Refined hyper-contractivity and uniqueness for the Keller–Segel equations

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ABSTRACT

This note considers the hyper-contractivity and the uniqueness of the weak solutions to the two dimensional Keller–Segel equations. We prove a refined hyper-contractive property and consequently obtain the uniqueness of global weak solutions provided that the initial data satisfy $\int_{\mathbb{R}^2} \rho_0\,dx < 8\pi$, $\int_{\mathbb{R}^2} |x|\rho_0\,dx < \infty$ and $\int_{\mathbb{R}^2} \rho_0 \log \rho_0\,dx < \infty$. We also extend the result to higher dimensions.

1. Introduction

In this note, we deal with the refined hyper-contractive property and the uniqueness to the following classical two dimensional Keller–Segel equations,

$$\begin{cases} \rho_t = \Delta\rho - \operatorname{div}(\rho\nabla c), & x \in \mathbb{R}^2,\ t > 0, \\ -\Delta c = \rho, & x \in \mathbb{R}^2,\ t > 0, \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where the initial datum $\rho_0 \in L^1_+(\mathbb{R}^2)$. Here ρ denotes the density of cells, c represents the chemical concentration. The nonlocal aggregation comes from the Newtonian potential, i.e.,

$$c = -\frac{1}{2\pi}\log|x| * \rho. \tag{1.2}$$

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The solution of Eqs. (1.1) conserves the mass, i.e., $\int_{\mathbb{R}^2} \rho\, dx \equiv \int_{\mathbb{R}^2} \rho_0\, dx =: M_0$. If $\rho \in L^1_+(\mathbb{R}^2) \cap L\log L(\mathbb{R}^2)$, then the associated free energy of the model (1.1) is given by

$$\mathcal{F}(\rho(\cdot,t)) = \int_{\mathbb{R}^2} \rho \log \rho\, dx - \frac{1}{2}\int_{\mathbb{R}^2} \rho c dx. \tag{1.3}$$

Formally the following entropy-dissipation equality holds

$$\frac{d}{dt}\mathcal{F}(\rho(\cdot,t)) + \int_{\mathbb{R}^2} \rho \Big|\nabla(\log\rho - c)\Big|^2 dx = 0. \tag{1.4}$$

It is well known that $M_c = 8\pi$ is the critical mass for the global existence by utilizing the logarithmic Hardy–Littlewood Sobolev inequality, see [1,2]. If $M_0 < 8\pi$, $\int_{\mathbb{R}^2} \rho_0 \log \rho_0\, dx < \infty$ and $M_k(0) := \int_{\mathbb{R}^2} |x|^k \rho_0\, dx < \infty$ for some real number $k > 0$ (for simplicity, we will take $k = 1$ below), then for any $T > 0$, there exists a global weak solution ρ satisfying the following estimate in [0,T) (c.f. [3, Lemma 2.4])

$$\int_{\mathbb{R}^2} \rho(x,t)|\log\rho(x,t)|\, dx < C_T, \tag{1.5}$$

where the constant C_T depends on T, M_0, the initial entropy $\int_{\mathbb{R}^2} \rho_0 \log\rho_0\, dx$ and $M_k(0)$. If $M_0 > 8\pi$ and the initial second moment is finite, then weak solutions to (1.1) blow up in a finite time. For the case $M_0 = 8\pi$, weak solutions exist globally and tend to a steady state in L^1 norm provided that the initial relative entropy is finite [3].

The main motivation for writing this short note is from some researches on the uniqueness for the 2D-Navier–Stokes equations with L^1 initial vorticity. The 2D-Navier–Stokes equations in the vorticity-stream function formulation read

$$\begin{cases} \omega_t = \Delta\omega - \operatorname{div}(\omega\nabla^{\perp}\psi), & x \in \mathbb{R}^2,\ t > 0, \\ -\Delta\psi = \omega, & x \in \mathbb{R}^2,\ t > 0, \\ \omega(x,0) = \omega_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.6}$$

where the notation $\nabla^{\perp} := (-\partial_{x_2}, \partial_{x_1})$. When initial vorticity $\omega_0 \in L^1$, the global existence is first proved by Giga et al. in [4], and the results are improved in [5] and [6]. Kato [6,7] and Ben-Artzi [5], Brezis [8] proved that the solution is existent and unique if $\omega_0 \in X$, where X is a relatively compact subset in L^1. The following hyper-contractive property plays an important role in the proof of the uniqueness for the 2D-Navier–Stokes equations

$$t^{1-\frac{1}{p}}\|\omega\|_{L^p} \to 0, \quad \text{as } t \to 0. \tag{1.7}$$

Notice that for 2-D Keller–Segel equations (1.1), the standard hyper-contractive estimate is given by (for completeness, see Appendix for a proof)

$$t^{1-\frac{1}{p}}\|\rho\|_{L^p} \le C(p,T), \quad \text{for } 0 < t < T,\ p \ge 1. \tag{1.8}$$

Since (1.8) only shows the bounds, we need a refined hyper-contractivity as (1.7) to ensure the uniqueness for the global weak solutions to the 2D-Keller–Segel equations. This is the task of this note. Our main results are stated below.

Theorem 1.1 (*Refined Hyper-Contractivity*). *Assume initial density $\rho_0 \in L^1_+(\mathbb{R}^2)$, $M_0 < 8\pi$, $\int_{\mathbb{R}^2} \rho_0 \log\rho_0\, dx < \infty$ and $\int_{\mathbb{R}^2} |x|\rho_0\, dx < \infty$, then for any fixed $T > 0$, $0 < \varepsilon < 1$, $q > 1$, there exists some $C = C(T; C_T; q; \varepsilon) > 0$, where C_T is defined in (1.5), such that the following refined hyper-contractive estimate holds*

$$t^{1-\frac{1}{q}}|\log t|^{(1-\varepsilon)/q}\|\rho\|_{L^q} \le C, \quad \textit{for any } t \in (0,1]. \tag{1.9}$$

Combining (1.9) with the standard semigroup theory and some uniform estimates (3.9) and (3.10), we have the following uniqueness theorem.

Theorem 1.2 (*Global Existence and Uniqueness*)**.** *Assume initial density* $\rho_0 \in L^1_+(\mathbb{R}^2)$, $M_0 < 8\pi$, $\int_{\mathbb{R}^2} \rho_0 \log \rho_0 \, dx < \infty$ *and* $\int_{\mathbb{R}^2} |x|\rho_0 \, dx < \infty$. *Then there is a unique global weak solution for the Keller–Segel equations* (1.1).

Recently, there are many results on the uniqueness for the Keller–Segel equations using the classical PDE theory (see papers [9–11]), and using the Lagrangian coordinates method (refer to [12–15]). However these uniqueness results are all in the class of the bounded solutions except works [13,16]. In [13], Egaña and Mischler proved the uniqueness of the entropy weak solution for the 2D Keller–Segel equations in $L^\infty([0,T); L^1 \cap L \log L(\mathbb{R}^2)) \cap C([0,T); \mathcal{D}'(\mathbb{R}^2))$ by DiPerna–Lions renormalizing argument. In [16], Bedrossian and Masmoudi prove a local existence and uniqueness of mild solutions for initial measure only satisfying $\max_{x\in\mathbb{R}} \mu\{x\} < 8\pi$. This note provides a simple proof of the uniqueness by utilizing the refined hyper-contractive result in (1.9). This method is generalized to the higher dimension Keller–Segel equations in Section 4.

2. Refined hyper-contractivity

Proof of Theorem 1.1. Using interpolation inequality for $\int_{\mathbb{R}^2} \rho^q \, dx$ and taking $p = \frac{q}{1-\varepsilon} > q$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \rho^q \, dx &= \int_{\mathbb{R}^2} \rho^{q/p}(1+|\log\rho|)^{q/p} \rho^{q-q/p}(1+|\log\rho|)^{-q/p} \, dx \\ &\le \Big(\int_{\mathbb{R}^2} \rho(1+|\log\rho|) \, dx \Big)^{q/p} \Big(\int_{\mathbb{R}^2} \rho^{\frac{q(p-1)}{p-q}} (1+|\log\rho|)^{-\frac{q}{p-q}} \, dx \Big)^{\frac{p-q}{p}}. \end{aligned} \tag{2.1}$$

Hence, (2.1) and (1.5) imply

$$\|\rho\|_{L^q} \le (M_0 + C_T)^{1/p} \Big(\int_{\mathbb{R}^2} \rho^{\frac{q(p-1)}{p-q}} (1+|\log\rho|)^{-\frac{q}{p-q}} \, dx \Big)^{(p-q)/qp}. \tag{2.2}$$

Noticing $\frac{q(p-1)}{p-q} > 1$ due to $q > 1$, we know that there is a universal constant $\tilde{C}$ such that for any $R \ge 1$, $0 < r \le R$, it holds

$$\frac{r^{\frac{q(p-1)}{p-q}-1}}{(1+|\log r|)^{\frac{q}{p-q}}} \le \tilde{C} \frac{R^{\frac{q(p-1)}{p-q}-1}}{(1+\log R)^{\frac{q}{p-q}}}. \tag{2.3}$$

Indeed, there is an R_0 such that for any $R \ge R_0$ and $r \le R$

$$\frac{r^{\frac{q(p-1)}{p-q}-1}}{(1+|\log r|)^{\frac{q}{p-q}}} \le \frac{R^{\frac{q(p-1)}{p-q}-1}}{(1+\log R)^{\frac{q}{p-q}}}. \tag{2.4}$$

Hence if $R_0 \le 1$, we directly obtain (2.3) with $\tilde{C} = 1$. If $R_0 > 1$, there is a C_{R_0} such that for any $1 \le R \le R_0$, $0 < r < R$

$$\frac{r^{\frac{q(p-1)}{p-q}-1}}{(1+|\log r|)^{\frac{q}{p-q}}} \le \frac{R_0^{\frac{q(p-1)}{p-q}-1}}{(1+\log R_0)^{\frac{q}{p-q}}} \le C_{R_0} \frac{R^{\frac{q(p-1)}{p-q}-1}}{(1+\log R)^{\frac{q}{p-q}}}. \tag{2.5}$$

Here the first inequality is from (2.4) with $R = R_0$ and $C_{R_0} = R_0^{\frac{q(p-1)}{p-q}-1}$. Combining (2.4) and (2.5), we can obtain (2.3) by taking $\tilde{C} = \max\{1, C_{R_0}\}$. Applying (2.3), we get

$$\begin{aligned} & t^{\frac{q(p-1)}{p-q}-1} \int_{\mathbb{R}^2} \rho^{\frac{q(p-1)}{p-q}} (1+|\log\rho|)^{-\frac{q}{p-q}} \, dx \\ &= t^{\frac{q(p-1)}{p-q}-1} \left(\int_{\{\rho<R\}} + \int_{\{\rho\ge R\}} \right) \rho^{\frac{q(p-1)}{p-q}} (1+|\log\rho|)^{-\frac{q}{p-q}} \, dx \end{aligned}$$

$$\le \tilde{C}\, t^{\frac{q(p-1)}{p-q}-1} \frac{R^{\frac{q(p-1)}{p-q}-1}}{(1+|\log R|)^{\frac{q}{p-q}}} \int_{\{\rho<R\}} \rho\, dx + \frac{1}{(1+|\log R|)^{\frac{q}{p-q}}} t^{\frac{q(p-1)}{p-q}-1} \int_{\{\rho\ge R\}} \rho^{\frac{q(p-1)}{p-q}}\, dx$$
$$\le \tilde{C}\, t^{\frac{q(p-1)}{p-q}-1} \frac{R^{\frac{q(p-1)}{p-q}-1}}{(1+|\log R|)^{\frac{q}{p-q}}} M_0 + \frac{C(M_0, C_T, q, \varepsilon)}{(1+|\log R|)^{\frac{q}{p-q}}}$$

where we have used (1.8) in the last inequality.

Let $R = \frac{1}{t}$ in $t \in (0, 1]$, we have

$$t^{\frac{q(p-1)}{p-q}-1} \int_{\mathbb{R}^2} \rho^{\frac{q(p-1)}{p-q}} (1+|\log \rho|)^{-\frac{q}{p-q}}\, dx \le \frac{C(M_0, C_T, q, \varepsilon)}{(\log \frac{1}{t})^{q/(p-q)}}. \tag{2.6}$$

Substituting (2.6) into (2.2), we deduce

$$\|\rho\|_{L^q} \le C(M_0, C_T, q, \varepsilon)\Big(t^{\frac{q(p-1)}{p-q}-1} \int_{\mathbb{R}^2} \rho^{\frac{q(p-1)}{p-q}} (1+|\log \rho|)^{-\frac{q}{p-q}}\, dx\Big)^{(p-q)/qp} t^{-(\frac{q(p-1)}{p-q}-1)\frac{p-q}{pq}}$$
$$\le C(M_0, C_T, q, \varepsilon)|\log t|^{-\frac{1}{p}} t^{-\frac{q-1}{q}}.$$

Noticing that $p = \frac{q}{1-\varepsilon}$, then it holds

$$t^{q-1} |\log t|^{1-\varepsilon} \|\rho\|_q^q \le C(M_0, C_T, q, \varepsilon), \quad t \in (0, 1]. \tag{2.7}$$

This is a complete proof of Theorem 1.1. □

3. Uniqueness of solution

We recall that the heat semigroup operator $e^{t\Delta}$ is defined by $e^{t\Delta} f := G(x,t) * f$, where $G(x,t)$ is the heat kernel in $\mathbb{R}^2$ and is given by $G(x,t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}$. It follows immediately from the Young's inequality for the convolution that

$$\|e^{t\Delta} f\|_{L^p} \le A_{p,q} t^{-(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^q}, \qquad \|\nabla e^{t\Delta} f\|_{L^p} \le B_{p,q} t^{-\frac{1}{2}-(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^q} \tag{3.1}$$

for any $1 \le q \le p \le +\infty$, $f \in L^q(\mathbb{R}^2)$ and all $t > 0$. Here $A_{p,q},\ B_{p,q}$ are two universal constants.

Proof of Theorem 1.2. Let ρ_1 and ρ_2 be two weak solutions to the Keller–Segel equations (1.1) with the same initial data, and let $\bar{\rho} = \rho_1 - \rho_2$ and $\bar{c} = c_1 - c_2$. Then $\bar{\rho}$ and $\bar{c}$ satisfy the following equation

$$\bar{\rho}_t = \Delta \bar{\rho} - \mathrm{div}(\bar{\rho} \nabla c_1) - \mathrm{div}(\rho_2 \nabla \bar{c}).$$

By the fundamental solution representation of the heat equation, we have

$$\bar{\rho} = -\int_0^t \nabla e^{(t-s)\Delta} \cdot (\bar{\rho}(s) \nabla c_1(s)) ds - \int_0^t \nabla e^{(t-s)\Delta} \cdot (\rho_2(s) \nabla \bar{c}(s)) ds =: I_1 + I_2. \tag{3.2}$$

Next we will prove $\bar{\rho} \equiv 0$ in $t \in [0, t_1]$, t_1 is a constant only dependent of M_0 and C_T. For $r > 1$, $q > 1$ and $0 < k < \frac{1}{q}$, denote

$$F_{k,\ r}(t) = \sup_{0<s<t} s^k \|\bar{\rho}\|_{L^r}, \qquad G_{i,\ q}(t) = \sup_{0<s<t} s^{\frac{q-1}{q}} \|\rho_i\|_{L^q}, \quad i = 1,\ 2.$$

A computation for I_1 shows

$$t^k \|I_1\|_{L^r} = t^k \Big\| \int_0^t \nabla e^{(t-s)\Delta} \cdot (\bar{\rho}(s) \nabla c_1(s)) ds \Big\|_{L^r}.$$

By Bochner Theorem (see [17, p. 650]), we know

$$t^k\|I_1\|_{L^r} \le t^k \int_0^t \|\nabla e^{(t-s)\Delta}\cdot(\bar{\rho}(s)\nabla c_1(s))\|_{L^r} ds. \tag{3.3}$$

Hence, for any $1 \le \sigma \le r$, (3.3) together with (3.1) implies

$$t^k\|I_1\|_{L^r} \le B_{r,\sigma} t^k \int_0^t (t-s)^{-\frac{1}{2}-(\frac{1}{\sigma}-\frac{1}{r})}\|\bar{\rho}(s)\nabla c_1(s)\|_{L^\sigma} ds.$$

For any $r' > 1$ satisfying $\frac{1}{\sigma} = \frac{1}{r} + \frac{1}{r'}$, we have

$$t^k\|I_1\|_{L^r} \le B_{r,\sigma} t^k \int_0^t (t-s)^{-\frac{1}{2}-(\frac{1}{\sigma}-\frac{1}{r})}\|\bar{\rho}(s)\|_{L^r}\|\nabla c_1(s)\|_{L^{r'}} ds. \tag{3.4}$$

Using the weak Young's inequality for $\nabla c_1(s) = -\frac{1}{2\pi}\frac{x}{|x|^2} * \rho_1(x,s)$, and taking $r' > 1$ with $1 + \frac{1}{r'} = \frac{1}{2} + \frac{1}{q}$, we deduce

$$\|\nabla c_1(s)\|_{L^{r'}} \le \frac{1}{2\pi}\left\|\frac{x}{|x|^2}\right\|_{L^2_w}\|\rho_1\|_{L^q} \le \bar{C}\|\rho_1\|_{L^q}, \qquad \bar{C} = \frac{1}{2\pi}\left\|\frac{x}{|x|^2}\right\|_{L^2_w}. \tag{3.5}$$

Thus (3.4) and (3.5) tell us

$$\begin{aligned} t^k\|I_1\|_{L^r} &\le C(r,\sigma) t^k \int_0^t (t-s)^{-\frac{1}{q}}\|\bar{\rho}(s)\|_{L^r}\|\rho_1(s)\|_{L^q} ds \\ &\le C(r,\sigma) F_{k,\,r}(t) G_{1,\,q}(t) t^k \int_0^t (t-s)^{-\frac{1}{q}} s^{-k-\frac{q-1}{q}} ds. \end{aligned}$$

Let $u = s/t$, we easily know

$$t^k\|I_1\|_{L^r} \le C(r,\sigma) F_{k,\,r}(t) G_{1,\,q}(t) \int_0^1 \frac{1}{(1-u)^{\frac{1}{q}} u^{k+\frac{q-1}{q}}} du. \tag{3.6}$$

Similarly, we obtain

$$t^k\|I_2\|_{L^r} \le C(r,\sigma) F_{k,\,r}(t) G_{2,\,q}(t) \int_0^1 \frac{1}{(1-u)^{\frac{1}{q}} u^{k+\frac{q-1}{q}}} du. \tag{3.7}$$

Using

$$\int_0^1 (1-u)^{-\frac{1}{q}} u^{-(k+\frac{q-1}{q})} du = \mathcal{B}(\alpha,\beta) < \infty,$$

where $\alpha = \frac{1}{q} - k > 0$ due to $0 < k < \frac{1}{q}$, and $\beta = 1 - \frac{1}{q} > 0$ by $q > 1$, one has from (3.2), (3.6), (3.7) and the definition of $F_{k,\,r}(t)$

$$F_{k,\,r}(t) \le C(r,\sigma)\mathcal{B}(\alpha,\beta) F_{k,\,r}(t)\Big(G_{1,\,q}(t) + G_{2,\,q}(t)\Big). \tag{3.8}$$

By (2.7), we know that for any $t \in (0,1]$

$$G_{1,\,q}(t) + G_{2,\,q}(t) = \max_{0\le s\le t} s^{\frac{q-1}{q}}\|\rho_1\|_{L^q} + \max_{0\le s\le t} s^{\frac{q-1}{q}}\|\rho_2\|_{L^q} \le 2C(M_0, C_T, q, \varepsilon)|\log t|^{\frac{\varepsilon-1}{q}}.$$

Hence taking

$$t_1 = \exp\Big\{-\left(4C(r,\sigma)\mathcal{B}(\alpha,\beta)C(M_0,C_T,q,\varepsilon)\right)^{\frac{q}{1-\varepsilon}}\Big\}, \tag{3.9}$$

then for any $0 < t \le t_1$, we have

$$F_{k,\,r}(t) \le \frac{1}{2}F_{k,\,r}(t), \tag{3.10}$$

which implies $F_{k,\,r}(t) \equiv 0$ in the interval $[0, t_1]$.

Finally, since t_1 is a constant only depending on T, q, ε, M_0 and C_T. Taking t_1 as a new initial time, repeating the above process, we have that the model (1.1) has a unique weak solution in $t \in [t_1, 2t_1]$. We can continue this process and obtain a unique global solution in $[0, T)$. That is the proof of Theorem 1.2. □

4. Extension to high dimension case

It is well known that the Keller–Segel model (1.1) is critical for two dimension, and it is super-critical for higher dimension. In two dimension, the critical mass 8π gives sharp condition on the initial mass for the global existence and the finite-time blow-up. For $n \geq 3$ however, to our knowledge, there is not an exact criteria to distinguish the global existence and the finite-time blow-up for (1.1). Some results on global existence and blow-up can be found in [18, Section 5]. For the uniqueness in the class of bounded solutions, see [14]. Notice that the diffusion term and the aggregation term are exactly balanced in the $L^{n/2}$-invariant scaling. In this section, we will show a uniqueness result under a natural condition for the initial data: $\|\rho_0\|_{L^{n/2}} < \frac{8S_n}{n}$, where S_n is the best constant of Sobolev inequality, see [14,18,19]. The main result is as follows

Theorem 4.1 (*Refined Hyper-Contractivity and Uniqueness*). *For any given $0 < \varepsilon \ll 1$, $n \geq 3$, $p \geq \frac{n}{2} + \varepsilon$, if the initial density $\rho_0 \in L^1_+(\mathbb{R}^n)$ and satisfies $\|\rho_0\|_{L^{n/2}} < \frac{4S_n}{p}$, and $\|\rho_0\|_{L^{n/2+\varepsilon}} < \infty$, then there is a unique global weak solution for the Keller–Segel equation in $\mathbb{R}^n$, and there is a $C > 0$ such that the following refined hyper-contractivity holds*

$$t^{p-\frac{n}{2}-\frac{2\varepsilon p}{n+2\varepsilon}} \|\rho\|^p_{L^p} \leq C, \quad \text{for any } t > 0. \tag{4.1}$$

Proof. The proof is similar to the proof for two dimension case. We only give a sketch here. First, a hyper-contractive property can be obtained for any $p \geq \frac{n}{2}$, i.e., if $\|\rho_0\|_{L^{n/2}} < \frac{4S_n}{p}$, then there is a positive constant C only depending on p, n and $\|\rho_0\|_{L^{n/2}}$ such that it holds

$$t^{p-\frac{n}{2}} \|\rho\|^p_{L^p} \leq C.$$

Next, for any given $\varepsilon > 0$, $p \geq \frac{n}{2} + \varepsilon$, if $\|\rho_0\|_{L^{n/2}} < \frac{4S_n}{p}$, and $\|\rho_0\|_{L^{n/2+\varepsilon}} < \infty$, we can show that $\|\rho(\cdot,t)\|_{L^{n/2+\varepsilon}} \leq \|\rho_0\|_{L^{n/2+\varepsilon}}$. Using the boundedness of the $\|\rho(\cdot,t)\|_{L^{n/2+\varepsilon}}$ and following the same method to the proof of Theorem 1.1, one can obtain the refined hyper-contractive property (4.1) for the high dimension Keller–Segel model (1.1). Finally, the proof of the uniqueness is exactly the same as the proof of Theorem 1.2. □

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Appendix

To prove (1.8), for $K > 0$ to be determined later, taking $p(\rho - K)_+^{p-1}$, $p \geq 2$, as a test function in the first equation of Eqs. (1.1), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} (\rho - K)_+^p \, dx = &- \frac{4(p-1)}{p} \int_{\mathbb{R}^n} |\nabla (\rho - K)_+^{\frac{p}{2}}|^2 \, dx + (p-1) \int_{\mathbb{R}^n} (\rho - K)_+^{p+1} \, dx \\ &+ K(2p-1) \int_{\mathbb{R}^n} (\rho - K)_+^p \, dx + pK^2 \int_{\mathbb{R}^n} (\rho - K)_+^{p-1} \, dx. \end{aligned} \tag{A.1}$$

Using the interpolation inequality and the Young's inequality for the last two terms on right-hand side, we deduce

$$\frac{d}{dt}\int_{\mathbb{R}^n}(\rho-K)_+^p\,dx \le -\frac{4(p-1)}{p}\int_{\mathbb{R}^n}|\nabla(\rho-K)_+^{\frac{p}{2}}|^2\,dx + 2(2p-1)\int_{\mathbb{R}^n}(\rho-K)_+^{p+1}\,dx + \Big(K(2p-1)C_1(1/K)+pK^2C_2(1/K^2)\Big)M_0. \tag{A.2}$$

Utilizing (1.5) and the inequality

$$\int_{\mathbb{R}^n}|u|^{2\gamma}\,dx \le \gamma^2\int_{\mathbb{R}^n}|u|^{2(\gamma-1)}\,dx\int_{\mathbb{R}^n}|\nabla u|^2\,dx$$

with $u=(\rho-K)_+^{p/2}$, $\gamma=\frac{p+1}{p}$, we can get

$$2(2p-1)\int_{\mathbb{R}^n}(\rho-K)_+^{p+1}\,dx \le \frac{\bar{C}(p,T)}{\log K}\int_{\mathbb{R}^n}|\nabla(\rho-K)_+^{\frac{p}{2}}|^2\,dx. \tag{A.3}$$

Taking $K=e^{\frac{p\bar{C}(p,T)}{2(p-1)}}$, then (A.2) and (A.3) imply

$$\frac{d}{dt}\int_{\mathbb{R}^n}(\rho-K)_+^p\,dx + \frac{2(p-1)}{p}\int_{\mathbb{R}^n}|\nabla(\rho-K)_+^{\frac{p}{2}}|^2\,dx \le C(p,T). \tag{A.4}$$

On the other hand, by interpolation inequality and (A.3), we have

$$\Big(\int_{\mathbb{R}^n}(\rho-K)_+^p\,dx\Big)^{p/(p-1)} \le C(p,T)\int_{\mathbb{R}^n}|\nabla(\rho-K)_+^{\frac{p}{2}}|^2\,dx. \tag{A.5}$$

Therefore, (A.4) and (A.5) imply

$$\frac{d}{dt}\int_{\mathbb{R}^n}(\rho-K)_+^p\,dx + \widetilde{C}\Big(\int_{\mathbb{R}^n}(\rho-K)_+^p\,dx\Big)^{p/(p-1)} \le C(p,T). \tag{A.6}$$

Solving the ordinary differential inequality (A.6), it holds

$$t^{p-1}\int_{\mathbb{R}^n}(\rho-K)_+^p\,dx \le C(p,T), \quad \text{for any } t\in(0,T). \tag{A.7}$$

Hence for above fixed constant K, (A.7) gives the following hyper-contractive property

$$t^{p-1}\|\rho\|_{L^p}^p \le C(p,T), \quad \text{for any } p\ge 2. \tag{A.8}$$

Finally, we show below hyper-contractivity for the case $1<p<2$. For any $r\ge 2$, a simple interpolation inequality implies

$$\|\rho\|_{L^p} \le \|\rho\|_{L^1}^{1-\theta}\|\rho\|_{L^r}^{\theta}, \tag{A.9}$$

where $\theta=\frac{1-1/p}{1-1/r}$. Thus (A.8) and (A.9) give

$$t^{p-1}\|\rho\|_{L^p}^p \le C(p,T), \quad \text{for any } 1<p<2. \tag{A.10}$$

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