A Note on L^∞-Bound and Uniqueness to a Degenerate Keller-Segel Model

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Abstract In this note we establish the uniform L^∞-bound for the weak solutions to a degenerate Keller-Segel equation with the diffusion exponent $\frac{2n}{n+2} < m < 2 - \frac{2}{n}$ under a sharp condition on the initial data for the global existence. As a consequence, the uniqueness of the weak solutions is also proved.

Keywords Displacement convexity · Log-Lipschitz · Yudovich's type theorem · Stability in Wasserstein metric

1 Introduction

In this note, we consider the following degenerate Keller-Segel equations

$$\begin{cases} \rho_t = \Delta\rho^m - \operatorname{div}(\rho\nabla c), & x \in \mathbb{R}^n,\ t \geq 0, \\ -\Delta c = \rho, & x \in \mathbb{R}^n,\ t \geq 0, \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

with the diffusion exponent $m \in (\frac{2n}{n+2}, 2 - 2/n)$, $n \geq 3$, and the initial data $\rho_0 \in L^1_+(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Here $\rho(x,t)$ represents the bacteria density and $c(x,t)$ represents the chemical

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substance concentration and it is given by

$$c = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \rho(y) dy, \tag{1.2}$$

where $\alpha(n)$ is the volume of the n-dimension unit ball.

If $\rho \in L^1(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$, then the associated free energy of the model (1.1) can be written as

$$\mathcal{F}(\rho) = \frac{1}{m-1} \int_{\mathbb{R}^n} \rho^m(x,t) dx - \frac{1}{2} \int_{\mathbb{R}^n} \rho(x,t) c(x,t) dx. \tag{1.3}$$

In fact, the Hardy-Letterwood-Sobolev inequality and (1.2) imply that the second term on the right side of (1.3) is bounded. And formally the following entropy-dissipation equality holds

$$\frac{d}{dt} \mathcal{F}\big(\rho(\cdot,t)\big) + \int_{\mathbb{R}^n} \rho \left| \nabla \left(\frac{m}{m-1} \rho^{m-1} - c \right) \right|^2 dx = 0. \tag{1.4}$$

The model (1.1) can be written in the gradient flow form:

$$\rho_t + \operatorname{div}(\rho v) = 0, \tag{1.5}$$

where the drift velocity $v = -\nabla \mu$, μ is the chemical potential giving by

$$\mu = \frac{\delta \mathcal{F}}{\delta \rho} = \frac{m}{m-1} \rho^{m-1} - c.$$

The entropy-dissipation equality (1.4) implies that the velocity field v satisfies the estimate

$$\int_0^T \|v\|^2_{L^2(\mathbb{R}^n, \rho dx; \mathbb{R}^n)} dt = \int_0^T \int_{\mathbb{R}^n} \left| \nabla \left(\frac{m}{m-1} \rho^{m-1} - c \right) \right|^2 \rho dx dt < +\infty, \tag{1.6}$$

if the initial free energy $\mathcal{F}(\rho_0)$ is finite.

There are many results on the existence and the blow-up for the degenerate Keller-Segel models in literatures, c.f. [3, 4, 8, 20, 21]. For the degenerate system, there exist two important diffusion exponents. One is $m = m^* = 2 - 2/n$ and it produces an exact balance between the diffusion and the nonlocal aggregation in the mass invariant scaling. This exponent is usually referred to as the critical exponent and it is connected to the Fujita exponent. The global existence for small initial data, and blow-up behavior for large initial data to the parabolic-elliptic model (1.1) with $m > 1$ were established by Sugiyama and her collaborators in [20, 21]. The global existence to the parabolic-parabolic system with $m = m^*$ was given by [5] under a sharp condition on the initial data. Another important diffusion exponent is $m = m_c = \frac{2n}{n+2}$ and it was referred to as the energy critical in [4] and it is the critical exponent for the existence of positive stationary solution to the associated Lane-Emden equation. At this exponent the free energy is conformal invariant and the steady solution is unstable [8]. Some results on the global existence and the blow-up at this exponent under a sharp initial condition was established in [8].

For the model (1.1) with $m_c < m < m^*$, the following exact criteria of the initial data on the global existence and the blow-up in a finite time was established in [9].

Assumption 1

$$\rho_0 \in L^1_+\big(\mathbb{R}^n\big) \cap L^m\big(\mathbb{R}^n\big), \qquad \mathcal{F}(\rho_0) < \mathcal{F}^*, \tag{1.7}$$

where $\mathcal{F}^*$ is given by

$$\mathcal{F}^* = \frac{2 - \frac{2}{n} - m}{(m-1)(1-\frac{2}{n})} \left(\frac{2n^2\alpha(n)}{C(n)}\right)^{\frac{n(m-1)}{2n-2-mn}} M_0^{\frac{2n-m(n+2)}{2n-2-mn}} > 0, \tag{1.8}$$

M_0 is the initial mass $\|\rho_0\|_{L^1(\mathbb{R}^n)}$ and $C(n)$ is the best constant of the Hardy-Littlewood-Sobolev inequality, see [15, *pp.* 106].

Now we recall the results of the paper [9], which is helpful for proving our main results.

Proposition 1.1 *Assume the initial density ρ_0 satisfies Assumption 1. Let*

$$s^* = \left(\frac{2n^2\alpha(n) M_0^{\frac{2n-m(n+2)}{n-2}}}{C(n)}\right)^{\frac{n(m-1)}{2n-2-mn}} > 0. \tag{1.9}$$

Then the following statements are true

(1) (*global existence*) *if*

$$\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < \big(s^*\big)^{\frac{n-2}{2n(m-1)}}, \tag{1.10}$$

then there exists a global weak solution to Eqs. (1.1) *and it satisfies*

(i) *there exists a constant $\mu_1 < 1$ such that*

$$\|\rho(\cdot,t)\|_{\frac{2n}{n+2}} < \big(\mu_1 s^*\big)^{\frac{n-2}{2n(m-1)}}, \quad \text{for all } t > 0. \tag{1.11}$$

(ii) *Time and space derivative regularities, for any $T > 0$ and for any $1 < p < \infty$,*

$$\nabla\rho^{\frac{m+p-1}{2}} \in L^2\big(0,T; L^2\big(\mathbb{R}^n\big)\big), \tag{1.12}$$

$$\partial_t\rho \in L^2\big(0,T; W^{-1,s}_{\mathrm{loc}}\big(\mathbb{R}^n\big)\big), \tag{1.13}$$

where $s = \min\{\frac{2m}{m+1}, \frac{nm(m+1)}{nm+(n-m)(m+1)}\} > 1$.

(2) (*finite time blow-up*) *if*

$$\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} > \big(s^*\big)^{\frac{n-2}{2n(m-1)}} \tag{1.14}$$

and $m_2(0) < \infty$, then any weak solution must blow up in a finite time, i.e. $\exists T^* > 0$ such that $\|\rho(\cdot,t)\|_{L^{\frac{2n}{n+2}}} \to \infty$ as $t \to T^*$.

Remark 1.1 One can see clearly from Proposition 1.1 that the exact criterion $(s^*)^{\frac{n-2}{2n(m-1)}}$ given above is actually a relation between $L^{\frac{2n}{n+2}}$ norm and the total mass of the weak solution. Moreover, the condition $\mathcal{F}(\rho_0) < \mathcal{F}^*$ exclude the case $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} = (s^*)^{\frac{n-2}{2n(m-1)}}$.

A more general model is given by

$$\begin{cases} \rho_t = \Delta\rho^m - \mathrm{div}(\rho\nabla c), & x\in\mathbb{R}^n,\ t\geq 0,\\ \alpha c_t - \Delta c + \beta c = \rho, & x\in\mathbb{R}^n,\ t\geq 0,\\ \rho(x,0)=\rho_0(x),\ c(x,0)=c_0(x) & x\in\mathbb{R}^n, \end{cases} \tag{1.15}$$

where $\alpha, \beta \geq 0, m > 0$.

Remark 1.2 Proposition 1.1 shows that there exists a large mass global weak solution to Eqs. (1.1) with $m_c < m < m^*$. Recently, Bedrossian [2] proved that some large mass global weak solutions exist for the Keller-Segel equations (1.15) with $m = m^*$, $\alpha = 0$ and $\beta > 0$.

The uniform L^∞-bounds for weak solutions were obtained by using a bootstrap iterative technique in [6]. This note will give a uniform L^∞-bound under Assumption 1 and the sharp condition (1.10). As a consequence, the uniqueness will be proved in Theorem 1.1.

The uniqueness of the Keller-Segel model (1.15) has been concerned by many scholars recently. The uniqueness of weak solutions to the model (1.15) with $m = 1$ can be obtained easily from an energy method for the case $\alpha = 0, \beta > 0$. While for the high dimensional degenerate Keller-Segel model (1.15) with $m \neq 1$, it seems that the uniqueness can not be obtained directly from standard energy estimates or comparison methods in the solution space $L^\infty([0,T);\ L^1 \cup L^\infty(\mathbb{R}^n))$. However, with some additional regularities, there are some results on the uniqueness given by using the classical PDE theory. For example, Sugiyama [19] proved the uniqueness for 1-D Keller-Segel model (1.15) with $m > 1, \alpha = 0, \beta > 0$, and additional assumption on the regularities $\partial_t\rho \in L^1_{\mathrm{loc}}(\mathbb{R}\times(0,T))$, $\partial_x\rho \in L^1(\mathbb{R}\times(0,T))$. Miura-Sugiyama in [17] and Kagei-Kawakami-Sugiyama in [18] respectively proved the uniqueness of weak solutions to the model (1.15) with $\alpha = 1$ or 0, with additional Hölder regularity by adapting the duality method coupled with the vanishing viscosity duality method. However, it is still an open problem for the Hölder regularity for the degenerate Keller-Segel equations.

Notice that there exists a weak comparison principle for the radial symmetric solution to the Keller-Segel system with dimension $n \geq 2$ [8, 13]. However, these systems don't have the comparison principle in the classical sense due to the non-local aggregation with a log-Lip singular potential. This kind of log-Lip singularity also appeared in the two dimensional incompressible Euler equation and the uniqueness was proved by Yudovich in 1963 [22] in the class of the bounded L^∞ solution.

Three methods were recently developed to adapt the Yudovich's method to the Keller-Segel models. The first method is the optimal transport method, see Carrillo, Lisini and Mainini in [7], they proved the uniqueness in the class of bounded solution and bounded Fisher information. This method may also be suitable for other systems with the gradient flow structure. The second method is the DiPerna-Lions' renormalizing argument [10], which can give a uniqueness of the solution in the L^p space. The method can be used to equations with linear diffusion or the model with the semigroup structure. Recently, Egana and Mischler proved the uniqueness of the entropy weak solution for the two dimensional Keller-Segel equation [11] in $L^\infty([0,T);L^1(\mathbb{R}^2)) \cap C([0,T);\mathcal{D}'(\mathbb{R}^2))$. However, this method can not be used for the degenerate Keller-Segel system. The final method is the Brownian motion method. The uniqueness for Eqs. (1.15) with $m = 1, \alpha = \beta = 0$ was also proved [16] by using this method. All these three methods are based on Lagrangian coordinates. This note is based on the first method mentioned above and the main result is given below.

Define the space

$$X_2(M_0) := \left\{ \rho \in L^1\big(\mathbb{R}^n\big) : \rho \geq 0, \int_{\mathbb{R}^n} \rho(x)dx = M_0, \int_{\mathbb{R}^n} |x|^2 \rho(x)dx < \infty \right\}$$

with Wasserstein metric W_2.

Theorem 1.1 (Space-Time Uniform L^∞ Estimate and Uniqueness) *Assume that the initial density satisfies Assumption 1 and the sharp condition (1.10), then we have*

(i) *If the initial data $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^n)$, then the model (1.1) has a global weak solution with regularity,*

$$\rho \in L^\infty\big(\mathbb{R}_+; L^1_+\big(\mathbb{R}^n\big) \cap L^\infty\big(\mathbb{R}^n\big)\big).$$

Furthermore, if $\rho_0 \in X_2(M_0)$, then for any $t > 0$, $\rho \in X_2(M_0)$.

(ii) *If $\rho_0,\ u_0 \in X_2(M_0) \cap L^\infty(\mathbb{R}^n)$ and $\rho,\ u$ are two solutions as already obtained in (i) with initial data $\rho_0,\ u_0$, then for any fixed $T > 0$, there exist positive constants C and $\bar{C}$ depending only on T, M_0 and $L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^n))$ norms of ρ and u such that for any $0 < t < T$ the following inequality holds*

$$W_2\big(\rho(\cdot,t), u(\cdot,t)\big) \leq \bar{C} \max\big\{ W_2(\rho_0, u_0), \big(W_2(\rho_0, u_0)\big)^{e^{-Ct}} \big\},$$

which implies that the weak solutions to the model (1.1) are unique and stable with respective to the initial data.

This note is organized as follows. Section 2 gives the uniform L^∞-estimate of the weak solutions to the model (1.1) under Assumption 1 and the sharp condition (1.10). In Section 3, some basic properties to the model (1.1) are given. Using these properties and the L^∞ estimate for the weak solutions, we prove the uniqueness for the weak solutions to the degenerate Keller-Segel equations.

2 Uniform L^∞ Estimate for the Weak Solutions

In this section we will extend the result of Proposition 1.1 to a uniform L^∞ bound by utilizing a bootstrap iterative technique [6] under Assumption 1, the sharp condition (1.10), and $\rho_0 \in L^\infty$.

Lemma 2.1 (The L^{p_k} Estimate) *Assume the initial density $\rho_0 \in L^\infty(\mathbb{R}^n)$, and satisfies Assumption 1 and the sharp condition (1.10). Let (ρ, c) be a weak solution with initial data ρ_0. Let $p_k = 2^k + \frac{2n}{n+2} + 1$ for $k \in \mathbb{N}$. Then we have*

$$\frac{d}{dt}\|\rho\|^{p_k}_{L^{p_k}} \leq -\|\rho\|^{p_k}_{L^{p_k}} + C p_k^{q_2}\Big(\big(\|\rho\|^{p_{k-1}}_{L^{p_{k-1}}}\big)^{\eta_1} + \big(\|\rho\|^{p_{k-1}}_{L^{p_{k-1}}}\big)^{\eta_2}\Big), \quad k = 1, 2, \cdots. \quad (2.1)$$

where C is a fixed constant independent of p_k, and η_1, η_2 are constants satisfying $\eta_1, \eta_2 \leq 2$.

Proof Taking $p_k\rho^{p_k-1}$ as a test function in the first equation of (1.1), we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\rho^{p_k}dx &= -\frac{4p_k m(p_k-1)}{(m+p_k-1)^2}\int_{\mathbb{R}^n}|\nabla\rho^{\frac{m+p_k-1}{2}}|^2dx + (p_k-1)\int_{\mathbb{R}^n}\rho^{p_k+1}dx\\ &\le -2C_1\int_{\mathbb{R}^n}|\nabla\rho^{\frac{m+p_k-1}{2}}|^2dx + p_k\int_{\mathbb{R}^n}\rho^{p_k+1}dx,\end{aligned}$$

where $0 < C_1 \le \frac{2p_k m(p_k-1)}{(m+p_k-1)^2}$ is a fixed constant.

Now we will focus on estimating the last term $\int_{\mathbb{R}^n}\rho^{p_k+1}dx$,

$$\begin{aligned}\int_{\mathbb{R}^n}\rho^{p_k+1}dx &= \|\rho^{\frac{m+p_k-1}{2}}\|_{L^{\frac{2(p_k+1)}{m+p_k-1}}}^{\frac{2(p_k+1)}{m+p_k-1}}\\ &\le G_1^{\alpha\frac{2(p_k+1)}{m+p_k-1}}\|\nabla\rho^{\frac{m+p_k-1}{2}}\|_{L^2}^{\alpha\frac{2(p_k+1)}{m+p_k-1}}\cdot\|\rho^{\frac{m+p_k-1}{2}}\|_{L^r}^{(1-\alpha)\frac{2(p_k+1)}{m+p_k-1}},\end{aligned}\tag{2.2}$$

where $G_1 = S_n^{-\frac{\alpha}{2}}$, S_n is the best constant for the Sobolev inequality, and

$$\frac{m+p_k-1}{2}r = p_{k-1},\qquad \alpha = \frac{\frac{1}{r}-\frac{m+p_k-1}{2(p_k+1)}}{\frac{1}{r}-\frac{n-2}{2n}},\qquad 1-\alpha = \frac{\frac{m+p_k-1}{2(p_k+1)}-\frac{n-2}{2n}}{\frac{1}{r}-\frac{n-2}{2n}}.\tag{2.3}$$

The Young's inequality implies that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\rho^{p_k}dx &\le -2C_1\int_{\mathbb{R}^n}|\nabla\rho^{\frac{m+p_k-1}{2}}|^2dx + \sigma_1\|\nabla\rho^{\frac{m+p_k-1}{2}}\|_{L^2}^{q_1\alpha\frac{2(p_k+1)}{m+p_k-1}}\\ &\quad + C(\sigma_1)(p_k)^{q_2}G_1^{\alpha q_2\frac{2(p_k+1)}{m+p_k-1}}\|\rho^{\frac{m+p_k-1}{2}}\|_{L^r}^{q_2(1-\alpha)\frac{2(p_k+1)}{m+p_k-1}},\end{aligned}\tag{2.4}$$

where $C(\sigma_1) = (\sigma_1 q_1)^{-q_2/q_1}q_2^{-1}$, $q_1 = \frac{m+p_k-1}{\alpha(p_k+1)}$, i.e., $q_1\alpha\frac{2(p_k+1)}{m+p_k-1} = 2$, and

$$q_2 = \frac{m+p_k-1}{(m+p_k-1)-\alpha(p_k+1)} \le 1+n.\tag{2.5}$$

Thus, taking $\sigma_1 = C_1$ in (2.4), we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho^{p_k}dx \le -C_1\int_{\mathbb{R}^n}|\nabla\rho^{\frac{m+p_k-1}{2}}|^2dx + C(\sigma_1)(p_k)^{q_2}G_1^{\alpha q_2\frac{2(p_k+1)}{m+p_k-1}}\left(\|\rho\|_{L^{p_{k-1}}}^{p_{k-1}}\right)^{\eta_1},\tag{2.6}$$

where

$$\eta_1 := \frac{2(p_k+1)q_2(1-\alpha)}{r(m+p_k-1)} = \frac{m-2+\frac{2}{n}(p_k+1)}{m-2+\frac{2}{n}p_{k-1}} \le 2.\tag{2.7}$$

On the other hand,

$$\begin{aligned}\|\rho\|_{L^{p_k}}^{p_k} &= \|\rho^{\frac{m+p_k-1}{2}}\|_{L^{\frac{2p_k}{m+p_k-1}}}^{\frac{2p_k}{m+p_k-1}}\\ &\le G_2^{\frac{2p_k}{m+p_k-1}}\|\nabla\rho^{\frac{m+p_k-1}{2}}\|_{L^2}^{\theta\frac{2p_k}{m+p_k-1}}\|\rho^{\frac{m+p_k-1}{2}}\|_{L^r}^{(1-\theta)\frac{2p_k}{m+p_k-1}},\end{aligned}\tag{2.8}$$

where $G_2 = S_n^{-\frac{\theta}{2}}$, r is the same constant as one in (2.2), and

$$\theta = \frac{\frac{1}{r} - \frac{m+p_k-1}{2p_k}}{\frac{1}{r} - \frac{n-2}{2n}}, \quad 1-\theta = \frac{\frac{m+p_k-1}{2p_k} - \frac{n-2}{2n}}{\frac{1}{r} - \frac{n-2}{2n}}.$$

Similar to (2.4), we have

$$\|\rho\|_{L^{p_k}}^{p_k} \le G_2^{\theta\ell_2 \frac{2p_k}{m+p_k-1}} \bar{C}(\sigma_1)\|\rho\|_{L^{p_{k-1}}}^{p_k((1-\theta))\ell_2} + \sigma_1 \left\|\nabla \rho^{\frac{m+p_k-1}{2}}\right\|_{L^2}^2, \tag{2.9}$$

where $\bar{C}(\sigma_1) = (\sigma_1\ell_1)^{-\ell_2/\ell_1}\ell_2^{-1}$, $\ell_1 = \frac{m+p_k-1}{\theta p_k}$, and $\ell_2 = \frac{m+p_k-1}{(m+p_k-1)-\theta p_k}$. Hence from (2.6) and (2.9), we deduce

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} \rho^{p_k} dx \le &-\int_{\mathbb{R}^n} \rho^{p_k} dx + G_2^{\theta\ell_2 \frac{2p_k}{m+p_k-1}} \bar{C}(\sigma_1)\|\rho\|_{L^{p_{k-1}}}^{p_{k-1}\eta_2} \\ &+ C(\sigma_1) p_k^{q_2} G_1^{\alpha q_2 \frac{2(p_k+1)}{m+p_k-1}} \|\rho\|_{L^{p_{k-1}}}^{p_{k-1}\eta_1}.\end{aligned}$$

where $\eta_2 := \frac{(m+p_k-1)p_k(1-\theta)}{[(m+p_k-1)-\theta p_k]p_{k-1}} = \frac{m-1+\frac{2}{n}p_k}{m-1+\frac{2}{n}p_{k-1}} \le 2$. Define

$$C_1(p_k) := C(\sigma_1) G_1^{\alpha q_2 \frac{2(p_k+1)}{m+p_k-1}}, \quad C_2(p_k) := G_2^{\theta\ell_2 \frac{2p_k}{m+p_k-1}} \bar{C}(\sigma_1).$$

A simple computation gives

$$C_1(p_k) \to C(m,n,M_0), \quad \text{as } p_k \to \infty. \tag{2.10}$$

In fact,

$$C_1(p_k) = \frac{(m+p_k-1)-\alpha(p_k+1)}{m+p_k-1}\left(C_1 \frac{m+p_k-1}{\alpha(p_k+1)}\right)^{-\frac{\alpha(p_k+1)}{(m+p_k-1)-\alpha(p_k+1)}} S_n^{-\frac{\alpha(p_k+1)}{(m+p_k-1)-\alpha(p_k+1)}}.$$

Here $\alpha \sim O(1)$, $1-\alpha \sim O(1)$ as $p_k \to \infty$, and S_n is independent of p_k. So, we can get the limit relation (2.10).

Hence $C_1(p_k)$ is uniformly bounded for any $k \ge 1$. A similar discussion gives that $C_2(p_k)$ is also uniformly bounded. So, let $C(n,m) > 1$ be a common upper bound of $C_1(p_k)$ and $C_2(p_k)$, we obtain the following differential inequality

$$\frac{d}{dt}\|\rho\|_{L^{p_k}}^{p_k} \le -\|\rho\|_{L^{p_k}}^{p_k} + C(n,m) p_k^{q_2}\left(\left(\|\rho\|_{L^{p_{k-1}}}^{p_{k-1}}\right)^{\eta_1} + \left(\|\rho\|_{L^{p_{k-1}}}^{p_{k-1}}\right)^{\eta_2}\right). \qquad \square$$

Proof of (i) of Theorem 1.1 First, we need to estimate

$$y_0(t) := \int_{\mathbb{R}^n} \rho^{2+\frac{2n}{n+2}}(x,t)dx = \int_{\mathbb{R}^n} \rho^{p_0}(x,t)dx.$$

Similar to the proof obtaining (2.1), we have

$$\frac{d}{dt}\|\rho\|_{L^{p_0}}^{p_0} \le -\|\rho\|_{L^{p_0}}^{p_0} + \widetilde{C} p_0^{\widetilde{q}_2}\left(\left(\|\rho\|_{L^{\frac{2n}{n+2}}}^{\frac{2n}{n+2}}\right)^{\widetilde{\eta}_1} + \left(\|\rho\|_{L^{\frac{2n}{n+2}}}^{\frac{2n}{n+2}}\right)^{\widetilde{\eta}_2}\right),$$

where $\widetilde{C}$, $\widetilde{q}_2$ and $\widetilde{\eta}_1$, $\widetilde{\eta}_2$ are constants independent of k. So, by using the uniform upper bound (1.11) of $\|\rho\|_{L^{\frac{2n}{n+2}}}$, in Proposition 1.1, we deduce

$$y_0(t) = \|\rho\|_{L^{p_0}}^{p_0} \leq C(n, m, M_0). \tag{2.11}$$

Let $y_k(t) := \|\rho(\cdot, t)\|_{L^{p_k}}^{p_k}$, solving the differential inequality (2.1), we obtain

$$\begin{aligned} \left(e^t y_k(t)\right)' &\leq C(m, n) p_k^{q_2} \left(y_{k-1}^{\eta_2} + y_{k-1}^{\eta_1}\right) e^t \\ &\leq 2C(m, n) 4^{1+n} 2^{(n+1)k} \max\left\{1, \sup_{t\geq 0} y_{k-1}^2(t)\right\} e^t, \end{aligned} \tag{2.12}$$

where the last inequality used $1 < q_2 \leq n+1$, see (2.5). Let $a_k := 2C(m, n)4^{1+n}2^{(n+1)k} > 1$, $K = \max\{y_k(0), 1\}$ for all $k \geq 1$. Integrating (2.12), it holds that

$$\begin{aligned} y_k(t) &\leq a_k \max\left\{1, \sup_{t\geq 0} y_{k-1}^2(t)\right\}\left(1 - e^{-t}\right) + y_k(0)e^{-t} \\ &\leq a_k \max\left\{1, \sup_{t\geq 0} y_{k-1}^2(t), y_k(0)\right\} \\ &\leq a_k \max\left\{\sup_{t\geq 0} y_{k-1}^2(t), K\right\}, \end{aligned} \tag{2.13}$$

From (2.13) after some iterative steps, we have

$$\begin{aligned} y_k(t) &\leq a_k (a_{k-1})^2 (a_{k-2})^{2^2} \cdots (a_1)^{2^{k-1}} \max\left\{\sup_{t\geq 0} y_0^{2^k}(t), K^{2^{k-1}}\right\} \\ &\leq \left(2C(m, n)\right)^{2^k-1} \left(4^{n+1}\right)^{2^k-1} \left(2^{n+1}\right)^{k+2(k-1)+2^2(k-2)+\cdots+2^{k-1}(k-(k-1))} \\ &\quad \times \max\left\{\sup_{t\geq 0} y_0^{2^k}(t), K^{2^{k-1}}\right\} \\ &= \left(2C(m, n)4^{n+1}\right)^{2^k-1} \left(2^{n+1}\right)^{2\cdot 2^k-k-2} \max\left\{\sup_{t\geq 0} y_0^{2^k}(t), K^{2^{k-1}}\right\}, \end{aligned}$$

Taking the power $1/p_k$ to the above inequality, then

$$\|\rho\|_{L^{p_k}} \leq 2C(m, n) 4^{n+1} 2^{2(n+1)} \max\left\{\sup_{t\geq 0} y_0(t), K\right\}. \tag{2.14}$$

Thus, (2.11) and (2.14) imply that $\|\rho\|_{L^\infty}$ is uniformly bounded.

Now we will prove if $\rho_0 \in X_2(M_0)$, then for any $t > 0$, $\rho \in X_2(M_0)$.

In fact, from [9], there exists a global weak solution to the model (1.1) under Assumption 1 and the sharp condition (1.10). Using a similar proof to the step 7 and the step 11 of [4, Theorem 2.11], we can get the weak solution ρ satisfies mass conservation, and it's second moment satisfies the following equation:

$$\frac{dm_2(t)}{dt} = \left(2n - \frac{2(n-2)}{m-1}\right) \int_{\mathbb{R}^n} \rho^m dx + 2(n-2)\mathcal{F}(\rho).$$

Noticing that the condition $m < 2 - 2/n$ implies the coefficient $2n - \frac{2(n-2)}{m-1} < 0$, we can get that the second moment is bounded for any fixed T. So, the weak solution ρ belongs to the space $X_2(M_0)$. □

3 Stability and Uniqueness of the Weak Solutions

In this section, we use the result on L^∞-bound of the weak solutions ρ obtained in the previous section to prove the uniqueness of the weak solution to the model (1.1) under Assumption 1 and the sharp condition (1.10). Moreover, the stability in the Wasserstein metric is obtained by using the method given in [7]. In fact, all the conditions proposed in [7, Definition 1.1] are true for our model (1.1). For completeness, we will provide the details of the stability proof.

3.1 Preliminaries

Let $\rho_1, \rho_2 \in X_2(M_0)$. Define the W_2-metric between ρ_1 and ρ_2 as

$$W_2(\rho_1, \rho_2) = \left(\int_{\mathbb{R}^n} |x - \tau(x)|^2 \rho_1(x) dx \right)^{1/2}.$$

Here τ is the unique optimal transport map between ρ_1 and ρ_2, $\tau_\sharp \rho_1 = \rho_2$, i.e., for every continuous and bounded function $\phi : \mathbb{R}^n \to \mathbb{R}$, it holds

$$\int_{\mathbb{R}^n} \phi(x) \rho_2(x) dx = \int_{\mathbb{R}^n} \phi\big(\tau(x)\big) \rho_1(x) dx.$$

Now we recall the following two important results for proving the uniqueness. One is the Log-Lipschitz property of ∇c. It was proved by Kato [12] for the 2D case. Carrillo et al. in [7] stated that the generalization to high dimension is almost identical. We recall it in the following lemma.

Lemma 3.1 *Assume that* $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *and* c *satisfies* (1.2), *then we have*

$$|\nabla c(x) - \nabla c(y)| \le C|x - y|\big(1 + \log^- |x - y|\big) \|\rho\|_{L^1 \cap L^\infty}, \tag{3.1}$$

and

$$|\nabla c(x) - \nabla c(y)|^2 \le C^2 \phi\big(|x - y|^2\big) \|\rho\|^2_{L^1 \cap L^\infty}, \tag{3.2}$$

where

$$\phi(r) := \begin{cases} r \log^2 r, & r \le e^{-1-\sqrt{2}}, \\ r + 2(1 + \sqrt{2}) e^{-1-\sqrt{2}}, & r > e^{-1-\sqrt{2}} \end{cases} \tag{3.3}$$

is a continuous and differentiable concave function on $(0, \infty)$.

Following the paper [14, Theorem 2.9], we can get the second key result, which is the estimate on $\|\nabla \bar{c} - \nabla c\|_{L^2}$ in terms of the Wasserstein metric.

Lemma 3.2 *Let* $\bar{\rho},\ \rho \in X_2(M_0)$, *and* $(\bar{\rho}, \bar{c})$ *and* (ρ, c) *satisfy Eq.* (1.2). *Then*

$$\|\nabla \bar{c} - \nabla c\|_{L^2} \le \big(\max\{\|\bar{\rho}\|_{L^\infty}, \|\rho\|_{L^\infty}\}\big)^{1/2} W_2(\bar{\rho}, \rho). \tag{3.4}$$

For completeness to the reader, we will outline proofs of Lemmas 3.1 and 3.2 in the Appendix.

3.2 Stability and Uniqueness

Proposition 3.1 (Stability in Wasserstein Metric) *Let ρ be a weak solution to the problem* (1.1) *with initial data $\rho_0 \in X_2(M_0) \cap L^\infty(\mathbb{R}^n)$. Then the following properties hold*

(i) *For any $\bar{\rho} \in X_2(M_0) \cap L^\infty(\mathbb{R}^n)$, the map $t \to W_2(\rho(\cdot,t), \bar{\rho}(\cdot,t))$ is absolutely continuous and there exists a positive constant C depending only on M_0, $\|\rho\|_\infty$, and $\|\bar{\rho}\|_\infty$ such that for a.e. $t \in (0,+\infty)$, it holds*

$$\frac{1}{2}\frac{d}{dt}W_2^2(\rho,\bar{\rho}) \le \mathcal{F}(\bar{\rho}) - \mathcal{F}(\rho) + C\omega\big(W_2^2(\rho,\bar{\rho})\big), \tag{3.5}$$

where $\phi(r)$ is defined in (3.3), *and*

$$\omega(r) = \sqrt{M_0 r \phi\big(M_0^{-1} r\big)}, \quad \text{for } r > 0. \tag{3.6}$$

(ii) *Let u be another bounded weak solution with the initial condition $u_0 \in X_2(M_0) \cap L^\infty(\mathbb{R}^n)$. Then for any fixed $T > 0$, there exists a ε_0 depending only on T, M_0, $\|\rho\|_\infty$ and $\|\bar{\rho}\|_\infty$ such that*

(a) *if $W_2(\rho_0, u_0) < \varepsilon_0$, we have*

$$W_2\big(\rho(\cdot,t), u(\cdot,t)\big) \le \sqrt{\max\{M_0, 1\}}\big(W_2(\rho_0,u_0)\big)^{e^{-Ct}} \quad \text{for any } t < T; \tag{3.7}$$

(b) *if $W_2(\rho_0,u_0) \ge \varepsilon_0$, then for any $t < T$, the following relation holds*

$$W_2\big(\rho(\cdot,t), u(\cdot,t)\big) \le \bar{C}\max\Big\{W_2(\rho_0,u_0), \big(W_2(\rho_0,u_0)\big)^{e^{-Ct}}\Big\}, \tag{3.8}$$

where C is the same with the case (*i*), *$\bar{C}$ is a constant depending only on T, M_0, $\|\rho\|_\infty$ and $\|\bar{\rho}\|_\infty$.*

Remark 3.1 Notice that the results of Proposition 3.1 were proved in [7] for the Keller-Segel models in any dimension both parabolic-elliptic and fully parabolic. Moreover, their techniques applied equally well for degenerate cases once the L^∞ bounds are known. Here, we include a proof for the sake of completeness.

Proof Step 1. Absolutely continuous.

Since $\rho(x,t)$ is a weak solution to Eq. (1.5) and it has the time regularity (1.13), and the velocity field v satisfies (1.6), then [1, Theorem 8.3.1] implies that $\rho(\cdot,t)$ is an absolutely continuous curve in $X_2(M_0)$. So, from [1, Theorem 8.4.7], we have that for any fixed $\bar{\rho} \in X_2(M_0)$, the derivative of $W_2^2(\rho(\cdot,t),\bar{\rho})$ respect to time t exists almost everywhere and satisfies the following equation

$$\frac{1}{2}\frac{d}{dt}W_2^2\big(\rho(\cdot,t),\bar{\rho}\big) = \int_{\mathbb{R}^n} v(x,t)\cdot\big(x - \tau_1(x)\big)\rho(x,t)dx, \quad \text{for a.e. } t \in [0,T), \tag{3.9}$$

where τ_1 is the optimal map between $\rho(x,t)$ and $\bar{\rho}(x)$, and $v = -\nabla(\frac{m}{m-1}\rho^{m-1} - c)$. Moreover, the Hölder inequality gives

$$\int_0^T \int_{\mathbb{R}^n} v(x,t)\cdot\big(x-\tau_1(x)\big)\rho(x,t)dxdt$$
$$\le \int_0^T \left(\int_{\mathbb{R}^n} |v(x,t)|^2\rho dx\right)^{1/2}\left(\int_{\mathbb{R}^n} |x-\tau_1(x)|^2\rho dx\right)^{1/2} dt.$$

From (1.6) with finite second moments of ρ and $\bar\rho$, we know $\frac{d}{dt}W_2^2(\rho(\cdot,t),\bar\rho)\in L^1([0,T])$. Then the fundamental theorem of Calculus for Lebesgue integrals implies that $W_2^2(\rho(\cdot,t),\bar\rho)$ is absolutely continuous on $[0,T)$.

Step 2. Displacement convexity.

According to [1, Subsection.9.3], we know that the power functional

$$\frac{1}{m-1}\int_{\mathbb{R}^n}\rho^m dx,\quad m\ge 1-\frac{1}{n}$$

has the displacement convexity property, and it is a lower semi-continuous functional. Thus, from [1, pp. 231, (10.1.7)], it holds

$$\int_{\mathbb{R}^n}\bar\rho^m dx-\int_{\mathbb{R}^n}\rho^m dx\ge\int_{\mathbb{R}^n}\xi(x)\cdot\big(\tau_1(x)-x\big)\rho dx, \tag{3.10}$$

where $\xi(x)=m\nabla\rho^{m-1}\in L^2(\mu;\mathbb{R}^n)$ (it was given by (1.12) with $p=m$) belongs to the Fréchet sub-differential of $\int_{\mathbb{R}^n}\rho^m dx$ at $\mu:=\rho dx$.

Hence, (3.9) and (3.10) imply that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}W_2^2\big(\rho(\cdot,t),\bar\rho\big)&=\int_{\mathbb{R}^n}\left(\frac{m}{m-1}\nabla\rho^{m-1}-\nabla c\right)\cdot\big(\tau_1(x)-x\big)\rho(x,t)dx\\&\le\frac{1}{m-1}\int_{\mathbb{R}^n}\big(\bar\rho^m-\rho^m\big)dx-\int_{\mathbb{R}^n}\nabla c(x,t)\cdot\big(\tau_1(x)-x\big)\rho(x,t)dx\\&=\mathcal{F}(\bar\rho)-\mathcal{F}(\rho)+\frac{1}{2}\int_{\mathbb{R}^n}\big(|\nabla\bar c|^2-|\nabla c|^2\big)dx\\&\quad-\int_{\mathbb{R}^n}\nabla c(x,t)\cdot\big(\tau_1(x)-x\big)\rho(x,t)dx.\end{aligned}\tag{3.11}$$

Notice that the following two equalities hold true

(i) Identity

$$|\nabla\bar c|^2-|\nabla c|^2=|\nabla\bar c-\nabla c|^2+2\nabla c\cdot(\nabla\bar c-\nabla c). \tag{3.12}$$

(ii) Since τ_1 is the optimal transport plan between ρ and $\bar\rho$, we can get

$$\begin{aligned}\int_{\mathbb{R}^n}\nabla c\cdot(\nabla\bar c-\nabla c)dx&=\int_{\mathbb{R}^n}(c\bar\rho-c\rho)dx\\&=\int_{\mathbb{R}^n}\Big(c\big(\tau_1(x),t\big)-c(x,t)\Big)\rho dx\\&=\int_{\mathbb{R}^n}\int_0^1\frac{d}{ds}c\big(\tau_s(x),t\big)ds\rho dx\\&=\int_{\mathbb{R}^n}\int_0^1\nabla c\big(\tau_s(x),t\big)\cdot\big(\tau_1(x)-x\big)ds\rho dx.\end{aligned}\tag{3.13}$$

So, (3.11), (3.12) and (3.13) give that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}W_2^2\big(\rho(\cdot,t),\bar\rho\big)&\le\mathcal{F}(\bar\rho)-\mathcal{F}(\rho)+\frac{1}{2}\int_{\mathbb{R}^n}\big(|\nabla\bar c-\nabla c|^2\big)dx\\&\quad+\int_{\mathbb{R}^n}\int_0^1\Big(\nabla c\big(\tau_s(x),t\big)-\nabla c(x,t)\Big)\cdot\big(\tau_1(x)-x\big)\rho(x,t)dsdx.\end{aligned}\tag{3.14}$$

Step 3. Log-Lipschitz estimate.

Denote

$$I := \int_{\mathbb{R}^n} \int_0^1 \big(\nabla c\big(\tau_s(x), t\big) - \nabla c(x, t)\big) \cdot \big(\tau_1(x) - x\big) ds \rho dx.$$

Using Lemma 3.1 with the concavity and the increase property of ϕ given by (3.3), we have

$$\begin{aligned} |I| &\leq W_2(\rho, \bar{\rho}) \int_0^1 \left(\int_{\mathbb{R}^n} |\nabla c\big(\tau_s(x), t\big) - \nabla c(x, t)|^2 \rho dx \right)^{1/2} ds \\ &\leq W_2(\rho, \bar{\rho}) \int_0^1 \left(\int_{\mathbb{R}^n} C^2 \phi\big(|\tau_s(x) - x|^2\big) \rho dx \right)^{1/2} ds \\ &\leq \sqrt{M_0} C W_2(\rho, \bar{\rho}) \int_0^1 \sqrt{\phi\big(M_0^{-1} W_2^2(\rho, \rho_s)\big)} ds \\ &\leq \sqrt{M_0} C W_2(\rho, \bar{\rho}) \int_0^1 \sqrt{\phi\big(M_0^{-1} W_2^2(\rho, \bar{\rho})\big)} ds. \end{aligned}$$

The last inequality is valid since for any $s \in [0, 1]$, it holds

$$W_2^2(\rho, \rho_s) = s^2 W_2^2(\rho, \bar{\rho}).$$

Thus by the definition of the function $\omega(x)$ in (3.6), we know that

$$|I| \leq C\omega\big(W_2^2(\rho, \bar{\rho})\big). \tag{3.15}$$

So, from (3.14) together with (3.4) and (3.15), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} W_2^2(\rho, \bar{\rho}) &\leq \mathcal{F}(\bar{\rho}) - \mathcal{F}(\rho) + \frac{1}{2}\max\{\|\rho\|_{L^\infty}, \|\bar{\rho}\|_{L^\infty}\} W_2^2(\rho, \bar{\rho}) + C\omega\big(W_2^2(\rho, \bar{\rho})\big) \\ &\leq \mathcal{F}(\bar{\rho}) - \mathcal{F}(\rho) + C\omega\big(W_2^2(\rho, \bar{\rho})\big), \end{aligned}$$

where the last inequality used the fact that $\omega(r) \geq r$ for any $r > 0$. Hence we obtain (3.5).

Step 4. Stability.

Using (3.5) and [1, Lemma 4.3.4], it holds for a.e. $t \in (0, T)$

$$\begin{aligned} \frac{1}{2}\frac{d}{ds} W_2^2\big(\rho(\cdot, s), u(\cdot, s)\big)\big|_{s=t} &\leq \frac{1}{2}\frac{d}{ds} W_2^2\big(\rho(\cdot, s), u(\cdot, t)\big)\big|_{s=t} + \frac{1}{2}\frac{d}{ds} W_2^2\big(\rho(\cdot, t), u(\cdot, s)\big)\big|_{s=t} \\ &\leq 2C\omega\big(W_2^2\big(\rho(\cdot, t), u(\cdot, t)\big)\big), \end{aligned}$$

i.e.,

$$\frac{d}{dt} W_2^2\big(\rho(\cdot, t), u(\cdot, t)\big) \leq 4C\omega\big(W_2^2\big(\rho(\cdot, t), u(\cdot, t)\big)\big), \quad \text{for a.e., } t \in (0, T), \tag{3.16}$$

where C only depends on $\|\rho\|_\infty$, $\|u\|_\infty$, and the initial mass.

First, we will prove the argument (a). Denote

$$y(t) := W_2^2\big(\rho(\cdot, t), u(\cdot, t)\big),$$

and let

$$\mathcal{M}(s) = \int_s^{M_0 e^{-1-\sqrt{2}}} \frac{1}{\omega(r)} dr,$$

where $\omega(r) = -r \ln \frac{r}{M_0}$. For $0 \le s \le M_0 e^{-1-\sqrt{2}}$, it holds

$$\mathcal{M}(s) = \ln \left| \ln \frac{s}{M_0} \right| - \ln(1+\sqrt{2}). \tag{3.17}$$

Moreover, a simple computation with (3.16) gives

$$\frac{d}{dt}\mathcal{M}\big(y(t)\big) = -\frac{1}{\omega(y(t))}\frac{d}{dt}y(t) \ge -4C,$$

i.e.,

$$-\mathcal{M}\big(y(t)\big) + \mathcal{M}\big(y(0)\big) \le 4Ct, \quad \text{for any } t \in [0, T). \tag{3.18}$$

On the other hand, choose $0 < \varepsilon_0 < 1$ depending only on T and M_0 such that it satisfies

$$0 < \varepsilon_0 < \sqrt{M_0 e^{-1-\sqrt{2}}}, \tag{3.19}$$

and

$$0 < \varepsilon_0 < \left(\frac{M_0 e^{-1-\sqrt{2}}}{\max\{M_0, 1\}} \right)^{\frac{1}{2}e^{4CT}}. \tag{3.20}$$

We first consider the case $y(0) < \varepsilon_0^2$. Since $y(t)$ is absolutely continuous in t, and $y(0) < \varepsilon_0^2$, (3.19) implies that $y(t) < M_0 e^{-1-\sqrt{2}}$ for $t \ll 1$. Assume T_1 is the first time such that $y(T_1) = M_0 e^{-1-\sqrt{2}}$, then for any $t \in [0, T_1]$, (3.17) and (3.18) give that $y(t)$ satisfies

$$y(t) \le M_0^{1-e^{-4Ct}} y(0)^{e^{-4Ct}} \le \max\{M_0, 1\} y(0)^{e^{-4Ct}}, \quad \text{for any } t \in [0, T_1]. \tag{3.21}$$

From (3.21), we know that

$$M_0 e^{-1-\sqrt{2}} = y(T_1) \le \max\{M_0, 1\} y(0)^{e^{-4CT_1}}.$$

That is

$$T_1 \ge -\frac{1}{4C} \ln\left(\ln \frac{M_0 e^{-1-\sqrt{2}}}{\max\{M_0, 1\}} \Big/ \ln\big(y(0)\big) \right) > T,$$

where the last inequality used the condition (3.20) and $y(0) < \varepsilon_0^2$. Thus for any fixed $T > 0$, we have

$$y(t) \le \max\{M_0, 1\}\big(y(0)\big)^{e^{-Ct}}, \quad \text{for any } t \in [0, T), \tag{3.22}$$

This gives (3.7).

Now we prove the case $y(0) \ge \varepsilon_0^2$. If $y(0) \ge \varepsilon_0^2$, there exist two subcases: (i) for any $t \in [0, T)$, $y(t) \ge \varepsilon_0^2$; (ii) there exists a t_0 such that $y(t_0) < \varepsilon_0^2$.

For the subcase (i), by the definition of $\omega(r)$, we know that there exists a $C(\varepsilon_0)$ such that $\omega(r) \le C(\varepsilon_0) r$ for $r \ge \varepsilon_0^2$. Hence using (3.16), we get

$$\frac{d}{dt}y(t) \le 4C\omega\big(y(t)\big) \le C(\varepsilon_0) y(t),$$

which implies

$$y(t) \le y(0)e^{C(\varepsilon_0)t} \le C(\varepsilon_0, T)y(0), \quad \text{for any } t \in [0, T). \tag{3.23}$$

For the subcase (ii), there exists a t_0 such that $y(t_0) < \varepsilon_0^2$. Choosing t_0 as a initial time, a similar process to the case $y(0) < \varepsilon_0^2$ gives the following inequality

$$y(t) \le \max\{M_0, 1\}\big(y(t_0)\big)^{e^{-C(t-t_0)}}, \quad \text{for any } t \in [t_0, T).$$

Noticing $0 \le y(t_0) < \varepsilon_0^2 < 1$, we have

$$y(t) \le \max\{M_0, 1\}\big(\varepsilon_0^2\big)^{e^{-Ct}} \le \max\{M_0, 1\}\big(y(0)\big)^{e^{-Ct}}, \quad \text{for any } t \in [0, T). \tag{3.24}$$

Hence (3.23) and (3.24) imply (3.8). □

Proof of Theorem 1.1 (i) of Theorem 1.1 was proved in Section 2. (ii) of Theorem 1.1 is a directly consequence of Proposition 3.1. □

Appendix

For the convenience of the reader, this section will outline proofs of Lemmas 3.1 and 3.2.

Proof of Lemma 3.1 From (1.2), we know that

$$\begin{aligned} |\nabla c(x) - \nabla c(y)| &= \left| -\frac{1}{n\alpha(n)} \int_{\mathbb{R}^n} \left(\frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right) \rho dz \right| \\ &\le \frac{1}{n\alpha(n)} \int_{\mathbb{R}^n} \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right| \rho dz. \end{aligned} \tag{4.1}$$

Let $r = |x-y|$. If $r \ge 1$, using the boundedness of $\|\rho\|_{L^1}$ and $\|\rho\|_{L^\infty}$, we know (3.1) is true. Next we need only to consider the case $0 < r < 1$. By dividing the region in (4.1), we have

$$\begin{aligned} &|\nabla c(x) - \nabla c(y)| \\ &\le \frac{1}{n\alpha(n)} \left(\int_{|x-z|\ge 2r} \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right| \rho dz + \int_{|x-z|<2r} \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right| \rho dz \right) \\ &=: I_1 + I_2. \end{aligned}$$

For I_2, from the boundedness of $\|\rho\|_{L^\infty}$, we have

$$\begin{aligned} I_2 &= \frac{1}{n\alpha(n)} \int_{|x-z|<2r} \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right| \rho dz \\ &\le C\|\rho\|_{L^\infty} \left(\int_{|x-z|<2r} \frac{1}{|x-z|^{n-1}} dz + \int_{|y-z|<3r} \frac{1}{|y-z|^{n-1}} dz \right) \\ &\le C\|\rho\|_{L^1\cap L^\infty} r. \end{aligned}$$

For I_1, we have directly

$$
\begin{aligned}
I_1 &= \frac{1}{n\alpha(n)} \int_{|x-z|\geq 2r} \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right| \rho dz \\
&\leq C \int_{|x-z|\geq 2r} \max\left\{ \frac{1}{|x-z|^n}, \frac{1}{|y-z|^n} \right\} |x-y| \rho dz \\
&\leq C|x-y| \bigg(\int_{2r\leq |x-z|<2} \max\left\{ \frac{1}{|x-z|^n}, \frac{1}{|y-z|^n} \right\} \rho dz \\
&\qquad + \int_{|x-z|\geq 2} \max\left\{ \frac{1}{|x-z|^n}, \frac{1}{|y-z|^n} \right\} \rho dz \bigg) \\
&\leq Cr \left(\|\rho\|_{L^\infty} \left(\int_{2r\leq |x-z|<2} \frac{1}{|x-z|^n} dz + \int_{r\leq |y-z|<3} \frac{1}{|y-z|^n} dz \right) + M_0 \right) \\
&\leq C\|\rho\|_{L^1\cap L^\infty} r \left(\int_{2r}^{2} \frac{1}{s^n} s^{n-1} ds + \int_r^3 \frac{1}{s^n} s^{n-1} ds \right) + \bar{C}\|\rho\|_{L^1\cap L^\infty} r \\
&\leq C\|\rho\|_{L^1\cap L^\infty} r(1-\log r).
\end{aligned}
$$

In summary, the relation (3.1) holds for any $r>0$, and a simple computation gives (3.2). □

Proof of Lemma 3.2 Let $\rho_s := \tau_{s\sharp}\rho$ be the interpolation along the optimal transport map τ_1 between ρ and $\bar{\rho}$, where $\tau_s = (1-s)I + s\tau_1$. Taking $\psi \in C_c^\infty(\mathbb{R}^n)$, we obtain

$$
\begin{aligned}
&\frac{d}{ds} \int_{\mathbb{R}^n} \psi(x)\rho_s(x)dx \\
&= \frac{d}{ds} \int_{\mathbb{R}^n} \psi\big((1-s)x + s\tau_1(x)\big)\rho(x)dx \\
&= \int_{\mathbb{R}^n} \nabla\psi\big((1-s)x + s\tau_1(x)\big) \cdot \big(\tau_1(x) - x\big)\rho(x)dx \\
&\leq \left(\int_{\mathbb{R}^n} \left|\nabla\psi\big((1-s)x + s\tau_1(x)\big)\right|^2 \rho(x)dx \right)^{1/2} \left(\int_{\mathbb{R}^n} |\tau_1(x) - x|^2 \rho(x)dx \right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^n} |\nabla\psi(x)|^2 \rho_s(x)dx \right)^{1/2} W_2(\rho,\bar{\rho}).
\end{aligned} \tag{4.2}
$$

Integrating (4.2) respect to s from 0 to 1, we get

$$
\int_{\mathbb{R}^n} \psi(x)\big(\bar{\rho}(x) - \rho\big)dx = \int_0^1 \left(\int_{\mathbb{R}^n} |\nabla\psi(x)|^2 \rho_s(x)dx \right)^{1/2} ds\, W_2(\bar{\rho},\rho) \tag{4.3}
$$

$$
\leq \left(\sup_{s\in[0,1]} \|\rho_s\|_{L^\infty} \right)^{1/2} \left(\int_{\mathbb{R}^n} |\nabla\psi(x)|^2 dx \right)^{1/2} W_2(\rho,\bar{\rho}). \tag{4.4}
$$

On the other hand, taking $\psi(x)$ as a test function in the second equation of (1.1), we have

$$
\int_{\mathbb{R}^n} \psi(x)(\bar{\rho} - \rho)dx = -\int_{\mathbb{R}^n} \psi(x)\Delta(\bar{c} - c)dx = \int_{\mathbb{R}^n} \nabla\psi(x) \cdot (\nabla\bar{c} - \nabla c)dx. \tag{4.5}
$$

From (4.3) and (4.5), we can deduce

$$\int_{\mathbb{R}^n} \nabla\psi(x)\cdot(\nabla\bar{c}-\nabla c)dx \le \Big(\sup_{s\in[0,1]}\|\rho_s\|_{L^\infty}\Big)^{1/2}\|\nabla\psi(x)\|_{L^2}W_2(\rho,\bar{\rho}).$$

From [14, Corollary 2.7]), one has $\|\rho_s\|_{L^\infty}\le\max\{\|\bar{\rho}\|_{L^\infty},\|\rho\|_{L^\infty}\}$. Thus

$$\|\nabla\bar{c}-\nabla c\|_{L^2}\le\big(\max\{\|\bar{\rho}\|_{L^\infty},\|\rho\|_{L^\infty}\}\big)^{1/2}W_2(\rho,\bar{\rho}). \qquad \square$$

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