}x}{\left(\int_{\mathbb{R}^d} (h^*)^{m+1} \mathrm{d}x\right)^{\frac{a+p/2}{m+1}}} \leqslant \frac{\left(\int_{\mathbb{R}^d} h^{q+1} \mathrm{d}x\right)^{\frac{a-p/2}{q+1}} \int_{\mathbb{R}^d} |\nabla h|^p \,\mathrm{d}x}{\left(\int_{\mathbb{R}^d} h^{m+1} \mathrm{d}x\right)^{\frac{a+p/2}{m+1}}} = J(h).$$

Hence (2.7) holds. □

Next we will use the compactness argument and the Strauss inequality for the radial function in $W^{1,p}(\mathbb{R}^d)$ space [24, lemma II.1] to prove existence of a minimizer of $J(h)$ in $X^*_{\rm rad}$. The proof is indeed rather standard and we provide a proof below for completeness.

Lemma 2.2 (Strauss' inequality). *Assume that* $d \geqslant 2$, $1 \leqslant p < \infty$ *and* u *is a radial function in* $W^{1,p}(\mathbb{R}^d)$ *space. Then for a.e.* $x \in \mathbb{R}^d \setminus \{0\}$, *the following inequality holds*

$$|u(x)| \leqslant C(d,p) |x|^{\frac{1-d}{p}} \|u\|_{W^{1,p}(\mathbb{R}^d)}. \tag{2.11}$$

Proposition 2.1. *There exists a minimizer* h_0 *of* $J(h)$ *in* $X^*_{\rm rad}$ *such that*

$$J(h_0) = \beta = \inf_{h \in X^*_{\rm rad}} J(h), \quad \|h_0\|_{L^{q+1}} = \|h_0\|_{L^{m+1}} = 1, \ \|\nabla h_0\|^p_{L^p} = \beta. \tag{2.12}$$

Proof. First, we know that $J(h)$ has the following scaling invariance:

$$J(h_{\mu,\lambda}) = J(h), \quad h_{\mu,\lambda} = \mu h(\lambda x), \quad \forall\ \mu, \lambda > 0. \tag{2.13}$$

In fact, it is a direct consequence of the following equalities with a given by (2.5)

$$\|h_{\mu,\lambda}\|_{L^{q+1}}^{a-p/2} = \mu^{a-p/2}\lambda^{-\frac{d(a-p/2)}{q+1}}\|h\|_{L^{q+1}}^{a-p/2},\ \ \|\nabla h_{\mu,\lambda}\|_{L^p}^p = \mu^p\lambda^{p-d}\|\nabla h\|_{L^p}^p,$$
$$\|h_{\mu,\lambda}\|_{L^{m+1}}^{a+p/2} = \mu^{a+p/2}\lambda^{-\frac{d(a+p/2)}{m+1}}\|h\|_{L^{m+1}}^{a+p/2}.$$

By (2.7) and (2.13), there exists a minimizing sequence $\{h_k\} \in X_{\rm rad}^*$ satisfying $\int_{\mathbb{R}^d} h_k^{q+1}\,\mathrm{d}x = \int_{\mathbb{R}^d} h_k^{m+1}\,\mathrm{d}x = 1$ such that

$$\lim_{k\to\infty} J(h_k) = \lim_{k\to\infty}\int_{\mathbb{R}^d} |\nabla h_k|^p\,\mathrm{d}x = \beta = \inf_{h\in X_{\rm rad}^*} J(h). \tag{2.14}$$

For $d\geqslant 2$, there exist a subsequence (still denoted by h_k) and $h_0 \in W^{1,p}(\mathbb{R}^d)\cap L^{m+1}(\mathbb{R}^d)$ such that as $k\to\infty$

$$h_k \rightharpoonup h_0, \quad \text{in } L^{q+1}(\mathbb{R}^d)\cap L^{m+1}(\mathbb{R}^d), \tag{2.15}$$

$$\nabla h_k \rightharpoonup \nabla h_0, \quad \text{in } L^p(\mathbb{R}^d). \tag{2.16}$$

Hence by the Fatou lemma, we have

$$\|\nabla h_0\|_{L^p} \leqslant \liminf_{k\to\infty}\|\nabla h_k\|_{L^p} = \beta^{1/p}, \tag{2.17}$$

$$\|h_0\|_{L^{q+1}} \leqslant \liminf_{k\to\infty}\|h_k\|_{L^{q+1}} = 1. \tag{2.18}$$

On the other hand, the formula (2.11) indicates that for any radial function $h_k \in W^{1,p}(\mathbb{R}^d)$, it holds that

$$|h_k| \leqslant C|x|^{\frac{1-d}{p}}\|h_k\|_{W^{1,p}(\mathbb{R}^d)}, \text{ for } |x|>0,\ d\geqslant 2, \tag{2.19}$$

where the constant C is only dependent of d and p. By (2.19), we know that for $s > \frac{pd}{d-1}$, it holds that

$$\int_{|x|>R} h_k^s\,\mathrm{d}x \leqslant C\int_{|x|>R}|x|^{-\frac{s(d-1)}{p}}\,\mathrm{d}x = C\int_R^{+\infty} r^{-\frac{s(d-1)}{p}+(d-1)}\,\mathrm{d}r = CR^{d-\frac{s(d-1)}{p}}.$$

Since $d - \frac{s(d-1)}{p} < 0$, for any $\varepsilon > 0$, there is R_ε^1 such that

$$\int_{|x|>R_\varepsilon^1} h_k^s\,\mathrm{d}x \leqslant \varepsilon.$$

If $q+1 < s \leqslant \frac{pd}{d-1}$, using the interpolation inequality, it holds that

$$\|h_k\|_{L^s(B^c(0,R))}^s \leqslant \|h_k\|_{L^{q+1}(B^c(0,R))}^{s(1-\theta)}\|h_k\|_{L^\gamma(B^c(0,R))}^{s\theta}, \quad \text{for } \gamma > \frac{pd}{d-1}.$$

So, there exists a R_ε^2 such that

$$\|h_k\|_{L^s(B^c(0,R_\varepsilon^2))}^s \leqslant \varepsilon.$$

Taking $R = 2\max\{R_\varepsilon^1, R_\varepsilon^2\}$, one has

$$\|h_k\|_{L^s(B^c(0,R))}^s \leqslant \varepsilon. \tag{2.20}$$

The Sobolev embedding theorem gives

$$W^{1,p}(B(0,R+1)) \hookrightarrow\hookrightarrow L^s(B(0,R+1)),$$

provided that $1 < s < p^* = \frac{pd}{d-p}$ if $p < d$, and $s \geqslant 1$ for $p \geqslant d$. Together with (2.20) implies that there is a strong convergence subsequence of h_k (still denoted by h_k)

$$h_k \to h_0, \quad \text{in } L^s(\mathbb{R}^d), \quad \text{as } k \to \infty. \tag{2.21}$$

Since $1 < m+1 < \frac{pd}{d-p}$ for $p < d$, and $1 < m+1 < \infty$ for $p \geqslant d$, we know from (2.21) that

$$\int_{\mathbb{R}^d} |h_k - h_0|^{m+1}\,\mathrm{d}x \to 0, \text{ as } k \to \infty, \tag{2.22}$$

$$\|h_0\|_{L^{m+1}} = \lim_{k\to\infty} \|h_k\|_{L^{m+1}} = 1. \tag{2.23}$$

From (2.17), (2.18) and (2.23), we deduce

$$J(h_0) = \frac{\left(\int_{\mathbb{R}^d} h_0^{q+1}\,\mathrm{d}x\right)^{\frac{a-p/2}{q+1}} \int_{\mathbb{R}^d} |\nabla h_0|^p\,\mathrm{d}x}{\left(\int_{\mathbb{R}^d} h_0^{m+1}\,\mathrm{d}x\right)^{\frac{a+p/2}{m+1}}} \leqslant \beta.$$

Noticing that $J(h_0) \geqslant \beta$ by the definition of β, one knows that

$$J(h_0) = \beta, \quad \|h_0\|_{L^{q+1}} = \|h_0\|_{L^{m+1}} = 1, \quad \|\nabla h_0\|_{L^p}^p = \beta. \tag{2.24}$$

For $d = 1$, Sz. Nagy [27] proved existence of a minimizer of $J(h)$ that can be represented in terms of an incomplete Beta function and obtained the celebrated Sz. Nagy inequalities. Hence for $d \geqslant 1$, there exists an optimal radial decreasing function h_0 such that

$$\beta = \inf_{h\in X_{\mathrm{rad}}^*} J(h) = J(h_0).$$

□

3. Euler–Lagrange equation, contact angle, and free boundary problem

First, using a variational method, we show that any critical points of $\mathcal{J}(h)$ in the non-negative radial decreasing function space X_{rad}^* defined by (1.8) satisfy the free boundary problem (1.9) and (1.10) up to a re-scaling. Next, we use a well known result on uniqueness given by Pucci and Serrin [31] to prove uniqueness of solutions to the free boundary problem (1.9) and (1.10).

Proposition 3.1. *Let $h \in X^*_{\mathrm{rad}}$ be a critical point of $\mathcal{J}(h)$. Then there exist $\mu, \lambda > 0$ such that $\bar{h}(x) = \frac{1}{\mu} h\left(\frac{1}{\lambda} x\right)$ satisfies that*

(i) *$\bar{h}$ is a solution to the initial value problem, for some $\alpha > 1$*

$$\bar{h}'' + \frac{d-1}{r}\bar{h}' + \bar{h}^m = 1, \quad \text{in supp } \bar{h} \cap \{r > 0\}, \tag{3.1}$$

$$\bar{h}(0) = \alpha, \quad \bar{h}'(0) = 0. \tag{3.2}$$

(ii) *$\bar{h}$ satisfies that*

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla \bar{h}|^2 \, \mathrm{d}x = \frac{1}{a+1}\int_{\mathbb{R}^d} \bar{h}^{m+1} \, \mathrm{d}x, \tag{3.3}$$

where $a = \frac{d+2(m+1)}{md}$. Denote

$$\bar{\mathcal{F}}(\bar{h}) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla \bar{h}|^2 \, \mathrm{d}x - \frac{1}{a+1}\int_{\mathbb{R}^d} \bar{h}^{m+1} \, \mathrm{d}x. \tag{3.4}$$

Hence $\bar{\mathcal{F}}(\bar{h}) = 0$.

(iii) *there is a finite point $R \in (0, \infty)$ such that $\bar{h}(R) = 0$.*

Proof.

Step 1. Re-scaling, admissible variation and the proof of (i).

Let $\lambda_1, \mu_1 > 0$ be two re-scaling parameters, to be determined in (3.5). Since h is a critical point, $h_1(y) := \frac{1}{\mu_1} h(\frac{1}{\lambda_1} y)$, $\mathcal{J}(h_1) = \mathcal{J}(h)$ due to (2.13), hence h_1 is also a critical point ($\frac{\delta \mathcal{J}(h_1)}{\delta h} = 0$). Choose μ_1 and λ_1 such that the following two equalities hold

$$\|h_1\|_{L^1} = \|h_1\|_{L^{m+1}} = 1, \quad \text{and denote } a_1 := \|\nabla h_1\|^2_{L^2}. \tag{3.5}$$

Since $h_1 \in H^1_{\mathrm{rad}}$, it is observed that $h_1(r)$ is continuous in $(0, \infty)$. Denote the support of h_1 as $\Omega := \{x \in \mathbb{R}^d | \, h_1 > 0\}$. Since $h_1(r)$ is a radial decreasing function, one knows that $0 \in \Omega$ and Ω is an open ball $B(0, R)$ for some $0 < R \leqslant \infty$. For any $\phi \in C_0^\infty(\Omega)$, we can show that ϕ is an admissible variation at h_1, i.e. there is an $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$ one has $h_1 + \varepsilon\phi \geqslant 0$. Then from a direct computation and using (3.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{J}(h_1 + \varepsilon\phi) = \int_{\mathbb{R}^d} (-2\Delta h_1 + (a-1)a_1 - (a+1)a_1 h_1^m)\phi \, \mathrm{d}x = 0, \quad \forall \phi \in C_0^\infty(\Omega).$$

This implies that h_1 satisfies the following generalized Lane–Emden equation

$$\Delta h_1 - \frac{a-1}{2}a_1 + \frac{a+1}{2}a_1 h_1^m = 0, \quad \text{in } \mathcal{D}'(B(0, R)). \tag{3.6}$$

Again re-scale function h_1 as $\bar{h}(y) = \frac{1}{\mu} h_1(\frac{y}{\lambda})$, where $\lambda, \mu > 0$ are given by (3.8) below. Similar to h_1, we know that $\bar{h}$ is also a critical point of $\mathcal{J}(h)$ in X^*_{rad}. Moreover, (3.6) implies that $\bar{h}$ satisfies the following equation

$$\mu\lambda^2 \Delta \bar{h} - \frac{a-1}{2}a_1 + \frac{a+1}{2}a_1 \mu^m \bar{h}^m = 0, \quad \text{in } \mathcal{D}'(B(0, \lambda R)). \tag{3.7}$$

Taking

$$\mu = \left(\frac{a-1}{a+1}\right)^{1/m}, \quad \lambda^d = \left(\frac{a-1}{2\mu}\right)^{d/2} a_1^{d/2}, \tag{3.8}$$

one has

$$\mu\lambda^2 = \frac{a-1}{2}a_1 = \frac{a+1}{2}a_1\mu^m. \tag{3.9}$$

Then above equation (3.7) becomes

$$\Delta\bar{h} - 1 + \bar{h}^m = 0, \quad \text{in } \mathcal{D}'(B(0, \lambda R)). \tag{3.10}$$

Using the elliptic regularity iteratively and $1-\bar{h}^m \in L^{\frac{m+1}{m}}(B(0,R_0))$ with $0 < R_0 < \lambda R$, it holds that

$$\bar{h} \in W^{k,\,q}(B(0,R_0)) \text{ for some } q > 1,\ (k-1)q > d.$$

Hence $\bar{h}$ is C^1- function in $B(0,R_0)$. Thus we denote the peak value as $\alpha = \bar{h}(0)$, and we have $\bar{h}'(0) = 0$.

Now we prove that $\bar{h}(0) > 1$ by using a contradiction method. We assume that $\bar{h}(0) \leqslant 1$. Decreasing property of $\bar{h}(r)$ in r implies that for any fixed $R > 0$, $x \in B(0,R)$, $\bar{h}(x) \leqslant 1$. Thus we have

$$\Delta\bar{h} = 1 - \bar{h}^m \geqslant 0, \quad \text{in } B(0,R). \tag{3.11}$$

From the maximum principle, we know that the maximum of $\bar{h}$ is reached at $|x| = R$. Since $\bar{h}$ is a radial decreasing continuous function, it holds that $\bar{h}(x) \equiv \bar{h}(R)$ in $B(0,R)$ for any $R > 0$. Plugging this constant solution into (3.11), one knows that $\bar{h}(x) \equiv 1$ in $B(0,R)$ for any $R > 0$. Hence $\bar{h}(x) \equiv 1$ in $\mathbb{R}^d$. It is contradictory to the integrability of $\bar{h}$. Hence $\bar{h}$ satisfies (3.1) and (3.2). This completes the proof of (i).

Step 2. The proof of (ii): $\bar{\mathcal{F}}(\bar{h}) = 0$.

From (3.5) and the definition of $\bar{h}$, we know that

$$1 = \int_{\mathbb{R}^d} h_1 \,\mathrm{d}x = \frac{\mu}{\lambda^d}\int_{\mathbb{R}^d} \bar{h}\,\mathrm{d}x,$$
$$1 = \int_{\mathbb{R}^d} h_1^{m+1} \,\mathrm{d}x = \frac{\mu^{m+1}}{\lambda^d}\int_{\mathbb{R}^d} \bar{h}^{m+1}\,\mathrm{d}x,$$
$$a_1 = \int_{\mathbb{R}^d} |\nabla h_1|^2 \,\mathrm{d}x = \frac{\mu^2\lambda^2}{\lambda^d}\int_{\mathbb{R}^d} |\nabla\bar{h}|^2\,\mathrm{d}x.$$

Hence (3.9) gives

$$\int_{\mathbb{R}^d} \bar{h}\,\mathrm{d}x = \frac{\lambda^d}{\mu}, \quad \int_{\mathbb{R}^d} \bar{h}^{m+1}\,\mathrm{d}x = \frac{a+1}{a-1}\frac{\lambda^d}{\mu}, \quad \int_{\mathbb{R}^d} |\nabla\bar{h}|^2\,\mathrm{d}x = \frac{2}{a-1}\frac{\lambda^d}{\mu}.$$

Together with (3.8), a simple computation gives

$$\int_{\mathbb{R}^d} \bar{h}\,\mathrm{d}x = 2^{-\frac{d}{2}}(a-1)^{\frac{d}{2}-\frac{1}{m}(\frac{d}{2}+1)}(a+1)^{\frac{1}{m}(\frac{d}{2}+1)}a_1^{\frac{d}{2}}, \tag{3.12}$$

$$\int_{\mathbb{R}^d} \bar{h}^{m+1}\,\mathrm{d}x = 2^{-\frac{d}{2}}(a-1)^{\frac{d}{2}-1-\frac{1}{m}(\frac{d}{2}+1)}(a+1)^{1+\frac{1}{m}(\frac{d}{2}+1)}a_1^{\frac{d}{2}}, \tag{3.13}$$

$$\int_{\mathbb{R}^d} |\nabla\bar{h}|^2\,\mathrm{d}x = 2^{1-\frac{d}{2}}(a-1)^{\frac{d}{2}-1-\frac{1}{m}(\frac{d}{2}+1)}(a+1)^{\frac{1}{m}(\frac{d}{2}+1)}a_1^{\frac{d}{2}}. \tag{3.14}$$

Thus (3.13) and (3.14) imply (3.3). This completes the proof of (ii).

Step 3. The proof of (iii): there exists a finite R such that $\bar{h}(R) = 0$.

For a radial decreasing non-negative H^1-function, there only exist two cases: (a) there exists a finite R such that $\bar{h}(R) = 0$; (b) $\bar{h}(r) > 0$ for all $r \geqslant 0$, and hence $\bar{h}(r) \to 0$, $\bar{h}'(r) \to 0$ as $r \to \infty$.

Now we show that the second case can not happen. In fact, if (b) holds, then for any $r > 1$, integrating (3.1) from 0 to r and using $\bar{h}'(0) = 0$, we get

$$\bar{h}'(r) + \int_0^r \frac{d-1}{s}\bar{h}'(s)\mathrm{d}s + \int_0^1 \bar{h}(s)^m\,\mathrm{d}s + \int_1^r \bar{h}(s)^m\,\mathrm{d}s = r.$$

Noticing that $\bar{h}'(s) \leqslant 0$ for $s \in (0, \infty)$, we have

$$\lim_{r\to\infty} \int_1^r \bar{h}(s)^m\,\mathrm{d}s = \infty.$$

Therefore

$$\int_1^\infty \bar{h}(s)^m s^{d-1}\,\mathrm{d}s \geqslant \int_1^\infty \bar{h}(s)^m\,\mathrm{d}s = \infty,$$

which is contradictory to boundedness of the L^m-norm of $\bar{h}$ due to $\bar{h} \in L^1(\mathbb{R}^d) \cap L^{m+1}(\mathbb{R}^d)$. Hence there exists a finite R such that $\bar{h}(R) = 0$. This completes the proof of (iii) and hence proposition 3.1 is proved. □

Lemma 3.1. *Let h be a solution to the initial value problem* (3.1) *and* (3.2) *with a contact point R* ($h(R) = 0$)*. Then the following Pohozaev type identity between the free energy and the contact angle holds*

$$\bar{\mathcal{F}}(h) = \frac{d|B(0,R)|}{2(d+2)}h'(R)^2. \tag{3.15}$$

Proof. We can view (3.1) as an equation for a nonlinear oscillator with damping in $(0, R)$, and introduce the energy function

$$H(r) := \frac{1}{2}(h'(r))^2 + \frac{1}{m+1}h^{m+1}(r) - h(r). \tag{3.16}$$

Then by multiplying h' to (3.1), we have the following energy-dissipation relation

$$\frac{\mathrm{d}H(r)}{\mathrm{d}r} + \frac{d-1}{r}(h')^2 = 0. \tag{3.17}$$

Multiplying r^d to (3.17) and integrating from 0 to R, one has

$$\int_0^R r^d\left(\frac{\mathrm{d}H(r)}{\mathrm{d}r}+\frac{d-1}{r}(h'(r))^2\right)\mathrm{d}r=0,$$

i.e.

$$R^dH(R)-d\int_0^R r^{d-1}H(r)\,\mathrm{d}r+(d-1)\int_0^R (h'(r))^2r^{d-1}\,\mathrm{d}r=0.$$

Notice that $H(R)=\frac{1}{2}h'(R)^2$ from (3.16). Then with some simple computations, it holds that

$$\begin{aligned}\frac{1}{2}R^dh'(R)^2&=d\int_0^R r^{d-1}H(r)\,\mathrm{d}r-(d-1)\int_0^R (h'(r))^2r^{d-1}\,\mathrm{d}r\\&=(1-d/2)\int_0^R r^{d-1}(h'(r))^2\,\mathrm{d}r+\frac{d}{(m+1)}\int_0^R r^{d-1}h^{m+1}\,\mathrm{d}r-d\int_0^R r^{d-1}h\,\mathrm{d}r\\&=(1-d/2)\frac{1}{S_d}\int_{B(0,R)}|\nabla h|^2\,\mathrm{d}x+\frac{d}{(m+1)}\frac{1}{S_d}\int_{B(0,R)}h^{m+1}\,\mathrm{d}x-d\frac{1}{S_d}\int_{B(0,R)}h\,\mathrm{d}x,\end{aligned}$$

where S_d is the surface area of the unit ball in $\mathbb{R}^d$.

On the other hand, multiplying h to (3.1) and integrating in $B(0, R)$, we obtain

$$\int_{B(0,R)}h\,\mathrm{d}x=-\int_{B(0,R)}|\nabla h|^2\,\mathrm{d}x+\int_{B(0,R)}h^{m+1}\,\mathrm{d}x. \tag{3.18}$$

Hence, using $a=\frac{d+2(m+1)}{md}$, we have

$$\frac{1}{2}S_dR^d|h'(R)|^2=(1+d/2)\int_{B(0,R)}|\nabla h|^2-\frac{dm}{(m+1)}\int_{B(0,R)}h^{m+1}=(d+2)\bar{\mathcal{F}}(h).$$

This gives (3.15). □

Corollary 3.1. *Let $h\in X^*_{\mathrm{rad}}$ be a critical point of $\mathcal{J}(h)$. Then there exist re-scaling parameters $\lambda,\mu>0$ such that $\bar{h}=\frac{1}{\mu}h(\frac{1}{\lambda}x)$ satisfies the free boundary problem (1.9) and (1.10).*

Proof. As a direct consequence of (3.15) and $\bar{\mathcal{F}}(\bar{h})=0$, one knows that $\bar{h}'(R)=0$. In other words, the contact angle is zero. This case is the so-called complete wetting regime in Young's law [20]. □

Remark 3.1. A simple computation gives that $\bar{h}$ satisfies

$$\bar{h}''(R^-)=1,\quad \bar{h}'''(R^-)=-\frac{d-1}{R}. \tag{3.19}$$

Uniqueness of solutions to the free boundary problem (1.9) and (1.10) can be proved by a direct verification for conditions in a uniqueness theorem of Pucci-Serrin [31]. We recall it below.

Lemma 3.2 ([31, theorem 3]). *The free boundary problem*

$$\begin{aligned} &h'' + \frac{d-1}{r}h' + f(h) = 0, \quad \text{in } 0<r<R,\\ &h'(0)=0, \quad h(R)=h'(R)=0 \end{aligned}$$

has a unique radial solution if $f(h)$ satisfies the following conditions

(i) *f is locally integrable on $[0,\infty)$. In particular, the integral $F(h)=\int_0^h f(\tau)\,d\tau$ exists, and $F(h)\to 0$ as $h\to 0$;*
(ii) *f is continuously differentiable on $(0,\infty)$;*
(iii) *there exists $a>0$ such that $f(a)=0$ and*

$$\begin{aligned} &f(h)<0 \quad \text{for } 0<h<a,\\ &f(h)>0 \quad \text{for } a<h<\infty; \end{aligned}$$

(iv) *$F(h)$ and $f(h)$ satisfy the following relation*

$$\frac{\mathrm{d}}{\mathrm{d}h}\left[\frac{F(h)}{f(h)}\right] \geqslant \frac{d-2}{2d}. \tag{3.20}$$

Proposition 3.2. *Assume $m>0$ for $d=1,2$, $0<m<\frac{d+2}{d-2}$ for $d\geqslant 3$. Then there is a unique solution $h(r)$ to the free boundary problem (1.9) and (1.10) in X^*_{rad} and it satisfies $\alpha=h(0)>1$, $h'(r)<0$ for $0<r<R$.*

Proof. By proposition 2.1, we know that there is a minimizer h of the functional $\mathcal{J}(h)$ in X^*_{rad}, and hence h is a critical point of the functional $\mathcal{J}(h)$. From corollary 3.1, we know that the minimizer h of the functional $\mathcal{J}(h)$ in X^*_{rad} is a solution to the free boundary problem (1.9) and (1.10). Hence existence was proved.

To prove uniqueness, we only need to verify that $f(h):=h^m-1$ satisfies the conditions (i)–(iv) in lemma 3.2. The conditions (i)–(iii) are obvious for $f(h)$ with $m>0$.

Now we verify the condition (iv) for $f(h)$ with $m>0$ for $d=1,2$, $0<m<\frac{d+2}{d-2}$ for $d\geqslant 3$. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}h}\left[\frac{F(h)}{f(h)}\right] - \frac{d-2}{2d} = 1 - \frac{mh^{m-1}\left(\frac{h^{m+1}}{m+1}-h\right)}{(h^m-1)^2} - \frac{d-2}{2d}.$$

Obviously, if $d=1,2$, then $\frac{\mathrm{d}}{\mathrm{d}h}\left[\frac{F(h)}{f(h)}\right]-\frac{d-2}{2d}\geqslant 0$ for $h\geqslant 0$ and $m>0$. For the case $0<m<\frac{d+2}{d-2}$ with $d\geqslant 3$, a simple computation gives

$$\frac{\mathrm{d}}{\mathrm{d}h}\left[\frac{F(h)}{f(h)}\right] - \frac{d-2}{2d} = \frac{d-2}{2d}\frac{\left(\frac{2d}{(m+1)(d-2)}-1\right)h^{2m} + \left(\frac{2d(m-2)}{d-2}+2\right)h^m + \left(\frac{2d}{d-2}-1\right)}{(h^m-1)^2}.$$

Denote $v:=h^m$, $\xi:=\frac{2d}{(m+1)(d-2)}-1$, $\zeta:=\frac{2d(m-2)}{d-2}+2$, and $\gamma:=\frac{2d}{d-2}-1$. To verify (3.20), it is equivalent to prove

$$\xi v^2 + \zeta v + \gamma \geqslant 0.$$

Since $0 < m < \frac{d+2}{d-2}$, we have $\xi > 0$. Thus if $\zeta \geqslant 0$, i.e. $m \geqslant \frac{d+2}{d}$, then (iv) holds. If $m < \frac{d+2}{d}$, we only need to prove $\zeta^2 - 4\xi\gamma \leqslant 0$. So, we compute

$$\zeta^2 - 4\xi\gamma = -\frac{2dm}{(m+1)(d-2)}\left((3m - m^2)\frac{d+2}{d-2} - m(m+1)\right). \tag{3.21}$$

Due to $m < \frac{d+2}{d}$, we deduce $\zeta^2 - 4\xi\gamma \leqslant 0$, i.e. (iv) of lemma 3.2 holds.

From Step 1 in the proof of proposition 3.1, we have $\alpha > 1$.

Finally, we use a contradiction method to prove $h'(r) < 0$ for $0 < r < R$. If it is not true, then there exists (choice to be the first one) $r_0 \in (0, R)$ such that $h'(r_0) = 0$.

Since $h \in X^*_{\mathrm{rad}}$ and $h(R) = 0$ for a finite $R > 0$, we can show that extremum points are never reached in the set $\{r | h(r) = 1\}$. Therefore all extremum points must be either local minimum points ($h''(r) > 0$) or local maximum points ($h''(r) < 0$). From the equation (1.9), at extremum points it holds that

$$h''(r) = 1 - h^m(r),$$

which implies that $h(r) > 1$ for maximum points, $h(r) < 1$ for minimum points.

Solving (1.9) and (1.10), we have

$$h'(r) = -\frac{1}{r^{d-1}}\int_0^r s^{d-1}(h(s)^m - 1)\,\mathrm{d}s, \text{ for any } r \in (0, R). \tag{3.22}$$

Thus for any r satisfying $h(r) \geqslant 1$, (3.22) implies $h'(r) < 0$. The first extremum point r_0 must be a local minimum point, and it satisfies $h(r_0) < 1$.

Since extremum points must be either local minimum points or local maximum points, the next extremum point after r_0 must be a local maximum point, denoted by r_1, which satisfies $h(r_1) > 1$. Inductively, we order all extremum points as a sequence $r_0, r_1, r_2, \cdots$, and at these points $h(r)$ satisfies

$$h(r_{2(k-1)}) < 1, \text{ for } k = 1, 2, \cdots, \tag{3.23}$$

$$h(r_{2k-1}) > 1, \text{ for } k = 1, 2, \cdots. \tag{3.24}$$

Now we claim that the sequence $\{h(r_{2(k-1)})\}_{k=1}^{\infty}$ of local minimum is increasing and the sequence $\{h(r_{2k-1})\}_{k=1}^{\infty}$ of local maximum is decreasing.

From (3.17), we deduce

$$H(r_{2(k-1)}) > H(r_{2k}),$$

i.e.

$$\frac{1}{m+1}h(r_{2(k-1)})^{m+1} - h(r_{2(k-1)}) > \frac{1}{m+1}h(r_{2k})^{m+1} - h(r_{2k}).$$

Notice that $f(s) = \frac{1}{m+1}s^{m+1} - s$ is decreasing for $s \in (0, 1)$, and is increasing for $s \in (1, \infty)$. Hence (3.23) indicates that $h(r_{2(k-1)}) < h(r_{2k})$. In a similar way, we have $h(r_{2k+1}) < h(r_{2k-1})$ by (3.24). In other words, the solution is oscillatory around the value $h = 1$ and has decreasing amplitude, and hence it will never touchdown. This is a contradiction to $h(R) = 0$ for a finite $R > 0$. Hence $h > 0,\ h'(r) < 0$ before touching down at $r = R$. □

As an additional result, all critical points of $\mathcal{J}(h)$ in non-radial case also satisfy a free boundary problem.

Proposition 3.3. *Let a non-negative function $h \in C(\mathbb{R}^d)$ be a critical point of $\mathcal{J}(h)$ with support set $\Omega := \operatorname{supp} h$. Assume Ω is a bounded open star domain with the vantage point 0 and $\Gamma := \partial\Omega$ is its smooth boundary. Then h satisfies the following free boundary problem up to a re-scaling*

$$\Delta h + h^m = 1, \quad \text{in } \Omega, \tag{3.25}$$

$$h = \partial_{\vec{n}} h = 0, \ \text{on } \Gamma. \tag{3.26}$$

Proof. Similar to Step 1 and Step 2 in the proof of proposition 3.1, there exist $\lambda, \mu > 0$ such that $\bar{h}(x) = \frac{1}{\mu}h(\frac{x}{\lambda})$, $\Omega_0 = \operatorname{supp} \bar{h}$, and $\bar{h}$ satisfies

$$\Delta\bar{h} + \bar{h}^m = 1, \ \text{in } \Omega_0, \tag{3.27}$$

$$\bar{\mathcal{F}}(\bar{h}) = 0. \tag{3.28}$$

We also know that Ω_0 is a bounded open star domain with the vantage point 0. Let $\vec{n}$ be out normal of $\partial\Omega_0$ so that $x \cdot \vec{n} > 0$ on $\partial\Omega_0$.

Noticing that $\bar{h} = 0$ on $\partial\Omega_0$, $\bar{h} > 0$ in Ω_0, one has

$$\nabla\bar{h} = -|\nabla\bar{h}|\vec{n}. \tag{3.29}$$

Below we show a Pohozaev type identity connecting the free energy to the contact angle

$$\bar{\mathcal{F}}(\bar{h}) = \frac{1}{2(d+2)} \int_{\partial\Omega_0} (x \cdot \vec{n})|\nabla\bar{h}|^2 \, ds. \tag{3.30}$$

Indeed, multiplying $\nabla \cdot (x\bar{h})$ to (3.27), one has

$$\int_{\Omega_0} \nabla \cdot (x\bar{h})\Delta\bar{h} \, dx = \int_{\Omega_0} \nabla \cdot (x\bar{h})(1 - \bar{h}^m) \, dx. \tag{3.31}$$

Notice that

$$\int_{\Omega_0} \nabla \cdot (x\bar{h})(1 - \bar{h}^m) \, dx = -\int_{\Omega_0} (x\bar{h})\nabla(1 - \bar{h}^m) \, dx = -\frac{dm}{m+1} \int_{\Omega_0} \bar{h}^{m+1} \, dx. \tag{3.32}$$

Using (3.29), we have

$$\begin{aligned} \int_{\Omega_0} \nabla \cdot (x\bar{h})\Delta\bar{h} \, dx &= \int_{\Omega_0} \nabla(\nabla \cdot (x\bar{h})) \cdot \nabla\bar{h} \, dx + \int_{\partial\Omega_0} \nabla \cdot (x\bar{h})\partial_{\vec{n}}\bar{h} \, ds \\ &= -\frac{d+2}{2} \int_{\Omega_0} |\nabla\bar{h}|^2 \, dx + \frac{1}{2} \int_{\partial\Omega_0} (x \cdot \vec{n})|\nabla\bar{h}|^2 \, ds. \end{aligned} \tag{3.33}$$

Hence from (3.31)–(3.33), we have

$$(d+2)F(\bar{h}) = \frac{1}{2}\int_{\partial\Omega_0} (\vec{n}\cdot x)|\nabla \bar{h}|^2 \, ds,$$

i.e. (3.30) holds true.

Since $\bar{\mathcal{F}}(\bar{h}) = 0$ and $\vec{n}\cdot x > 0$, we know that

$$\partial_{\vec{n}}\bar{h} = \nabla\bar{h}\cdot\vec{n} = 0 \text{ a.e. on } \partial\Omega_0 . \tag{3.34}$$

Summarizing above process, $\bar{h}$ satisfies the free boundary problem

$$\Delta\bar{h} + \bar{h}^m = 1 \text{ in } \Omega_0 , \tag{3.35}$$

$$\bar{h} = \partial_{\vec{n}}\bar{h} = 0 \text{ on } \partial\Omega_0 . \tag{3.36}$$

Hence we complete the proof of proposition 3.3. □

Proposition 3.4. *Let h be a solution to the free boundary problem (3.25) and (3.26). Denote $\Omega := \{x|h(x) > 0\}$. Assume that Ω is a bounded open domain with C^2 boundary (not be assumed simply connected) and $h \in C^2(\bar{\Omega})$. Then Ω is a ball and h is radial symmetric.*

Proof. The proof of proposition 3.4 is a direct application of [30, theorem 8.3.2] with $A(z, s) = 1, f(z, s) = z^m - 1$ (here we use the same notations as these used in [30]), because $A(z, s)$ and $f(z, s)$ satisfy all the conditions in theorem 8.3.2. □

4. Proof of theorem 1.1

Existence of a minimizer h_c for the functional $\mathcal{J}(h)$ was given in proposition 2.1 with $q = 0$ and $p = 2$, i.e.

$$\mathcal{J}(h_c) = \inf_{h\in X} \mathcal{J}(h) = \beta_0.$$

The minimizer h_c is also a critical point of $\mathcal{J}(h)$, hence corollary 3.1, proposition 3.2 tell us that h_c is a unique solution to the free boundary problem (1.9) and (1.10) and satisfies $h_c'(r) < 0$ for $r \in (0, R)$. Similar to Step 1 and Step 2 in the proof of proposition 3.1, we can deduce that h_c satisfies

$$\int_{\mathbb{R}^d} h_c \, dx = 2^{-\frac{d}{2}}(a-1)^{\frac{d}{2}-\frac{1}{m}(\frac{d}{2}+1)}(a+1)^{\frac{1}{m}(\frac{d}{2}+1)}\beta_0^{\frac{d}{2}}, \tag{4.1}$$

$$\int_{\mathbb{R}^d} h_c^{m+1} \, dx = 2^{-\frac{d}{2}}(a-1)^{\frac{d}{2}-1-\frac{1}{m}(\frac{d}{2}+1)}(a+1)^{1+\frac{1}{m}(\frac{d}{2}+1)}\beta_0^{\frac{d}{2}}, \tag{4.2}$$

$$\int_{\mathbb{R}^d} |\nabla h_c|^2 \, dx = 2^{1-\frac{d}{2}}(a-1)^{\frac{d}{2}-1-\frac{1}{m}(\frac{d}{2}+1)}(a+1)^{\frac{1}{m}(\frac{d}{2}+1)}\beta_0^{\frac{d}{2}}. \tag{4.3}$$

Define $M_c = \int_{\mathbb{R}^d} h_c \, dx$. From (4.1), one has

$$\beta_0 = 2(a-1)^{\frac{1}{m}(1+\frac{2}{d})-1}(a+1)^{-\frac{1}{m}(1+\frac{2}{d})}M_c^{\frac{2}{d}}. \tag{4.4}$$

Hence for any $h \in X$ the following inequality holds

$$\left(\int_{\mathbb{R}^d} |f|^{m+1}\,\mathrm{d}x\right)^{\frac{(a+1)}{m+1}} \leqslant \frac{1}{\beta_0}\left(\int_{\mathbb{R}^d} |f|\,\mathrm{d}x\right)^{a-1} \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}x. \tag{4.5}$$

Moreover from (2.13), the invariance of $\mathcal{J}(h)$ under a re-scaling $h_{\mu,\lambda}(x) = \mu h(\lambda x)$ implies that the above equality holds if $h = Ah_c(\lambda(x - x_0))$ for any $A > 0,\ \lambda > 0,\ x_0 \in \mathbb{R}^d$.

Finally, we derive the closed-form solution h_c for (1.9) and (1.10) for $d = 1$. We recall (1.9) and (1.10) for $d = 1$ as

$$h'' + h^m = 1, \quad \text{in supp } h \cap \{r > 0\}, \tag{4.6}$$

$$h'(0) = 0, \quad h(R) = h'(R) = 0. \tag{4.7}$$

By the energy functional (3.16) and the energy-dissipation relation (3.17), we know that the following equality holds

$$\frac{(h')^2}{2} + \frac{h^{m+1}}{m+1} - h = C. \tag{4.8}$$

Since $h(R) = h'(R) = 0$, we have $C = 0$. Hence the conditions $h(0) = \alpha$ and $h'(0) = 0$ imply $\frac{\alpha^{m+1}}{m+1} - \alpha = 0$, i.e. $\alpha = (m+1)^{\frac{1}{m}} > 1$. Solving (4.8), one has

$$h'(r) = -\sqrt{2\left(h - \frac{h^{m+1}}{m+1}\right)}. \tag{4.9}$$

Integrating (4.9) with respect to r in $(0, R)$, a series of computations give

$$R = 2^{-1/2} m^{-1} (m+1)^{\frac{1}{2m}} \mathcal{B}\left(\frac{1}{2}, \frac{1}{2m}\right).$$

Moreover, integrating (4.9) with respect to r from r to R for any $r \in (0, R)$, we deduce

$$h_c(r) = \left((m+1)B^{-1}\left(\sqrt{2}\,m(m+1)^{-\frac{1}{2m}}(R - r); \frac{1}{2}, \frac{1}{2m}\right)\right)^{\frac{1}{m}},$$

where B^{-1} is the inverse of the incomplete Beta function $B(x;\text{a },b)$, which is defined as

$$B(x; a, b) := \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}x.$$

Now we compute the minimum β_0 of the functional $\mathcal{J}(h)$. By (4.3) and $a = \frac{d+2(m+1)}{md}$, we know that

$$2^{-\frac{1}{2}}\left(\frac{m+3}{m}\right)^{-\frac{1}{2}-\frac{3}{2m}}\left(\frac{3(m+1)}{m}\right)^{\frac{3}{2m}}\beta_0^{\frac{1}{2}} = \frac{1}{2}\int_{-R}^{R} h_c' \cdot h_c'\,\mathrm{d}x = \int_0^R h_c' \cdot h_c'\,\mathrm{d}x. \tag{4.10}$$

On the other hand, we compute the right hand side of the above equality

$$\begin{aligned}\int_0^R h_c' \cdot h_c'\,\mathrm{d}x = \int_0^\alpha h_c'\,\mathrm{d}h_c = \int_0^\alpha \sqrt{2(h - \frac{h^{m+1}}{m+1})}\,\mathrm{d}h \\ = 2^{\frac{1}{2}} m^{-1}(m+1)^{\frac{3}{2m}}\mathcal{B}\left(\frac{3}{2}, \frac{3}{2m}\right).\end{aligned} \tag{4.11}$$

Then (4.10) and (4.11) imply

$$\beta_0 = 4 \cdot 3^{-\frac{3}{m}} m^{-3} (m+3)^{1+\frac{3}{m}} \left(\mathcal{B}\left(\frac{3}{2}, \frac{3}{2m} \right) \right)^2. \tag{4.12}$$

Furthermore, from (4.4) we obtain

$$M_c = \|h_c\|_{L^1} = 2^{\frac{1}{2}} m^{-2} (m+1)^{\frac{3}{2m}} (m+3) \mathcal{B}\left(\frac{3}{2}, \frac{3}{2m} \right).$$

Therefore we have finished all the proofs of theorem 1.1.

Remark 4.1. The results in theorem 1.1 agree with the following classical results for some special cases:

- In 1941, Sz. Nagy [27] obtained the best constant $\beta_0 = \frac{4\pi^2}{9}$ for $m = 3$ in (4.12) (this case is known as the Sz. Nagy inequality).
- In 1958, Nash deduced the best constant $\beta_0 = \frac{16\pi^2}{27}$ for $m = 1$ in (4.12) (this case is known as the Nash inequality [28]).

5. Finite time blow-up and spreading phenomenon for thin film equation

In this section, we show that M_c is the critical mass to the following higher dimensional thin film equation

$$h_t + \nabla \cdot (h \nabla \Delta h) + \nabla \cdot (h \nabla h^m) = 0, \quad x \in \mathbb{R}^d$$

with $d \geqslant 2$ and the critical exponent $m = 1 + \frac{2}{d}$. We impose the following initial condition

$$h(x, 0) = h_0(x), \ x \in \mathbb{R}^d. \tag{5.1}$$

Here we consider the following initial data:

$$h_0 \geqslant 0, \ \text{supp}\ h_0 \in B(0, a) \text{ for some } a > 0, \ h_0(x) \in L^1(\mathbb{R}^d). \tag{5.2}$$

Notice that non-negative solutions $h(x, t)$ to (1.1) satisfy conservation of mass, i.e. formula

$$\int_{\mathbb{R}^d} h(x, t)\, \mathrm{d}x \equiv \int_{\mathbb{R}^d} h_0(x)\, \mathrm{d}x =: m_0.$$

Following Bernis and Friedman [4], we define an entropy weak solution.

Definition 5.1. We say that a non-negative function

$$\begin{aligned} &h \in L^\infty(0, T; H^1(\mathbb{R}^d)), \ \Delta h \in L^2(0, T; L^2(\mathbb{R}^d)), \\ &\partial_t h \in L^2(0, T; H^{-1}(\mathbb{R}^d)), \ h^{1/2} \nabla \Delta h \in L^2(P_T), \\ &P_T := \Omega \times (0, T] \setminus \{(x, t) | h(x, t) = 0\} \end{aligned}$$

is an entropy weak solution to (1.1) and (5.1) in $[0, T)$ provided that

- The weak form holds for any function $\phi \in C_0^\infty(\mathbb{R}^d \times [0, T])$,

$$\int_0^T \int_{\mathbb{R}^d} \phi h_t \, \mathrm{d}x\mathrm{d}t = \iint_{P_T} \nabla\phi \cdot \ h \nabla \Delta h \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \nabla \phi \ \cdot h \nabla h^m \, \mathrm{d}x\mathrm{d}t.$$

- $\mathcal{F}(h(\cdot,t))$ is a non-increasing function in t and satisfies the following entropy-dissipation inequality

$$\mathcal{F}(h(\cdot,t))+\iint_{P_T} h|\nabla(\Delta h+h^m)|^2\ \mathrm{d}x\mathrm{d}t\leqslant\mathcal{F}(h_0),\quad \text{for any } 0\leqslant t<T. \tag{5.3}$$

Definition 5.2. We say that a non-negative solution $h(x,t)$ to the model (1.1) blows up at $T^r_{\max}$ in $L^r(\mathbb{R}^d)$, $r>1$, if it satisfies

$$\|h(\cdot,t)\|_{L^r}<\infty,\ \text{for all } 0<t<T^r_{\max},\quad \limsup_{t\to T^r_{\max}}\|h(\cdot,t)\|_{L^r}=\infty.$$

$T^r_{\max}$ is called the blow-up time of $\|h(\cdot,t)\|_{L^r}$.

Theorem 5.1. *For any $m_0>M_c$, there exists h_0 satisfying $\int_{\mathbb{R}^d}h_0(x)\,\mathrm{d}x=m_0$ and $\int_{\mathbb{R}^d}|x|^2h_0(x)\,\mathrm{d}x<\infty$ such that any weak solution to (1.1) with the initial datum h_0 has a finite time blow-up in $L^{m+1}(\mathbb{R}^d)$.*

Proof. Recall that the critical mass $M_c:=\|h_c\|_{L^1}$ for $d\geqslant 2$. We construct an initial datum

$$h_0(x):=(1+\varepsilon)h_c(x),\quad \varepsilon=\frac{m_0-M_c}{M_c}>0. \tag{5.4}$$

It satisfies $\int_{\mathbb{R}^d}h_0(x)\,\mathrm{d}x=m_0>M_c$, and $\int_{\mathbb{R}^d}|x|^2h_0(x)\,\mathrm{d}x<\infty$ because h_c has a compact support. Moreover, a simple computation gives

$$\begin{aligned}\mathcal{F}(h_0)&=\frac{(1+\varepsilon)^2}{2}\int_{\mathbb{R}^d}|\nabla h_c(x)|^2\,\mathrm{d}x-\frac{(1+\varepsilon)^{m+1}}{m+1}\int_{\mathbb{R}^d}h_c^{m+1}(x)\,\mathrm{d}x\\&=\frac{(1+\varepsilon)^2}{2}\int_{\mathbb{R}^d}|\nabla h_c(x)|^2\,\mathrm{d}x-\frac{(1+\varepsilon)^2}{2}(1+\varepsilon)^{m-1}\int_{\mathbb{R}^d}|\nabla h_c(x)|^2\,\mathrm{d}x\\&\quad+(1+\varepsilon)^{m+1}\left(\int_{\mathbb{R}^d}\frac{1}{2}|\nabla h_c(x)|^2\,\mathrm{d}x-\frac{1}{m+1}\int_{\mathbb{R}^d}h_c^{m+1}(x)\,\mathrm{d}x\right).\end{aligned} \tag{5.5}$$

Since $m=1+\frac{2}{d}$ implies that $a=\frac{d+2(m+1)}{md}=m$, from the property (3.3) we easily obtain

$$\frac{1}{2}\int_{\mathbb{R}^d}|\nabla h_c(x)|^2\,\mathrm{d}x=\frac{1}{m+1}\int_{\mathbb{R}^d}h_c^{m+1}(x)\,\mathrm{d}x.$$

Hence we have that

$$\mathcal{F}(h_0)=\frac{(1+\varepsilon)^2}{2}(1-(1+\varepsilon)^{m-1})\int_{\mathbb{R}^d}|\nabla h_c(x)|^2\,\mathrm{d}x<0. \tag{5.6}$$

Now we need to prove that the L^{m+1}-norm of solutions blows up in finite time $T^{m+1}_{\max}$. If not, then for any $t>0$, an entropy solution exists in $L^{m+1}(\mathbb{R}^d)$. So, we compute the second moment (see [34] given by Witelski *et al* for the one-dimensional case)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m_2(t) &= 2\int_{\mathbb{R}^d} hx\cdot\nabla(\Delta h + h^m)\,\mathrm{d}x\\ &= -2\int_{\mathbb{R}^d} dh(\Delta h + h^m)\,\mathrm{d}x - 2\int_{\mathbb{R}^d} x\cdot\nabla h(\Delta h + h^m)\,\mathrm{d}x\\ &= 2d\int_{\mathbb{R}^d}|\nabla h|^2\,\mathrm{d}x - 2d\int_{\mathbb{R}^d} h^{m+1}\,\mathrm{d}x + 2\int_{\mathbb{R}^d}\nabla\left(\sum_{i=1}^{\infty} x_i h_{x_i}\right)\cdot\nabla h\,\mathrm{d}x - 2\int_{\mathbb{R}^d}\frac{x\cdot\nabla h^{m+1}}{m+1}\,\mathrm{d}x\\ &= (d+2)\int_{\mathbb{R}^d}|\nabla h|^2\,\mathrm{d}x - \frac{2dm}{m+1}\int_{\mathbb{R}^d} h^{m+1}\,\mathrm{d}x.\end{aligned}$$

Noticing that $m = 1 + \frac{2}{d}$ and (5.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}m_2(t) = 2(d+2)\mathcal{F}(h(\cdot,t)) \leqslant 2(d+2)\mathcal{F}(h_0) < 0. \tag{5.7}$$

Since the initial second moment is finite, then there exists a finite time t^* such that $m_2(t^*) = 0$.

On the other hand, a simple computation shows that

$$\begin{aligned}\int_{\mathbb{R}^d} h\,\mathrm{d}x &= \int_{|x|\leqslant R} h\,\mathrm{d}x + \int_{|x|>R} h\,\mathrm{d}x\\ &\leqslant \|h\|_{L^{m+1}}(\alpha_d R^d)^{\frac{m}{m+1}} + \frac{1}{R^2}\int_{|x|>R}|x|^2 h\,\mathrm{d}x\\ &\leqslant \|h\|_{L^{m+1}}\alpha_d^{\frac{m}{m+1}} R^{\frac{dm}{m+1}} + \frac{1}{R^2}m_2(t).\end{aligned} \tag{5.8}$$

Taking

$$R = \alpha_d^{-\frac{m}{dm+2(m+1)}}\left(\frac{m_2(t)}{\|h\|_{L^{m+1}}}\right)^{\frac{m+1}{dm+2(m+1)}},$$

we have $\|h\|_{L^{m+1}}\alpha_d^{\frac{m}{m+1}}R^{\frac{dm}{m+1}} = \frac{1}{R^2}m_2(t)$. Hence we obtain from (5.8)

$$\int_{\mathbb{R}^d} h\,\mathrm{d}x \leqslant 2\|h\|_{L^{m+1}}\alpha_d^{\frac{m}{m+1}}R^{\frac{dm}{m+1}} = 2\alpha_d^{\frac{2m}{dm+2(m+1)}}(\|h\|_{L^{m+1}})^{\frac{2(m+1)}{dm+2(m+1)}}(m_2(t))^{\frac{dm}{dm+2(m+1)}},$$

which implies

$$\|h(\cdot,t)\|_{L^{m+1}} \geqslant \left(\frac{m_0}{2}\right)^{\frac{dm+2(m+1)}{2(m+1)}}\alpha_d^{-\frac{m}{m+1}}(m_2(t))^{-\frac{dm}{2(m+1)}}. \tag{5.9}$$

Hence from (5.9) and the fact $m_2(t^*) = 0$, we know that there is $T_{\max}^{m+1} \leqslant t^*$ such that

$$\limsup_{t\to T_{\max}^{m+1}}\|h(\cdot,t)\|_{L^{m+1}} = \infty,$$

which is a contradiction to global existence of entropy weak solutions in $L^{m+1}(\mathbb{R}^d)$. Hence solutions blow up in finite time in $L^{m+1}(\mathbb{R}^d)$. □

Remark 5.1. Using the interpolation inequality $\|h\|_{m+1}^{m+1} \leqslant \|h\|_{L^1} \|h\|_{L^\infty}^m$, we know that there is $T_{\max} \leqslant T_{\max}^{m+1}$ such that

$$\limsup_{t \to T_{\max}} \|h(\cdot, t)\|_{L^\infty} = \infty.$$

Furthermore, if the initial mass is less than M_c, an entropy weak solution exists globally [25] for the one-dimensional case. For a multi-dimensional thin film equation with an unstable diffusion term, Taranets and King [32] showed short-time existence of solutions for the problem (1.1) and (5.1) with $d = 2, 3$ in a bounded domain with the boundary condition (1.13). Moreover, they proved global existence of weak solutions to (1.1) and (5.1) for all initial conditions with sufficiently small mass, refer to [32, theorem 4]. For the whole space, existence of weak solutions in multi-dimension is still an open problem. However, if an entropy weak solution to (1.1) and (5.1) exists globally, then the second moment goes to infinity as $t \to \infty$ or $h(\cdot, t_k) \rightharpoonup 0$ in $L^1(\mathbb{R}^d)$ for some subsequence $t_k \to \infty$ if $m_0 < M_c$ as stated in theorem 5.2 below. This shows that a part of the mass spreads to infinity as $t \to \infty$.

Lemma 5.1. *Assume $h \in L^1_+(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$. Denoting $m_0 := \int_{\mathbb{R}^d} h \, \mathrm{d}x$, we have*

$$\mathcal{F}(h) \geqslant \frac{1}{2}\left(1 - \left(\frac{m_0}{M_c}\right)^{2/d}\right) \int_{\mathbb{R}^d} |\nabla h|^2 \, \mathrm{d}x \tag{5.10}$$

$$\geqslant \frac{1}{m+1}\left(\left(\frac{M_c}{m_0}\right)^{\frac{2}{d}} - 1\right) \int_{\mathbb{R}^d} h^{m+1} \, \mathrm{d}x. \tag{5.11}$$

Proof. From the generalized Sz. Nagy inequality (1.12) with $m = 1 + \frac{2}{d}$, one easily gets

$$\frac{1}{m+1} \int_{\mathbb{R}^d} |h|^{m+1} \, \mathrm{d}x \leqslant 2^{-1}\left(\frac{m_0}{M_c}\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} |\nabla h|^2 \, \mathrm{d}x, \tag{5.12}$$

which implies

$$\mathcal{F}(h) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla h|^2 \, \mathrm{d}x - \frac{1}{m+1} \int_{\mathbb{R}^d} h^{m+1} \, \mathrm{d}x \geqslant \frac{1}{2}\left(1 - \left(\frac{m_0}{M_c}\right)^{2/d}\right) \int_{\mathbb{R}^d} |\nabla h|^2 \, \mathrm{d}x. \tag{5.13}$$

Hence (5.10) holds. So, (5.12) and (5.10) imply (5.11). □

Theorem 5.2. *Assume that initial data h_0 satisfy (5.2), $m_0 < M_c$ and $\mathcal{F}(h_0) < \infty$. Let $h(x, t)$ be a global non-negative entropy weak solution of (1.1) with the initial condition (5.1) given by definition 5.1. Then*

$$\sup_{0<t<\infty} \{\|h(\cdot, t)\|_{L^{m+1}} + \|h(\cdot, t)\|_{H^1}\} \leqslant C(\|h_0\|_{H^1}, \mathcal{F}(h_0)), \tag{5.14}$$

and at least one of the following results holds

$$(a)\ \lim_{t\to\infty} m_2(t) = \infty, \tag{5.15}$$

$$(b)\ h(\cdot, t_k) \rightharpoonup 0 \text{ in } L^1(\mathbb{R}^d)\ \textit{for some subsequence}\ t_k \to \infty. \tag{5.16}$$

Proof. Since $m_0 < M_c$, the inequality (5.10) and $\mathcal{F}(h(\cdot,t)) \leqslant \mathcal{F}(h_0)$ indicate that

$$\mathcal{F}(h(\cdot,t)) \geqslant 0, \quad \int_{\mathbb{R}^d} |\nabla h(\cdot,t)|^2 \,\mathrm{d}x \leqslant \frac{2\mathcal{F}(h_0)}{1-\left(\frac{m_0}{M_c}\right)^{2/d}} =: C_0, \quad \text{for any } t > 0. \tag{5.17}$$

Here we used the fact that the free energy is decreasing in time t. So, (5.17) implies that (5.14) holds. And from $\mathcal{F}(h(\cdot,t)) \geqslant 0$, we know that there is a $\mathcal{F}_\infty$ such that

$$\lim_{t\to\infty} \mathcal{F}(h(\cdot,t)) = \mathcal{F}_\infty \geqslant 0.$$

On the other hand, a simple computation gives

$$\frac{\mathrm{d}}{\mathrm{d}t} m_2(t) = 2(d+2)\mathcal{F}(h(\cdot,t)) \geqslant 2(d+2)\mathcal{F}_\infty \geqslant 0, \tag{5.18}$$

which says that the second moment is increasing in t.

Now we prove that (a) or (b) holds. Suppose that

$$m_2(t) \not\to +\infty, \text{ as } t\to\infty. \tag{5.19}$$

By (5.18), we have that there exists a constant $\tilde{C} > 0$ such that $m_2(t) \leqslant \tilde{C}$ for any $t \in (0,\infty)$. In this case, we claim that there is a sequence t_k and h_∞ such that as $t_k \to \infty$, it holds that

$$h(t_k) \to h_\infty, \text{ strongly in } L^{m+1}(\mathbb{R}^d). \tag{5.20}$$

In fact, from (5.12) and (5.17), we have $h \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d))$. Noticing that the second moment is finite, we deduce that

(1) $\forall\, \varepsilon > 0$, there exists a $R_\varepsilon > 0$ such that

$$\begin{aligned}\int_{|x|>R_\varepsilon} h^{m+1}\,\mathrm{d}x &\leqslant \frac{(d+1)}{d} M_c^{-2/d} \left(\int_{|x|>R_\varepsilon} h\,\mathrm{d}x\right)^{2/d} \int_{\mathbb{R}^d} |\nabla h|^2\,\mathrm{d}x \\ &\leqslant C(C_0,d)\frac{1}{R_\varepsilon^2}\int_{|x|>R_\varepsilon} |x|^2 h\,\mathrm{d}x \\ &\leqslant C(C_0,d)\frac{m_2(t)}{R_\varepsilon^2}.\end{aligned}$$

Hence taking

$$R_\varepsilon \geqslant \left(\frac{2C(C_0,d)\tilde{C}}{\varepsilon}\right)^{1/2} \geqslant \left(\frac{2C(C_0,d)m_2(t)}{\varepsilon}\right)^{1/2}, \tag{5.21}$$

we obtain

$$\int_{|x|>R_\varepsilon} h^{m+1}\,\mathrm{d}x < \varepsilon, \text{ for any } t>0. \tag{5.22}$$

Then there is a subsequence t_k (without relabel) and $h_{1,\infty}$ such that

$$h(\cdot,t_k) \rightharpoonup h_{1,\infty}, \quad \text{as } t_k\to\infty \quad \text{in } L^{m+1}(|x|\geqslant R_\varepsilon)$$

and

$$\int_{|x|>R_\varepsilon} h_{1,\infty}^{m+1}\,\mathrm{d}x \leqslant \liminf_{k\to\infty}\int_{|x|>R_\varepsilon} h(x,t_k)^{m+1}\,\mathrm{d}x \leqslant \varepsilon. \tag{5.23}$$

(2) For a fixed R_ε satisfying (5.21), we know that $h(x,t_k)\in L^\infty(\mathbb{R}_+;H^1(B(0,R_\varepsilon)))$ by (5.12) and (5.17). Thus by the Sobolev embedding theorem, one obtains that there is a strong convergent subsequence, still denoted by $h(x,t_k)$, and $h_{2,\infty}$ such that

$$h(\cdot,t_k)\to h_{2,\infty}, \text{ strongly in } L^{m+1}(B(0,R_\varepsilon)). \tag{5.24}$$

Let h_∞ be the combination of $h_{1,\infty}$ and $h_{2,\infty}$ defined in $\mathbb{R}^d$. Hence, from (5.22)–(5.24), there is a K such that if $k\geqslant K$, then

$$\begin{aligned}\int_{\mathbb{R}^d}|h(x,t_k)-h_\infty|^{m+1}\,\mathrm{d}x &= \int_{|x|>R_\varepsilon}|h(x,t_k)-h_\infty|^{m+1}\,\mathrm{d}x + \int_{|x|\leqslant R_\varepsilon}|h(x,t_k)-h_\infty|^{m+1}\,\mathrm{d}x\\ &\leqslant C\int_{|x|>R_\varepsilon}|h(x,t_k)|^{m+1}+|h_\infty|^{m+1}\,\mathrm{d}x + \int_{|x|\leqslant R_\varepsilon}|h(x,t_k)-h_\infty|^{m+1}\,\mathrm{d}x\\ &< C\varepsilon,\end{aligned}$$

which proves our claim (5.20). Thus we have

$$\lim_{k\to\infty}\int_{\mathbb{R}^d} h^{m+1}(x,t_k)\,\mathrm{d}x = \int_{\mathbb{R}^d} h_\infty^{m+1}\,\mathrm{d}x. \tag{5.25}$$

On the other hand, by Fatou's lemma with (5.17), we know that

$$\nabla h(x,t_k)\rightharpoonup \nabla h_\infty, \text{ in } L^2(\mathbb{R}^d)$$

implies

$$\int_{\mathbb{R}^d}|\nabla h_\infty|^2\,\mathrm{d}x \leqslant \liminf_{k\to\infty}\int_{\mathbb{R}^d}|\nabla h(x,t_k)|^2\,\mathrm{d}x. \tag{5.26}$$

The formulas (5.25) and (5.26) give

$$\begin{aligned}\mathcal{F}(h_\infty) &= \frac{1}{2}\int_{\mathbb{R}^d}|\nabla h_\infty|^2\,\mathrm{d}x - \frac{1}{m+1}\int_{\mathbb{R}^d} h_\infty^{m+1}\mathrm{d}x\\ &\leqslant \liminf_{k\to\infty}\frac{1}{2}\int_{\mathbb{R}^d}|\nabla h(x,t_k)|^2\,\mathrm{d}x - \lim_{k\to\infty}\frac{1}{m+1}\int_{\mathbb{R}^d} h^{m+1}(x,t_k)\,\mathrm{d}x\\ &= \liminf_{k\to\infty}\mathcal{F}(h(\cdot,t_k)) = \mathcal{F}_\infty.\end{aligned} \tag{5.27}$$

Finally, noticing that $\|h(\cdot,t_k)\|_{L^1} = m_0$, $\|h(\cdot,t_k)\|_{L^{m+1}} \leqslant C$ and the second moment is finite, we have by the Dunford–Pettis theorem that as $k \to \infty$

$$h(\cdot,t_k) \rightharpoonup h_\infty, \text{ in } L^1(\mathbb{R}^d). \tag{5.28}$$

Hence Fatou's lemma implies

$$\int_{\mathbb{R}^d} h_\infty \,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^d} h(x,t_k)\,\mathrm{d}x = m_0 < M_c. \tag{5.29}$$

We have two cases: (i) $h_\infty = 0$, (ii) $h_\infty \neq 0$. In the case (i), by (5.28) there exists a subsequence t_k such that $h(\cdot,t_k) \rightharpoonup 0$ as $t_k \to \infty$. Thus (5.16) holds. In the case (ii), by the inequality (5.12), we know $\mathcal{F}(h_\infty) > 0$. Hence (5.27) gives $\mathcal{F}_\infty > 0$. Notice that

$$m_2(t) \geqslant m_2(0) + 2(d+2)\mathcal{F}_\infty t \to +\infty, \text{ as } t \to \infty,$$

which contradicts with (5.19). That implies (5.15). This finishes the proof of theorem 5.2. □

Acknowledgment

We would like to acknowledge some insightful discussions with Tom Witelski and Hangjie Ji. The work of J-G Liu was partially supported by KI-Net NSF RNMS grant No. 1107291, NSF DMS grant No. 1514826. Jinhuan Wang was partially supported by National Natural Science Foundation of China (Grant No. 11301243) and Program for Liaoning Excellent Talents in University (Grant No. LJQ2015041).

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