A DEGENERATE p-LAPLACIAN KELLER-SEGEL MODEL

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Abstract. This paper investigates the existence of a uniform in time L^∞ bounded weak solution for the p-Laplacian Keller-Segel system with the supercritical diffusion exponent $1 < p < \frac{3d}{d+1}$ in the multi-dimensional space $\mathbb{R}^d$ under the condition that the $L^{\frac{d(3-p)}{p}}$ norm of initial data is smaller than a universal constant. We also prove the local existence of weak solutions and a blow-up criterion for general $L^1 \cap L^\infty$ initial data.

1. **Introduction.** In this paper, we study the following p-Laplacian Keller-Segel model in $d \geq 3$:

$$\begin{cases} \partial_t u = \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^d,\ t > 0, \\ -\Delta v = u, & x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $p > 1$. $1 < p < 2$ is called the fast p-Laplacian diffusion, while $p > 2$ is called the slow p-Laplacian diffusion. Especially, the p-Laplacian Keller-Segel model turns to the original model when $p = 2$.

The Keller-Segel model was firstly presented in 1970 to describe the chemotaxis of cellular slime molds [13, 14]. The original model was considered in 2D,

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^2,\ t > 0, \\ -\Delta v = u, & x \in \mathbb{R}^2,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{2}$$

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$u(x,t)$ represents the cell density, and $v(x,t)$ represents the concentration of the chemical substance which is given by the fundamental solution

$$v(x,t) = \Phi(x) * u(x,t),$$

where

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & d=2, \\ \frac{1}{d(d-2)\alpha(d)}\frac{1}{|x|^{d-2}}, & d \geq 3, \end{cases}$$

$\alpha(d)$ is the volume of the d-dimensional unit ball. In this model, cells are attracted by the chemical substance and also able to emit it.

One natural extension of the original Keller-Segel model is the degenerate Keller-Segel model in the multi-dimension with $m > 1$,

$$\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^d, \ t > 0, \\ -\Delta v = u, & x \in \mathbb{R}^d, \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{3}$$

which has been widely studied [2, 4, 7, 8, 15, 22, 23, 24, 25]. Another natural extension is the degenerate p-Laplacian Keller-Segel model in the multi-dimension since the porous medium equation and the p-Laplacian equation are all called nonlinear diffusion equations. Work in these two models has frequent overlaps both in phenomena to be described, results to be proved and techniques to be used. The porous medium equation and the p-Laplacian equation are different territories with some important traits in common. The evolution p-Laplacian equation is also called the non-Newtonian filtration equation which describes the diffusion with the diffusivity depending on the gradient of the unknown. The comprehensive and systematic study for these two equations can be found in Vázquez [27], DiBenedetto [10] and Wu, Zhao, Yin and Li [28].

In the p-Laplacian Keller-Segel model, the exponent p plays an important role. When $p = \frac{3d}{d+1}$, if (u,v) is a solution of (1), constructing the following mass invariant scaling for u and a corresponding scaling for v

$$\begin{aligned} u_\lambda(x,t) &= \lambda u\left(\lambda^{\frac{1}{d}}x, \lambda t\right), \\ v_\lambda(x,t) &= \lambda^{1-\frac{2}{d}} v\left(\lambda^{\frac{1}{d}}x, \lambda t\right), \end{aligned} \tag{4}$$

then (u_λ, v_λ) is also a solution for (1) and hence $p = \frac{3d}{d+1}$ is referred to the critical exponent. For the general exponent p, (u_λ, v_λ) satisfies the following equation

$$\begin{cases} u_t = \lambda^{\left(1+\frac{1}{d}\right)p-3} \nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right) - \nabla \cdot (u \nabla v), \\ -\Delta v = u. \end{cases} \tag{5}$$

If $\left(1+\frac{1}{d}\right)p - 3 < 0$ which is called the supercritical case, the aggregation dominates the diffusion for high density(large λ) which leads to the finite-time blow-up, and the diffusion dominates the aggregation for low density(small λ) which leads to the infinite-time spreading. If $\left(1+\frac{1}{d}\right)p - 3 > 0$ which is called the subcritical case, the aggregation dominates the diffusion for low density(small λ) which prevents spreading, while the diffusion dominates the aggregation for high density(large λ) which prevents blow-up. At the end of Section 5, we have the theorem of the existence of a global weak solution for (1) in the subcritical case.

In the supercritical case, there is a L^q space, where $q = \frac{d(3-p)}{p}$. The q is crucial when studying the existence and blow-up results of the p-Laplacian Keller-Segel model and almost all the results are related to the initial data $\|u_0(\cdot)\|_{L^q(\mathbb{R}^d)}$. Also considering model (1), if (u, v) is a solution, then

$$u_\lambda(x,t) = \lambda u\left(\lambda^{\frac{3-p}{p}}x, \lambda t\right),$$

$$v_\lambda(x,t) = \lambda^{3-\frac{6}{p}} v\left(\lambda^{\frac{3-p}{p}}x, \lambda t\right),$$

is also a solution of (1). Furthermore, the scaling of $u(x,t)$ preserves the L^q norm $\|u_\lambda\|_{L^q} = \|u\|_{L^q}$. For $1 < p < \frac{3d}{d+1}$, if $\|u_0\|_{L^q(\mathbb{R}^d)} < C_{d,p}$, where $C_{d,p}$ is a universal constant depending on d and p, then we will show that there exists a global weak solution. Since the initial condition $u_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$, we can prove that weak solutions are bounded uniformly in time by using the bootstrap iterative method(See [3], [19]). With no restriction of the L^q norm on initial data, we prove the local existence of a weak solution. This result also provides a natural blow-up criterion for $1 < p < \frac{3d}{d+1}$ that all $\|u\|_{L^h(\mathbb{R}^d)}$ blow up at exactly the same time for $h \in (q, +\infty)$. In the subcritical case $p > \frac{3d}{d+1}$, there exists a global weak solution of (1) without any restriction of the size of initial data.

In the process of proving the existence of a global weak solution of (1), we combine the Aubin-Lions Lemma with the monotone operator theory. The theory of monotone operators was proposed by Minty [20, 21]. Then the theory was used to obtain the existence results for quasi-linear elliptic and parabolic partial differential equations by Browder [5, 6], Leray and Lions [17], Hartman and Stampacchia [12], DiBenedetto and Herrero [11].

The paper is organized as follows. In Section 2, we define a weak solution, introduce a Sobolev inequality with the best constant and some lemmas. In Section 3, we give the *a priori* estimates of our weak solution. In Section 4, we prove the theorem about uniformly in time L^∞ bound of weak solutions using a bootstrap iterative method. In Section 5, we construct a regularized problem to prove the existence of a global weak solution. Finally, in Section 6, we discuss the local existence of weak solutions and a blow-up criterion.

2. Preliminaries.

The generic constant will be denoted by C, even if it is different from line to line. At the beginning, we define a weak solution of (1) in this paper.

Definition 2.1. (Weak solution) Let $u_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$ be initial data and $T \in (0, \infty)$. $v(x,t)$ is given by the fundamental solution

$$v(x,t) = \frac{1}{d(d-2)\alpha(d)} \int_{\mathbb{R}^d} \frac{u(y,t)}{|x-y|^{d-2}}\, dy.$$

Then (u, v) is a weak solution to (1) if u satisfies

(i) Regularity:

$$u \in L^\infty\left(0,T; L^1_+(\mathbb{R}^d)\right) \cap L^p\left(0,T; W^{1,p}(\mathbb{R}^d)\right) \cap L^2\left(0,T; L^{\frac{2d}{d+2}}(\mathbb{R}^d)\right),$$

$$\partial_t u \in L^{\frac{p}{p-1}}\left(0,T; W^{-2,\frac{p}{p-1}}(\mathbb{R}^d)\right).$$

(ii) $\forall\ \psi(x,t) \in C_c^\infty\left([0,T) \times \mathbb{R}^d\right)$,

$$\int_0^T \int_{\mathbb{R}^d} u(x,t)\psi_t(x,t)\, dxdt = \int_0^T \int_{\mathbb{R}^d} |\nabla u(x,t)|^{p-2}\nabla u(x,t) \cdot \nabla\psi(x,t)\, dxdt$$

$$
\begin{aligned}
&- \frac{1}{2d\alpha(d)} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left[\nabla\psi(x,t) - \nabla\psi(y,t)\right] \cdot (x-y)}{|x-y|^2} \frac{u(x,t)u(y,t)}{|x-y|^{d-2}} \, dxdydt \\
&- \int_{\mathbb{R}^d} u_0(x)\psi(x,0)dx. \qquad (6)
\end{aligned}
$$

The following lemma is a Sobolev inequality with the best constant which was identified by Talenti [26] and Aubin [1].

Lemma 2.2. *(Sobolev inequality) Let $1 < p < d$. If the function $u \in W^{1,p}(\mathbb{R}^d)$, then*

$$
\|u\|_{L^{p*}(\mathbb{R}^d)} \leq K(d,p)\|\nabla u\|_{L^p(\mathbb{R}^d)}, \qquad (7)
$$

where $p* = \frac{dp}{d-p}$ and

$$
K(d,p) = \pi^{-\frac{1}{2}} d^{-\frac{1}{p}} \left(\frac{p-1}{d-p}\right)^{1-\frac{1}{p}} \left[\frac{\Gamma(1+\frac{d}{2})\Gamma(d)}{\Gamma(\frac{d}{p})\Gamma(1+d-\frac{d}{p})}\right]^{\frac{1}{d}}. \qquad (8)
$$

Next two lemmas are proposed by Bian and Liu [2].

Lemma 2.3. *Assume $y(t) \geq 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \leq \gamma - \beta y(t)^a$ for $\gamma \geq 0, \beta > 0$ and $a > 0$. Then*

(i) for $a > 1$, $y(t)$ has the following hyper-contractive property:

$$
y(t) \leq \left(\frac{\gamma}{\beta}\right)^{\frac{1}{a}} + \left[\frac{1}{\beta(a-1)t}\right]^{\frac{1}{a-1}}, \quad t > 0,
$$

(ii) for $a = 1$, $y(t)$ decays as

$$
y(t) \leq \frac{\gamma}{\beta} + y(0)e^{-\beta t},
$$

(iii) for $a < 1$, $\gamma = 0$, $y(t)$ has the finite time extinction, which means that there exists a T_{ext} satisfying $0 < T_{ext} \leq \frac{y^{1-a}(0)}{\beta(1-a)}$ such that $y(t) = 0$ for all $t > T_{ext}$.

Lemma 2.4. *Assume $f(t) \geq 0$ is a non-increasing function for $t > 0$, $y(t) \geq 0$ is a C^1 function for $t > 0$ and satisfies $y'(t) \leq f(t) - \beta y(t)^a$ for some constants $a > 1$ and $\beta > 0$, then for any $t_0 > 0$ one has*

$$
y(t) \leq \left(\frac{f(t_0)}{\beta}\right)^{\frac{1}{a}} + \Big(\beta(a-1)(t-t_0)\Big)^{-\frac{1}{a-1}}, \quad \text{for} \ \ t > t_0.
$$

With the additional condition that $y(0)$ is bounded, we have Lemma 2.5 which can be proved by contradiction arguments.

Lemma 2.5. *Assume $y(t) \geq 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \leq \gamma - \beta y(t)^a$ for $\gamma > 0, \beta > 0$ and $a > 0$. If $y(0)$ is bounded, then*

$$
y(t) \leq \max\left(y(0), \left(\frac{\gamma}{\beta}\right)^{\frac{1}{a}}\right), \ t > 0.
$$

3. *A priori* **estimates of weak solutions.** In this section, we prove Theorem 3.1 which is concerning *a priori* estimates of weak solutions of (1).

Theorem 3.1. *Let $d \geq 3$, $1 < p < \frac{3d}{d+1}$ and $q = \frac{d(3-p)}{p}$. Under the assumption that $u_0 \in L^1_+(\mathbb{R}^d)$ and $A(d,p) = C_{p,d}^{3-p} - \|u_0\|_{L^q}^{3-p} > 0$, where $C_{p,d} = \left[\frac{qp^p}{K^p(d,p)(q-2+p)^p}\right]^{\frac{1}{3-p}}$ is a universal constant, let (u,v) be a non-negative weak solution of (1). Then $u \in L^\infty\big(\mathbb{R}_+; L^q(\mathbb{R}^d)\big)$, $u \in L^{q+1}\big(\mathbb{R}_+; L^{q+1}(\mathbb{R}^d)\big)$ and $\nabla u^{\frac{q-2+p}{p}} \in L^p\big(\mathbb{R}_+; L^p(\mathbb{R}^d)\big)$. Furthermore, following a priori estimates hold true:*

(i) For $1 < p < \frac{2d}{d+1}$, $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}$ has the finite time extinction. The extinct time T_{ext} satisfies

$$0 < T_{ext} \leq T_0,$$

where T_0 depends on $d, p, A(d,p), \|u_0\|_{L^1(\mathbb{R}^d)}$ and $\|u_0\|_{L^q(\mathbb{R}^d)}$.

(ii) For $p = \frac{2d}{d+1}$, $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}$ decays exponentially in time

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leq \|u_0\|_{L^q} e^{-Ct},$$

where C is a constant depending on $d, p, A(d,p), \|u_0\|_{L^1(\mathbb{R}^d)}$ and $\|u_0\|_{L^q(\mathbb{R}^d)}$.

(iii) For $\frac{2d}{d+1} < p < \frac{3d}{d+1}$, $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}$ decays in time

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leq \frac{\|u_0\|_{L^q}}{(1+Ct)^{\frac{q-1}{q\left(p-2+\frac{p}{d}\right)}}},$$

where C depends on $d, p, A(d,p), \|u_0\|_{L^1(\mathbb{R}^d)}$ and $\|u_0\|_{L^q(\mathbb{R}^d)}$. And for any $1 \leq h \leq q$, $\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}$ decays in time

$$\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)} \leq \frac{\|u_0\|_{L^q}^{\frac{q(h-1)}{h(q-1)}} \|u_0\|_{L^1}^{\frac{q-h}{h(q-1)}}}{(1+Ct)^{\frac{h-1}{h\left(p-2+\frac{p}{d}\right)}}}.$$

For any $q < h < \infty$, $u(x,t)$ has hyper-contractive property

$$\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h \leq C\left(t^{-\frac{(q+\epsilon-1)(h-q+1)(h-1)}{\epsilon\left(p-2+\frac{p}{d}\right)\left(h+p-3+\frac{p}{d}\right)}} + t^{-\frac{h-1}{p-2+\frac{p}{d}}}\right),$$

where C is a constant depending on $h, d, p, A(d,p)$ and $\|u_0\|_{L^1}$, $\epsilon > 0$ satisfies $\frac{(q+\epsilon)p^p}{K^p(d,p)(q+\epsilon-2+p)^p} - \|u_0\|_{L^q}^{3-p} \geq \frac{A(d,p)}{2}$.

Proof. **Step 1.** (The L^q estimate for $1 < p < \frac{3d}{d+1}$). Multiplying the first equation in problem (1) by qu^{q-1} and integrating it over $\mathbb{R}^d$, we obtain

$$\begin{aligned}\frac{d}{dt}\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^q &= \int_{\mathbb{R}^d} \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) qu^{q-1}\,dx - \int_{\mathbb{R}^d} \nabla\cdot(u\nabla v)qu^{q-1}\,dx \\ &= -q(q-1)\int_{\mathbb{R}^d} u^{q-2}|\nabla u|^p\,dx + (q-1)\int_{\mathbb{R}^d}\nabla u^q\cdot\nabla v\,dx \\ &= -\frac{q(q-1)p^p}{(q-2+p)^p}\left\|\nabla u^{\frac{q-2+p}{p}}(t)\right\|_{L^p(\mathbb{R}^d)}^p + (q-1)\|u\|_{L^{q+1}(\mathbb{R}^d)}^{q+1}. \end{aligned} \quad (9)$$

Now we estimate the second term on the right hand side. Firstly, by using the interpolation inequality, we obtain that

$$
\begin{aligned}
\|u\|_{L^{q+1}(\mathbb{R}^d)}^{q+1} &\le \|u\|_{L^{\frac{(q-2+p)d}{d-p}}}^{\frac{d(q-2+p)}{pd+pq-2d}} \|u\|_{L^q}^{\frac{q(pd+pq+p-3d)}{pd+pq-2d}} \\
&= \left\| u^{\frac{q-2+p}{p}} \right\|_{L^{\frac{dp}{d-p}}}^{\frac{dp}{pd+pq-2d}} \|u\|_{L^q}^{\frac{q(pd+pq+p-3d)}{pd+pq-2d}} \\
&= \left\| u^{\frac{q-2+p}{p}} \right\|_{L^{\frac{dp}{d-p}}}^{p} \|u\|_{L^q}^{3-p},
\end{aligned} \tag{10}
$$

where the last equality holds since $\frac{d}{pd+pq-2d} = 1$ and $\frac{q(pd+pq+p-3d)}{pd+pq-2d} = 3-p$ from $q = \frac{d(3-p)}{p}$. Then using the Sobolev inequality (7), (10) turns to

$$
\|u\|_{L^{q+1}(\mathbb{R}^d)}^{q+1} \le K^p(d,p) \left\| \nabla u^{\frac{q-2+p}{p}} \right\|_{L^p}^{p} \|u\|_{L^q}^{3-p}, \tag{11}
$$

where $K(d,p)$ is given by (8). Substituting (11) into (9), we have

$$
\frac{d}{dt}\|u\|_{L^q}^q + (q-1)\left(\frac{qp^p}{(q-2+p)^p} - K^p(d,p)\|u\|_{L^q}^{3-p} \right) \left\| \nabla u^{\frac{q-2+p}{p}} \right\|_{L^p}^{p} \le 0. \tag{12}
$$

Since $\|u_0(\cdot)\|_{L^q(\mathbb{R}^d)} < \left[\frac{qp^p}{K^p(d,p)(q-2+p)^p} \right]^{\frac{1}{3-p}} =: C_{p,d}$, following two estimates hold true

$$
\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} < \|u_0(\cdot)\|_{L^q(\mathbb{R}^d)} < C_{p,d}, \tag{13}
$$

$$
(q-1)K^p(d,p)\left(C_{p,d}^{3-p} - \|u_0\|_{L^q}^{3-p} \right) \int_0^\infty \left\| \nabla u^{\frac{q-2+p}{p}} \right\|_{L^p}^{p} \, ds \le C_{p,d}.
$$

Combining (11) with two estimates above, we obtain

$$
u(x,t) \in L^\infty\left(\mathbb{R}_+; L^q(\mathbb{R}^d)\right), \tag{14}
$$

$$
u(x,t) \in L^{q+1}\left(\mathbb{R}_+; L^{q+1}(\mathbb{R}^d)\right), \tag{15}
$$

$$
\nabla u^{\frac{q-2+p}{p}}(x,t) \in L^p\left(\mathbb{R}_+; L^p(\mathbb{R}^d)\right). \tag{16}
$$

Step 2. (The L^q decay estimate). By using the interpolation inequality and (11), we have

$$
\begin{aligned}
\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^q &\le \|u\|_{L^{q+1}}^{\frac{(q+1)(q-1)}{q}} \|u\|_{L^1}^{\frac{1}{q}} \\
&\le \left[K^p(d,p) \left\| \nabla u^{\frac{q-2+p}{p}} \right\|_{L^p}^{p} \|u\|_{L^q}^{3-p} \right]^{\frac{q-1}{q}} \|u\|_{L^1}^{\frac{1}{q}},
\end{aligned} \tag{17}
$$

i.e.

$$
\left\| \nabla u^{\frac{q-2+p}{p}} \right\|_{L^p}^{p} \ge \frac{\|u\|_{L^q}^{\frac{q^2}{q-1}-3+p}}{K^p(d,p)\|u_0\|_{L^1}^{\frac{1}{q-1}}} = \frac{\left(\|u\|_{L^q}^q\right)^{1+\frac{p-2+\frac{p}{d}}{q-1}}}{K^p(d,p)\|u_0\|_{L^1}^{\frac{1}{q-1}}}, \tag{18}
$$

since $\|u(\cdot,t)\|_{L^1} \le \|u_0\|_{L^1}$. Substituting (18) into (12) yields that

$$
\frac{d}{dt}\|u(\cdot,t)\|_{L^q}^q + \frac{(q-1)A(d,p)}{\|u_0\|_{L^1}^{\frac{1}{q-1}}} \left(\|u\|_{L^q}^q\right)^{1+\frac{p-2+\frac{p}{d}}{q-1}} \le 0, \tag{19}
$$

where we denote $A(d,p) := C_{p,d}^{3-p} - \|u_0\|_{L^q}^{3-p}$.

Next we discuss the inequality (19) in three different situations.

(a) If $1+\frac{p-2+\frac{p}{d}}{q-1}>1$, i.e. $\frac{2d}{d+1}<p<\frac{3d}{d+1}$, we can prove that $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}$ decays in time

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le \frac{\|u_0\|_{L^q}}{(1+Ct)^{\frac{q-1}{q\left(p-2+\frac{p}{d}\right)}}}, \tag{20}$$

where $C=\frac{A(d,p)\left(p-2+\frac{p}{d}\right)\left(\|u_0\|_{L^q}^q\right)^{\frac{p-2+\frac{p}{d}}{q-1}}}{\|u_0\|_{L^1}^{\frac{1}{q-1}}}$.

(b) If $1+\frac{p-2+\frac{p}{d}}{q-1}=1$, i.e. $p=\frac{2d}{d+1}$, $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}$ decays exponentially in time

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le \|u_0\|_{L^q(\mathbb{R}^d)}e^{-Ct},$$

where $C=\frac{(q-1)A(d,p)}{q\|u_0\|_{L^1(\mathbb{R}^d)}^{1/(q-1)}}$.

(c) If $0<1+\frac{p-2+\frac{p}{d}}{q-1}<1$, i.e. $1<p<\frac{2d}{d+1}$, $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}$ has the finite time extinction. The extinct time T_{ext} satisfies $0<T_{ext}\le T_0$, where $T_0=\frac{\|u_0\|_{L^q(\mathbb{R}^d)}^{-\frac{q\left(p-2+\frac{p}{d}\right)}{q-1}}\|u_0\|_{L^1(\mathbb{R}^d)}^{1/(q-1)}}{-A(d,p)\left(p-2+\frac{p}{d}\right)}$.

Step 3. (The L^h decay estimate for any $1\le h\le q$ when $\frac{2d}{d+1}<p<\frac{3d}{d+1}$). Using the interpolation inequality and (20), we have

$$\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)} \le \|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^{\frac{q(h-1)}{h(q-1)}}\|u(\cdot,t)\|_{L^1(\mathbb{R}^d)}^{\frac{q-h}{h(q-1)}} \le \frac{\|u_0\|_{L^q}^{\frac{q(h-1)}{h(q-1)}}\|u_0\|_{L^1}^{\frac{q-h}{h(q-1)}}}{(1+Ct)^{\frac{h-1}{h\left(p-2+\frac{p}{d}\right)}}}. \tag{21}$$

Step 4. (The hyper-contractive property for any $q<h<\infty$ when $\frac{2d}{d+1}<p<\frac{3d}{d+1}$). **$L^r$ estimates with $r=q+\epsilon$ for ϵ small enough.** Since $A(d,p)=C_{p,d}^{3-p}-\|u_0\|_{L^q}^{3-p}$ where $C_{p,d}=\left[\frac{qp^p}{K^p(d,p)(q-2+p)^p}\right]^{\frac{1}{3-p}}$, there exists $\epsilon>0$ such that

$$\frac{(q+\epsilon)p^p}{K^p(d,p)(q+\epsilon-2+p)^p}-\|u_0\|_{L^q}^{3-p}\ge\frac{A(d,p)}{2}. \tag{22}$$

In the same way of obtaining (9)-(11), we obtain

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r=-\frac{r(r-1)p^p}{(r-2+p)^p}\left\|\nabla u^{\frac{r-2+p}{p}}(t)\right\|_{L^p}^p+(r-1)\|u\|_{L^{r+1}}^{r+1}, \tag{23}$$

and

$$\begin{aligned}\|u\|_{L^{r+1}(\mathbb{R}^d)}^{r+1} &\le \|u\|_{L^{\frac{(r-2+p)d}{d-p}}}^{\frac{d(r-2+p)(r+1-q)}{rd+pd+pq-qd-2d}}\|u\|_{L^q}^{\frac{q(pd+pr+p-3d)}{rd+pd+pq-qd-2d}}\\ &=\left\|u^{\frac{r-2+p}{p}}\right\|_{L^{\frac{dp}{d-p}}}^{\frac{dp(r+1-q)}{rd+pd+pq-qd-2d}}\|u\|_{L^q}^{\frac{q(pd+pr+p-3d)}{rd+pd+pq-qd-2d}}\\ &=\left\|u^{\frac{r-2+p}{p}}\right\|_{L^{\frac{dp}{d-p}}}^{p}\|u\|_{L^q}^{3-p}\\ &\le K^p(d,p)\left\|\nabla u^{\frac{r-2+p}{p}}\right\|_{L^p}^p\|u\|_{L^q}^{3-p},\end{aligned} \tag{24}$$

where the third equality holds since $\frac{d(r+1-q)}{rd+pd+pq-qd-2d}=1$ and $\frac{q(pd+pr+p-3d)}{rd+pd+pq-qd-2d}=3-p$, and the last inequality holds from the Sobolev inequality. Then combining

(22), (23) and (24) together, we have

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^r}^r+\frac{(r-1)K^p(d,p)A(d,p)}{2}\left\|\nabla u^{\frac{r-2+p}{p}}\right\|_{L^p}^p\le 0. \tag{25}$$

By using the interpolation inequality and (24), we have

$$\begin{aligned}\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r&\le\|u\|_{L^{r+1}}^{\frac{(r+1)(r-1)}{r}}\|u\|_{L^1}^{\frac{1}{r}}\\&\le\left[K^p(d,p)\left\|\nabla u^{\frac{r-2+p}{p}}\right\|_{L^p}^p\|u\|_{L^q}^{3-p}\right]^{\frac{r-1}{r}}\|u\|_{L^1}^{\frac{1}{r}}\\&\le\left[K^p(d,p)\left\|\nabla u^{\frac{r-2+p}{p}}\right\|_{L^p}^p\|u\|_{L^r}^{\frac{r(3-p)(q-1)}{q(r-1)}}\|u_0\|_{L^1}^{\frac{(3-p)(r-q)}{q(r-1)}}\right]^{\frac{r-1}{r}}\|u_0\|_{L^1}^{\frac{1}{r}},\end{aligned}$$

i.e.

$$\left\|\nabla u^{\frac{r-2+p}{p}}\right\|_{L^p}^p\ge\frac{\left(\|u\|_{L^r}^r\right)^{1+\frac{p-2+\frac{p}{d}}{r-1}}}{K^p(d,p)\|u_0\|_{L^1}^{\frac{1}{r-1}\left(1+\frac{p(r-q)}{d}\right)}}, \tag{26}$$

since $\|u\|_{L^1(\mathbb{R}^d)}\le\|u_0\|_{L^1(\mathbb{R}^d)}$. Substituting (26) into (25) yields that

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^r}^r+\beta_1\left(\|u\|_{L^r}^r\right)^{1+\frac{p-2+\frac{p}{d}}{r-1}}\le 0,\quad \beta_1:=\frac{(r-1)A(d,p)}{2\|u_0\|_{L^1}^{\frac{1}{r-1}\left(1+\frac{p(r-q)}{d}\right)}}. \tag{27}$$

Solving this inequality by using Lemma 2.3, we have

$$\|u(\cdot,t)\|_{L^r}^r\le C(r)t^{-\frac{r-1}{p-2+\frac{p}{d}}}. \tag{28}$$

Hyper-contractive estimates of L^h norm for $h\ge r$. For $h\ge r>q$, using the interpolation inequality, Sobolev inequality and Young's inequality together, we obtain

$$\begin{aligned}\|u\|_{L^{h+1}(\mathbb{R}^d)}^{h+1}&\le\|u\|_{L^{\frac{(h-2+p)d}{d-p}}}^{\frac{d(h-2+p)(h+1-r)}{hd+pd+pr-rd-2d}}\|u\|_{L^r}^{\frac{r(pd+ph+p-3d)}{hd+pd+pr-rd-2d}}\\&=\left\|u^{\frac{h-2+p}{p}}\right\|_{L^{\frac{dp}{d-p}}}^{\frac{dp(h+1-r)}{hd+pd+pr-rd-2d}}\|u\|_{L^r}^{\frac{r(pd+ph+p-3d)}{hd+pd+pr-rd-2d}}\\&\le K^{\frac{dp(h+1-r)}{hd+pd+pr-rd-2d}}(d,p)\left\|\nabla u^{\frac{h-2+p}{p}}\right\|_{L^p}^{\frac{dp(h+1-r)}{hd+pd+pr-rd-2d}}\|u\|_{L^r}^{\frac{r(pd+ph+p-3d)}{hd+pd+pr-rd-2d}}\\&\le\frac{hp^p}{2(h-2+p)^p}\left\|\nabla u^{\frac{h-2+p}{p}}\right\|_{L^p}^p+C(h,r)\left(\|u\|_{L^r}^r\right)^{1+\frac{h-r+1}{r-q}},\end{aligned} \tag{29}$$

where

$$\frac{dp(h+1-r)}{hd+pd+pr-rd-2d}=\frac{pd(h+1-r)}{d(h+1-r)+p(r-q)}<p.$$

Considering (9) with $h=q$, we have

$$\begin{aligned}\frac{d}{dt}\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h&=-\frac{h(h-1)p^p}{(h-2+p)^p}\left\|\nabla u^{\frac{h-2+p}{p}}(t)\right\|_{L^p(\mathbb{R}^d)}^p+(h-1)\|u\|_{L^{h+1}(\mathbb{R}^d)}^{h+1}\\&\le-\frac{h(h-1)p^p}{2(h-2+p)^p}\left\|\nabla u^{\frac{h-2+p}{p}}(t)\right\|_{L^p(\mathbb{R}^d)}^p+C(h,r)\left(\|u\|_{L^r}^r\right)^{1+\frac{h-r+1}{r-q}}.\end{aligned} \tag{30}$$

Substituting (28) into (30) yields that

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h\le-\frac{h(h-1)p^p}{2(h-2+p)^p}\left\|\nabla u^{\frac{h-2+p}{p}}\right\|_{L^p(\mathbb{R}^d)}^p+C(h,r)t^{-\frac{(r-1)(h-q+1)}{\left(p-2+\frac{p}{d}\right)(r-q)}}. \tag{31}$$

By the same way of obtaining (26), we obtain

$$\left\|\nabla u^{\frac{h-2+p}{p}}\right\|_{L^p}^p \geq \frac{\left(\|u\|_{L^h}^h\right)^{1+\frac{p-2+\frac{p}{d}}{h-1}}}{K^p(d,p)\|u_0\|_{L^1}^{\frac{1}{h-1}\left(1+\frac{p(h-q)}{d}\right)}}. \tag{32}$$

Then (31) turns to

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h \leq -\beta_2\left(\|u\|_{L^h}^h\right)^{1+\frac{p-2+\frac{p}{d}}{h-1}} + C(h,r)t^{-\frac{(r-1)(h-q+1)}{\left(p-2+\frac{p}{d}\right)(r-q)}}, \tag{33}$$

where $\beta_2 = \frac{h(h-1)p^p}{2(h-2+p)^p K^p(d,p)\|u_0\|_{L^1}^{\frac{1}{h-1}\left(1+\frac{p(h-q)}{d}\right)}}$.

Using Lemma 2.4 with $y(t) = \|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h$, $a = 1 + \frac{p-2+\frac{p}{d}}{h-1} > 1$, $\beta = \beta_2 > 0$ and $f(t) = C(h,r)t^{-\frac{(r-1)(h-q+1)}{\left(p-2+\frac{p}{d}\right)(r-q)}}$, for any $t > t_0 > 0$, we have

$$\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h \leq C(h,r)t_0^{-\frac{(q+\epsilon-1)(h-q+1)(h-1)}{\epsilon\left(p-2+\frac{p}{d}\right)\left(h+p-3+\frac{p}{d}\right)}} + C(h)(t-t_0)^{-\frac{h-1}{p-2+\frac{p}{d}}}. \tag{34}$$

By choosing $t_0 = \frac{t}{2}$, we obtain that for any $t > 0$

$$\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h \leq C\left(t^{-\frac{(q+\epsilon-1)(h-q+1)(h-1)}{\epsilon\left(p-2+\frac{p}{d}\right)\left(h+p-3+\frac{p}{d}\right)}} + t^{-\frac{h-1}{p-2+\frac{p}{d}}}\right), \tag{35}$$

where C is a constant depending on $h, d, p, A(d,p)$ and $\|u_0\|_{L^1}$, ϵ satisfies (22). $\square$

4. **The uniformly in time L^∞ estimate of weak solutions.** In this section, we prove our theorem about uniformly in time L^∞ boundness of weak solutions by using a bootstrap iterative method. At the beginning of this section, we prove the following proposition concerning L^h norm estimates of weak solutions for $1 < h < \infty$.

Proposition 1. *Let $d \geq 3$, $1 < p < \frac{3d}{d+1}$ and $q = \frac{d(3-p)}{p}$. If $u_0 \in L^1_+(\mathbb{R}^d) \cap L^h(\mathbb{R}^d)$ for $1 < h < \infty$ and $A(d,p) = C_{p,d}^{3-p} - \|u_0\|_{L^q}^{3-p} > 0$, where $C_{p,d} = \left[\frac{qp^p}{K^p(d,p)(q-2+p)^p}\right]^{\frac{1}{3-p}}$ is a universal constant, let (u,v) be a non-negative weak solution of (1). Then $u(x,t)$ satisfies for any $t > 0$*

$$\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h \leq C\|u_0\|_{L^q(\mathbb{R}^d)}^{\frac{q(h-1)}{q-1}}, \quad 1 < h \leq q, \tag{36}$$

where C depends on h, q, and $\|u_0\|_{L^1}$, and

$$\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h \leq C_u^h, \quad q < h < \infty, \tag{37}$$

where C_u^h is a constant depending on $d, p, h, \|u_0\|_{L^1}$ and $\|u_0\|_{L^h}$, $\epsilon > 0$ satisfies $\frac{(q+\epsilon)p^p}{K^p(d,p)(q+\epsilon-2+p)^p} - \|u_0\|_{L^q}^{3-p} \geq \frac{A(d,p)}{2}$.

Actually, the proof of Proposition 1 is almost the same as the proof of Theorem 3.1, except for the different initial condition $u_0 \in L^1_+(\mathbb{R}^d) \cap L^h(\mathbb{R}^d)$ for $1 < h < \infty$.

Proof. Using the same method in Step 1 of Theorem 3.1, we have for all $t > 0$

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} < \|u_0(\cdot)\|_{L^q(\mathbb{R}^d)} < C_{p,d}.$$

Then we discuss in two different situations with respect to h.

For $1 < h \le q$, using the interpolation inequality, we have

$$\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h \le \|u_0(\cdot)\|_{L^1(\mathbb{R}^d)}^{\frac{q-h}{q-1}} \|u_0(\cdot)\|_{L^q(\mathbb{R}^d)}^{\frac{q(h-1)}{q-1}}. \tag{38}$$

For $q < h < \infty$, letting $r := q+\epsilon \le h < \infty$, there exists $\epsilon > 0$ small enough such that

$$\frac{(q+\epsilon)p^p}{K^p(d,p)(q+\epsilon-2+p)^p} - \|u_0\|_{L^q}^{3-p} \ge \frac{A(d,p)}{2}.$$

Then (25) also holds true, i.e.

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^r}^r + \frac{(r-1)K^p(d,p)A(d,p)}{2}\left\|\nabla u^{\frac{r-2+p}{p}}\right\|_{L^p}^p \le 0.$$

Since $q < r \le h$, we have $u_0 \in L^r(\mathbb{R}^d)$ and

$$\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} \le \|u_0(\cdot)\|_{L^r(\mathbb{R}^d)}, \tag{39}$$

for all $t > 0$. Combining (30), (32) and (39) together, we obtain

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)}^h \le -\beta_3\left(\|u\|_{L^h}^h\right)^{1+\frac{p-2+\frac{p}{d}}{h-1}} + C(h,r)\left(\|u_0\|_{L^r}^r\right)^{1+\frac{h-r+1}{r-q}}, \tag{40}$$

where $\beta_3 := \frac{h(h-1)p^p}{2(h-2+p)^p K^p(d,p)\|u_0\|_{L^1}^{\frac{1}{h-1}\left(1+\frac{p(h-q)}{d}\right)}} > 0$. Using Lemma 2.5 with $y(t) = \|u(\cdot,t)\|_{L^h}^h$, $a = 1+\frac{p-2+\frac{p}{d}}{h-1} > 0$, $\beta = \beta_3 > 0$ and $\gamma = C(h,r)\left(\|u_0\|_{L^r}^r\right)^{1+\frac{h-r+1}{r-q}} > 0$, for any $t > 0$, we have

$$\begin{aligned}\|u(\cdot,t)\|_{L^h}^h &\le \max\left\{\|u_0\|_{L^h}^h,\ C(h,r)\left(\|u_0\|_{L^r}^r\right)^{\frac{(h-q+1)(h-1)}{\epsilon(h+p-3+\frac{p}{d})}}\right\} \\ &\le \max\left\{\|u_0\|_{L^h}^h,\ C(h)\left(\|u_0\|_{L^h}^h\right)^{\frac{(h-q+1)(q+\epsilon-1)}{\epsilon(h+p-3+\frac{p}{d})}}\right\} =: C_u^h,\end{aligned} \tag{41}$$

where ϵ satisfies $\frac{(q+\epsilon)p^p}{K^p(d,p)(q+\epsilon-2+p)^p} - \|u_0\|_{L^q}^{3-p} \ge \frac{A(d,p)}{2}$. □

Next, we prove the uniformly in time L^∞ boundness of $u(x,t)$ by using a bootstrap iterative technique [3, 19] with Proposition 1 and an additional initial condition $u_0 \in L^\infty(\mathbb{R}^d)$.

Theorem 4.1. *Let $d \ge 3$, $1 < p < \frac{3d}{d+1}$ and $q = \frac{d(3-p)}{p}$. If $u_0 \in L_+^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $A(d,p) = C_{p,d}^{3-p} - \|u_0\|_{L^q}^{3-p} > 0$, where $C_{p,d} = \left[\frac{qp^p}{K^p(d,p)(q-2+p)^p}\right]^{\frac{1}{3-p}}$ is a universal constant, let (u,v) be a non-negative weak solution of (1). Then for any $t > 0$,*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C(d,p,K_0),$$

where $K_0 = \max\left\{1,\ \|u_0\|_{L^1(\mathbb{R}^d)},\ \|u_0\|_{L^\infty(\mathbb{R}^d)}\right\}$.

Proof. We denote

$$h_k = 3^k + \frac{d(3-p)}{p} + 1, \quad \text{for } k \ge 1.$$

Multiplying the first equation in (1) by $h_k u^{h_k-1}$ and integrating, we have

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^{h_k}(\mathbb{R}^d)}^{h_k} = -\frac{h_k(h_k-1)p^p}{(h_k-2+p)^p}\left\|\nabla u^{\frac{h_k-2+p}{p}}(t)\right\|_{L^p}^p + (h_k-1)\|u\|_{L^{h_k+1}}^{h_k+1}. \tag{42}$$

Step 1. (The L^{h_k} estimate for $1 < p \le 2$) Taking $0 < C_1 \le \frac{h_k(h_k-1)p^p}{2(h_k-2+p)^p}$ is a fixed constant, then (42) turns to

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^{h_k}(\mathbb{R}^d)}^{h_k} \le -2C_1 \left\|\nabla u^{\frac{h_k-2+p}{p}}(t)\right\|_{L^p}^p + h_k\|u\|_{L^{h_k+1}}^{h_k+1}. \tag{43}$$

Using the interpolation inequality and Sobolev inequality together, we obtain

$$\begin{aligned}
\|u\|_{L^{h_k+1}(\mathbb{R}^d)}^{h_k+1} &\le \|u\|_{L^{\frac{(h_k-2+p)d}{d-p}}}^{(h_k+1)\theta} \|u\|_{L^{h_{k-1}}}^{(h_k+1)(1-\theta)} \\
&= \left\|u^{\frac{h_k-2+p}{p}}\right\|_{L^{\frac{dp}{d-p}}}^{\frac{p(h_k+1)\theta}{h_k-2+p}} \|u\|_{L^{h_{k-1}}}^{(h_k+1)(1-\theta)} \\
&\le K^{\frac{p(h_k+1)\theta}{h_k-2+p}}(d,p)\left\|\nabla u^{\frac{h_k-2+p}{p}}\right\|_{L^p}^{\frac{p(h_k+1)\theta}{h_k-2+p}} \|u\|_{L^{h_{k-1}}}^{(h_k+1)(1-\theta)},
\end{aligned} \tag{44}$$

where

$$\theta = \frac{d(h_k-2+p)(h_k-h_{k-1}+1)}{(h_k+1)\big((h_k-2+p)d - h_{k-1}(d-p)\big)},$$

$$1-\theta = \frac{h_{k-1}(h_kp+pd-3d+p)}{(h_k+1)\big((h_k-2+p)d - h_{k-1}(d-p)\big)}.$$

Since $h_{k-1} = 3^{k-1} + \frac{d(3-p)}{p} + 1 > \frac{d(3-p)}{p}$, it is easy to see that $\frac{p(h_k+1)\theta}{h_k-2+p} < p$. Then using Young's inequality and (44), we have

$$\begin{aligned}
h_k\|u\|_{L^{h_k+1}(\mathbb{R}^d)}^{h_k+1} &\le \frac{1}{a}\delta_1^a\left\|\nabla u^{\frac{h_k-2+p}{p}}\right\|_{L^p}^p + \frac{1}{b}\delta_1^{-b}K^{\frac{p(h_k+1)\theta b}{h_k-2+p}}(d,p)h_k^b\|u\|_{L^{h_{k-1}}}^{(h_k+1)(1-\theta)b} \\
&\le C_1\left\|\nabla u^{\frac{h_k-2+p}{p}}\right\|_{L^p}^p + C_2(h_k)h_k^b\|u\|_{L^{h_{k-1}}}^{(h_k+1)(1-\theta)b},
\end{aligned} \tag{45}$$

where

$$a = \frac{h_k-2+p}{(h_k+1)\theta} = \frac{d(h_k-h_{k-1}+1)+h_{k-1}p+pd-3d}{d(h_k-h_{k-1}+1)} > 1,$$

$$b = \frac{h_k-2+p}{h_k-2+p-(h_k+1)\theta} = \frac{d(h_k-h_{k-1}+1)+h_{k-1}p+pd-3d}{h_{k-1}p+pd-3d} > 1,$$

$$\delta_1 = (C_1a)^{\frac{1}{a}}, \quad C_2(h_k) = \frac{1}{b}(C_1a)^{-\frac{b}{a}}K^{\frac{p(h_k+1)\theta b}{h_k-2+p}}(d,p).$$

We can see that $C_2(h_k)$ is uniformly bounded since $a \to \frac{2d+p}{2d}$ and $b \to \frac{2d+p}{p}$ as $k \to \infty$. Substituting (45) into (43) yields to

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^{h_k}(\mathbb{R}^d)}^{h_k} \le -C_1\left\|\nabla u^{\frac{h_k-2+p}{p}}(t)\right\|_{L^p}^p + C_2(h_k)h_k^b\left(\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right)^{\gamma_1}, \tag{46}$$

where

$$\gamma_1 = \frac{(h_k+1)(1-\theta)b}{h_{k-1}} = \frac{h_kp+pd-3d+p}{h_{k-1}p+pd-3d} < 3.$$

Next, we estimate $\left\|\nabla u^{\frac{h_k-2+p}{p}}(t)\right\|_{L^p}^p$. By using the interpolation inequality and Sobolev inequality, we have

$$\begin{aligned}
\|u\|_{L^{h_k}(\mathbb{R}^d)}^{h_k} &\le \|u\|_{L^{\frac{(h_k-2+p)d}{d-p}}}^{h_k\beta}\|u\|_{L^{h_{k-1}}}^{h_k(1-\beta)} \\
&= \left\|u^{\frac{h_k-2+p}{p}}\right\|_{L^{\frac{dp}{d-p}}}^{\frac{ph_k\beta}{h_k-2+p}}\|u\|_{L^{h_{k-1}}}^{h_k(1-\beta)} \\
&\le K^{\frac{ph_k\beta}{h_k-2+p}}(d,p)\left\|\nabla u^{\frac{h_k-2+p}{p}}\right\|_{L^p}^{\frac{ph_k\beta}{h_k-2+p}}\|u\|_{L^{h_{k-1}}}^{h_k(1-\beta)},
\end{aligned} \tag{47}$$

where

$$\beta = \frac{d(h_k - 2 + p)(h_k - h_{k-1})}{h_k\big((h_k - 2 + p)d - h_{k-1}(d-p)\big)},$$

$$1 - \beta = \frac{h_{k-1}(h_k p + pd - 2d)}{h_k\big((h_k - 2 + p)d - h_{k-1}(d-p)\big)}.$$

Since it is easy to see that $\frac{p h_k \beta}{h_k - 2 + p} < p$, then using Young's inequality, we have

$$\begin{aligned}\|u\|_{L^{h_k}(\mathbb{R}^d)}^{h_k} &\le \frac{1}{a'}\delta_2^{a'} \left\|\nabla u^{\frac{h_k-2+p}{p}}\right\|_{L^p}^p + \frac{1}{b'}\delta_2^{-b'} K^{\frac{p h_k \beta b'}{h_k - 2+p}}(d,p)\|u\|_{L^{h_{k-1}}}^{h_k(1-\beta)b'} \\ &\le C_1 \left\|\nabla u^{\frac{h_k-2+p}{p}}\right\|_{L^p}^p + C_3(h_k)\left(\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right)^{\gamma_2}, \qquad (48)\end{aligned}$$

where

$$a' = \frac{h_k - 2 + p}{h_k \beta} = \frac{d(h_k - h_{k-1}) + h_{k-1}p + pd - 2d}{d(h_k - h_{k-1})} > 1,$$

$$b' = \frac{h_k - 2 + p}{h_k - 2 + p - h_k\beta} = \frac{d(h_k - h_{k-1}) + h_{k-1}p + pd - 2d}{h_{k-1}p + pd - 2d} > 1,$$

$$\delta_2 = (C_1 a')^{\frac{1}{a'}}, \quad C_3(h_k) = \frac{1}{b'}(C_1 a')^{-\frac{b'}{a'}} K^{\frac{p h_k \beta b'}{h_k - 2 + p}}(d,p),$$

$$\gamma_2 = \frac{h_k(1-\beta)b'}{h_{k-1}} = \frac{h_k p + pd - 2d}{h_{k-1}p + pd - 2d} < 3.$$

We can also check that $C_3(h_k)$ is uniformly bounded as $k \to \infty$. Combining (46) and (48) together, we have

$$\frac{d}{dt}\|u\|_{L^{h_k}}^{h_k} \le -\|u\|_{L^{h_k}}^{h_k} + C_2(h_k)h_k^b\left(\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right)^{\gamma_1} + C_3(h_k)\left(\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right)^{\gamma_2}. \qquad (49)$$

Since $C_2(h_k)$ and $C_3(h_k)$ are both uniformly bounded as $k \to \infty$, we can choose a constant $C_4 > 1$ which is an upper bound of $C_2(h_k)$ and $C_3(h_k)$. Then by $h_k > 1$ and $b > 1$, we have for any $t > 0$,

$$\frac{d}{dt}\|u\|_{L^{h_k}}^{h_k} \le -\|u\|_{L^{h_k}}^{h_k} + C_4 h_k^b\left[\left(\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right)^{\gamma_1} + \left(\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right)^{\gamma_2}\right]. \qquad (50)$$

Step 2. (The L^{h_k} estimate for $2 < p < \frac{3d}{d+1}$) By changing form of (42), we have

$$\begin{aligned}\frac{d}{dt}\left[(h_k - 2 + p)^{p-2}\|u\|_{L^{h_k}}^{h_k}\right] &= -\frac{h_k(h_k-1)p^p}{(h_k - 2 + p)^2}\left\|\nabla u^{\frac{h_k-2+p}{p}}(t)\right\|_{L^p}^p \\ &\quad + (h_k - 1)(h_k - 2 + p)^{p-2}\|u\|_{L^{h_k+1}}^{h_k+1} \\ &\le -2C_1'\left\|\nabla u^{\frac{h_k-2+p}{p}}(t)\right\|_{L^p}^p + C_5 h_k^2\|u\|_{L^{h_k+1}}^{h_k+1}, \qquad (51)\end{aligned}$$

where $0 < C_1' \le \frac{h_k(h_k-1)p^p}{2(h_k-2+p)^2}$ is a fixed constant and C_5 is also a fixed constant satisfying $(h_k - 1)(h_k - 2 + p)^{p-2} \le C_5 h_k^2$ since $h_k > 1$ and $p < 3$. Using Young's inequality and (44), we have

$$\begin{aligned}C_5 h_k^2\|u\|_{L^{h_k+1}}^{h_k+1} &\le \frac{1}{a}\delta_3^a\left\|\nabla u^{\frac{h_k-2+p}{p}}\right\|_{L^p}^p + \frac{1}{b}\delta_3^{-b}(C_5)^b K^{\frac{p(h_k+1)\theta b}{h_k-2+p}} h_k^{2b}\|u\|_{L^{h_{k-1}}}^{(h_k+1)(1-\theta)b} \\ &\le C_1'\left\|\nabla u^{\frac{h_k-2+p}{p}}(t)\right\|_{L^p}^p + C_2'(h_k)h_k^{2b}\|u\|_{L^{h_{k-1}}}^{(h_k+1)(1-\theta)b}, \qquad (52)\end{aligned}$$

where

$$a = \frac{h_k - 2 + p}{(h_k+1)\theta} = \frac{d(h_k - h_{k-1} + 1) + h_{k-1}p + pd - 3d}{d(h_k - h_{k-1} + 1)} > 1,$$

$$b = \frac{h_k - 2 + p}{h_k - 2 + p - (h_k + 1)\theta} = \frac{d(h_k - h_{k-1} + 1) + h_{k-1}p + pd - 3d}{h_{k-1}p + pd - 3d} > 1,$$

$$\delta_3 = (C_1' a)^{\frac{1}{a}}, \quad C_2'(h_k) = \frac{1}{b}(C_1' a)^{-\frac{b}{a}} (C_5)^b K^{\frac{p(h_k+1)\theta b}{h_k - 2 + p}}(d, p).$$

We can see that $C_2'(h_k)$ is uniformly bounded as $k \to \infty$. Since $(h_{k-1} - 2 + p)^{\frac{(p-2)(h_k+1)(1-\theta)b}{h_{k-1}}} \geq 1$, (52) turns to

$$C_5 h_k^2 \|u\|_{L^{h_k+1}}^{h_k+1} \leq C_1' \left\| \nabla u^{\frac{h_k - 2+p}{p}} \right\|_{L^p}^p + C_2'(h_k) h_k^{2b} \left[(h_{k-1} - 2 + p)^{p-2} \|u\|_{L^{h_{k-1}}}^{h_{k-1}} \right]^{\gamma_1}, \tag{53}$$

where

$$\gamma_1 = \frac{(h_k + 1)(1 - \theta)b}{h_{k-1}} = \frac{h_k p + pd - 3d + p}{h_{k-1}p + pd - 3d} < 3.$$

Substituting (53) into (51), we obtain

$$\begin{aligned} \frac{d}{dt} \left[(h_k - 2 + p)^{p-2} \|u\|_{L^{h_k}}^{h_k} \right] \leq &-C_1' \left\| \nabla u^{\frac{h_k - 2 + p}{p}}(t) \right\|_{L^p}^p \\ &+ C_2'(h_k) h_k^{2b} \left[(h_{k-1} - 2 + p)^{p-2} \|u\|_{L^{h_{k-1}}}^{h_{k-1}} \right]^{\gamma_1}. \end{aligned} \tag{54}$$

Next, by using Young's inequality and (47), we have

$$\begin{aligned} (h_k - 2 + p)^{p-2} \|u\|_{L^{h_k}}^{h_k} &\leq \frac{1}{a'} \delta_4^{a'} \left\| \nabla u^{\frac{h_k - 2+p}{p}} \right\|_{L^p}^p \\ &\quad + \frac{1}{b'} \delta_4^{-b'} K^{\frac{p h_k \beta b'}{h_k - 2 + p}}(d, p)(h_k - 2 + p)^{(p-2)b'} \|u\|_{L^{h_{k-1}}}^{h_k(1-\beta)b'} \\ &\leq C_1' \left\| \nabla u^{\frac{h_k - 2+p}{p}}(t) \right\|_{L^p}^p + C_3'(h_k) h_k^{b'} \|u\|_{L^{h_{k-1}}}^{h_k(1-\beta)b'}, \end{aligned} \tag{55}$$

where

$$a' = \frac{h_k - 2 + p}{h_k \beta} = \frac{d(h_k - h_{k-1}) + h_{k-1}p + pd - 2d}{d(h_k - h_{k-1})} > 1,$$

$$b' = \frac{h_k - 2 + p}{h_k - 2 + p - h_k \beta} = \frac{d(h_k - h_{k-1}) + h_{k-1}p + pd - 2d}{h_{k-1}p + pd - 2d} > 1,$$

$$\delta_4 = (C_1' a')^{\frac{1}{a'}}, \quad C_3'(h_k) = \frac{C_6}{b'} (C_1' a')^{-\frac{b'}{a'}} K^{\frac{p h_k \beta b'}{h_k - 2 + p}}(d, p),$$

and C_6 is a fixed constant such that $(h_k - 2 + p)^{(p-2)b'} \leq C_6 h_k^{b'}$. We can see that $C_3'(h_k)$ is uniformly bounded as $k \to \infty$. Also since $(h_{k-1} - 2 + p)^{\frac{(p-2)h_k(1-\beta)b'}{h_{k-1}}} \geq 1$, (55) turns to

$$\begin{aligned} (h_k - 2 + p)^{p-2} \|u\|_{L^{h_k}}^{h_k} \leq &C_1' \left\| \nabla u^{\frac{h_k - 2+p}{p}}(t) \right\|_{L^p}^p \\ &+ C_3'(h_k) h_k^{b'} \left[(h_{k-1} - 2 + p)^{p-2} \|u\|_{L^{h_{k-1}}}^{h_{k-1}} \right]^{\gamma_2}, \end{aligned} \tag{56}$$

where

$$\gamma_2 = \frac{h_k(1 - \beta)b'}{h_{k-1}} = \frac{h_k p + pd - 2d}{h_{k-1}p + pd - 2d} < 3.$$

Combining (54) and (56) together, we have

$$\begin{aligned}
&\frac{d}{dt}\left[(h_k-2+p)^{p-2}\|u\|_{L^{h_k}}^{h_k}\right] \\
&\le -(h_k-2+p)^{p-2}\|u\|_{L^{h_k}}^{h_k} + C_2'(h_k)h_k^{2b}\left[(h_{k-1}-2+p)^{p-2}\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right]^{\gamma_1} \\
&\quad + C_3'(h_k)h_k^{b'}\left[(h_{k-1}-2+p)^{p-2}\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right]^{\gamma_2}. \qquad (57)
\end{aligned}$$

Since $C_2'(h_k)$ and $C_3'(h_k)$ are both uniformly bounded as $k\to\infty$, we can choose a constant $C_7>1$ which is an upper bound of $C_2'(h_k)$ and $C_3'(h_k)$. Then by $h_k>1$ and $2b>b'>1$, we have for any $t>0$,

$$\begin{aligned}
&\frac{d}{dt}\left[(h_k-2+p)^{p-2}\|u\|_{L^{h_k}}^{h_k}\right] \le -(h_k-2+p)^{p-2}\|u\|_{L^{h_k}}^{h_k} \\
&\quad + C_7h_k^{2b}\left\{\left[(h_{k-1}-2+p)^{p-2}\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right]^{\gamma_1} + \left[(h_{k-1}-2+p)^{p-2}\|u\|_{L^{h_{k-1}}}^{h_{k-1}}\right]^{\gamma_2}\right\}. \qquad (58)
\end{aligned}$$

Step 3. (The uniform L^∞ estimate for $1<p<\frac{3d}{d+1}$) Let

$$y_k(t)=\begin{cases} \|u(\cdot,t)\|_{L^{h_k}}^{h_k}, & 1<p\le 2,\\ (h_k-2+p)^{p-2}\|u(\cdot,t)\|_{L^{h_k}}^{h_k}, & 2<p<\frac{3d}{d+1},\end{cases}$$

and $C_8>1$ is an upper bound of C_4 and C_7. Then (50) and (58) turn to

$$\frac{d}{dt}y_k(t)\le -y_k(t)+C_8h_k^{2b}\left(y_{k-1}^{\gamma_1}+y_{k-1}^{\gamma_2}\right). \qquad (59)$$

Multiplying e^t to both sides of (59), we have

$$\frac{d}{dt}\left(e^ty_k(t)\right)\le C_8{h_k}^{2b}\left(y_{k-1}^{\gamma_1}+y_{k-1}^{\gamma_2}\right)e^t\le 2C_8{h_k}^{2b}\max\left\{1,\ \sup_{t\ge0}y_{k-1}^3(t)\right\}e^t. \qquad (60)$$

Solving this ODE, we obtain for $t\ge 0$,

$$\begin{aligned}
y_k(t)&\le e^{-t}y_k(0)+2C_8{h_k}^{2b}\max\left\{1,\ \sup_{t\ge0}y_{k-1}^3(t)\right\}\left(1-e^{-t}\right)\\
&\le 2C_8{h_k}^{2b}\max\left\{1,\ y_k(0),\ \sup_{t\ge0}y_{k-1}^3(t)\right\}. \qquad (61)
\end{aligned}$$

It is easy to see that

$${h_k}^{2b}=\left(3^k+\frac{d(3-p)}{p}+1\right)^{2b}\le C_0 3^{2bk}\left(\frac{d(3-p)}{p}+1\right)^{2b}, \qquad (62)$$

where C_0 is an appropriate positive constant. Combining (61) and (62) together, we can see

$$y_k(t)\le C_9 3^{2bk}\left(\frac{d(3-p)}{p}+1\right)^{2b}\max\left\{1,\ y_k(0),\ \sup_{t\ge0}y_{k-1}^3(t)\right\},$$

for all $1 < p < \frac{3d}{d+1}$, where $C_9 = 2C_0C_8$. Then after some iterative steps, we have

$$y_k(t) \le \left(C_9 \left(\frac{d(2-m)}{2} + 1 \right)^{2b} \right)^{\frac{3^k-1}{2}} 3^{2b(\frac{3^{k+1}}{4} - \frac{k}{2} - \frac{3}{4})} \cdot \max \left\{ 1,\ \sum_{i=0}^{k-1} \sup_{t\ge 0} y_{k-i}^{3^i}(0),\ \sup_{t\ge 0} y_0^{3^k}(t) \right\}. \tag{63}$$

Denote $K_0 = \max \left\{ 1,\ \|u_0\|_{L^1(\mathbb{R}^d)},\ \|u_0\|_{L^\infty(\mathbb{R}^d)} \right\}$, then

$$y_k(0) = \begin{cases} \|u_0\|_{L^{h_k}}^{h_k} \le \max\left\{ \|u_0\|_{L^1}^{h_k},\ \|u_0\|_{L^\infty}^{h_k} \right\}, & 1 < p \le 2, \\ (h_k - 2 + p)^{p-2} \|u_0\|_{L^{h_k}}^{h_k}, & 2 < p < \frac{3d}{d+1}, \end{cases}$$

i.e.

$$y_k(0) \le \begin{cases} K_0^{h_k}, & 1 < p \le 2, \\ (h_k - 2 + p)^{p-2} K_0^{h_k}, & 2 < p < \frac{3d}{d+1}, \end{cases} \tag{64}$$

and for any $1 < p < \frac{3d}{d+1}$,

$$\lim_{k\to\infty} y_k^{\frac{1}{h_k}}(0) \le K_0, \tag{65}$$

since $\lim\limits_{k\to\infty} (h_k - 2 + p)^{\frac{p-2}{h_k}} = 1$ for $2 < p < \frac{3d}{d+1}$. Furthermore, we also have

$$\max \left\{ 1,\ \sum_{i=0}^{k-1} \sup_{t\ge 0} y_{k-i}^{3^i}(0) \right\} \le k K_0^{q_k}.$$

Taking the power $\frac{1}{h_k}$ to both sides of (63) and letting $k \to \infty$, we obtain

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C \max \left\{ \sup_{t\ge 0} y_0(t),\ K_0 \right\}, \tag{66}$$

where $C = 3^{\frac{3(2d+p)}{2p}} C_9^{\frac{1}{2}} \left(\frac{d(2-m)}{2} + 1 \right)^{d+1}$ since $b \to \frac{2d+p}{p}$ as $k \to \infty$. Recalling (37) in Proposition 1, it shows that

$$y_0(t) = \|u(\cdot,t)\|_{L^{q+2}(\mathbb{R}^d)}^{q+2} \le C_u^{q+2}. \tag{67}$$

Then (66) turns to

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C(d,p,K_0).$$

□

5. **Global existence of weak solutions.** The following Lemma proved in [9, Lemma 2.1] is necessary for the existence of weak solutions of problem (1) in the supercritical case.

Lemma 5.1. *For any $\eta, \eta' \in \mathbb{R}^d$, there exists*

$$\left(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta' \right) \cdot (\eta - \eta') \ge \begin{cases} C_1 \left(|\eta| + |\eta'| \right)^{p-2} |\eta - \eta'|^2, & p > 1, \\ C_2 |\eta - \eta'|^p, & p \ge 2, \end{cases}$$

where C_1 and C_2 are two positive constants only depending on p.

Theorem 5.2. *Let $d \geq 3$, $1 < p < \frac{3d}{d+1}$ and $q = \frac{d(3-p)}{p}$. If $u_0 \in L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $A(d,p) = C_{p,d}^{3-p} - \|u_0\|_{L^q}^{3-p} > 0$, where $C_{p,d} = \left[\frac{qp^p}{K^p(d,p)(q-2+p)^p}\right]^{\frac{1}{3-p}}$ is a universal constant. Then there exists a non-negative global weak solution (u,v) of (1), such that all a priori estimates in Theorem 3.1 and the uniform L^∞ estimate in Theorem 4.1 hold true.*

Proof. We separate the proof of Theorem 5.2 into four steps. In Step 1, we construct the regularized problem of (1) and show that all *a priori* estimates in Theorem 3.1 and the uniform L^∞ estimate in Theorem 4.1 hold true. Furthermore, we obtain the uniform estimate of ∇u_ϵ. In Step 2 and 3, by applying the Aubin-Lions Lemma, we prove that a non-negative weak solution of the regularized problem (68) converges strongly to a non-negative weak solution of (1) in a bounded domain. Finally, in Step 4, using the weak convergence and strong convergence estimates obtained in Step 1-3, we prove the existence of a global weak solution of (1) with monotone operators.

Step 1. (The regularized problem and *a priori* estimates) We consider the regularized problem of (1) for $\epsilon > 0$,

$$\begin{cases} \partial_t u_\epsilon = \nabla \cdot \left(|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon\right) + \epsilon \Delta u_\epsilon - \nabla \cdot (u_\epsilon \nabla v_\epsilon), & x \in \mathbb{R}^d,\ t > 0, \\ -\Delta v_\epsilon = J_\epsilon * u_\epsilon, & x \in \mathbb{R}^d,\ t > 0, \\ u_\epsilon(x,0) = u_{0\epsilon}(x), & x \in \mathbb{R}^d, \end{cases} \tag{68}$$

where $d \geq 3$, $1 < p < \frac{3d}{d+1}$ and $J_\epsilon(x) = \frac{1}{\epsilon^d} J\left(\frac{x}{\epsilon}\right)$, $J(x) = \frac{1}{\alpha(d)}\left(1 + |x|^2\right)^{-\frac{d+2}{2}}$ satisfying $\int_{\mathbb{R}^d} J_\epsilon(x)\, dx = 1$. A simple computation shows that v_ϵ can be expressed by

$$v_\epsilon(x,t) = \frac{1}{d(d-2)\alpha(d)} \int_{\mathbb{R}^d} \frac{u_\epsilon(y,t)}{\left(|x-y|^2 + \epsilon^2\right)^{\frac{d-2}{2}}}\, dy, \tag{69}$$

where $\alpha(d)$ is the volume of the d-dimensional unit ball. The initial condition $u_{0\epsilon}(x) \in C^\infty(\mathbb{R}^d)$ is a sequence of approximation for $u_0(x)$, which satisfies that there exists $\delta > 0$ such that for all $0 < \epsilon < \delta$,

$$\begin{gathered} u_{0\epsilon}(x) > 0, x \in \mathbb{R}^d, \\ u_{0\epsilon}(x) \in L^r(\mathbb{R}^d),\ \text{for all } r \geq 1, \\ \|u_{0\epsilon}(\cdot)\|_{L^1(\mathbb{R}^d)} = \|u_0(\cdot)\|_{L^1(\mathbb{R}^d)}, \\ \|u_{0\epsilon}(\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq C. \end{gathered}$$

According to the classical theory for parabolic equations [16], the regularized problem has a global smooth non-negative solution (u_ϵ, v_ϵ) with the regularity for all $r \geq 1$,

$$u_\epsilon \in L^\infty\left(\mathbb{R}_+; L^r(\mathbb{R}^d)\right) \cap L^{r+1}\left(\mathbb{R}_+; L^{r+1}(\mathbb{R}^d)\right).$$

Then we want to show that all *a priori* estimates in Theorem 3.1 hold true for our regularized problem. we take a cut-off function $0 \leq \psi_1(x) \leq 1$, satisfying

$$\psi_1(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2, \end{cases}$$

where $\psi_1(x) \in C_c^\infty(\mathbb{R}^d)$. Define $\psi_R(x) := \psi_1(\frac{x}{R})$, then we know that $\lim_{R\to\infty} \psi_R(x) = 1$, $|\nabla \psi_R(x)| \leq \frac{C_1}{R}$ and $|\Delta \psi_R(x)| \leq \frac{C_2}{R^2}$ for $x \in \mathbb{R}^d$, C_1, C_2 are constants. We can also define $\psi_R^{\frac{1}{p}}(x) := \psi_1^{\frac{1}{p}}(\frac{x}{R})$ and choose a constant C_3, such that $\left|\nabla \psi_R^{\frac{1}{p}}(x)\right| \le \frac{C_3}{R}$. Multiplying the first equation of (68) by $qu_\epsilon^{q-1}\psi_R(x)$ and integrating over $\mathbb{R}^d$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u_\epsilon^q \psi_R(x)\,dx &+ \frac{q(q-1)p^p}{(q-2+p)^p}\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q-2+p}{p}}\right|^p \psi_R(x)\,dx \\ &+ \frac{4\epsilon(q-1)}{q}\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q}{2}}\right|^2\psi_R(x)\,dx \\ &= -(q-1)\int_{\mathbb{R}^d} u_\epsilon^q \Delta v_\epsilon \psi_R(x)\,dx - q\int_{\mathbb{R}^d}|\nabla u_\epsilon|^{p-2}u_\epsilon^{q-1}\nabla u_\epsilon\cdot\nabla\psi_R(x)\,dx \\ &+ \int_{\mathbb{R}^d} u_\epsilon^q \nabla v_\epsilon\cdot\nabla\psi_R(x)\,dx + \epsilon\int_{\mathbb{R}^d} u_\epsilon^q\Delta\psi_R(x)\,dx. \end{aligned} \tag{70}$$

Integrating (70) from 0 to t yields that

$$\begin{aligned}\int_{\mathbb{R}^d} u_\epsilon^q(x,t)\psi_R(x)\,dx &+ \frac{q(q-1)p^p}{(q-2+p)^p}\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q-2+p}{p}}\right|^p\psi_R(x)\,dxds \\ &+ \frac{4\epsilon(q-1)}{q}\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q}{2}}\right|^2\psi_R(x)\,dxds \\ &= \int_{\mathbb{R}^d} u_{0\epsilon}^q(x)\psi_R(x)\,dx - (q-1)\int_0^t\int_{\mathbb{R}^d} u_\epsilon^q\Delta v_\epsilon\psi_R(x)\,dxds \\ &- q\int_0^t\int_{\mathbb{R}^d}|\nabla u_\epsilon|^{p-2}u_\epsilon^{q-1}\nabla u_\epsilon\cdot\nabla\psi_R(x)\,dxds \\ &+ \int_0^t\int_{\mathbb{R}^d} u_\epsilon^q\nabla v_\epsilon\cdot\nabla\psi_R(x)\,dxds + \epsilon\int_0^t\int_{\mathbb{R}^d} u_\epsilon^q\Delta\psi_R(x)\,dxds. \end{aligned} \tag{71}$$

For the second term on the right hand side of (71), by using Hölder's inequality, we have

$$\begin{aligned} -(q-1)\int_0^t\int_{\mathbb{R}^d} u_\epsilon^q\Delta v_\epsilon\psi_R(x)\,dxds &\le (q-1)\int_0^t\int_{\mathbb{R}^d} u_\epsilon^q J_\epsilon * u_\epsilon\,dxds \\ &\le (q-1)\int_0^t \|u_\epsilon\|_{L^{q+1}}^q\|J_\epsilon * u_\epsilon\|_{L^{q+1}}\,ds \\ &\le (q-1)\int_0^t\|u_\epsilon\|_{L^{q+1}}^{q+1}\,ds \le C(\epsilon), \end{aligned} \tag{72}$$

since $u_\epsilon \in L^{r+1}\left(\mathbb{R}_+; L^{r+1}(\mathbb{R}^d)\right)$ for any $r \ge 1$. Then we can use the dominated convergence theorem as $R \to \infty$ for any small $\epsilon > 0$ later. Next, we want to prove that last three terms on the RHS of (71) go to 0 as $R \to \infty$.

First, by using Hölder's inequality and Young's inequality of convolution [18, pp.107], we obtain

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^d} u_\epsilon^q\nabla v_\epsilon\cdot\nabla\psi_R(x)\,dxds &\le \frac{C_1}{R}\int_0^t\int_{\mathbb{R}^d} u_\epsilon^q\left|\nabla v_\epsilon\right|\,dxds \\ &\le \frac{C_1}{R}\int_0^t\|u_\epsilon\|^q_{L^{\frac{d(q+1)}{d+1}}}\|\nabla v_\epsilon\|_{L^{\frac{d(q+1)}{d-q}}}\,ds \\ &\le \frac{C}{R}\int_0^t\|u_\epsilon\|^q_{L^{\frac{d(q+1)}{d+1}}}\left\|\frac{x}{|x|^d}\right\|_{L_w^{\frac{d}{d-1}}}\|u_\epsilon\|_{L^{\frac{d(q+1)}{d+1}}}\,ds \end{aligned}$$

$$\leq \frac{C}{R}\int_0^t \|u_\epsilon\|^{q+1}_{L^{\frac{d(q+1)}{d+1}}}\,ds. \tag{73}$$

Then using the interpolation inequality, (73) yields to

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^d} u_\epsilon^q \nabla v_\epsilon\cdot\nabla\psi_R(x)\,dxds &\leq \frac{C}{R}\int_0^t \|u_\epsilon\|_{L^1}^{\frac{q+1}{dq}}\|u_\epsilon\|_{L^{q+1}}^{\frac{(q+1)(dq-1)}{dq}}\,ds\\ &\leq \frac{C\left(\|u_0\|_{L^1}\right)}{R}\left(\int_0^t \|u_\epsilon\|^{q+1}_{L^{q+1}}\,ds\right)^{\frac{dq-1}{dq}} \leq \frac{C(\epsilon)}{R},\end{aligned} \tag{74}$$

since $u_\epsilon \in L^{q+1}\left(\mathbb{R}_+; L^{q+1}(\mathbb{R}^d)\right)$ for $q > 1$.

Second, from $u_\epsilon \in L^\infty\left(\mathbb{R}_+; L^r(\mathbb{R}^d)\right)$ for all $r \geq 1$, we have

$$\epsilon\int_0^t\int_{\mathbb{R}^d} u_\epsilon^q\Delta\psi_R(x)\,dxds \leq \frac{C(t,\epsilon)}{R^2}. \tag{75}$$

Third, by using Hölder's inequality, we have

$$\begin{aligned}&-q\int_0^t\int_{\mathbb{R}^d}|\nabla u_\epsilon|^{p-2}u_\epsilon^{q-1}\nabla u_\epsilon\cdot\nabla\psi_R(x)\,dxds\\ &\leq q\int_0^t\int_{\mathbb{R}^d}|\nabla u_\epsilon|^{p-1}u_\epsilon^{q-1}\left|\nabla\psi_R(x)\right|\,dxds\\ &= q\int_0^t\int_{\mathbb{R}^d}|\nabla u_\epsilon|^{p-1}u_\epsilon^{\frac{(q-2)(p-1)}{p}}\psi_R^{\frac{p-1}{p}}(x)u_\epsilon^{\frac{q-2+p}{p}}\psi_R^{-\frac{p-1}{p}}(x)\left|\nabla\psi_R(x)\right|\,dxds\\ &= C\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q-2+p}{p}}\right|^{p-1}\psi_R^{\frac{p-1}{p}}(x)u_\epsilon^{\frac{q-2+p}{p}}\left|\nabla\psi_R^{\frac{1}{p}}(x)\right|\,dxds\\ &\leq \frac{C}{R}\left(\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q-2+p}{p}}\right|^p\psi_R(x)\,dxds\right)^{\frac{p-1}{p}}\left(\int_0^t\|u_\epsilon\|^{q-2+p}_{L^{q-2+p}}\,ds\right)^{\frac{1}{p}}.\end{aligned} \tag{76}$$

Then we should prove $\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q-2+p}{p}}\right|^p\psi_R(x)\,dxds$ is bounded in order to show (76) goes to 0 as $R\to\infty$. Using Young's inequality for (76), we obtain

$$\begin{aligned}&-q\int_0^t\int_{\mathbb{R}^d}|\nabla u_\epsilon|^{p-2}u_\epsilon^{q-1}\nabla u_\epsilon\cdot\nabla\psi_R(x)\,dxds\\ &\leq \frac{C}{R^{\frac{p}{p-1}}}\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q-2+p}{p}}\right|^p\psi_R(x)\,dxds + \frac{1}{p}\int_0^t\|u_\epsilon\|^{q-2+p}_{L^{q-2+p}}\,ds\\ &\leq \frac{C}{R^{\frac{p}{p-1}}}\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q-2+p}{p}}\right|^p\psi_R(x)\,dxds + C(t,\epsilon),\end{aligned} \tag{77}$$

since $q-2+p\geq 1$. Combining (71), (72), (74), (75) and (77) together, we have

$$\begin{aligned}&\int_{\mathbb{R}^d}u_\epsilon^q(x,t)\psi_R(x)\,dx + \left[\frac{q(q-1)p^p}{(q-2+p)^p} - \frac{C}{R^{\frac{p}{p-1}}}\right]\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q-2+p}{p}}\right|^p\psi_R(x)\,dxds\\ &\quad + \frac{4\epsilon(q-1)}{q}\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{q}{2}}\right|^2\psi_R(x)\,dxds\\ &\leq \int_{\mathbb{R}^d}u_{0\epsilon}^q(x)\psi_R(x)\,dx + C(t,\epsilon) + \frac{C(t,\epsilon)}{R} + \frac{C(t,\epsilon)}{R^2}.\end{aligned} \tag{78}$$

Taking R large enough, we can see that $\frac{q(q-1)p^p}{(q-2+p)^p} - \frac{C}{R^{\frac{p}{p-1}}} > 0$. Then (78) shows that

$$\int_0^t \int_{\mathbb{R}^d} \left| \nabla u_\epsilon^{\frac{q-2+p}{p}} \right|^p \psi_R(x)\ dxds \le C(t,\epsilon), \tag{79}$$

since $u_{0\epsilon} \in L^q(\mathbb{R}^d)$ when R is large enough. Substituting (79) into (76), we have

$$-q \int_0^t \int_{\mathbb{R}^d} |\nabla u_\epsilon|^{p-2} u_\epsilon^{q-1} \nabla u_\epsilon \cdot \nabla \psi_R(x)\ dxds \le \frac{C(t,\epsilon)}{R}. \tag{80}$$

Until now, we have proved that last three terms on the RHS of (71) go to 0 as $R \to \infty$. Using the dominated convergence theorem, when $R \to \infty$, (71) turns to

$$\begin{aligned} \int_{\mathbb{R}^d} u_\epsilon^q(x,t)\ dx &+ \frac{q(q-1)p^p}{(q-2+p)^p} \int_0^t \int_{\mathbb{R}^d} \left| \nabla u_\epsilon^{\frac{q-2+p}{p}} \right|^p\ dxds \\ &+ \frac{4\epsilon(q-1)}{q} \int_0^t \int_{\mathbb{R}^d} \left| \nabla u_\epsilon^{\frac{q}{2}} \right|^2\ dxds \\ &= \int_{\mathbb{R}^d} u_{0\epsilon}^q(x)dx - (q-1) \int_0^t \int_{\mathbb{R}^d} u_\epsilon^q \Delta v_\epsilon\ dxds, \end{aligned} \tag{81}$$

i.e., for any $t > 0$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} u_\epsilon^q(x,t)\ dx &+ \frac{q(q-1)p^p}{(q-2+p)^p} \int_{\mathbb{R}^d} \left| \nabla u_\epsilon^{\frac{q-2+p}{p}} \right|^p\ dx + \frac{4\epsilon(q-1)}{q} \int_{\mathbb{R}^d} \left| \nabla u_\epsilon^{\frac{q}{2}} \right|^2\ dx \\ &= (q-1) \int_{\mathbb{R}^d} u_\epsilon^q J_\epsilon * u_\epsilon\ dx \le (q-1) \|u_\epsilon\|_{L^{q+1}}^{q+1} \\ &\le (q-1) K^p(d,p) \left\| \nabla u_\epsilon^{\frac{q-2+p}{p}} \right\|_{L^p}^p \|u_\epsilon\|_{L^q}^{3-p}, \end{aligned} \tag{82}$$

where last two inequalities can be obtained by the same method of (72) and (11). Then we have

$$\frac{d}{dt} \|u_\epsilon\|_{L^q}^q + (q-1) \left(\frac{qp^p}{(q-2+p)^p} - K^p(d,p) \|u_\epsilon\|_{L^q}^{3-p} \right) \left\| \nabla u_\epsilon^{\frac{q-2+p}{p}} \right\|_{L^p}^p \le 0, \tag{83}$$

which is same to (12), and all *a priori* estimates in Theorem 3.1 hold true for our solution of the regularized problem. We also have following uniformly bounded estimates,

$$\|u_\epsilon\|_{L^\infty\left(\mathbb{R}_+; L_+^1 \cap L^q(\mathbb{R}^d)\right)} \le C, \tag{84}$$

$$\|u_\epsilon\|_{L^{q+1}(\mathbb{R}_+; L^{q+1}(\mathbb{R}^d))} \le C, \tag{85}$$

$$\left\| \nabla u_\epsilon^{\frac{r-2+p}{p}} \right\|_{L^p(\mathbb{R}_+; L^p(\mathbb{R}^d))} \le C, \quad 1 < r \le q. \tag{86}$$

Additionally, since $u_0 \in L^\infty(\mathbb{R}^d)$, we let $u_{0\epsilon}(x)$ also satisfy $\|u_{0\epsilon}\|_{L^\infty(\mathbb{R}^d)} \le C$, where C is a positive constant independent of ϵ. Then from the Theorem 4.1, we have the uniformly bounded estimate

$$\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C. \tag{87}$$

For $q \ge 2$, i.e. $1 < p \le \frac{3d}{d+2}$, by taking $r = 2$ in (86), we have

$$\|\nabla u_\epsilon\|_{L^p(\mathbb{R}_+; L^p(\mathbb{R}^d))} \le C.$$

For $1 < r \le q < 2$, i.e. $\frac{3d}{d+2} < p \le \frac{3d}{d+1}$, by using (87), we obtain

$$C \ge \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left| \nabla u_\epsilon^{\frac{r-2+p}{p}} \right|^p \, dxdt = C \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} u_\epsilon^{r-2} |\nabla u_\epsilon|^p \, dxdt$$
$$\ge C\big(\|u_\epsilon\|_{L^\infty}\big) \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} |\nabla u_\epsilon|^p \, dxdt,$$

where C is a positive constant. From two estimates above, we have

$$\|\nabla u_\epsilon\|_{L^p\left(\mathbb{R}_+; L^p(\mathbb{R}^d)\right)} \le C, \tag{88}$$

for all $1 < p < \frac{3d}{d+1}$.

Step 2. (The time regularity of u_ϵ) In this step, we want to estimate $\partial_t u_\epsilon$ in any bounded domain in order to use the Aubin-Lions Lemma. For any test function $\varphi(x)$ which satisfies $\varphi \in W^{2,p}(\Omega)$ and $\|\varphi\|_{W^{2,p}(\Omega)} \le 1$, we have

$$|\langle \partial_t u_\epsilon, \varphi \rangle| \le |\langle |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon, \nabla \varphi \rangle| + \epsilon |\langle u_\epsilon, \Delta \varphi \rangle| + |\langle u_\epsilon \nabla v_\epsilon, \nabla \varphi \rangle|$$
$$\le \|\nabla u_\epsilon\|_{L^p(\Omega)}^{p-1} + \epsilon \|u_\epsilon\|_{L^{\frac{p}{p-1}}(\Omega)} + \|u_\epsilon \nabla v_\epsilon\|_{L^{\frac{p}{p-1}}(\Omega)}. \tag{89}$$

Then for any $T > 0$, since $\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C$ and $\|\nabla u_\epsilon\|_{L^p(\mathbb{R}_+; L^p(\mathbb{R}^d))} \le C$, using Sobolev inequality, we obtain

$$\int_0^T \|\partial_t u_\epsilon\|_{W^{-2,\frac{p}{p-1}}(\Omega)}^{\frac{p}{p-1}} \, dt \le C\bigg(\int_0^T \|\nabla u_\epsilon\|_{L^p(\Omega)}^p \, dt + \epsilon^{\frac{p}{p-1}} \int_0^T \|u_\epsilon\|_{L^{\frac{p}{p-1}}(\Omega)}^{\frac{p}{p-1}} \, dt$$
$$+ \int_0^T \|u_\epsilon \nabla v_\epsilon\|_{L^{\frac{p}{p-1}}(\Omega)}^{\frac{p}{p-1}} \, dt \bigg)$$
$$\le C(T)(1 + \epsilon^{\frac{p}{p-1}}) + C \int_0^T \|\nabla v_\epsilon\|_{L^{\frac{p}{p-1}}(\Omega)}^{\frac{p}{p-1}} \, dt$$
$$\le 2C(T) + C \int_0^T \|\Delta v_\epsilon\|_{L^{\frac{dp}{dp-d+p}}(\Omega)}^{\frac{p}{p-1}} \, dt$$
$$\le C(T). \tag{90}$$

Then we have $\|\partial_t u_\epsilon\|_{L^{\frac{p}{p-1}}\left(0,T; W^{-2,\frac{p}{p-1}}(\Omega)\right)} \le C$.

Step 3. (The application of the Aubin-Lions Lemma) It is easy to see that

$$\|u_\epsilon\|_{L^p(0,T;L^p(\Omega))} \le C(\Omega, T),$$

from $\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C$, where Ω is any bounded domain. Then we obtain that $\|u_\epsilon\|_{L^p\left(0,T; W^{1,p}(\Omega)\right)} \le C$. By the Sobolev Embedding Theorem, we have $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ since $p < d$ and $p < \frac{dp}{d-p}$. Until now, we already have

$$\|u_\epsilon\|_{L^p\left(0,T; W^{1,p}(\Omega)\right)} \le C,$$

$$\|\partial_t u_\epsilon\|_{L^{\frac{p}{p-1}}\left(0,T; W^{-2,\frac{p}{p-1}}(\Omega)\right)} \le C,$$

and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega) \hookrightarrow W^{-2,\frac{p}{p-1}}(\Omega)$. By the Aubin-Lions Lemma, there existes a subsequence of $\{u_\epsilon\}$ without relabeling such that

$$u_\epsilon \to u, \text{ in } L^p\big(0,T; L^p(\Omega)\big). \tag{91}$$

Step 4. (The existence of a global weak solution) Next, we will show that (u,v) is a weak solution of the problem (1). The crucial idea in this step follows the proof of Theorem 2.2.1 in [28, p171]. The weak formulation for u_ϵ is that for any test function $\psi(x) \in C_c^\infty\left([0,T)\times\mathbb{R}^d\right)$ and any $0<T<\infty$,

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d} u_\epsilon(x,t)\psi_t(x,t)\ dxdt + \epsilon\int_0^T\int_{\mathbb{R}^d} u_\epsilon(x,t)\Delta\psi(x,t)\ dxdt\\&=\int_0^T\int_{\mathbb{R}^d}|\nabla u_\epsilon(x,t)|^{p-2}\nabla u_\epsilon(x,t)\cdot\nabla\psi(x,t)\ dxdt-\int_{\mathbb{R}^d}u_{0\epsilon}(x)\psi(x,0)dx\\&-\frac{1}{2d\alpha(d)}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{[\nabla\psi(x,t)-\nabla\psi(y,t)]\cdot(x-y)}{|x-y|^2+\epsilon^2}\frac{u_\epsilon(x,t)u_\epsilon(y,t)}{\left(|x-y|^2+\epsilon^2\right)^{\frac{d-2}{2}}}\ dxdydt.\end{aligned}\tag{92}$$

Next, we separate the proof of this step into three parts.

(i) Since $u_\epsilon\to u$ in $L^p\left(0,T;L^p(\Omega)\right)$, using Hölder's inequality, we have

$$\begin{aligned}\int_0^T\int_\Omega|u_\epsilon-u|\ dxdt&\le C(\Omega)\int_0^T\|u_\epsilon-u\|_{L^p(\Omega)}\ dt\\&\le C(\Omega,T)\|u_\epsilon-u\|_{L^p\left(0,T;L^p(\Omega)\right)}\\&\to 0,\ \text{ as }\epsilon\to 0,\end{aligned}$$

i.e.

$$u_\epsilon\to u,\quad\text{in } L^1\left(0,T;L^1(\Omega)\right).\tag{93}$$

(ii) Next we have

$$\begin{aligned}&\Bigg|\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\big[\nabla\psi(x,t)-\nabla\psi(y,t)\big]\cdot(x-y)}{|x-y|^2+\epsilon^2}\frac{u_\epsilon(x,t)u_\epsilon(y,t)}{\left(|x-y|^2+\epsilon^2\right)^{\frac{d-2}{2}}}\ dxdydt\\&-\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\big[\nabla\psi(x,t)-\nabla\psi(y,t)\big]\cdot(x-y)}{|x-y|^2}\frac{u(x,t)u(y,t)}{|x-y|^{d-2}}\ dxdydt\Bigg|\\&\le\Bigg|\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[\nabla\psi(x,t)-\nabla\psi(y,t)\big]\cdot(x-y)u_\epsilon(x,t)u_\epsilon(y,t)\\&\qquad\cdot\left(\frac{1}{|x-y|^d}-\frac{1}{\left(|x-y|^2+\epsilon^2\right)^{\frac{d}{2}}}\right)dxdydt\Bigg|\\&+\Bigg|\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{[\nabla\psi(x,t)-\nabla\psi(y,t)]\cdot(x-y)}{|x-y|^d}\\&\qquad\cdot\Big(u_\epsilon(x,t)u_\epsilon(y,t)-u(x,t)u(y,t)\Big)\ dxdydt\Bigg|\\&=:I_1+I_2.\end{aligned}\tag{94}$$

In order to estimate I_1, we have

$$\frac{1}{|x-y|^d}-\frac{1}{\left(|x-y|^2+\epsilon^2\right)^{\frac{d}{2}}}=\frac{d\epsilon^2}{\left(|x-y|^2+\epsilon^2\right)^{\frac{d+2}{2}}}$$

$$\leq \frac{d\epsilon}{|x-y|^{d+1}} \frac{\epsilon}{|x-y|} \leq \frac{d\epsilon}{|x-y|^{d+1}}, \tag{95}$$

since ϵ is small enough. Then using Hardy-Littlewood-Sobolev inequality, I_1 satisfies

$$\begin{aligned}
I_1 &\leq d\epsilon \left| \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\big[\nabla\psi(x,t) - \nabla\psi(y,t)\big] \cdot (x-y) u_\epsilon(x,t) u_\epsilon(y,t)}{|x-y|^{d+1}} \, dxdydt \right| \\
&\leq C\epsilon \int_0^T \int_\Omega \int_\Omega \frac{u_\epsilon(x,t) u_\epsilon(y,t)}{|x-y|^{d-1}} \, dxdydt \leq C\epsilon \int_0^T \|u_\epsilon\|^2_{L^{\frac{2d}{d+1}}(\Omega)} \, dt \\
&\leq C\epsilon T \to 0, \text{ as } \epsilon \to 0.
\end{aligned} \tag{96}$$

For I_2, also using Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
I_2 &\leq C \left| \int_0^T \int_\Omega \int_\Omega \frac{\big[u_\epsilon(x,t) - u(x,t)\big] u_\epsilon(y,t)}{|x-y|^{d-2}} \, dxdydt \right| \\
&\quad + C \left| \int_0^T \int_\Omega \int_\Omega \frac{\big[u_\epsilon(y,t) - u(y,t)\big] u(x,t)}{|x-y|^{d-2}} \, dxdydt \right| \\
&\leq C \int_0^T \|u_\epsilon - u\|_{L^{\frac{2d}{d+2}}} \|u_\epsilon\|_{L^{\frac{2d}{d+2}}} \, dt + C \int_0^T \|u_\epsilon - u\|_{L^{\frac{2d}{d+2}}} \|u\|_{L^{\frac{2d}{d+2}}} \, dt.
\end{aligned} \tag{97}$$

For $\frac{2d}{d+2} \leq p < \frac{3d}{d+1}$, by using the interpolation inequality and Hölder's inequality, we obtain

$$\begin{aligned}
\int_0^T \|u_\epsilon - u\|_{L^{\frac{2d}{d+2}}(\Omega)} \, dt &\leq \int_0^T \|u_\epsilon - u\|_{L^1(\Omega)}^{\frac{dp+2p-2d}{2d(p-1)}} \|u_\epsilon - u\|_{L^p(\Omega)}^{\frac{dp-2p}{2d(p-1)}} \, dt \\
&\leq C(T) \left(\int_0^T \|u_\epsilon - u\|^p_{L^p(\Omega)} \, dt \right)^{\frac{d-2}{2d(p-1)}} \to 0, \text{ as } \epsilon \to 0.
\end{aligned} \tag{98}$$

For $1 < p < \frac{2d}{d+2}$, since $\|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq C$ and $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq C$, we have

$$\begin{aligned}
\int_0^T \|u_\epsilon - u\|_{L^{\frac{2d}{d+2}}(\Omega)} \, dt &= \int_0^T \left(\int_\Omega |u_\epsilon - u|^p |u_\epsilon - u|^{\frac{2d}{d+2} - p} dx \right)^{\frac{d+2}{2d}} \, dt \\
&\leq \int_0^T \left(\int_\Omega |u_\epsilon - u|^p \left(\|u_\epsilon\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)} \right)^{\frac{2d}{d+2} - p} dx \right)^{\frac{d+2}{2d}} \, dt \\
&\leq C(T) \left(\int_0^T \|u_\epsilon - u\|^p_{L^p(\Omega)} \, dt \right)^{\frac{d+2}{2d}} \to 0, \text{ as } \epsilon \to 0.
\end{aligned} \tag{99}$$

Combining (98) and (99) shows that, for all $1 < p < \frac{3d}{d+1}$,

$$u_\epsilon \to u, \text{ in } L^1\left(0, T; L^{\frac{2d}{d+2}}(\Omega)\right). \tag{100}$$

Then we have $I_2 \to 0,$ as $\epsilon \to 0$. Until now, we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\big[\nabla\psi(x,t) - \nabla\psi(y,t)\big] \cdot (x-y)}{|x-y|^2 + \epsilon^2} \frac{u_\epsilon(x,t) u_\epsilon(y,t)}{\left(|x-y|^2 + \epsilon^2\right)^{\frac{d-2}{2}}} \, dxdydt \\
&\to \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\big[\nabla\psi(x,t) - \nabla\psi(y,t)\big] \cdot (x-y)}{|x-y|^2} \frac{u(x,t) u(y,t)}{|x-y|^{d-2}} \, dxdydt,
\end{aligned} \tag{101}$$

as $\epsilon \to 0$.

(iii) Finally, we will prove

$$\int_0^T \int_{\mathbb{R}^d} |\nabla u_\epsilon(x,t)|^{p-2} \nabla u_\epsilon(x,t) \cdot \nabla \psi(x,t) \, dxdt \\ \to \int_0^T \int_{\mathbb{R}^d} |\nabla u(x,t)|^{p-2} \nabla u(x,t) \cdot \nabla \psi(x,t) \, dxdt, \tag{102}$$

as $\epsilon \to 0$. Since $\|\nabla u_\epsilon\|_{L^p\left(\mathbb{R}_+; L^p(\mathbb{R}^d)\right)} \leq C$ in (88), we have

$$\int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^d} \left| |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \right|^{\frac{p}{p-1}} dx \right) dt = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} |\nabla u_\epsilon|^p \, dxdt \leq C.$$

There exists a χ such that

$$|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \rightharpoonup \chi, \quad \text{in } L^{\frac{p}{p-1}} \left(\mathbb{R}_+; L^{\frac{p}{p-1}}(\mathbb{R}^d) \right). \tag{103}$$

Letting $\epsilon \to 0$ in (92), we have

$$\int_0^T \int_{\mathbb{R}^d} u \psi_t \, dxdt - \int_0^T \int_{\mathbb{R}^d} \chi \cdot \nabla \psi \, dxdt + \int_0^T \int_{\mathbb{R}^d} u \nabla v \cdot \nabla \psi \, dxdt \\ + \int_{\mathbb{R}^d} u_0 \psi(0) dx = 0. \tag{104}$$

Then we will prove

$$\int_0^T \int_{\mathbb{R}^d} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dxdt = \int_0^T \int_{\mathbb{R}^d} \chi \cdot \nabla \psi \, dxdt, \tag{105}$$

to finish the proof of the existence of a weak solution for (1). Choosing $\phi(x,t) \in C_c^\infty\left([0,T) \times \mathbb{R}^d\right)$ with $0 \leq \phi \leq 1$, multiplying the first equation in (68) by $u_\epsilon \phi$ and integrating, we obtain

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u_\epsilon^2 \phi_t \, dxdt - \int_0^T \int_{\mathbb{R}^d} \left(|\nabla u_\epsilon|^{p-2} + \epsilon \right) |\nabla u_\epsilon|^2 \phi \, dxdt \\ - \int_0^T \int_{\mathbb{R}^d} u_\epsilon \left(|\nabla u_\epsilon|^{p-2} + \epsilon \right) \nabla u_\epsilon \cdot \nabla \phi \, dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u_\epsilon^2 \nabla v_\epsilon \cdot \nabla \phi \, dxdt \\ + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u_\epsilon^2 J_\epsilon * u_\epsilon \, \phi \, dxdt + \frac{1}{2} \int_{\mathbb{R}^d} u_{0\epsilon}^2(x) \phi(x,0) dx = 0. \tag{106}$$

Since $\left(|\nabla u_\epsilon|^{p-2} + \epsilon \right) |\nabla u_\epsilon|^2 \geq |\nabla u_\epsilon|^p$, then (106) turns to

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u_\epsilon^2 \phi_t \, dxdt - \int_0^T \int_{\mathbb{R}^d} |\nabla u_\epsilon|^p \phi \, dxdt \\ - \int_0^T \int_{\mathbb{R}^d} u_\epsilon \left(|\nabla u_\epsilon|^{p-2} + \epsilon \right) \nabla u_\epsilon \cdot \nabla \phi \, dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u_\epsilon^2 \nabla v_\epsilon \cdot \nabla \phi \, dxdt \\ + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u_\epsilon^2 J_\epsilon * u_\epsilon \, \phi \, dxdt + \frac{1}{2} \int_{\mathbb{R}^d} u_{0\epsilon}^2(x) \phi(x,0) dx \geq 0. \tag{107}$$

For any $\omega \in L^p\left(0,T; W^{1,p}(\mathbb{R}^d)\right)$ to be determined later, we can obtain the following inequality by using Lemma 5.1

$$\int_0^T \int_{\mathbb{R}^d} \left(|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon - |\nabla \omega|^{p-2} \nabla \omega \right) \cdot \nabla (u_\epsilon - \omega) \phi(x,t) \, dxdt \geq 0, \tag{108}$$

i.e.

$$\int_0^T\int_{\mathbb{R}^d}|\nabla u_\epsilon|^p\phi\ dxdt-\int_0^T\int_{\mathbb{R}^d}|\nabla\omega|^{p-2}\nabla\omega\cdot\nabla u_\epsilon\phi\ dxdt$$
$$-\int_0^T\int_{\mathbb{R}^d}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\cdot\nabla\omega\phi\ dxdt+\int_0^T\int_{\mathbb{R}^d}|\nabla\omega|^p\phi\ dxdt\geq 0. \tag{109}$$

Combining (107) and (109) together, we have

$$\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}u_\epsilon^2\phi_t\ dxdt-\int_0^T\int_{\mathbb{R}^d}|\nabla\omega|^{p-2}\nabla\omega\cdot\nabla(u_\epsilon-\omega)\phi\ dxdt$$
$$-\int_0^T\int_{\mathbb{R}^d}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\cdot\nabla\omega\phi\ dxdt$$
$$-\int_0^T\int_{\mathbb{R}^d}u_\epsilon\left(|\nabla u_\epsilon|^{p-2}+\epsilon\right)\nabla u_\epsilon\cdot\nabla\phi\ dxdt+\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}u_\epsilon^2\nabla v_\epsilon\cdot\nabla\phi\ dxdt$$
$$+\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}u_\epsilon^2 J_\epsilon*u_\epsilon\ \phi\ dxdt+\frac{1}{2}\int_{\mathbb{R}^d}u_{0\epsilon}^2(x)\phi(x,0)dx\geq 0. \tag{110}$$

Next we estimate terms in (110) one by one. Since $u_0\in L^1\cap L^\infty(\mathbb{R}^d)$, $\|u_{0\epsilon}\|_{L^1(\mathbb{R}^d)}=\|u_0\|_{L^1(\mathbb{R}^d)}$, $\|u_{0\epsilon}\|_{L^\infty(\mathbb{R}^d)}\leq C$ and $u_\epsilon\to u$ in $L^p\left(0,T;L^p(\Omega)\right)$, we have

$$\left|\int_0^T\int_{\mathbb{R}^d}u_\epsilon^2\phi_t\ dxdt-\int_0^T\int_{\mathbb{R}^d}u^2\phi_t\ dxdt\right|\leq C\int_0^T\int_\Omega|u_\epsilon-u|(u_\epsilon+u)\ dxdt$$
$$\leq C(\Omega,T)\|u_\epsilon-u\|_{L^p\left(0,T;L^p(\Omega)\right)}\to 0,\ \text{ as }\epsilon\to 0, \tag{111}$$

and

$$\left|\int_0^T\int_{\mathbb{R}^d}u_\epsilon^2 J_\epsilon*u_\epsilon\phi\ dxdt-\int_0^T\int_{\mathbb{R}^d}u^3\phi\ dxdt\right|$$
$$\leq\int_0^T\int_\Omega(u_\epsilon+u)|u_\epsilon-u|J_\epsilon*u_\epsilon\ dxdt+\int_0^T\int_\Omega u^2(J_\epsilon*u_\epsilon-u)\ dxdt$$
$$\leq C\int_0^T\|u_\epsilon-u\|_{L^p}\|J_\epsilon*u_\epsilon\|_{L^{\frac{p}{p-1}}}\ dt+C(\Omega,T)\int_0^T\|J_\epsilon*u_\epsilon-u\|_{L^p}^p\ dt$$
$$\leq C(\Omega,T)\|u_\epsilon-u\|_{L^p\left(0,T;L^p(\Omega)\right)}\to 0,\ \text{ as }\epsilon\to 0. \tag{112}$$

Since

$$\nabla u_\epsilon\rightharpoonup\nabla u,\ \text{ in }L^p\left(\mathbb{R}_+;L^p(\mathbb{R}^d)\right),$$
$$|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\rightharpoonup\chi,\ \text{ in }L^{\frac{p}{p-1}}\left(\mathbb{R}_+;L^{\frac{p}{p-1}}(\mathbb{R}^d)\right),$$

it is easy to see that

$$\left|\int_0^T\int_{\mathbb{R}^d}|\nabla\omega|^{p-2}\nabla\omega\cdot\left[\nabla(u_\epsilon-\omega)-\nabla(u-\omega)\right]\phi\ dxdt\right|$$
$$=\left|\int_0^T\int_{\mathbb{R}^d}|\nabla\omega|^{p-2}\nabla\omega\cdot\nabla(u_\epsilon-u)\phi\ dxdt\right|\to 0,\ \text{ as }\epsilon\to 0, \tag{113}$$

and

$$\left|\int_0^T\int_{\mathbb{R}^d}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\cdot\nabla\omega\phi\ dxdt-\int_0^T\int_{\mathbb{R}^d}\chi\cdot\nabla\omega\phi\ dxdt\right|\to 0, \tag{114}$$

as $\epsilon \to 0$, and

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^d} u_\epsilon \left(|\nabla u_\epsilon|^{p-2} + \epsilon \right) \nabla u_\epsilon \cdot \nabla \phi \; dxdt - \int_0^T \int_{\mathbb{R}^d} u\chi \cdot \nabla \phi \; dxdt \right| \\
&\leq C\epsilon \int_0^T \int_\Omega u_\epsilon |\nabla u_\epsilon| \; dxdt + \left| \int_0^T \int_{\mathbb{R}^d} u_\epsilon \left(|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon - \chi \right) \cdot \nabla \phi \; dxdt \right| \\
&\quad + \left| \int_0^T \int_{\mathbb{R}^d} (u_\epsilon - u) \chi \cdot \nabla \phi \; dxdt \right| \to 0, \text{ as } \epsilon \to 0,
\end{aligned} \tag{115}$$

where (115) holds from $\|u_\epsilon\|_{L^\infty(\mathbb{R}^d)} \leq C$. From (101) and $\|u_\epsilon\|_{L^\infty(\mathbb{R}^d)} \leq C$, we can easily obtain

$$\left| \int_0^T \int_{\mathbb{R}^d} u_\epsilon^2 \nabla v_\epsilon \cdot \nabla \phi \; dxdt - \int_0^T \int_{\mathbb{R}^d} u^2 \nabla v \cdot \nabla \phi \; dxdt \right| \to 0, \text{ as } \epsilon \to 0. \tag{116}$$

Then from (111)-(116), letting $\epsilon \to 0$, (110) turns to

$$\begin{aligned}
&\frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u^2 \phi_t \; dxdt - \int_0^T \int_{\mathbb{R}^d} |\nabla \omega|^{p-2} \nabla \omega \cdot \nabla (u - \omega) \phi \; dxdt \\
&- \int_0^T \int_{\mathbb{R}^d} \chi \cdot \nabla \omega \phi \; dxdt - \int_0^T \int_{\mathbb{R}^d} u\chi \cdot \nabla \phi \; dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u^2 \nabla v \cdot \nabla \phi \; dxdt \\
&+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u^3 \phi \; dxdt + \frac{1}{2} \int_{\mathbb{R}^d} u_0^2 \phi(0) dx \geq 0.
\end{aligned} \tag{117}$$

Taking $\psi = u\phi$ in (104), we have

$$\begin{aligned}
&\frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u^2 \phi_t \; dxdt - \int_0^T \int_{\mathbb{R}^d} \chi \cdot \nabla u \phi \; dxdt - \int_0^T \int_{\mathbb{R}^d} u\chi \cdot \nabla \phi \; dxdt \\
&+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u^2 \nabla v \cdot \nabla \phi \; dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u^3 \phi \; dxdt + \frac{1}{2} \int_{\mathbb{R}^d} u_0^2 \phi(0) dx = 0.
\end{aligned} \tag{118}$$

Combining (117) and (118) together, we obtain

$$\int_0^T \int_{\mathbb{R}^d} \chi \cdot \nabla (u - \omega) \phi \; dxdt - \int_0^T \int_{\mathbb{R}^d} |\nabla \omega|^{p-2} \nabla \omega \cdot \nabla (u - \omega) \phi \; dxdt \geq 0, \tag{119}$$

i.e.

$$\int_0^T \int_{\mathbb{R}^d} \left(\chi - |\nabla \omega|^{p-2} \nabla \omega \right) \cdot \nabla (u - \omega) \phi \; dxdt \geq 0. \tag{120}$$

Taking $\omega = u - \lambda \psi$ with $\lambda \geq 0$ yields that

$$\int_0^T \int_{\mathbb{R}^d} \left(\chi - |\nabla (u - \lambda \psi)|^{p-2} \nabla (u - \lambda \psi) \right) \cdot \nabla \psi \; \phi \; dxdt \geq 0. \tag{121}$$

Choosing ϕ such that $\operatorname{supp} \psi \subset \operatorname{supp} \phi$ and $\phi = 1$ on $\operatorname{supp} \psi$, letting $\lambda \to 0$, we obtain

$$\int_0^T \int_{\mathbb{R}^d} \left(\chi - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla \psi \; dxdt \geq 0. \tag{122}$$

Using the same method with $\lambda \leq 0$, we have

$$\int_0^T \int_{\mathbb{R}^d} \left(\chi - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla \psi \; dxdt \leq 0. \tag{123}$$

Then for any $\psi \in C_c^\infty\left([0,T)\times\mathbb{R}^d\right)$, we have

$$\int_0^T\int_{\mathbb{R}^d}|\nabla u|^{p-2}\nabla u\cdot\nabla\psi\ dxdt=\int_0^T\int_{\mathbb{R}^d}\chi\cdot\nabla\psi\ dxdt, \tag{124}$$

i.e. (102) holds.

Then combining (i)-(iii) and letting $\epsilon\to 0$, for any $0<T<\infty$, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}u(x,t)\psi_t(x,t)\ dxdt&=\int_0^T\int_{\mathbb{R}^d}|\nabla u(x,t)|^{p-2}\nabla u(x,t)\cdot\nabla\psi(x,t)\ dxdt\\&-\frac{1}{2d\alpha(d)}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\left[\nabla\psi(x,t)-\nabla\psi(y,t)\right]\cdot(x-y)}{|x-y|^2}\frac{u(x,t)u(y,t)}{|x-y|^{d-2}}\ dxdydt\\&-\int_{\mathbb{R}^d}u_0(x)\psi(x,0)dx,\end{aligned} \tag{125}$$

which means that (u,v) is a global weak solution of (1). □

For the subcritical case, we have the following theorem of the existence of a global weak solution. Since the proof is almost identical as that for the supercritical case, we omit details.

Theorem 5.3. *Let $d\geq 3$ and $p>\frac{3d}{d+1}$. If $u_0\in L_+^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, then there exists a non-negative global weak solution (u,v) of (1).*

6. Local existence of a weak solution and a blow-up criterion.

In this section, we prove that for $u_0\in L_+^1\cap L^\infty(\mathbb{R}^d)$, a weak solution of (1) exists locally without any restriction for the size of initial data. Furthermore, we also prove that if a weak solution blow up in finite time, then all L^h-norms of the weak solution blow up at the same time for $h>q$.

Theorem 6.1. *(Local existence of a weak solution) Let $d\geq 3$, $1<p<\frac{3d}{d+1}$ and $q=\frac{d(3-p)}{p}$. Assume $u_0\in L_+^1\cap L^\infty(\mathbb{R}^d)$. Then there are $T>0$, such that (1) has a weak solution in $0<t<T$.*

Proof. Take any fixed $r>q$. Using the same way of obtaining (30) and taking $h=r>q$, we have

$$\begin{aligned}\frac{d}{dt}\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r&=-\frac{r(r-1)p^p}{(r-2+p)^p}\left\|\nabla u^{\frac{r-2+p}{p}}\right\|_{L^p(\mathbb{R}^d)}^p+(r-1)\|u\|_{L^{r+1}(\mathbb{R}^d)}^{r+1}\\&\leq-\frac{r(r-1)p^p}{2(r-2+p)^p}\left\|\nabla u^{\frac{r-2+p}{p}}\right\|_{L^p(\mathbb{R}^d)}^p+C(r)\Big(\|u\|_{L^r(\mathbb{R}^d)}^r\Big)^{1+\frac{1}{r-q}},\end{aligned}$$

i.e.

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r\leq C(r)\Big(\|u\|_{L^r(\mathbb{R}^d)}^r\Big)^{1+\frac{1}{r-q}}. \tag{126}$$

Solving the inequality (126) shows that

$$\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r\leq\left[\frac{\frac{r-q}{C(r)}}{\frac{r-q}{C(r)}\left(\|u_0\|_{L^r(\mathbb{R}^d)}^r\right)^{\frac{1}{q-r}}-t}\right]^{r-q}. \tag{127}$$

Denoting $T_r:=\frac{r-q}{C(r)}\left(\|u_0\|_{L^r(\mathbb{R}^d)}^r\right)^{\frac{1}{q-r}}$, then for any fixed r, we choose $0<T<T_r$. Next by the same way of proving Theorem 5.2, (1) has a local in time weak solution in $0<t<T$. □

Proposition 2. *(Blow-up criterion) Under the same assumptions as Theorem 6.1 and $r = q + \epsilon$ where ϵ is small enough, let $T^r_{\max}$ be the largest L^r-norm existence time of a weak solution, i.e.*

$$\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} < \infty, \quad \text{for all } 0 < t < T^r_{\max},$$
$$\limsup_{t\to T^r_{\max}} \|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} = \infty,$$

and $T^h_{\max}$ be the largest L^h-norm existence time of a weak solution for $h \geq r > q$. Then if $T^h_{\max} < \infty$ for any h,

$$T^h_{\max} = T^r_{\max}, \quad \text{for all } h \geq r.$$

Proof. Since $\|u(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}$, by interpolation inequality, we know that for $h \geq r$, $T^h_{\max} \leq T^r_{\max}$. If $T^h_{\max} < T^r_{\max}$ for any $h \geq r$, then we will have contradiction arguments. $T^h_{\max} < T^r_{\max}$ implies

$$\limsup_{t\to T^h_{\max}} \|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} =: A < \infty.$$

Then for $h \geq r > q$, using (30), we have

$$\frac{d}{dt}\|u(\cdot,t)\|^h_{L^h(\mathbb{R}^d)} \leq C(h,r)\Big(\|u\|^r_{L^r(\mathbb{R}^d)}\Big)^{1+\frac{h-r+1}{r-q}} \leq C(h,r,A), \tag{128}$$

i.e.

$$\|u(\cdot,t)\|_{L^h(\mathbb{R}^d)} \leq C\left(h,r,A,\|u_0\|_{L^h(\mathbb{R}^d)},T^h_{\max}\right), \text{ for } t \in \left(0,T^h_{\max}\right),$$

which contradicts with

$$\limsup_{t\to T^h_{\max}} \|u(\cdot,t)\|_{L^h(\mathbb{R}^d)} = \infty.$$

Thus we have the conclusion that $T^h_{\max} = T^r_{\max}$ for all $h \geq r > q$, i.e. L^h-norms blow up at the same time. ☐

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