A note on Monge–Ampère Keller–Segel equation

Hui Huang[a,b,*], Jian-Guo Liu[b]

[a] *Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, People's Republic of China*
[b] *Departments of Physics and Mathematics, Duke University, Durham, NC 27708, USA*

ARTICLE INFO

Article history:
Received 8 April 2016
Received in revised form 5 May 2016
Accepted 5 May 2016
Available online 13 May 2016

Keywords:
Chemotaxis
Polar factorization
Convex potential
Brenier map
Global existence

ABSTRACT

This note studies the Monge–Ampère Keller–Segel equation in a periodic domain $\mathbb{T}^d (d \geq 2)$, a fully nonlinear modification of the Keller–Segel equation where the Monge–Ampère equation $\det(I + \nabla^2 v) = u + 1$ substitutes for the usual Poisson equation $\Delta v = u$. The existence of global weak solutions is obtained for this modified equation. Moreover, we prove the regularity in $L^\infty\left(0, T; L^\infty \cap W^{1,1+\gamma}(\mathbb{T}^d)\right)$ for some $\gamma > 0$.

1. Introduction

Keller–Segel (KS) model was firstly presented in 1970 to describe the chemotaxis of cellular slime molds [1]. The original model was considered in 2-dimension,

$$\begin{cases} \partial_t u = \triangle u + \nabla \cdot (u \nabla v), \quad x \in \mathbb{R}^2, \ t > 0, \\ \Delta v = u(t, x), \\ u(0, x) = u_0(x). \end{cases} \tag{1}$$

In the context of biological aggregation, $u(t, x)$ represents the bacteria density, and $v(t, x)$ represents the chemical substance concentration.

In this note, we study the Monge–Ampère Keller–Segel (MAKS) model in a periodic domain $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ $(d \geq 2)$:

$$\begin{cases} \partial_t u = \Delta u + \nabla \cdot (u \nabla v), \quad x \in \mathbb{T}^d, \ t > 0, \\ \det(I + \nabla^2 v) = u + 1, \\ u(0, x) = u_0(x). \end{cases} \tag{2}$$

* Corresponding author at: Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, People's Republic of China.

E-mail addresses: huanghui12@mails.tsinghua.edu.cn (H. Huang), jliu@phy.duke.edu (J.-G. Liu).

http://dx.doi.org/10.1016/j.aml.2016.05.003

where I is the identity matrix. In the absence of Δu term in (2), this model was introduced by Brenier [2, (5.34), (5.36)] as a fully nonlinear version of popular models in chemotaxis theory, such as the celebrated Keller–Segel model or similar models in astrophysics. We will prove the global existence of weak solutions to MAKS model (2) in a weak sense, which is made precise in Section 2.

Monge–Ampère Keller–Segel system (2) is an approximation of the original KS system (1) in the following re-scaling. Let us recast the equation (2) by introducing the new unknowns:

$$u^\delta(t,x) = \frac{1}{\delta} u\left(\frac{t}{\delta}, \frac{x}{\sqrt{\delta}}\right); \qquad v^\delta(t,x) = v\left(\frac{t}{\delta}, \frac{x}{\sqrt{\delta}}\right).$$

Then we have

$$u(t,x) = \delta u^\delta(\delta t, \sqrt{\delta} x); \qquad v(t,x) = v^\delta(\delta t, \sqrt{\delta} x).$$

Moreover, these new unknowns should be governed by the following MAKS system

$$\begin{cases} \partial_t u^\delta = \Delta u^\delta + \nabla \cdot (u^\delta \nabla v^\delta), \\ \det(I + \delta \nabla^2 v^\delta) = 1 + \delta u^\delta. \end{cases} \tag{3}$$

We formally linearize the determinant $\det(I + \delta\nabla^2 v^\delta)$ around the identity matrix and obtain

$$1 + \delta u^\delta = \det(I + \delta \nabla^2 v^\delta) = 1 + \delta \Delta v^\delta + O(\delta^2). \tag{4}$$

Then the Monge–Ampère equation turns into the Poisson equation $\Delta v^\delta = u^\delta + O(\delta)$, from which, when we set $O(\delta) = 0$, we recognize the MAKS system (3) as the original KS system showed in (1).

The density u in the original KS system (1) is driven by the gradient of Newtonian potential $\nabla v = \nabla N * u$, where N is the fundamental solution of Laplacian equation, and potential v has the superposition principle relation with u. Moreover, it has an important property: if $0 \le u \in L^\infty(\mathbb{T}^d)$, then ∇v is log-Lipschitz continuous. However, for MAKS model (2), the Newtonian potential is replaced by a convex potential $V[u]$ discovered by Brenier [3]. The advantage is that $\nabla v = \nabla V[u] - x$ is globally convex and has uniform L^∞ bound if $0 \le u \in L^1(\mathbb{T}^d)$. But the convex potential will lose the superposition principle relation with the density.

There are many mathematical models involved substituting the fully nonlinear Monge–Ampère equation for the Poisson equation. For example, the semigeotrophic equations in meteorology have a long history. After suitable changes of variables, they can be reformulated as a coupled Monge–Ampère/transport problem [4], which appear as a variant of the two-dimensional incompressible Euler equations in vorticity form, where the Poisson equation that relates to the stream function and the vorticity field is replaced by the Monge–Ampère equation [4–7]. Moreover, in [8], Brenier and Loeper studied the Vlasov–Monge–Ampère system, a fully non-linear version of the Vlasov–Poisson system. Similarly, Brenier [9], by substituting the Monge–Ampère equation for the linear Poisson equation to model gravitation, he introduced a modified Zeldovich approximate model related to the early universe reconstruction problem.

2. The polar decomposition theorem

The polar factorization of maps has been discovered by Brenier [3]. It was later extended to the general case of Riemannian manifolds by McCann in [10].

Let us consider a mapping $X : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that for all $\overrightarrow{p} \in \mathbb{Z}^d$, $X(\cdot + \overrightarrow{p}) = X + \overrightarrow{p}$. We use the push-forward of Lebesgue measure of $\mathbb{R}^d$ by X, and it is denoted by $u = X_\sharp dx$. Then u is a probability measure on $\mathbb{T}^d$ and we have the following theorem:

Theorem 1 (*Theorem 1.2 [6]*). *Let $X : \mathbb{R}^d \mapsto \mathbb{R}^d$ be described as above with $u = X_\sharp dx$.*

1. *Up to a constant, there exists a unique convex function* $\widehat{V}[u]$ *such that* $\widehat{V}[u] - x^2/2$ *is* $\mathbb{Z}^d$*-periodic (and thus* $\nabla\widehat{V}[u] - x$ *is* $\mathbb{Z}^d$*-periodic), and*

$$\forall \varphi \in C^0(\mathbb{T}^d), \quad \int_{\mathbb{T}^d} \varphi(\nabla\widehat{V}[u](x))dx = \int_{\mathbb{T}^d} \varphi(x)du(x). \tag{5}$$

2. *Let* $V[u]$ *be the Legendre transform of* $\widehat{V}[u]$*. If* u *is Lebesgue integrable, then* $V[u]$ *is a convex function satisfying that* $V[u] - x^2/2$ *is* $\mathbb{Z}^d$*-periodic (and thus* $\nabla V[u] - x$ *is* $\mathbb{Z}^d$*-periodic), unique up to a constant, and*

$$\forall \varphi \in C^0(\mathbb{T}^d), \quad \int_{\mathbb{T}^d} \varphi(\nabla V[u](x))du(x) = \int_{\mathbb{T}^d} \varphi(x)dx. \tag{6}$$

Moreover we have the bound $\|\nabla V[u] - x\|_{L^\infty(\mathbb{T}^d)} \le \sqrt{d}/2$.

Link with the Monge–Ampère equation. We can interpret (5) as a weak version of the Monge–Ampère equation

$$u(\nabla\widehat{V}) \det \nabla^2\widehat{V} = 1, \tag{7}$$

and (6) can be seen as a weak version of another Monge–Ampère equation

$$\det \nabla^2 V = u. \tag{8}$$

Moreover, we will also use the following result originally from [3]. The first one establishes the continuity of the polar decomposition.

Theorem 2 (*Theorem 2.6 [6]*). *Let* u_n *be a sequence of Lebesgue integrable positive measures on* $\mathbb{T}^d$*, such that for all* n*,* $\int_{\mathbb{T}^d} du_n \le C$ *and let* $\widehat{V}_n = \widehat{V}[u_n]$*,* $V_n = V[u_n]$ *be as defined in Theorem 1. If for any* $\varphi \in C^0(\mathbb{T}^d)$ *such that* $\int \varphi du_n$ *converges to* $\int \varphi du$*, then the sequence* $\widehat{V}_n$ *can be chosen in such a way that* $\widehat{V}_n$ *converges to* $\widehat{V}[u]$ *uniformly on* $\mathbb{T}^d$ *and strongly in* $W^{1,1}(\mathbb{T}^d)$*, and* V_n *converges to* $V[u]$ *uniformly on* $\mathbb{T}^d$ *and strongly in* $W^{1,1}(\mathbb{T}^d)$.

Theorem 1 allows us to recast MAKS equation (2) as

$$\partial_t u = \Delta u + \nabla \cdot (u(\nabla V[u+1] - x)), \quad x \in \mathbb{T}^d,\ t > 0, \tag{9a}$$

$$u(0, x) = u_0(x). \tag{9b}$$

where $V[u+1]$ is as defined in Theorem 1. For simplicity, we denote $V[u+1]$ as $V[u]$.

Remark 1. If u is continuous and satisfies $0 \le u \le C_1$, it has been proved in [11] that $\nabla V[u](x)$ is log-Lipschitz continuous. The log-Lipschitz continuity usually ensures the uniqueness and stability in the Wasserstein distance. Moreover, according to [8, Theorem 4.4], if $u \in C^\alpha(\mathbb{T}^d)$, $\alpha \in (0, 1)$, then $V[u]$ is a classical solution of

$$\det \nabla^2 V[u] = u + 1. \tag{10}$$

3. Existence of global weak solutions

To begin this section, we give the following definition of the weak solution to the MAKS equation (9).

Definition 1. Let initial data $0 \le u_0 \in L^1(\mathbb{T}^d)$. Then (u, V) is a global weak solution to (9) if it satisfies for any $T > 0$:

1. $u \in L^\infty\left(0, T; L^1(\mathbb{T}^d)\right) \cap L^2\left(0, T; H^1(\mathbb{T}^d)\right)$ and $\partial_t u \in L^2\left(0, T; H^{-1}(\mathbb{T}^d)\right)$.
2. $\forall\ \varphi \in C_c^\infty\left([0, T) \times \mathbb{T}^d\right)$,

$$\int_0^T \int_{\mathbb{T}^d} u \partial_t \varphi dx dt = \int_0^T \int_{\mathbb{T}^d} (\nabla u \nabla \varphi + u(\nabla V - x) \cdot \nabla \varphi) dx dt - \int_{\mathbb{T}^d} u_0 \varphi(0, x) dx,$$

where V is defined as in Theorem 1.

The main result of this note is as follows:

Theorem 3. *Let initial data $0 \le u_0 \in L^2(\mathbb{T}^d)$. Then the MAKS system (2) admits a global non-negative weak solution (u, V) in $t \in [0, T]$ for any $T > 0$. And the conservation of mass holds: $\int_{\mathbb{T}^d} u(t, x) dx = \int_{\mathbb{T}^d} u_0(x) dx$.*

Proof. We build a sequence of approximate solutions $(u_\varepsilon, V_\varepsilon)_{\varepsilon > 0}$ by regularization and let ε goes to zero. To do the limiting process, the non-linear term will be treated with the help of Theorem 2.

Step 1: Construction of a sequence of approximate solutions. We consider a mollifier $\psi(x) \in C_c^\infty(\mathbb{R}^d)$ such that $\psi(x) \ge 0$, $\int_{\mathbb{T}^d} \psi(x) dx = 1$ and $\psi_\varepsilon(x) = \varepsilon^{-d} \psi(x/\varepsilon)$. And we can define the mollification as $\psi_\varepsilon * u_0 := \int_{\mathbb{T}^d} \psi_\varepsilon(x - y) u_0(y) dy$. Then we study solutions to the following approximate problem

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \nabla \cdot \left(u_\varepsilon(\nabla V_\varepsilon(x) - x)\right), \quad x \in \mathbb{T}^d,\ t > 0, \tag{11a}$$

$$u_{\varepsilon,0}(x) = \psi_\varepsilon * u_0(x), \tag{11b}$$

$$V_\varepsilon(x) = \psi_\varepsilon * V[u_\varepsilon]. \tag{11c}$$

Since V_ε given by (11c) is bounded in $H^k(\mathbb{T}^d)$ for any k and $\varepsilon > 0$, the estimate for Eq. (11a) for any fixed $\varepsilon > 0$ is basically same as that for the heat equation. Hence, the solvability of the regularized problem (11) can be obtained by using the technique in Majda and Bertozzi [12, Section 3.2.2], where it proved the global existence of the solution to a regularization of the Euler and Navier–Stokes equation by using the Picard theorem and continuation property of ODEs on a Banach space. We omit the detail here.

Step 2: Weak convergence of u_ε and ∇u_ε. Multiplying Eq. (11a) by $2u_\varepsilon$ and integrating over $\mathbb{T}^d$, we obtain

$$\frac{d}{dt} \|u_\varepsilon\|_2^2 + 2 \|\nabla u_\varepsilon\|_2^2 = - \int_{\mathbb{T}^d} u_\varepsilon (\nabla V_\varepsilon - x) \cdot \nabla u_\varepsilon dx \le \|\nabla u_\varepsilon\|_2^2 + C \|u_\varepsilon\|_2^2. \tag{12}$$

where we have used $\|\nabla V_\varepsilon - x\|_\infty \le \sqrt{d}/2$.

Hence for any $T > 0$, the following estimates hold

$$\|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \le C_T, \qquad \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le C_T. \tag{13}$$

According to the above estimates, there is a subsequence (still denote u_ε), such that as $\varepsilon \to 0$, the following weak convergence results hold

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0, T; L^2(\mathbb{T}^d)\right), \qquad \nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2\left(0, T; L^2(\mathbb{T}^d)\right). \tag{14}$$

Step 3: Strong convergence of $\nabla V_\varepsilon(t, \cdot)$ a.e. t. We claim that for any $p \in [1, \infty)$,

$$\nabla V_\varepsilon(t, \cdot) \to \nabla V(t, \cdot) \quad \text{in } L^p(\mathbb{T}^d),\ \text{a.e. } t \in [0, T]. \tag{15}$$

Indeed, such strong convergence of ∇V_ε follows from Theorem 2 provided that we have for a.e. $t \in [0, T]$,

$$\int_{\mathbb{T}^d} \varphi(x) u_\varepsilon(t, x) dx \to \int_{\mathbb{T}^d} \varphi(x) u(t, x) dx, \tag{16}$$

for any $\varphi \in C^0(\mathbb{T}^d)$. To verify (16), we need to prove that there is a subsequence (still denote u_ε)

$$u_\varepsilon \to u \quad \text{in } L^2(\mathbb{T}^d) \text{ a.e. } t \in [0, T], \text{ as } \varepsilon \to 0. \tag{17}$$

Indeed, it is easy to check that $\|\partial_t u_\varepsilon\|_{L^2(0,T;H^{-1}(\mathbb{T}^d))} \le C_T$, which leads to

$$u_\varepsilon \to u \quad \text{in } L^2\big(0, T; L^2(\mathbb{T}^d)\big), \text{ as } \varepsilon \to 0, \tag{18}$$

by using Aubin–Lions lemma as $H^1(\mathbb{T}^d) \hookrightarrow\hookrightarrow L^2(\mathbb{T}^d) \hookrightarrow H^{-1}(\mathbb{T}^d)$. Then (17) follows from (18), which completes the proof of (15).

Step 4: Existence of a global weak solution. Next, we will show that $(u, V[u])$ is a weak solution to (9). The weak formulation for u_ε is that for any test function $\varphi \in C_c^\infty\big([0, T) \times \mathbb{T}^d\big)$,

$$\int_0^T \int_{\mathbb{T}^d} u_\varepsilon \partial_t \varphi dx dt = \int_0^T \int_{\mathbb{T}^d} (\nabla u_\varepsilon \nabla \varphi + u_\varepsilon(\nabla V_\varepsilon - x) \cdot \nabla \varphi) dx dt - \int_{\mathbb{T}^d} u_{\varepsilon,0} \varphi(0, x) dx. \tag{19}$$

Recall that (14), (15), (18) and $\|\nabla V_\varepsilon - x\|_\infty \le \sqrt{d}/2$. Then by using the dominant convergence theorem, one concludes that by passing limit $\varepsilon \to 0$ in (19)

$$\int_0^T \int_{\mathbb{T}^d} u \partial_t \varphi dx dt = \int_0^T \int_{\mathbb{T}^d} (\nabla u \nabla \varphi + u(\nabla V - x) \cdot \nabla \varphi) dx dt - \int_{\mathbb{T}^d} u_0 \varphi(0, x) dx. \tag{20}$$

We finished the proof of the existence of global weak solutions.

Step 5: Positivity preserving. By using Lemma 7.6 in [13], if we define the negative part of the function u as $u_- := \min\{u, 0\}$, then one can easily prove that

$$\frac{d}{dt}\|u_-\|_2^2 + 2\,\|\nabla u_-\|_2^2 = -\int_{\mathbb{T}^d} u_-(\nabla V - x) \cdot \nabla u_- dx \le \|\nabla u_-\|_2^2 + C\|u_-\|_2^2. \tag{21}$$

Applying Gronwall's inequality to

$$\frac{d}{dt}\|u_-\|_2^2 \le C\|u_-\|_2^2; \qquad \|u_{0-}\|_2^2 = 0, \tag{22}$$

one has $\|u_-\|_2^2 \equiv 0$, which leads to $u(t, x) \ge 0$.

Step 6: Conservation of mass. Integrating (9a) over $\mathbb{T}^d$ and using the fact that ∇u, $\nabla V - x$ are periodic, one has

$$\frac{d}{dt}\int_{\mathbb{T}^d} u dx = \int_{\mathbb{T}^d} \Delta u dx + \int_{\mathbb{T}^d} \nabla \cdot (u(\nabla V - x)) dx = 0. \tag{23}$$

Thus, we conclude that

$$\int_{\mathbb{T}^d} u(t, x) dx = \int_{\mathbb{T}^d} u_0(x) dx. \quad \square \tag{24}$$

4. Regularity in $L^\infty(0, T; L^\infty \cap W^{1,1+\gamma}(\mathbb{T}^d))$

Theorem 4. *Let initial data $0 \le u_0 \in L^\infty(\mathbb{T}^d)$ and $\nabla u_0 \in L^{1+\gamma}(\mathbb{T}^d)$ for some $\gamma > 0$. Suppose (u, V) be a weak solution to MAKS equation (9), then for any $T > 0$ and $t \in [0, T]$,*

$$u(t, x) \in L^\infty\left(0, T; L^\infty \cap W^{1,1+\gamma}(\mathbb{T}^d)\right). \tag{25}$$

Proof. First we will prove that

$$\|u(\cdot,t)\|_\infty \le C(T,d,A_0), \tag{26}$$

with $A_0 = \max\{1, \|u_0\|_{L^1(\mathbb{T}^d)}, \|u_0\|_{L^\infty(\mathbb{T}^d)}\}$. Multiplying (9a) with pu^{p-1}, $p \ge 2$ and integrating over $\mathbb{T}^d$, we have

$$\begin{aligned}\frac{d}{dt}\|u\|_p^p + \frac{4(p-1)}{p}\left\|\nabla u^{\frac{p}{2}}\right\|_2^2 &= -(p-1)\int_{\mathbb{T}^d}(\nabla V - x)\nabla u^p dx = -2(p-1)\int_{\mathbb{T}^d}(\nabla V - x)u^{\frac{p}{2}}\nabla u^{\frac{p}{2}}dx \\ &\le C\left\|u^{\frac{p}{2}}\right\|_2\left\|\nabla u^{\frac{p}{2}}\right\|_2 \le \frac{2(p-1)}{p}\left\|\nabla u^{\frac{p}{2}}\right\|_2^2 + C\|u\|_p^p. \end{aligned}\tag{27}$$

Then the L^∞ bound can be obtained directly by the standard Moser iteration after getting (27). For example, one can check the paper by Alikakos [14], formula (3.20) and the computation afterwards. For completeness, we put these detail computation in Appendix.

From Theorem 3, we know that u is positivity preserving and the conservation of mass hold:

$$u \ge 0; \quad \|u(t,x)\|_1 = \|u_0\|_1, \text{ for } t \in [0,T]. \tag{28}$$

By the construction of V in Theorem 1, one concludes that

$$\begin{cases} 1 \le \det(\nabla^2 V) \le \|u\|_\infty + 1, \\ V \text{ convex}, \\ V - x^2/2 \text{ periodic}. \end{cases}\tag{29}$$

Recall the result in [15, P.16], for some $\gamma > 0$ we have

$$\|\nabla^2 V\|_{1+\gamma} \le C(T,d,\|u\|_\infty). \tag{30}$$

The heat semigroup operator $e^{t\Delta}$ defined by $e^{t\Delta}u := H(t,x) * u$, where $H(t,x) = \frac{1}{(4\pi t)^{d/2}}\sum_{k\in\mathbb{Z}^d} e^{-\frac{|x+k|^2}{4t}}$ is the periodic heat kernel. It follows immediately from Young's inequality for the convolution that

$$\|e^{t\Delta}u\|_p \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u\|_q, \qquad \|\nabla e^{t\Delta}u\|_p \le Ct^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u\|_q, \tag{31}$$

for any $1 \le q \le p \le +\infty$, $u \in L^p(\mathbb{T}^d)$ and all $t > 0$. Here C is constant dependent of p, q.

By the fundamental solution representation of the heat equation, the solution to the MAKS equation can be represented as

$$u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}(\nabla\cdot(u(\nabla V - x)))ds, \tag{32}$$

for any $T > 0$, $0 < t < T$.

By choosing $q = p = 1+\gamma$ in (31), a simple computation leads to

$$\|\nabla u\|_{1+\gamma} \le C\|\nabla u_0\|_{1+\gamma} + \int_0^t (t-s)^{-1/2}\|\nabla\cdot(u(\nabla V - x))\|_{1+\gamma}ds. \tag{33}$$

From Theorem 1, we have that $\|\nabla V - x\|_\infty \le \sqrt{d}/2$ and moreover $\|\nabla^2 V\|_{1+\gamma} \le C(T,d,\|u\|_\infty)$, then we conclude

$$\|\nabla\cdot(u(\nabla V - x))\|_{1+\gamma} \le \sqrt{d}/2\|\nabla u\|_{1+\gamma} + C\left(T,d,\|u\|_\infty,|\mathbb{T}^d|\right). \tag{34}$$

Hence we have

$$\|\nabla u\|_{1+\gamma} \le C_1 + C_2\int_0^t (t-s)^{-1/2}\|\nabla u\|_{1+\gamma}ds. \tag{35}$$

Applying a generalized Gronwall's inequality with weak singularities [16, Lemma 7.1.1], we have

$$\|\nabla u\|_{1+\gamma} \le C\left(T, d, \|\nabla u_0\|_{1+\gamma}, \|u_0\|_\infty\right), \tag{36}$$

which concludes the proof. □

Acknowledgments

The authors would like to thank Yann Brenier for suggesting this problem to us. The work of Jian-Guo Liu is partially supported by KI-Net NSF RNMS grant No. 1107291 and NSF grant DMS 1514826. Hui Huang is partially supported by National Natural Science Foundation of China (Grant number: 41390452, 11271118).

Appendix. The proof of L^∞ estimate in Theorem 4

Proof. Using Gronwall's inequality in (27), one concludes that

$$\|u(\cdot,t)\|_p^p \le e^{Ct}\|u_0\|_p^p \le C(T, d, A_0). \tag{A.1}$$

Define $p_k := 2^k + 2$ with $k \ge 0$. For $k = 0$, $p_0 = 3$, from (A.1) we have

$$\|u(\cdot,t)\|_{p_0}^{p_0} \le C(T, d, A_0). \tag{A.2}$$

For $k \ge 1$, take $p_k u^{p_k-1}$ as a test function in (9a), one has

$$\begin{aligned}\frac{d}{dt}\|u\|_{p_k}^{p_k} &= -\frac{4(p_k-1)}{p_k}\left\|\nabla u^{\frac{p_k}{2}}\right\|_2^2 - (p_k-1)\int_{\mathbb{T}^d}(\nabla V - x)\nabla u_k^p dx \\ &\le -2C_{p_k}\left\|\nabla u^{\frac{p_k}{2}}\right\|_2^2 + p_k\left\|u^{\frac{p_k}{2}}\right\|_2\left\|\nabla u^{\frac{p_k}{2}}\right\|_2. \end{aligned}\tag{A.3}$$

Now, we focus on estimating the term $\|u^{\frac{p_k}{2}}\|_2\|\nabla u^{\frac{p_k}{2}}\|_2$

$$\left\|u^{\frac{p_k}{2}}\right\|_2\left\|\nabla u^{\frac{p_k}{2}}\right\|_2 \le \left\|u^{\frac{p_k}{2}}\right\|_{\frac{2d}{d-2}}^{\theta}\left\|u^{\frac{p_k}{2}}\right\|_r^{1-\theta}\left\|\nabla u^{\frac{p_k}{2}}\right\|_2 \le S_d^\theta\left\|\nabla u^{\frac{p_k}{2}}\right\|_2^{1+\theta}\left\|u^{\frac{p_k}{2}}\right\|_r^{1-\theta}, \tag{A.4}$$

with $\frac{p_k}{2}r = p_{k-1}$, $\theta = \frac{\frac{1}{r}-\frac{1}{2}}{\frac{1}{r}-\frac{d-2}{2d}}$, where we have used the Sobolev inequality $\|u\|_{\frac{2d}{d-2}} \le S_d\|\nabla u\|_2$. The Young's inequality tells that

$$\frac{d}{dt}\|u\|_{p_k}^{p_k} \le -C_{p_k}\left\|\nabla u^{\frac{p_k}{2}}\right\|_2^2 + C(\sigma)p_k^{q_2}S_d^{\theta q_2}\|u\|_{p_{k-1}}^{p_k}, \tag{A.5}$$

where $\sigma = C_{p_k}$, $q_2 = \frac{2}{1-\theta} \le d+2$.

On the other hand,

$$\|u\|_{p_k}^{p_k} = \left\|u^{\frac{p_k}{2}}\right\|_2^2 \le S_d^{2\theta_1}\left\|\nabla u^{\frac{p_k}{2}}\right\|_2^{2\theta_1}\left\|u^{\frac{p_k}{2}}\right\|_r^{2(1-\theta_1)}, \tag{A.6}$$

where r is the same as before and $\theta_1 = \frac{\frac{1}{r}-\frac{1}{2}}{\frac{1}{r}-\frac{d-2}{2d}}$. Similar to (A.5), we have

$$\|u\|_{p_k}^{p_k} \le \sigma\left\|\nabla u^{\frac{p_k}{2}}\right\|_2^2 + \overline{C}(\sigma)S_d^{2\theta_1\ell_2}\|u\|_{p_{k-1}}^{p_k\ell_2(1-\theta_1)}, \tag{A.7}$$

where $\ell_2 = \frac{1}{1-\theta_1}$.

Hence from (A.5) and (A.7), we deduce

$$\frac{d}{dt}\|u\|_{p_k}^{p_k} \leq -\|u\|_{p_k}^{p_k} + C(\sigma)p_k^{q_2}S_d^{\theta q_2}\|u\|_{p_{k-1}}^{p_k} + \overline{C}(\sigma)S_d^{2\theta_1\ell_2}\|u\|_{p_{k-1}}^{p_k}. \tag{A.8}$$

Define

$$C_1(p_k) := C(\sigma)S_d^{\theta q_2}; \qquad C_2(p_k) := \overline{C}(\sigma)S_d^{2\theta_1\ell_2}.$$

It is easy to know that $C_1(p_k)$ and $C_2(p_k)$ is uniformly bounded for any $k \geq 1$. So, we let $C(d) > 1$ be a common upper bound of $C_1(p_k)$ and $C_2(p_k)$, we obtain the following inequality

$$\frac{d}{dt}\|u\|_{p_k}^{p_k} \leq -\|u\|_{p_k}^{p_k} + C(d)p_k^{q_2}\|u\|_{p_{k-1}}^{p_k}. \tag{A.9}$$

Solving the inequality (A.9), we get

$$(e^t\|u\|_{p_k}^{p_k})' \leq C(d)p_k^{q_2}\|u\|_{p_{k-1}}^{p_k}e^t \leq 2C(d)4^{d+2}2^{k(d+2)}\sup_{t\geq 0}\|u\|_{p_{k-1}}^{p_k}e^t, \tag{A.10}$$

where the last inequality used $1 < q_2 \leq d+2$.

Notice that $\|u_0\|_{p_k}^{p_k} \leq \|u_0\|_1\|u_0\|_\infty^{p_k-1}$, so we have

$$\max\{\|u_0\|_{p_k}^{p_k}, 1\} \leq A^{p_k}, \tag{A.11}$$

where constant $A > 1$ is independent of k but depends on $\|u_0\|_1$ and $\|u_0\|_\infty$. Let $a_k := 2C(d)4^{d+2}2^{k(d+2)} > 1$ and integrate (A.10), then one has

$$\|u\|_{p_k}^{p_k} \leq a_k \sup_{t\geq 0}\|u\|_{p_{k-1}}^{p_k}(1-e^{-t}) + \|u_0\|_{p_k}^{p_k}e^{-t} \leq a_k \max\left\{\sup_{t\geq 0}\|u\|_{p_{k-1}}^{p_k}, A^{p_k}\right\}. \tag{A.12}$$

Taking the power $\frac{1}{p_k}$ to above inequality, then

$$\|u\|_{p_k} \leq a_k^{1/p_k}\max\left\{\sup_{t\geq 0}\|u\|_{p_{k-1}}, A\right\}. \tag{A.13}$$

After some iterative steps, we have

$$\begin{aligned}\|u\|_{p_k} &\leq a_k^{1/p_k}a_{k-1}^{1/p_{k-1}}\cdots a_1^{1/p_1}\max\left\{\sup_{t\geq 0}\|u\|_{p_0}, A\right\}\\ &\leq (2C(d)4^{d+2})^{1-\frac{1}{2^k}}(2^{d+2})^{2-\frac{1}{2^{k-1}}-\frac{k}{2^k}}\max\left\{\sup_{t\geq 0}\|u\|_{p_0}, A\right\}.\end{aligned} \tag{A.14}$$

Recall $\|u\|_{p_0}^{p_0} \leq C(T,d,A_0)$, then the L^∞ estimate is obtained by passing to the limit $k \to \infty$ in (A.14). □

References

[1] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (3) (1970) 399–415. http://dx.doi.org/10.1016/0022-5193(70)90092-5.

[2] Y. Brenier, Hilbertian approaches to some non-linear conservation laws, Contemp. Math. 526 (2010) 19–34.

[3] Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions, Comm. Pure Appl. Math. 44 (4) (1991) 375–417. http://dx.doi.org/10.1002/cpa.3160440402.

[4] J. Benamou, Y. Brenier, Weak existence for the semigeostrophic equations formulated as a coupled Monge–Ampère transport problem, SIAM J. Appl. Math. 58 (5) (1998) 1450–1461. http://dx.doi.org/10.1137/S0036139995294111.

[5] M. Cullen, M. Feldman, Lagrangian solutions of semigeostrophic equations in physical space, SIAM J. Math. Anal. 37 (5) (2006) 1371–1395. http://dx.doi.org/10.1137/040615444.

[6] G. Loeper, A fully nonlinear version of the incompressible Euler equations: the semigeostrophic system, SIAM J. Math. Anal. 38 (3) (2006) 795–823. http://dx.doi.org/10.1137/050629070.

[7] M. Lopes Filho, H.N. Lopes, Existence of a weak solution for the semigeostrophic equation with integrable initial data, in: Proceedings of the Royal Society of Edinburgh-A-Mathematics, Vol. 132, Cambridge Univ Press, 2002, pp. 329–340.
[8] Y. Brenier, G. Loeper, A geometric approximation to the Euler equations: the Vlasov–Monge–Ampère system, Geom. Funct. Anal. GAFA 14 (6) (2004) 1182–1218. http://dx.doi.org/10.1007/s00039-004-0488-1.
[9] Y. Brenier, A modified least action principle allowing mass concentrations for the early universe reconstruction problem, Confluentes Math. 3 (03) (2011) 361–385. http://dx.doi.org/10.1142/S1793744211000400.
[10] R.J. McCann, Polar factorization of maps on riemannian manifolds, Geom. Funct. Anal. GAFA 11 (3) (2001) 589–608. http://dx.doi.org/10.1007/PL00001679.
[11] H.-Y. Jian, X.-J. Wang, Continuity estimates for the Monge–Ampere equation, SIAM J. Math. Anal. 39 (2) (2007) 608–626. http://dx.doi.org/10.1137/060669036.
[12] A.J. Majda, A.L. Bertozzi, Vorticity and Incompressible Flow, Vol. 27, Cambridge University Press, 2002, http://dx.doi.org/10.1017/CBO9780511613203.
[13] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, springer, 2015, http://dx.doi.org/10.1007/978-3-642-96379-7.
[14] N.D. Alikakos, l^p bounds of solutions of reaction–diffusion equations, Comm. Partial Differential Equations 4 (8) (1979) 827–868. http://dx.doi.org/10.1080/03605307908820113.
[15] A. Figalli, Global Existence for the Semigeostrophic Equations via Sobolev Estimates for Monge–Ampere, in: CIME Lecture Notes, Springer, to appear.
[16] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Vol. 840, Springer-Verlag, New York, 1981, http://dx.doi.org/10.1007/BFb0089647.