ERROR ESTIMATES OF THE AGGREGATION-DIFFUSION SPLITTING ALGORITHMS FOR THE KELLER-SEGEL EQUATIONS

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Abstract. In this paper, we discuss error estimates associated with three different aggregation-diffusion splitting schemes for the Keller-Segel equations. We start with one algorithm based on the Trotter product formula, and we show that the convergence rate is $C\Delta t$, where Δt is the time-step size. Secondly, we prove the convergence rate $C\Delta t^2$ for the Strang's splitting. Lastly, we study a splitting scheme with the linear transport approximation, and prove the convergence rate $C\Delta t$.

1. **Introduction.** In this paper we will consider the following Keller-Segel (KS) equations [8, 15] in $\mathbb{R}^d$ ($d \geq 2$):

$$\begin{cases} \partial_t \rho = \triangle \rho - \nabla \cdot (\rho \nabla c), \quad x \in \mathbb{R}^d,\ t > 0, \\ -\triangle c = \rho(t,x), \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{1}$$

This model is developed to describe the biological phenomenon chemotaxis. Here, $\rho(t,x)$ represents the bacteria density, and $c(t,x)$ represents the chemical substance concentration.

The most important feature of the KS model (1) is the competition between the aggregation term $-\nabla \cdot (\rho \nabla c)$ and the diffusion term $\Delta \rho$. In this paper, we develop three classes of positivity preserving aggregation-diffusion splitting algorithms for the Keller-Segel equations to handle the possible singularity. And we provide a

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rigorous proof of the fact that the solutions of these algorithms will converge to solutions of the Keller-Segel equations at a certain rate. The precise convergence rate will be given in Theorem 1.1 and Theorem 1.2 stated below after these algorithms have been defined. The convergence analysis for our aggregation-diffusion splitting algorithms are analog to that of the viscous splitting algorithms for the Navier-Stokes equations.

In fluid dynamics, the smooth solutions to the Euler equations are good approximations to the smooth solutions of the Navier-Stokes equations with small viscosity. This idea provides a method to approximate a solution to the Navier-Stokes equations by means of alternatively solving the inviscid Euler equations and a diffusion process over small time steps. Such approximations are called viscous splitting algorithms because they are forms of operator splitting in which the viscous term $\nu\Delta v$ is split from the inviscid part of the equations [12, Chap.3.4], where ν is the viscosity. In 1980, Beale and Majda [1] first proved the convergence rate $C\nu\Delta t^2$ of the viscous splitting method for the two-dimensional Navier-Stokes equations.

Generally speaking, there are two basic splitting techniques. The first one is based on the Trotter product formula [18, Chap.11, Appendix A] and the convergence rate has been showed to be $C\nu\Delta t$. The second algorithm is based on the Strang's splitting [17], which has the advantage of converging as $C\nu\Delta t^2$ with no additionally computational expense. These two basic splitting methods were considered for linear hyperbolic problems by Strang [17] in 1968. He deduced the order of convergence by comparing a Taylor expansion in time of the exact solution with the approximation. Operator splitting is a powerful method for numerical investigation of complex models. Fields of application where splitting is useful to apply include air pollution meteorology [2], fluid dynamic models [9], cloud physics [14] and biomathematics [4]. Lastly, we refer to [13] for theoretical and practical use of splitting methods.

For the KS equations (1), the splitting methods can be done as follows. Discretize time as $t_n = n\Delta t$ with time-step size Δt, and on each time step first solve the aggregation equation, then the heat equation to simulate effects of the diffusion term $\Delta\rho$. We will define this algorithm formally as below.

Denote the solution operator to an aggregation equation by $A(t)$, such that $u(t,x) = A(t)u_0(x)$ solves

$$\begin{cases} \partial_t u = -\nabla\cdot(u\nabla c), \quad x\in\mathbb{R}^d,\ t>0, \\ -\triangle c = u(t,x), \\ u(0,x) = u_0(x). \end{cases} \tag{2}$$

By using Lemma 7.6 in Gilbarg and Trudinger [5], if we define the negative part of the function u as $u_- := \min\{u, 0\}$, then one can easily prove that

$$\frac{d}{dt}\int_{\mathbb{R}^n} u_-^2\,dx = \int_{\mathbb{R}^n} u_-^3\,dx \le 0, \tag{3}$$

which leads to that u is nonnegative if u_0 is nonnegative.

Also denote the solution operator to the heat equation by $H(t)$, so that $\omega(t,x) = H(t)\omega_0(x)$ solves

$$\begin{cases} \partial_t w = \Delta\omega, \quad x\in\mathbb{R}^d,\ t>0, \\ \omega(0,x) = \omega_0(x). \end{cases} \tag{4}$$

Similarly, we can prove that

$$\frac{d}{dt}\int_{\mathbb{R}^n} \omega_-^2 \, dx = -\int_{\mathbb{R}^n} |\nabla \omega_-|^2 \, dx \leq 0, \tag{5}$$

which also leads to that ω is nonnegative if ω_0 is nonnegative.

Then we can define the first order splitting algorithm by means of the Trotter product formula [18]:

$$\rho^{(n)}(x) = [H(\Delta t)A(\Delta t)]^n \rho_0(x), \tag{6}$$

where $\rho^{(n)}(x)$ is the approximate value of the exact solution at time $t_n = n\Delta t$.

Furthermore, there is a second order splitting algorithm follows from Strang's method [17]:

$$\hat{\rho}^{(n)}(x) = [H(\frac{\Delta t}{2})A(\Delta t)H(\frac{\Delta t}{2})]^n \rho_0(x). \tag{7}$$

From the results of (3) and (5), we know that the splitting schemes (6) and (7) are positivity preserving.

Since the error estimates are valid when the solution of the KS equations is regular enough, we assume that

$$0 \leq \rho_0 \in L^1 \cap H^k(\mathbb{R}^d), \text{ with } k > \frac{d}{2},$$

then the KS system (1) has a unique local solution with the following regularity

$$\|\rho\|_{L^\infty(0,T;H^k(\mathbb{R}^d))} \leq C(\|\rho_0\|_{L^1\cap H^k(\mathbb{R}^d)}),$$

where $T > 0$ only depends on $\|\rho_0\|_{L^1\cap H^k(\mathbb{R}^d)}$. The proof of this result is a standard process and it is provided in [7, Appendix A]. As a direct result of the Sobolev imbedding theorem, one has

$$\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \leq C(\|\rho_0\|_{L^1\cap H^k(\mathbb{R}^d)}),$$

for $k > d/2$.

The convergence results of our splitting algorithms (6) and (7) can be described as follows:

Theorem 1.1. *Assume that $0 \leq \rho_0(x) \in L^1 \cap H^k(\mathbb{R}^d)$ with $k > \frac{d}{2} + 5$. Let $\rho(t,x)$ be the regular solution to the KS equations (1) with initial data $\rho_0(x)$. Then there exist some $C_*, T_* > 0$ depending on $\|\rho_0\|_{L^1\cap H^k}$, such that for $\Delta t \leq C_*$ and $(n+1)\Delta t \leq T_*$, the solutions to splitting algorithms*

$$\rho^{(n)}(x) = [H(\Delta t)A(\Delta t)]^n \rho_0(x); \quad \hat{\rho}^{(n)}(x) = [H(\frac{\Delta t}{2})A(\Delta t)H(\frac{\Delta t}{2})]^n \rho_0(x),$$

are convergent to $\rho(t_n, x)$ in L^2 norm. Moreover, the following estimates hold

$$\max_{0\leq t_n\leq T_*} \|\rho^{(n)} - \rho(t_n,\cdot)\|_2 \leq C(T_*, \|\rho_0\|_{L^1\cap H^k})\Delta t; \tag{8}$$

$$\max_{0\leq t_n\leq T_*} \|\hat{\rho}^{(n)} - \rho(t_n,\cdot)\|_2 \leq C(T_*, \|\rho_0\|_{L^1\cap H^k})\Delta t^2. \tag{9}$$

Next, we will set up an aggregation-diffusion splitting scheme with the linear transport approximation as in [6] and provide the error estimate of this method.

First, we recast $c(t,x) = \Phi * \rho(t,x)$ with the fundamental solution of the Laplacian equation $\Phi(x)$, which can be represented as

$$\Phi(x) = \begin{cases} \dfrac{C_d}{|x|^{d-2}}, & \text{if } d \geq 3, \\ -\dfrac{1}{2\pi} \ln |x|, & \text{if } d = 2, \end{cases} \tag{10}$$

where $C_d = \dfrac{1}{d(d-2)\alpha_d}$, $\alpha_d = \dfrac{\pi^{d/2}}{\Gamma(d/2+1)}$, i.e. α_d is the volume of the d-dimensional unit ball.

Furthermore, the $\Phi(x)$ in (10) is also called Newtonian potential, and we can take the gradient of $\Phi(x)$ as the attractive force $F(x)$. Thus we have

$$F(x) = \nabla\Phi(x) = -\frac{C_* x}{|x|^d}, \ \forall\, x \in \mathbb{R}^d \backslash \{0\}, \ d \geq 2, \tag{11}$$

where $C_* = \frac{\Gamma(d/2)}{2\pi^{d/2}}$ and $\nabla c = F * \rho$.

Suppose that $0 \leq s \leq \Delta\, t$ and solve (2) in $t \in [t_n, t_{n+1}]$. If we denote $v := \nabla c = F * u$, then $u(t_n + s, X(x,s))$ satisfies

$$u_s + \nabla \cdot (uv) = 0,$$

with flow map

$$\frac{dX(x,s)}{ds} = v(X(x,s), s); \quad X(x,0) = x, \tag{12}$$

which leads to

$$u\left(t_n + s, X(x,s)\right) \det \frac{dX(x,s)}{dx} = u(t_n, x).$$

By using Euler forward method, we have the linear approximation of (12)

$$X(x,s) \approx x + sv(x,0) = x + sF * u(t_n, x).$$

Then, one has

$$\frac{dX(x,s)}{ds} = F * u(t_n, x) =: V(X(x,s), s).$$

Let $L(t_n + s, X(x,s))$ satisfying

$$L_s + \nabla \cdot (LV) = 0,$$

with flow map

$$\frac{dX(x,s)}{ds} = V(X(x,s), s); \quad X(x,0) = x,$$

which leads to

$$L\left(t_n + s, X(x,s)\right) \det \frac{dX(x,s)}{dx} = L(t_n, x).$$

Then we can propose the following aggregation-diffusion splitting method with linear transport approximation:

$$G^{(n)}(x) = F * \tilde{\rho}^{(n)}(x), \tag{13}$$

$$L^{(n+1)}\big(x + \Delta t\, G^{(n)}(x)\big) = \det{}^{-1}\big(I + \Delta t\, DG^{(n)}(x)\big)\tilde{\rho}^{(n)}(x), \tag{14}$$

$$\tilde{\rho}^{(n+1)}(x) = H(\Delta t) L^{(n+1)}(x). \tag{15}$$

And here we require that $\Delta t < \frac{1}{\|DG^{(n)}\|_2}$ to make sure $\det^{-1}\left(I + \Delta t\, DG^{(n)}(x)\right)$ is non-singular.

The motivation of this scheme comes from the random particle blob method for the KS equations. As a future work, the results obtained in this article will be used to establish the error estimates of the random particle blob method for the KS equations.

One can write (13) to (15) in the symbolic form

$$\tilde{\rho}^{(n)}(x) = [H(\Delta t)\tilde{A}(\Delta t)]^n \rho_0(x), \tag{16}$$

and it is obvious that this scheme also has the positivity preserving property.

Moreover, we also prove the convergence theorem of the splitting algorithm (16) as below:

Theorem 1.2. *Assume that $0 \le \rho_0(x) \in L^1 \cap H^k(\mathbb{R}^d)$ with $k > \frac{d}{2} + 3$. Let $\rho(t,x)$ be the regular solution to the KS equations (1) with initial data $\rho_0(x)$. Then there exist some $C'_*, T'_* > 0$ depending on $\|\rho_0\|_{L^1 \cap H^k}$, such that for $\Delta t \le C'_*$ and $(n+1)\Delta t \le T'_*$, the solution to the splitting algorithm*

$$\tilde{\rho}^{(n)}(x) = [H(\Delta t)\tilde{A}(\Delta t)]^n \rho_0(x),$$

is convergent to $\rho(t_n, x)$ in L^2 norm. Moreover, the following estimate holds

$$\max_{0 \le t_n \le T'_*} \|\tilde{\rho}^{(n)} - \rho(t_n, \cdot)\|_2 \le C(T'_*, \|\rho_0\|_{L^1 \cap H^k})\Delta t. \tag{17}$$

In this article, we only present and analyze these semi-discrete splitting schemes and the spatial discretization is not considered. When the solution is regular, the standard spatial discretization such as finite element method, finite difference method and spectral method can be directly applied here and the numerical analysis for these three spatial discretization in the splitting schemes are standard, which is omitted here. However, for the KS equations, solutions can develop singularity. Computing such singular solutions is very challenging, and we refer to [11] for numerical results, where authors prove that the fully discrete scheme is conservative and positivity preserving. Another natural approach in spatial discretization is using the particle method. Actually, the main motivation of current paper is to develop a splitting scheme to analyze the random particle blob method for KS equations.

Notation. For convenience, in this article, we use $\|\cdot\|_p$ for L^p norm of a function. The generic constant will be denoted generically by C, even if it is different from line to line.

To conclude this introduction, we give the outline of this article. In Section 2, we establish the error estimates of the first and second order aggregation-diffusion splitting schemes through three steps: stability, consistency and convergence. Similarly, we provide the error estimate of a splitting scheme with the linear transport approximation in Section 3.

2. **The convergence analysis of the aggregation-diffusion splitting algorithms and the proof of Theorem 1.1.** Like always, we follow the Lax's equivalence theorem [16] to prove the convergence of a numerical algorithm, which is that stability and consistency of an algorithm imply its convergence. Therefore, we break the proof of Theorem 1.1 up into three steps.

Step 1. The first step is to prove the stability, which ensures that the solution of the splitting algorithm (6) is priori controlled in an appropriate norm. The following proposition shows that our splitting method is $H^k(\mathbb{R}^d)$ stable.

Proposition 1. *(Stability) Suppose that initial density $0 \le \rho_0(x) \in L^1 \cap H^k(\mathbb{R}^d)$ with $k > \frac{d}{2}$. There exists some $T_1 > 0$ depending on $\|\rho_0\|_{L^1\cap H^k}$, such that for the algorithms (6) and (7), we have*

$$\|\rho^{(n)}\|_{H^k} \le C(T_1, \|\rho_0\|_{L^1\cap H^k}), \quad \forall\ 0 \le n\Delta t \le T_1; \tag{18}$$

$$\|\hat{\rho}^{(n)}\|_{H^k} \le C(T_1, \|\rho_0\|_{L^1\cap H^k}), \quad \forall\ 0 \le n\Delta t \le T_1. \tag{19}$$

Proof. We will only prove (18) in detail and the proof of (19) is almost the same.

Suppose that $0 \le s \le \Delta t$, and we define

$$u(s + t_{n-1}) := A(s)\rho^{(n-1)},$$

and

$$\check{\rho}(s + t_{n-1}) := H(s)u(s + t_{n-1}) = H(s)A(s)\rho^{(n-1)}.$$

Notice that when $s = 0$, $\check{\rho}(t_{n-1}) = \rho^{(n-1)}$ and that when $s = \Delta t$, $\check{\rho}(t_n) = \rho^{(n)}$. The standard regularity of heat equation gives that

$$\|\check{\rho}(s + t_{n-1})\|_{H^k} = \|H(s)A(s)\rho^{(n-1)}\|_{H^k} \le \|A(s)\rho^{(n-1)}\|_{H^k}. \tag{20}$$

In order to give the estimate of $\|A(s)\rho^{(n-1)}\|_{H^k}$, we need to solve the hyperbolic equation (2).

Multiply (2) by $2u$ and integrate over $\mathbb{R}^d$, then for $k > \frac{d}{2}$, we have

$$\frac{d}{dt}\|u\|_2^2 = \int_{\mathbb{R}^d} \nabla(u^2)\nabla c\, dx = \int_{\mathbb{R}^d} u^3 dx \le \|u\|_\infty \|u\|_2^2 \le \|u\|_{H^k}\|u\|_2^2,$$

where $-\Delta c = u$ and the Soblev imbedding theorem have been used.

Now we multiply (2) by $2D^{2m}u$ with $1 \le |m| \le k$ and integrate over $\mathbb{R}^d$, then one has

$$\begin{aligned}\frac{d}{dt}\|D^m u\|_2^2 &= -2\int_{\mathbb{R}^d} \nabla\cdot(D^m(u\nabla c))D^m u\, dx\\ &= -2\int_{\mathbb{R}^d} \nabla\cdot[D^m(u\nabla c) - D^m u\nabla c]D^m u\, dx\\ &\quad - 2\int_{\mathbb{R}^d} \nabla\cdot(D^m u\nabla c)D^m u\, dx\\ &=: I_1 + I_2.\end{aligned}$$

Estimate I_1 first, then we have

$$\begin{aligned}|I_1| &\le 2\int_{\mathbb{R}^d} |\nabla\cdot[D^m(u\nabla c) - D^m u\nabla c]D^m u|\ dx\\ &= 2\int_{\mathbb{R}^d} \left|\nabla\cdot[\sum_{a+b=m, b>0} \binom{m}{b} D^b(\nabla c)D^a u]D^m u\right|\ dx\\ &\le C\sum_{|a|+|b|=|m|-1} \|D^m u\|_2 \left\|\nabla\cdot[D^b D_j(\nabla c)D^a u]\right\|_2,\end{aligned} \tag{21}$$

where we have used the same notation in formula (3.23) [19, Chap.13, P.11].

Now, we compute each component of $\left\|\nabla\cdot[D^b D_j(\nabla c)D^a u]\right\|_2$ with $|a| + |b| = |m| - 1$:

$$\begin{aligned}&\left\|D_i[D^b D_j(\nabla c)D^a u]\right\|_2\\ =&\left\|D^b D_i D_j(\nabla c)D^a u + D^b D_j(\nabla c)D^a D_i(u)\right\|_2\\ \le& C\|D_j(\nabla c)\|_\infty \|u\|_{H^{|m|}} + C\|D_j(\nabla c)\|_{H^{|m|}}\|u\|_\infty\end{aligned}$$

$$\leq C\|D_j(\nabla c)\|_{H^k}\|u\|_{H^k} + C\|D_j(\nabla c)\|_{H^k}\|u\|_{H^k} \leq C\|u\|_{H^k}^2,$$

by using Taylor [19, Proposition 3.6], Soblev imbedding theorem and $-\Delta c = u$.

Hence we have

$$|I_1| \leq C\|u\|_{H^k}^3. \tag{22}$$

For I_2, one has

$$I_2 = 2\int_{\mathbb{R}^d} D^m u \nabla(D^m u)\nabla c\, dx = \int_{\mathbb{R}^d} |D^m u|^2 u\, dx \leq \|u\|_{H^k}^2\|u\|_\infty \leq \|u\|_{H^k}^3. \tag{23}$$

Combining (22) and (23), it follows that

$$\frac{d}{dt}\|D^m u\|_2^2 \leq C\|u\|_{H^k}^3, \quad 1 \leq |m| \leq k,$$

which leads to

$$\frac{d}{dt}\|u\|_{H^k}^2 \leq C\|u\|_{H^k}^3.$$

Thus we have

$$\|u\|_{H^k} \leq \frac{1}{\|u_0\|_{H^k}^{-1} - Ct}, \tag{24}$$

and there exists some $T_1 > 0$ depending on $\|u_0\|_{L^1\cap H^k}$, such that for $0 \leq t \leq T_1$

$$\|u\|_{H^k} \leq C(T_1, \|u_0\|_{L^1\cap H^k}).$$

Moreover, one has

$$\|A(s)\rho^{(n-1)}\|_{H^k} \leq \frac{1}{\|\rho^{(n-1)}\|_{H^k}^{-1} - Cs}, \quad 0 \leq s \leq \Delta t. \tag{25}$$

Hence it follows from (20) and (25) by taking $s = \Delta t$

$$\|\rho^{(n)}\|_{H^k} \leq \frac{1}{\|\rho^{(n-1)}\|_{H^k}^{-1} - C\Delta t}. \tag{26}$$

Recasting (26), one has

$$\|\rho^{(n)}\|_{H^k}^{-1} \geq \|\rho^{(n-1)}\|_{H^k}^{-1} - C\Delta t.$$

By induction on n, we concludes that

$$\|\rho^{(n)}\|_{H^k} \leq \frac{1}{\|\rho_0\|_{H^k}^{-1} - Cn\Delta t}.$$

with $n\Delta t \leq T_1$.

Until now, we have finished the proof of (18) and we can prove (19) almost the same way. □

Step 2. In this step, we will prove our splitting algorithms (6) and (7) are consistent with the KS equations (1) by using the H^k stability in Proposition 1.

Proposition 2. *(Consistency) Assume that the initial data $0 \leq \rho_0(x) \in L^1 \cap H^k(\mathbb{R}^d)$ with $k > \frac{d}{2} + 5$. Let $\rho(t,x)$ be the regular solution to the KS equations (1) with local existence time T and T_1 is used in Proposition 1. If we define $T_* := \min\{T, T_1\}$, then the local errors*

$$r_n(s) = H(s)A(s)\rho^{(n-1)} - \rho(s + (n-1)\Delta t), \quad 0 \leq s \leq \Delta t,\ n\Delta t \leq T_*; \tag{27}$$

$$\hat{r}_n(s) = H(\frac{s}{2})A(s)H(\frac{s}{2})\hat{\rho}^{(n-1)} - \rho(s + (n-1)\Delta t), \quad 0 \leq s \leq \Delta t,\ n\Delta t \leq T_*,$$

satisfy

$$\|r_n(s)\|_2 \le e^{C_1 s}(\|r_n(0)\|_2 + C_2 s^2); \tag{28}$$

$$\|\hat{r}_n(s)\|_2 \le e^{C_1' s}(\|\hat{r}_n(0)\|_2 + C_2' s^3), \tag{29}$$

where $C_1,\ C_2,\ C_1',\ C_2'$ *depend on* $T_*,\ \|\rho_0\|_{L^1\cap H^k}$.

Proof. We start with proving (28). Recalling the definition of F in (11), we define the bilinear operator B as

$$B[u,v] := -\nabla\cdot(uF*v), \quad \text{with } -\nabla\cdot(F*v) = v,$$

and $\check{\rho}(s+t_{n-1}) = H(s)A(s)\rho^{(n-1)}$. Considering the time interval $0 \le s \le \Delta t$, it follows that

$$\begin{aligned}
\frac{\partial}{\partial s}\check{\rho} &= \Delta H(s)A(s)\rho^{(n-1)} + H(s)B[A(s)\rho^{(n-1)}, A(s)\rho^{(n-1)}] \\
&= \Delta\check{\rho} + B[\check{\rho},\check{\rho}] + H(s)B[A(s)\rho^{(n-1)}, A(s)\rho^{(n-1)}] \\
&\quad - B[H(s)A(s)\rho^{(n-1)}, H(s)A(s)\rho^{(n-1)}] \\
&= \Delta\check{\rho} + B[\check{\rho},\check{\rho}] + f_n(s),
\end{aligned}$$

where we denote

$$f_n(s) = H(s)B[A(s)\rho^{(n-1)}, A(s)\rho^{(n-1)}] - B[H(s)A(s)\rho^{(n-1)}, H(s)A(s)\rho^{(n-1)}]. \tag{30}$$

For the exact solution $\rho(s+t_{n-1})$ to (1), one has

$$\frac{\partial}{\partial s}\rho = \Delta\rho + B[\rho,\rho].$$

Thus the difference between $\check{\rho}(s+t_{n-1})$ and $\rho(s+t_{n-1})$ satisfies

$$\frac{\partial}{\partial s}r_n(s) = \Delta r_n(s) + B[r_n(s),\check{\rho}] + B[\rho, r_n(s)] + f_n(s), \quad 0 \le s \le \Delta t. \tag{31}$$

Take the L^2 inner product of (31) with $2r_n(s)$, then we have

$$\begin{aligned}
&\frac{d}{ds}\|r_n(s)\|_2^2 + 2\|\nabla r_n(s)\|_2^2 \\
=&2(B[r_n(s),\check{\rho}], r_n(s)) + 2(B[\rho, r_n(s)], r_n(s)) + 2(f_n(s), r_n(s)).
\end{aligned}$$

We compute that

$$\begin{aligned}
2(B[r_n(s),\check{\rho}], r_n(s)) &= -2\int_{\mathbb{R}^d}\nabla\cdot(r_nF*\check{\rho})r_n\,dx = \int_{\mathbb{R}^d}\nabla(r_n^2)F*\check{\rho}\,dx \\
&\le \|r_n(s)\|_2^2\|\check{\rho}\|_\infty \le C(T_*, \|\rho_0\|_{L^1\cap H^k})\|r_n(s)\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
2(B[\rho, r_n(s)], r_n(s)) &= -2\int_{\mathbb{R}^d}\nabla\cdot(\rho F*r_n)r_n\,dx = 2\int_{\mathbb{R}^d}\rho\nabla r_nF*r_n dx \\
&\le 2\|\rho\|_d\|\nabla r_n\|_2\|F*r_n\|_{\frac{2d}{d-2}} \\
&\le C(T_*, \|\rho_0\|_{L^1\cap H^k})\|\nabla r_n(s)\|_2\|r_n(s)\|_2 \\
&\le \varepsilon\|\nabla r_n(s)\|_2^2 + C(\varepsilon, T_*, \|\rho_0\|_{L^1\cap H^k})\|r_n(s)\|_2^2,
\end{aligned}$$

where we have used the weak Young's inequality [10, P.107] $\|F*r_n\|_{\frac{2d}{d-2}} \le C\|r_n\|_2$ and Young's inequality $ab \le \varepsilon a^2 + C(\varepsilon)b^2$ with ε small enough.

Moreover, we have

$$2(f_n(s), r_n(s)) \le 2\|f_n(s)\|_2\|r_n(s)\|_2,$$

which leads to

$$\frac{d}{ds}\|r_n(s)\|_2 \le C(T_*, \|\rho_0\|_{L^1\cap H^k})\|r_n(s)\|_2 + \|f_n(s)\|_2.$$

Next step is to estimate the function $f_n(s)$. By the definition of $H(s)$ in (4), it satisfies

$$H(s)\omega_0 = H(0)\omega_0 + \int_0^s \Delta H(\tau)\omega_0\, d\tau,$$

so that

$$H(s) = I + \int_0^s \Delta H(\tau)\, d\tau =: I + \bar{H}(s).$$

Rewrite $f_n(s)$ in (30), one has

$$\begin{aligned} &f_n(s)\\ =&(I+\bar{H}(s))B[A(s)\rho^{(n-1)}, A(s)\rho^{(n-1)}]\\ &- B[(I+\bar{H}(s))A(s)\rho^{(n-1)}, (I+\bar{H}(s))A(s)\rho^{(n-1)}]\\ =&\bar{H}(s)B[A(s)\rho^{(n-1)}, A(s)\rho^{(n-1)}] - B[\bar{H}(s)A(s)\rho^{(n-1)}, \bar{H}(s)A(s)\rho^{(n-1)}]\\ &- B[\bar{H}(s)A(s)\rho^{(n-1)}, A(s)\rho^{(n-1)}] - B[A(s)\rho^{(n-1)}, \bar{H}(s)A(s)\rho^{(n-1)}]. \end{aligned} \tag{32}$$

To estimate $f_n(s)$, we compute

$$\begin{aligned} &\|\bar{H}(s)B[A(s)\rho^{(n-1)}, A(s)\rho^{(n-1)}]\|_2\\ =&s\|\Delta H(-\nabla\cdot(A\rho^{(n-1)}F * A\rho^{(n-1)}))\|_2 \le s\|H(-\nabla\cdot(A\rho^{(n-1)}F * A\rho^{(n-1)}))\|_{H^{\frac{d}{2}+2}}\\ \le& sC\|A\rho^{(n-1)}F * A\rho^{(n-1)}\|_{H^{\frac{d}{2}+3}} \le C(T_*, \|\rho_0\|_{L^1\cap H^{\frac{d}{2}+3}})s. \end{aligned}$$

And similarly, we can compute other terms in (32). Thus for $k > \frac{d}{2}+3$, we have

$$\|f_n(s)\|_2 \le C(T_*, \|\rho_0\|_{L^1\cap H^k})s, \quad 0 \le s \le \Delta t.$$

Until now, we have got

$$\frac{d}{ds}\|r_n(s)\|_2 \le C_1\|r_n(s)\|_2 + C_2 s.$$

By using Gronwall's inequality [3, Appendix B, P.624], one concludes that

$$\|r_n(s)\|_2 \le e^{C_1 s}(\|r_n(0)\|_2 + C_2 s^2), \quad 0 \le s \le \Delta t,$$

where $C_1,\ C_2$ depends on $T_*,\ \|\rho_0\|_{L^1\cap H^k}$. Thus, (28) has been proved.

Next we are going to prove (29) by using the same procedure in the above arguments, and we can write

$$\frac{\partial}{\partial s}\hat{\rho} = \Delta\hat{\rho} + B[\hat{\rho}, \hat{\rho}] + \hat{f}_n(s),$$

where $\hat{\rho}(s+t_{n-1}) = H(\frac{s}{2})A(s)H(\frac{s}{2})\rho^{(n-1)}$ and

$$\begin{aligned} \hat{f}_n(s) = &HB[AH\rho^{(n-1)}, AH\rho^{(n-1)}] - B[HAH\rho^{(n-1)}, HAH\rho^{(n-1)}]\\ &+ \frac{1}{2}HA_H\Delta H\rho^{(n-1)} - \frac{1}{2}\Delta HAH\rho^{(n-1)}, \end{aligned} \tag{33}$$

with $H = H(\frac{s}{2})$, $A = A(s)$ and $A_H = \frac{\partial}{\partial H}A(s; H(\frac{s}{2})\rho^{(n-1)})$.

Thus, using the argument identical to that we have used to estimate $f_n(s)$, for $k > \frac{d}{2}+5$, we have

$$\|\hat{f}_n(s)\|_2 \le C(T_*, \|\rho_0\|_{L^1\cap H^k})s^2, \quad 0 \le s \le \Delta t,$$

and

$$\frac{d}{ds}\|\hat{r}_n(s)\|_2 \le C_1'\|\hat{r}_n(s)\|_2 + C_2's^2,$$

which leads to (29) by using Gronwall's inequality. □

Step 3. Finally, we can prove the convergence Theorem 1.1 by using Proposition 2. We estimate $r_n(\Delta t) = \rho^{(n)}(x) - \rho(t_n, x)$ as

$$\|\rho^{(n)} - \rho(t_n,\cdot)\|_2 \le e^{C_1\Delta t}(\|\rho^{(n-1)}(x) - \rho(t_{n-1},x)\|_2 + C_2(\Delta t)^2).$$

Standard induction implies that

$$\begin{aligned}\|\rho^{(n)} - \rho(t_n,\cdot)\|_2 &\le C_2(\Delta t)^2\sum_{j=1}^{n} e^{jC_1\Delta t} = C_2(\Delta t)^2\frac{e^{C_1\Delta t}(e^{nC_1\Delta t}-1)}{e^{C_1\Delta t}-1}\\ &\le \frac{C_2}{C_1}\Delta t(e^{C_1(n+1)\Delta t}-1) \le \frac{C_2}{C_1}\Delta t(e^{C_1T_*}-1), \qquad (34)\end{aligned}$$

for $(n+1)\Delta t \le T_*$, which concludes the proof of (8) in Theorem 1.1. A similar argument holds for (9). Until now, we have completed the proof of Theorem 1.1.

3. The convergence analysis of the splitting method with linear transport approximation and the proof of Theorem 1.2.

In this section, we will prove the convergence estimate of the spitting method with linear transport approximation. Recall this splitting method proposed in Introduction with the initial data $\tilde{\rho}^{(0)}(x) = \rho_0(x)$:

$$G^{(n)}(x) = F * \tilde{\rho}^{(n)}(x), \qquad (35)$$

$$L^{(n+1)}\big(x + \Delta t\, G^{(n)}(x)\big) = \det{}^{-1}\big(I + \Delta t\, DG^{(n)}(x)\big)\tilde{\rho}^{(n)}(x), \qquad (36)$$

$$\tilde{\rho}^{(n+1)}(x) = H(\Delta t)L^{(n+1)}(x). \qquad (37)$$

The proof of Theorem 1.2 can also be divided into three steps like Section 2.

Step 1. As we have done in the last section, firstly, we need to prove that the semi-discrete equations (35) to (37) are stable, i.e.

$$\|\tilde{\rho}^{(n)}\|_{H^k} \le C(\|\rho_0\|_{L^1\cap H^k}). \qquad (38)$$

In order to do this, we will need the following lemma:

Lemma 3.1. *Assume that $x_{n+1} \le x_n + \Delta t g(x_n)$ for some nonnegative and increasing function $g(x)$, then we have*

$$x_n \le y(n\Delta t), \quad \forall\ 0 \le n\Delta t \le T_2,$$

where $y(t)$ is a solution to the following ODE

$$\begin{cases} y'(t) = g(y(t)), \\ y(0) = x_0. \end{cases} \qquad (39)$$

in $[0, T_2]$.

Proof. We will prove this lemma by the induction on n. The case $n = 0$ can be obtained obviously by the initial condition. Since $g(x) \ge 0$, we have that $y(t)$ is a nondecreasing function, which leads to

$$y((n+1)\Delta t) = y(n\Delta t) + \int_{t_n}^{t_{n+1}} g(y(t))dt \ge y(n\Delta t) + \Delta t\, g(y(n\Delta t)).$$

By the assumption $x_{n+1} \le x_n + \Delta t g(x_n)$ and the induction hypothesis $x_n \le y(n\Delta t)$, one has

$$x_{n+1} \le y(n\Delta t) + \Delta t g(y(n\Delta t)) \le y((n+1)\Delta t).$$

Hence, we concludes our proof. □

To prove (38), if we set $x_n = \|\tilde{\rho}^{(n)}\|_{H^k}$ in Lemma 3.1, then we only need to find the nonnegative and increasing function $g(x)$ satisfying

$$\|\tilde{\rho}^{(n+1)}\|_{H^k} \le \|\tilde{\rho}^{(n)}\|_{H^k} + \Delta t\, g(\|\tilde{\rho}^{(n)}\|_{H^k}).$$

Proposition 3. *(Stability) Suppose that the initial density $0 \le \rho_0(x) \in L^1 \cap H^k(\mathbb{R}^d)$ with $k > \frac{d}{2} + 1$. Then there exists some $C_1, T_3 > 0$ depending on $\|\rho_0\|_{L^1\cap H^k}$, such that for the algorithm (16) with $\Delta t \le C_1$, we have*

$$\|\tilde{\rho}^{(n)}\|_{H^k} \le C(T_3, \|\rho_0\|_{L^1\cap H^k}), \quad \forall\ 0 \le n\Delta t \le T_3. \tag{40}$$

Proof. **Step 1.** (Estimate of the right handside of (36)) We begin with defining

$$W_1(u) := \frac{\det^{-1}(I + \Delta t\, u) - 1}{\Delta t},$$

with $\Delta t < \frac{1}{\|u\|_2}$ and

$$\eta(x) := \det{}^{-1}\big(I + \Delta t\, DG^{(n)}(x)\big)\tilde{\rho}^{(n)}(x) = \Delta t W_1\big(DG^{(n)}(x)\big)\tilde{\rho}^{(n)}(x) + \tilde{\rho}^{(n)}(x). \tag{41}$$

Then $W_1(0) = 0$ and $W_1(u)$ is a smooth function with a bound independent of Δt. According to [19, Proposition 3.9], we have

$$\big\|W_1(DG^{(n)}(\cdot))\big\|_{H^k} \le \omega_1(\|DG^{(n)}\|_\infty)(1 + \|DG^{(n)}\|_{H^k}),$$

and

$$\big\|W_1(DG^{(n)}(\cdot))\big\|_\infty \le \omega_2(\|DG^{(n)}\|_\infty),$$

where $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are increasing functions. We have to mention here that the functions $\omega_i(\cdot)$ in the following text are always increasing functions and we denote $\omega(\cdot)$ to be a generic function which maybe different from line to line. Moreover, we have

$$\|DG^{(n)}\|_\infty \le C\|DG^{(n)}\|_{H^k} \le C\|\tilde{\rho}^{(n)}\|_{H^k}, \tag{42}$$

where we have used the elliptic regularity of (35) in the second inequality. And (42) implies that

$$\omega_1(\|DG^{(n)}\|_\infty) \le \omega_1(C\|\tilde{\rho}^{(n)}\|_{H^k}) =: \omega_1'(\|\tilde{\rho}^{(n)}\|_{H^k});$$

$$\omega_2(\|DG^{(n)}\|_\infty) \le \omega_2'(\|\tilde{\rho}^{(n)}\|_{H^k}).$$

Hence, by Moser's inequality [19, Proposition 3.7], from (41) one concludes that

$$\begin{aligned}
&\|\eta\|_{H^k} = \|\tilde{\rho}^{(n)} + \Delta t\, W_1\tilde{\rho}^{(n)}\|_{H^k}\\
\le&\|\tilde{\rho}^{(n)}\|_{H^k} + C\Delta t\,(\|W_1\|_\infty\|\tilde{\rho}^{(n)}\|_{H^k} + \|W_1\|_{H^k}\|\tilde{\rho}^{(n)}\|_\infty)\\
\le&\Big(1 + C\Delta t\left(\omega_1'(\|\tilde{\rho}^{(n)}\|_{H^k}) + \omega_2'(\|\tilde{\rho}^{(n)}\|_{H^k})\right) + C\Delta t\,\omega_1'(\|\tilde{\rho}^{(n)}\|_{H^k})\|\tilde{\rho}^{(n)}\|_{H^k}\Big)\|\tilde{\rho}^{(n)}\|_{H^k}\\
\le&\Big(1 + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\,\|\tilde{\rho}^{(n)}\|_{H^k}\Big)\|\tilde{\rho}^{(n)}\|_{H^k}\\
\le&\|\tilde{\rho}^{(n)}\|_{H^k} + \Delta t\omega(\|\tilde{\rho}^{(n)}\|_{H^k}),
\end{aligned}$$

where in the second inequality we have used

$$\|DG^{(n)}\|_{H^k} \le C\|\tilde{\rho}^{(n)}\|_{H^k}; \quad \|\tilde{\rho}^{(n)}\|_\infty \le C\|\tilde{\rho}^{(n)}\|_{H^k}.$$

Step 2. (Estimate of the left handside of (14)) In this step, we consider the operation $\eta \to \bar{\eta}$ to study the left handside of (14), where $\bar{\eta}\big(x+\Delta t DG^{(n)}(x)\big) = \eta(x)$ for any function $\eta(x)$.

Like we have done before, we rewrite

$$\det\big(I + \Delta t DG^{(n)}(x)\big) = 1 + \Delta t W_2\big(DG^{(n)}(x)\big),$$

with

$$\big\|W_2(DG^{(n)}(\cdot))\big\|_\infty \le \omega_3(\|\tilde{\rho}^{(n)}\|_{H^k}).$$

Then, one can compute

$$\begin{aligned}\|\bar{\eta}\|_2^2 &= \int_{\mathbb{R}^d} \bar{\eta}(y)^2 dy = \int_{\mathbb{R}^d} \bar{\eta}^2\big(x + \Delta t G^{(n)}(x)\big)\det\big(I + \Delta t DG^{(n)}(x)\big)dx \\ &\le \big(1 + \omega_3(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\|\eta\|_2^2. \end{aligned} \tag{43}$$

Continue this process, we know $\partial_y \bar{\eta} = \overline{\partial_x \eta} \cdot \overline{(I + \Delta t DG^{(n)})^{-1}}$. Again let us recast $\big(I + \Delta t DG^{(n)}(x)\big)^{-1} = I + \Delta t W_3\big(DG^{(n)}(x)\big)$ with

$$\big\|W_3(DG^{(n)}(\cdot))\big\|_\infty \le \omega_4(\|\tilde{\rho}^{(n)}\|_{H^k});$$

$$\big\|W_3(DG^{(n)}(\cdot))\big\|_{H^k} \le \omega_5(\|\tilde{\rho}^{(n)}\|_{H^k})(1 + \|\tilde{\rho}^{(n)}\|_{H^k}).$$

Thus one has

$$\begin{aligned}\|\partial_y \bar{\eta}\|_2 &\le \|\overline{\partial_x \eta}\|_2 + \Delta t \|\overline{\partial_x \eta \cdot W_3}\|_2 \\ &\le \big(1 + \omega_3(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\big(\|\partial_x \eta\|_2 + \Delta t\|\partial_x \eta \cdot W_3\|_2\big) \\ &\le \big(1 + \omega_3(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\big(1 + \omega_4(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\|\partial_x \eta\|_2,\end{aligned}$$

which leads to

$$\|\bar{\eta}\|_{H^1} \le \big(1 + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\|\eta\|_{H^1}. \tag{44}$$

Next, we verify by induction on $1 \le s \le k$ such that

$$\|\bar{\eta}\|_{H^s} \le \big(1 + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\|\eta\|_{H^s} + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t. \tag{45}$$

Recall that we have proved the case $s = 1$ in (44). For $s \ge 1$, one has

$$\|\partial_y \bar{\eta}\|_{H^s} \le \|\overline{\partial_x \eta}\|_{H^s} + \Delta t\|\overline{\partial_x \eta \cdot W_3}\|_{H^s}.$$

By the induction hypothesis

$$\|\overline{\partial_x \eta}\|_{H^s} \le \big(1 + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\|\eta\|_{H^{s+1}} + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t.$$

Moreover,

$$\begin{aligned}&\Delta t\|\overline{\partial_x \eta \cdot W_3}\|_{H^s} \\ \le&\big(\Delta t + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\|\partial_x \eta \cdot W_3\|_{H^s} + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t \\ \le&\big(\Delta t + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)C\big(\|W_3\|_\infty\|\eta\|_{H^{s+1}} + \|\eta\|_{H^k}\|W_3\|_{H^s}\big) + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t \\ \le&\omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\|\eta\|_{H^{s+1}} + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t(\|\tilde{\rho}^{(n)}\|_{H^k} + \Delta t\omega(\|\tilde{\rho}^{(n)}\|_{H^k})) \\ &+ \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t \\ \le&\omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\|\eta\|_{H^{s+1}} + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t.\end{aligned}$$

Hence we have

$$\|\partial_y \bar{\eta}\|_{H^s} \le \big(1 + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\|\eta\|_{H^{s+1}} + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t,$$

which verifies that

$$\|\bar{\eta}\|_{H^{s+1}} \leq \big(1+\omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\|\eta\|_{H^{s+1}} + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t.$$

This completes the proof of (45). Finally since $L^{(n+1)} = \bar{\eta}$, the (45) specializes to the following

$$\begin{aligned}
&\|L^{(n+1)}\|_{H^k}\\
\leq&\big(1+\omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\big)\|\eta\|_{H^k} + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\\
\leq&(1+\omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t)\big(\|\tilde{\rho}^{(n)}\|_{H^k} + \Delta t\omega(\|\tilde{\rho}^{(n)}\|_{H^k})\big) + \omega(\|\tilde{\rho}^{(n)}\|_{H^k})\Delta t\\
\leq&\|\tilde{\rho}^{(n)}\|_{H^k} + \Delta t\omega(\|\tilde{\rho}^{(n)}\|_{H^k}).
\end{aligned} \tag{46}$$

Step 3. (Estimate of (15)) Finally, this step requires H^k norm bound for the linear heat equation, and we have

$$\|\tilde{\rho}^{(n+1)}\|_{H^k} \leq \|L^{(n+1)}\|_{H^k}.$$

Collecting (42), (46) and (47), one has

$$\|\tilde{\rho}^{(n+1)}\|_{H^k} \leq \|\tilde{\rho}^{(n)}\|_{H^k} + \Delta t\omega(\|\tilde{\rho}^{(n)}\|_{H^k}), \tag{47}$$

where ω is nonnegative and increasing. Now we can apply Lemma 3.1, and the following ODE

$$\begin{cases} y'(t) = \omega(y(t)),\\ y(0) = \|\rho_0\|_{H^k}, \end{cases} \tag{48}$$

has the solution $y(t)$ in $[0, T_3]$. By Lemma 3.1 and (47), one concludes that

$$\|\tilde{\rho}^{(n)}\|_{H^k} \leq y(n\Delta t) \leq y(T_3).$$

Until now, we have proved the stability result as follows

$$\|\tilde{\rho}^{(n)}\|_{H^k} \leq C(T_3, \|\rho_0\|_{L^1\cap H^k}), \quad \forall\ 0 \leq n\Delta t \leq T_3.$$

□

Step 2. In this step, we will prove the consistency of the algorithm (16) by using Proposition 3, which is described by the following proposition:

Proposition 4. *(Consistency) Assume that the initial data $0 \leq \rho_0(x) \in L^1 \cap H^k(\mathbb{R}^d)$ with $k > \frac{d}{2} + 3$. Let $\rho(t,x)$ be the regular solution to the KS equations (1) with local existence time T and T_3 is used in Proposition 3. Denote $T_*' := \min\{T, T_3\}$, then the local error*

$$\tilde{r}_n(s) = H(s)\tilde{A}(s)\tilde{\rho}^{(n)} - \rho(s+n\Delta t), \quad 0 \leq s \leq \Delta t,\ (n+1)\Delta t \leq T_*', \tag{49}$$

satisfies

$$\|\tilde{r}_n(s)\|_2 \leq e^{C_1 s}\left((1+C_3 s)\|\tilde{r}_n(0)\|_2 + C_2 s^2\right).$$

where $C_1,\ C_2,\ C_3$ depend on $T_*',\ \|\rho_0\|_{L^1\cap H^k}$.

Proof. Let us define $X := x + s\,G^{(n)}(x)$. Then for $L(t_n+s, X)$, it satisfies

$$\begin{aligned}
L(t_n+s, X) &= \det{}^{-1}\left(I + s\,DG^{(n)}(x(X,s))\right)\tilde{\rho}^{(n)}(x(X,s)),\\
\tilde{\rho}(t_n+s, X) &= H(s)L(t_n+s, X).
\end{aligned} \tag{50}$$

Denote $V(X(x,s),s) := G^{(n)}(x)$, then $L(t_n+s, X)$ is the solution to the following PDE

$$\partial_s L + \nabla\cdot(LV) = 0, \tag{51}$$

with initial data $L(t_n, X) = \tilde{\rho}^{(n)}(X)$.

Thus, it follows from (50) that

$$\partial_s \tilde{\rho} = \Delta\tilde{\rho} + H(s)[-\nabla\cdot(LV)].$$

For the exact solution $\rho(t_n+s, X)$ to (1), we have

$$\partial_s \rho = \Delta\rho - \nabla\cdot(\rho G).$$

Then the local error $\tilde{r}_n(s) = \tilde{\rho}(t_n+s, X) - \rho(t_n+s, X)$ satisfies

$$\partial_s \tilde{r}_n = \Delta\tilde{r}_n - \nabla\cdot(\tilde{r}_n V) - \nabla\cdot(\rho(V-G)) + \check{f}_n(s),$$

with

$$\check{f}_n(s) = \nabla\cdot(H(s)LV) - H(s)\nabla\cdot(LV).$$

As we have done in the Section 2, one has

$$\frac{d}{ds}\|\tilde{r}_n\|_2^2 + 2\|\nabla\tilde{r}_n\|_2^2 = -2(\nabla\cdot(\tilde{r}_n V), \tilde{r}_n) - 2(\nabla\cdot(\rho(V-G)), \tilde{r}_n) + 2(\check{f}_n, \tilde{r}_n).$$

We can compute that

$$-2(\nabla\cdot(\tilde{r}_n V), \tilde{r}_n) = -2\int_{\mathbb{R}^d} \nabla\cdot(\tilde{r}_n V)\tilde{r}_n dX \le \|\tilde{r}_n\|_2^2\|\nabla\cdot V\|_\infty \le C\|\tilde{r}_n\|_2^2,$$

and

$$\begin{aligned}&-2(\nabla\cdot(\rho(V-G)), \tilde{r}_n)\\ =&-2\int_{\mathbb{R}^d} \nabla\cdot(\rho(V-G))\tilde{r}_n dX\\ =&-2\int_{\mathbb{R}^d} \nabla\rho\cdot(V-G)\tilde{r}_n dX - 2\int_{\mathbb{R}^d} \rho\nabla\cdot(V-G)\tilde{r}_n dX.\end{aligned} \tag{52}$$

Applying the Hölder inequality, one has

$$\begin{aligned}-2\int_{\mathbb{R}^d} \nabla\rho\cdot(V-G)\tilde{r}_n dX &\le 2\|\nabla\rho\|_d\|V-G\|_{\frac{2d}{d-2}}\|\tilde{r}_n\|_2\\ &\le C\big(T'_*, \|\rho_0\|_{L^1\cap H^k(\mathbb{R}^d)}\big)\|V-G\|_{\frac{2d}{d-2}}\|\tilde{r}_n\|_2,\end{aligned} \tag{53}$$

where in the second inequality we have used the regularity of ρ.

Moreover, by using the weak Young's inequality, one concludes that

$$\begin{aligned}\|V-G\|_{\frac{2d}{d-2}} &= \big\|F*(\tilde{\rho}^{(n)}(x(\cdot,s)) - \rho(t_n+s,\cdot))\big\|_{\frac{2d}{d-2}}\\ &\le C\big\|\tilde{\rho}^{(n)}(x(\cdot,s)) - \rho(t_n+s,\cdot)\big\|_2\\ &\le C\big\|\tilde{\rho}^{(n)}(x(\cdot,s)) - \rho(t_n, x(\cdot,s))\big\|_2 + C\big\|\rho(t_n, x(\cdot,s)) - \rho(t_n+s,\cdot)\big\|_2\\ &\le C\|\tilde{r}_n(0)\|_2 + Cs.\end{aligned} \tag{54}$$

Next, we compute that

$$\begin{aligned}-2\int_{\mathbb{R}^d} \rho\nabla\cdot(V-G)\tilde{r}_n dX &\le 2\|-\rho\nabla\cdot(V-G)\|_2\|\tilde{r}_n\|_2\\ &= 2\|\rho(\tilde{\rho}^{(n)}(x(\cdot,s)) - \rho(t_n+s,\cdot))\|_2\|\tilde{r}_n\|_2\\ &\le (C\|\tilde{r}_n(0)\|_2 + Cs)\|\tilde{r}_n\|_2,\end{aligned} \tag{55}$$

where in the second inequality, (54) has been used.

Collecting (52) to (55), we have

$$-2(\nabla\cdot(\rho(V-G)), \tilde{r}_n) \le C\|\tilde{r}_n(0)\|_2\|\tilde{r}_n\|_2 + Cs\|\tilde{r}_n\|_2.$$

Additionally, like we have done in (32) and (33), for $k > \frac{d}{2} + 3$,

$$2(\check{f}_n, \tilde{r}_n) \leq 2\|\check{f}_n\|_2\|\tilde{r}_n\|_2 \leq Cs\|\tilde{r}_n\|_2.$$

Above all, we have got

$$\frac{d}{ds}\|\tilde{r}_n\|_2 \leq C_1\|\tilde{r}_n\|_2 + C_2 s + C_3\|\tilde{r}_n(0)\|_2,$$

which leads to

$$\|\tilde{r}_n\|_2 \leq e^{C_1 s}\left((1 + C_3 s)\|\tilde{r}_n(0)\|_2 + C_2 s^2\right),$$

by using Gronwall's inequality. □

Step 3. Now we can prove the convergence Theorem 1.2 by using Proposition 4. We estimate $\tilde{r}_n(\Delta t) = \tilde{\rho}^{(n+1)}(X) - \rho(t_{n+1}, X)$ as

$$\|\tilde{\rho}^{(n+1)} - \rho(t_{n+1}, \cdot)\|_2 \leq e^{C_1 \Delta t}\left((1 + C_3\Delta t)\|\tilde{\rho}^{(n)} - \rho(t_n, \cdot)\|_2 + C_2(\Delta t)^2\right).$$

Standard induction as we have done in (34) implies that

$$\|\tilde{\rho}^{(n)} - \rho(t_n, \cdot)\|_2 \leq \frac{C_2\Delta t}{C_1}\left((1 + C_3\Delta t)^n e^{C_1 T'_*} - 1\right),$$

for $(n+1)\Delta t \leq T'_*$, which concludes the proof of Theorem 1.2.

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