ANALYSIS OF POLYMERIC FLOW MODELS AND RELATED COMPACTNESS THEOREMS IN WEIGHTED SPACES*

XIUQING CHEN† AND JIAN-GUO LIU‡

Abstract. We studied coupled systems of the Fokker–Planck equation and the Navier–Stokes equation modeling the Hookean and the finitely extensible nonlinear elastic (FENE)-type polymeric flows. We proved the continuous embedding and compact embedding theorems in weighted spaces that naturally arise from related entropy estimates. These embedding estimates are shown to be sharp. For the Hookean polymeric system with a center-of-mass diffusion and a superlinear spring potential, we proved the existence of a global weak solution. Moreover, we were able to tackle the FENE model with L^2 initial data for the polymer density instead of the L^∞ counterpart in the literature.

Key words. Fokker–Planck equation, Navier–Stokes equation, polymer, compact embedding theorem, logarithmic Sobolev inequality, Hardy-type inequality, Hookean, FENE

AMS subject classifications. 35Q30, 35K55, 76D05, 65M06

DOI. 10.1137/120887850

1. Introduction. A special class of dilute polymer liquids can be modeled by the coupled system of the Fokker–Planck equation and the incompressible Navier–Stokes equation. Each polymer is represented by two beads connected through an extensible spring. These polymer liquids can be further classified according to the constitutive law of the springs, such as the Hookean dumbbell model and the finitely extensible nonlinear elastic (FENE) dumbbell model.

More precisely, let $\Omega \subset \mathbb{R}^d$ be a macroscopic, bounded physical domain with $\partial\Omega \in C^1$. The polymer distribution function $f(t, \mathbf{x}, \mathbf{n})$ and the fluid velocity $\mathbf{u}(t, \mathbf{x})$ satisfy the following equations (cf. Doi and Edwards [15]):

$$\partial_t f + \nabla_{\mathbf{x}} \cdot (\mathbf{u} f) + \nabla_{\mathbf{n}} \cdot \left(\nabla_{\mathbf{x}} \mathbf{u} \mathbf{n} f - \nabla_{\mathbf{n}} U f\right) = \varepsilon \Delta_{\mathbf{x}} f + \Delta_{\mathbf{n}} f, \tag{1.1}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}} p = \Delta_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} \cdot \sigma, \tag{1.2}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \tag{1.3}$$

where $(t, \mathbf{x}, \mathbf{n}) \in (0, \infty) \times \Omega \times D$, $D \subset \mathbb{R}^d$, p is the pressure, $U = U(|\mathbf{n}|)$ is the spring potential, and σ is the stress (in addition to the usual viscous stress) exerted by the polymer on fluids given by

$$\sigma = \int_D (\nabla_{\mathbf{n}} U \otimes \mathbf{n} - \mathrm{Id})\, f d\mathbf{n}, \tag{1.4}$$

where $\mathrm{Id} \in \mathbb{R}^{d\times d}$ is the unit tensor. Note that σ can be taken as symmetric and trace

*Received by the editors August 13, 2012; accepted for publication (in revised form) February 5, 2013; published electronically May 7, 2013.
http://www.siam.org/journals/sima/45-3/88785.html

†School of Sciences, Beijing University of Posts and Telecommunications, Beijing, 100876 China and Department of Physics and Department of Mathematics, Duke University, Durham, NC 27708 (buptxchen@yahoo.com). This author acknowledges support from the National Science Foundation of China (grant 11101049) and the Research Fund for the Doctoral Program of Higher Education of China (grant 20090005120009).

‡Department of Physics and Department of Mathematics, Duke University, Durham, NC 27708 (jian-guo.liu@duke.edu). This author acknowledges support from the National Science Foundation (NSF) of the USA, grant DMS 10-11738, and financial support from the Mathematical Sciences Center of Tsinghua University.

free,

$$\sigma = \int_D U'(|\mathbf{n}|)\left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{d}\mathrm{Id}\right) f d\mathbf{n}, \tag{1.5}$$

since the difference $\int_D (U'(|\mathbf{n}|)/d - 1) f d\mathbf{n}\,\mathrm{Id}$ can be merged into the pressure term in (1.2). For simplicity of presentation, we have taken all physical parameters to be 1 except for the center-of-mass diffusion coefficient ε.

There are three cases listed below with our results covering the first two cases.

Case 1. The Hookean dumbbell model with $D = \mathbb{R}^d$.

Since in practice, the linear Hookean law with $U(\mathbf{n}) = \frac{1}{2}|\mathbf{n}|^2$ is valid only for small $|\mathbf{n}|$, a superlinear Hookean law should be amended for large $|\mathbf{n}|$. Thus we take the spring potential $U = V(\frac{1}{2}|\mathbf{n}|^2)$, where $V \in W^{2,\infty}_{loc}([0,\infty); \mathbb{R}_{\geq 0})$ is a convex function in $[0,\infty)$ such that for some $s^* > 0$,

$$V(s) = s,\ s\in[0,s^*], \quad \lim_{s\to\infty}\frac{V(s)}{s} = \infty. \tag{1.6}$$

Case 2. The FENE dumbbell model with $D = B := \{\mathbf{n}\in\mathbb{R}^d : |\mathbf{n}| < 1\}$ and $U(\mathbf{n}) = -k\ln(1-|\mathbf{n}|^2)\,(k>0)$.

Case 3. The stiff limit of spring potential, or the inextensible spring.

In other words, we assume that the length of $\mathbf{n}$ is fixed, say $|\mathbf{n}| = 1$. Hence $D = \mathbb{S}^{d-1}$. The $\nabla_{\mathbf{n}} U f$ term in (1.1) is dropped, and the stress can be modeled by $\sigma = \int_{\mathbb{S}^{d-1}}(\mathbf{n}\otimes\mathbf{n} - \frac{1}{d}\mathrm{Id}) f d\mathbf{n}$. This is the Doi model for rod-like particle suspensions (see [15]).

In general, the center-of-mass diffusion coefficient ε is very small and it is often omitted in the mathematics literature (see Barrett and Süli [6] for the discussion of this term). The models with $\varepsilon = 0$ are much more difficult to analyze. We refer to a recent seminal work of Masmoudi [26] for the existence of a global weak solution for the FENE model in this case. For the Doi model with Stokes equation, Constantin [11] established the existence of the global smooth solution on a three dimensional period domain. In a series of papers [12]–[14], the authors proved the existence of the global smooth solution for several cases of coupled Navier–Stokes and Fokker–Planck equations including the Doi equation in both $\mathbb{R}^2$ and $\mathbb{T}^2$. Sun and Zhang [30] discussed some related problems in a two dimensional bounded domain. Using the propagation of compactness, Lions and Masmoudi [24] established the existence of a global weak solution for the Doi model in $\mathbb{T}^d\,(d = 2, 3)$. Recently, based on a quasi-compressible approximation of the pressure, Bae and Trivisa [4] investigated the global existence of weak solutions with a Dirichlet boundary condition in $\mathbb{R}^3$. However, for the Hookean model, the analysis for the case of $\varepsilon = 0$ is still open.

The models with $\varepsilon > 0$ were studied by Barrett and Süli [5], [6]. Barrett and Süli [5] used a cut-off function and a semi-implicit scheme to construct the approximate solutions, then they applied the compactness method to establish the global existence of weak solutions for the FENE model. The compactness argument they used is an improved version of the Dubinskiĭ lemma and Antoci's compactness embedding result (see Lemma 5.2, [1]). Using a similar method, Barrett and Süli [6] considered a special case of a superlinear Hookean dumbbell model, where V is assumed to have a power law growth for large $|\mathbf{n}|$ (see Example 1.2) and they obtained the existence of a global weak solution. In the compactness argument of [6], the authors followed the proof of Theorem 3.1 in Hooton [19] and obtained a similar compact embedding result. In addition, in [5] and [6], they also investigated the exponential decay of weak solutions to the equilibrium solution for the two models.

There are also some related works on the mathematical analysis of the FENE and modified Hookean models in the literature (see [2, 8, 21, 24, 25, 26, 31]). We refer the readers to [26] and the review articles [9], [23] for more references on these two models.

1.1. Initial-boundary problem with $\varepsilon > 0$. Denote the fluid rate-of-strain tensor $\frac{1}{2}(\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^\top)$ and the vorticity tensor $\frac{1}{2}(\nabla_{\mathbf{x}}\mathbf{u} - (\nabla_{\mathbf{x}}\mathbf{u})^\top)$ by E and W, respectively. $W\mathbf{n}$ can be rewritten as $\frac{1}{2}\boldsymbol{\omega} \times \mathbf{n}$, where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity. Then the interaction operator $\Delta_{\mathbf{n}} f + \nabla_{\mathbf{n}} \cdot [\nabla_{\mathbf{n}} U f - \nabla_{\mathbf{x}}\mathbf{u}\mathbf{n} f]$ in (1.1) can be recast as

$$\nabla_{\mathbf{n}} \cdot \left[\nabla_{\mathbf{n}} f + \nabla_{\mathbf{n}} (U + \phi(E)) f - \frac{1}{2}\boldsymbol{\omega} \times \mathbf{n} f\right], \tag{1.7}$$

where $\phi(E) = -\frac{1}{2}\mathbf{n} \cdot E\mathbf{n}$ is the straining potential which, together with the linear potential U, drives the polymer towards low total potential states. The main difficulty in analyzing the Fokker–Planck equation (1.1) is the co-existence of the nonlinear terms (the last two) in (1.7). If $\nabla_{\mathbf{x}}\mathbf{u}$ in (1.1) is replaced by its anti-symmetric part W, then (1.1) is called corotational (see Lions and Masmoudi [24]). In this case, $\phi(E)$ is absent in (1.7) and the problem becomes much simpler.

The linear part of (1.7) (which we shall refer to as the linear Fokker–Planck operator) can be rewritten as

$$\nabla_{\mathbf{n}} \cdot (\nabla_{\mathbf{n}} f + \nabla_{\mathbf{n}} U f) = \nabla_{\mathbf{n}} \cdot \left(M \nabla_{\mathbf{n}} \frac{f}{M}\right), \quad M(\mathbf{n}) := \frac{e^{-U(\mathbf{n})}}{\int_D e^{-U(\mathbf{n})} d\mathbf{n}}. \tag{1.8}$$

Here M is the Maxwellian (also known as the Gibbs measure) for the linear Fokker–Planck operator and is a natural weight function giving rise to the Banach spaces

$$L^p_M(\Omega \times D) := \left\{\varphi \in L^1_{loc}(\Omega \times D) : \|\varphi\|_{L^p_M(\Omega\times D)} < \infty\right\},$$

where

$$\|\varphi\|_{L^p_M(\Omega\times D)} := \left(\int_{\Omega\times D} M|\varphi|^p d\mathbf{n} d\mathbf{x}\right)^{1/p}$$

and $W^{1,p}_M(\Omega \times D) := \{\varphi \in L^1_{loc}(\Omega \times D) : \|\varphi\|_{W^{1,p}_M(\Omega\times D)} < \infty\}$ with

$$\|\varphi\|_{W^{1,p}_M(\Omega\times D)} := \left(\int_{\Omega\times D} M\left(|\varphi|^p + |\nabla_{\mathbf{x}}\varphi|^p + |\nabla_{\mathbf{n}}\varphi|^p\right) d\mathbf{n} d\mathbf{x}\right)^{1/p}.$$

Similarly, one can define the corresponding homogeneous spaces (independent of $\mathbf{x}$) $L^p_M(D)$ and $W^{1,p}_M(D)$, respectively.

We further impose the following initial conditions for (1.1)–(1.4):

$$f|_{t=0} = f_{in}, \quad \mathbf{u}|_{t=0} = \mathbf{u}_{in}, \tag{1.9}$$

and no-flux/no-slip boundary conditions

$$\nabla_{\mathbf{x}} f \cdot \nu|_{\partial\Omega} = 0, \quad \mathbf{u}|_{\partial\Omega} = 0. \tag{1.10}$$

For the FENE model, in addition to (1.9)–(1.10), we also impose the no-flux boundary condition

$$(\nabla_{\mathbf{n}} f + \nabla_{\mathbf{n}} U f - \nabla_{\mathbf{x}}\mathbf{u}\mathbf{n} f) \cdot \mathbf{n}\Big|_{\partial B} = 0, \tag{1.11}$$

on (1.1)–(1.4).

In view of (1.8) and setting $\hat{f} := \frac{f}{M}$, the system (1.1)–(1.4) admits the following relative entropy estimate

$$\frac{d}{dt}\int_\Omega \left(\int_D M\left[\hat{f}\left(\ln\hat{f}-1\right)+1\right]d\mathbf{n}+\frac{1}{2}|\mathbf{u}|^2\right)d\mathbf{x} + 4\int_{\Omega\times D} M\left(\varepsilon\left|\nabla_\mathbf{x}\sqrt{\hat{f}}\right|^2+\left|\nabla_\mathbf{n}\sqrt{\hat{f}}\right|^2\right)d\mathbf{n}d\mathbf{x}+\int_\Omega|\nabla_\mathbf{x}\mathbf{u}|^2d\mathbf{x}=0, \tag{1.12}$$

and its initial-boundary problem can be rewritten as

$$M[\partial_t\hat{f}+\mathbf{u}\cdot\nabla_\mathbf{x}\hat{f}-\varepsilon\Delta_\mathbf{x}\hat{f}]+\nabla_\mathbf{n}\cdot\left(M\nabla_\mathbf{x}\mathbf{u}\mathbf{n}\hat{f}\right)=\nabla_\mathbf{n}\cdot\left(M\nabla_\mathbf{n}\hat{f}\right), \tag{1.13}$$

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla_\mathbf{x})\mathbf{u}-\Delta_\mathbf{x}\mathbf{u}+\nabla_\mathbf{x}p=\nabla_\mathbf{x}\cdot\sigma, \tag{1.14}$$

$$\nabla_\mathbf{x}\cdot\mathbf{u}=0, \tag{1.15}$$

with initial and boundary conditions,

$$M\nabla_\mathbf{x}\hat{f}\cdot\nu|_{\partial\Omega}=0,\ \mathbf{u}|_{\partial\Omega}=0, \tag{1.16}$$

$$\hat{f}|_{t=0}=\frac{f_{in}}{M},\ \mathbf{u}|_{t=0}=\mathbf{u}_{in}. \tag{1.17}$$

For the FENE model, the boundary condition (1.11) translates to

$$\left(M\nabla_\mathbf{n}\hat{f}-M\nabla_\mathbf{x}\mathbf{u}\mathbf{n}\hat{f}\right)\cdot\mathbf{n}|_{\partial B}=0. \tag{1.18}$$

Here σ is given by

$$\sigma:=\int_D M(\nabla_\mathbf{n}U\otimes\mathbf{n}-\mathrm{Id})\hat{f}d\mathbf{n}. \tag{1.19}$$

In addition, by integrating (1.13) over D and letting $\rho := \int_D M\hat{f}d\mathbf{n}$, one has

$$\partial_t\rho+\mathbf{u}\cdot\nabla_\mathbf{x}\rho-\varepsilon\Delta_\mathbf{x}\rho=0, \tag{1.20}$$

which will be used in the uniform estimates for the density in sections 4 and 5.

We will investigate the initial-boundary problem with $\varepsilon > 0$. In the rest of this paper, we take $\varepsilon > 0$ unless otherwise specified.

1.2. Assumption on the spring potential U. To be specific, here we reiterate our requirements for the spring potential U.

(1) The Hookean model.

In the literature of mathematics, one ideal model is called the linear Hookean model, where $U(\mathbf{n}) = \frac{1}{2}|\mathbf{n}|^2$ for any $\mathbf{n} \in \mathbb{R}^d$. The corresponding Maxwellian is therefore $M(\mathbf{n}) = (2\pi)^{-\frac{d}{2}}e^{-\frac{|\mathbf{n}|^2}{2}}$. The model with a superlinear assumption for large $|\mathbf{n}|$, is called the superlinear Hookean model. Since the analysis, especially the compactness argument, does not rely on the spring potential $U(\mathbf{n})$ at bounded domain but depends on the superlinear assumption at far field, for simplicity in presentation, we assume that $U(\mathbf{n}) = V(\frac{1}{2}|\mathbf{n}|^2)$ for any $\mathbf{n} \in \mathbb{R}^d$. The corresponding Maxwellian is given by $M(\mathbf{n}) = \frac{e^{-V(\frac{1}{2}|\mathbf{n}|^2)}}{\int_B e^{-V(\frac{1}{2}|\mathbf{n}|^2)}d\mathbf{n}}$ and $\nabla_\mathbf{n}U(\mathbf{n}) = V'(\frac{1}{2}|\mathbf{n}|^2)\mathbf{n}$. Here $V \in W^{2,\infty}_{loc}([0,\infty);\mathbb{R}_{\geq 0})$ is assumed to be a convex function on $[0,\infty)$ satisfying the superlinear condition (1.6) and the assumption

$$V'(s)\leq e^{V(s)/4}\ \ (\forall s\gg 1). \tag{1.21}$$

Note that the restriction (1.21) is fairly loose. It holds for most of the C^1 superlinear convex functions in the literature. This condition will only be used in Lemma 2.1. It is not needed in the (compactness) embedding theorems (section 3.2).

From (1.6), one has $V'(s) \geq 1$ on $[0,\infty)$ and

$$\frac{V(s)}{s} \text{ is monotonically increasing in } [0,\infty). \tag{1.22}$$

Indeed, it follows directly from the mean value theorem on $[0,s]$ and the convexity of V that there exists $\theta = \theta(s) \in (0,1)$ such that

$$\left(\frac{V(s)}{s}\right)' = \frac{V'(s) - V(s)/s}{s} = \frac{V'(s) - V'(\theta s)}{s} \geq 0$$

and hence (1.22) holds.

The following are two examples of V satisfying (1.6) and (1.21).

Example 1.1.

$$V(s) = \begin{cases} s, & 0 \leq s \leq s^*, \\ s \ln\sqrt{\frac{e}{s^*}s} + \frac{1}{2}s^*, & s \geq s^*. \end{cases}$$

Here $\frac{V(s)}{s} = O(\ln s)$ as $s \to \infty$.

Example 1.2.

$$V(s) = \begin{cases} s, & 0 \leq s \leq s^*, \\ \frac{s^*}{\gamma+1}\left[\left(\frac{s}{s^*}\right)^{\gamma+1} - 1\right] + s^*, & s \geq s^* \end{cases} \quad (\gamma > 0).$$

Here $\frac{V(s)}{s} = O(s^\gamma)$ as $s \to \infty$ (see [6]).

(2) The FENE model.

Let $D = B := \{\mathbf{n} \in \mathbb{R}^d : |\mathbf{n}| < 1\}$ and $U(\mathbf{n}) = -k\ln(1 - |\mathbf{n}|^2)$, $k > 0$. Then the corresponding Maxwellian is $M(\mathbf{n}) = \frac{(1-|\mathbf{n}|^2)^k}{\int_B (1-|\mathbf{n}|^2)^k d\mathbf{n}}$. Since $1 - |\mathbf{n}|^2 = O(1 - |\mathbf{n}|)$ as $|\mathbf{n}| \to 1^-$, for simplicity in presentation, in the analysis of sections 3.1 and 5, we may use $1 - |\mathbf{n}|$ to replace $1 - |\mathbf{n}|^2$ and neglect the normalization constant, i.e.,

$$U(\mathbf{n}) = -k\ln(1 - |\mathbf{n}|)\,(k > 0) \quad\text{and}\quad M(\mathbf{n}) = (1 - |\mathbf{n}|)^k. \tag{1.23}$$

1.3. Main purpose of this paper. For the Hookean model with $\varepsilon > 0$, one of the main difficulties in proving the existence of a global weak solution is the weak compactness of the approximated stress tensors $\{\sigma_k\}$ in $L^1((0,T) \times \Omega)$ $(T > 0)$. In what follows, we outline our strategies. From integration by parts (see Lemma 2.2) and the property $G = G(t,\mathbf{x}) \in \mathbb{R}^{d\times d}$ with $tr(G) = 0$, we have that

$$\int_{\mathbb{R}^d} M\nabla_{\mathbf{n}} U \otimes \mathbf{n}\hat{f} : G d\mathbf{n} = \int_{\mathbb{R}^d} MG\mathbf{n} \cdot \nabla_{\mathbf{n}} \hat{f} d\mathbf{n} = 2\int_{\mathbb{R}^d} MG\mathbf{n}\sqrt{\hat{f}} \cdot \nabla_{\mathbf{n}} \sqrt{\hat{f}} d\mathbf{n}. \tag{1.24}$$

The entropy estimate (1.12) implies that the approximating sequence $\{\sqrt{M}\,\nabla_{\mathbf{n}}\sqrt{\hat{f}_k}\,\}$ is weakly compact in $L^2((0,T) \times \Omega \times \mathbb{R}^d)$. We only need to demonstrate compactness of $\{\sqrt{\hat{f}_k}\,\}$ in $L^2(0,T; L^2_{M(1+|\mathbf{n}|^2)}(\Omega \times \mathbb{R}^d))$. For this, the key point is to prove the compact embedding

$$H^1_M(\Omega \times \mathbb{R}^d) \hookrightarrow\hookrightarrow L^2_{M(1+|\mathbf{n}|^2)}(\Omega \times \mathbb{R}^d). \tag{1.25}$$

For the superlinear Hookean model, the compact embedding (1.25) holds (see Theorem 3.9). However for the linear Hookean model, (1.25) is no longer true (see Theorem 3.15), so the superlinear assumption (1.6) is a sharp condition for the compact embedding in (1.25). Therefore (1.6) is also a natural condition for the Hookean-type condition for the Hookean-type model from the viewpoint of analysis.

In the analysis of the FENE model with $\varepsilon > 0$, we also need some M-weighted compact embedding estimates.

The main contributions of this paper can be summarized as following.

(1) Starting from the relative entropy estimate (1.12), we systematically study the continuous embedding and (non)compact embedding theorems for some weighted spaces in the unit ball B and in $\mathbb{R}^d$, for any space dimension $d \in \mathbb{N}$.

One of the main difficulties in proving compact embedding in (weighted-)L^p spaces lies in obtaining uniform integrability estimates near the singularity of the weight function M at the boundary of D, or for large $|\mathbf{n}|$ when $D = \mathbb{R}^d$. Our key idea is to establish the continuous embedding into other weighted-L^p spaces with a larger weight function. This is done by means of a Hardy-type inequality in a hollow ball for the FENE model. As to the Hookean model, this is done by using the logarithmic Sobolev inequality and the Fenchel–Young inequality.

We should point out that our methods for obtaining the uniform integrability are different from those used in Lemma 5.2 in Antoci [1] and Theorem 3.1 in Hooton [19] as well as Theorem Appendix B.1 in Barrett and Süli [6]. Moreover, most of our compactness embedding results are sharp on the condition for the weight. In this sense, we have improved over the above-mentioned results.

(2) Following the method of Barrett and Süli [5], [6] with some improvements, we establish the global existence of weak solutions for the general superlinear Hookean model in dimension $d = 2, 3, 4$ with $\varepsilon > 0$. Compared with the results in Barrett and Süli [6], our contributions are listed below:

• Our results apply to the general superlinear Hookean model. The only assumptions are (1.6) and (1.21) whereas Barrett and Süli [6] dealt with a special case, where V is defined as in Example 1.2 with a power law growth at infinity.

• For the general superlinear Hookean model, we should point out that both the compact embedding (3.26) in Proposition 3.10 and its proof are quite different from the counterparts in Barrett and Süli [6] (see Appendices B, E, and F of [6]). For the linear Hookean model, the noncompact embedding result (Theorem 3.12) is new. The proof is based on a new Parseval-type identity in some intersection spaces. This noncompact embedding result indicates that the superlinear assumption (1.6) is sharp.

• In the construction of approximate solutions, our cut-off function is motivated by but different from that of Barrett and Süli [6]. First Barrett and Süli [6] used a cut-off only from above by $L > 1$, then they used another cut-off from below by $\delta > 0$. They established the uniform estimates for δ and took the limit $\delta \to 0$. It seems that their whole process is quite involved. However, we used a cut-off function by chopping off from above by $L > 1$ and from below by 0 for the drag term (see Definition 2.4). This single cut-off function is sufficient for the proof of existence for approximate solutions.

• Our a priori estimates for the approximate sequence are uniform in ε and time t, hence the weak solutions exist globally in time. The zero diffusion limit $\varepsilon \to 0$ is an open problem proposed in the recent work of Masmoudi [26]. It will be interesting to see if Masmoudi's $\log^2$-estimate can be carried out for our approximate solutions. We leave this problem for further study.

• In order to apply the time-space compactness theorems with assumptions on derivatives (such as the Aubin–Lions–Simon lemma, see [29],Theorem 5; the Dubinskiĭ lemma, see [7, Theorem 2.1] and [17, Theorem 1]), the traditional Rothe method for evolutionary PDEs (see [28] and [22]) is necessary and requires the construction of linear interpolation functions (also known as Rothe functions). However, the approach of the Rothe functions is fairly indirect and sometimes tedious, requiring more estimates and sometimes even more regularity assumptions on the initial data. In contrast, our approach is to apply Theorem 4.3 of Chen, Jüngel, and Liu [10] and Theorem 1 of Dreher and Jüngel [16], which consist of a nonlinear and a linear time-space compactness theorem with simple piecewise-constant functions of t, instead of the more complicated Rothe functions.

• Our compactness results for the approximate solutions are valid for $d = 2, 3, 4$ which lead to the existence of global weak solutions for the general superlinear Hookean model in $d = 2, 3, 4$ dimensions, while Barrett and Süli [6] only dealt with a special case of the spring potential in two and three space dimensions.

(3) Similarly to the proof of the superlinear Hookean model with $\varepsilon > 0$, we are also able to prove the existence of global weak solutions for the FENE model with L^2 initial data for the polymer density, in contrast to the L^∞ counterpart in Barrett and Süli [5] in both two and three space dimensions.

The rest of the paper is organized as follows. In section 2, we state some preliminary results for our analysis. In section 3, we prove some continuous and (non)-compact embedding theorems for the weighted spaces. Then in section 4, we establish the existence of global weak entropy solutions to the superlinear Hookean dumbbell model with $\varepsilon > 0$ in $d = 2, 3, 4$ dimensions. We use a semi-implicit scheme to construct approximate solutions and show their compactness. In section 5, we prove the existence of global entropy solutions to the FENE dumbbell model with $\varepsilon > 0$ in two and three space dimensions.

2. Preliminaries.

The following notations will be used in this paper:

$$\boldsymbol{L}^p(\Omega) = L^p(\Omega, \mathbb{R}^d),\ \boldsymbol{H}^n(\Omega) = H^n(\Omega, \mathbb{R}^d),\ \boldsymbol{C}_0^\infty(\Omega) = C_0^\infty(\Omega, \mathbb{R}^d),$$
$$\mathscr{V} = \{\mathbf{u} \in \boldsymbol{C}_0^\infty(\Omega) : \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0\},\quad \boldsymbol{H} = \{\mathbf{u} \in \boldsymbol{L}^2(\Omega) : \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{v}|_{\partial\Omega} = 0\},$$
$$\boldsymbol{V} = \{\mathbf{u} \in \boldsymbol{H}_0^1(\Omega) : \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0\},\quad \boldsymbol{V}^n = \boldsymbol{V} \cap \boldsymbol{H}^n(\Omega),$$

where $\mathscr{V}$ is dense in $\boldsymbol{H}$, $\boldsymbol{V}$, and $\boldsymbol{V}^n$. We also use the notations: $X \hookrightarrow Y$ (or $X \hookrightarrow\hookrightarrow Y$) denotes X is continuously (or compactly) embedded in Y; $X \not\hookrightarrow Y$ (or $X \hookrightarrow\not\hookrightarrow Y$) denotes X is not continuously (or continuously but not compactly) embedded in Y. $f_\tau \to (\rightharpoonup \text{ or } \overset{*}{\rightharpoonup}) f$ in X denotes a sequence $\{f_\tau\}_{\tau>0} \subset X$ converges strongly (weakly or weakly star) to f in X as $\tau \to 0$. $D^2 f$ denotes the Hessian matrix of f. $\mathscr{F}$ ($\mathscr{F}^{-1}$) denotes the Fourier's (inversion) transform. If $G \in \mathbb{R}^{d\times d}$ and $\mathbf{n} \cdot G\mathbf{n} \geq \lambda|\mathbf{n}|^2$ for all $\mathbf{n} \in \mathbb{R}^d$, we write $G \geq \lambda$ Id. $C(a, b, \cdots)$ denotes a constant only dependent on $a, b, \ldots$. $[s]$ denotes the maximum integral part of s.

Lemma 2.1. *Let M be the Maxwellian for the superlinear Hookean model; then*

$$\int_{\mathbb{R}^d} M|\mathbf{n}|^p |\nabla_{\mathbf{n}} U(\mathbf{n})|^2 d\mathbf{n} < \infty\ (\forall\, p \geq 0). \tag{2.1}$$

Proof. It follows from (1.21) that

$$\int_{\mathbb{R}^d} M|\mathbf{n}|^p |\nabla_{\mathbf{n}} U(\mathbf{n})|^2 d\mathbf{n} = C \int_{\mathbb{R}^d} \left(e^{-\frac{V\left(\frac{1}{2}|\mathbf{n}|^2\right)}{2}} |\mathbf{n}|^{p+2} \right) \left| e^{-\frac{V\left(\frac{1}{2}|\mathbf{n}|^2\right)}{4}} V'\left(\frac{1}{2}|\mathbf{n}|^2\right) \right|^2 d\mathbf{n}$$
$$\leq C \int_{\mathbb{R}^d} e^{-\frac{V\left(\frac{1}{2}|\mathbf{n}|^2\right)}{2}} |\mathbf{n}|^{p+2}\, d\mathbf{n}.$$

This finishes the proof of Lemma 2.1. ◻

Motivated by Lemma 3.1 in Barrett and Süli [6], one can directly show that the lemma below follows from the density of $C_0^\infty(\mathbb{R}^d)$ in $H_M^1(\mathbb{R}^d)$ and integration by parts.

LEMMA 2.2. *Let M be the Maxwellian for both the linear and superlinear Hookean model. Assume that $\hat{f} \in H_M^1(\mathbb{R}^d)$ and $G \in \mathbb{R}^{d\times d}$ is a constant matrix with $tr(G) = 0$. Then*

$$\int_{\mathbb{R}^d} MG\mathbf{n} \cdot \nabla_{\mathbf{n}} \hat{f} d\mathbf{n} = \int_{\mathbb{R}^d} M\nabla_{\mathbf{n}} U \otimes \mathbf{n}\hat{f} : Gd\mathbf{n}. \tag{2.2}$$

We first recall a definition in Barrett and Süli [7] and Dubinskiĭ [17]. Let B be a Banach space and $M_+ \subset B$. If $\forall\, u \in M_+, \forall c \in [0,\infty)$, $cu \in M_+$, then M_+ is called a nonnegative cone in B. If in addition, there exists a function $[u]_{M_+} : M_+ \to \mathbb{R}$ such that $[u]_{M_+} \geq 0; [u]_{M_+} = 0$ if and only if $u = 0; \forall c \in [0,\infty), [cu]_{M_+} = c[u]_{M_+}$, then M_+ is called a seminormed nonnegative cone in B. The definitions of continuous (or compact) embedding, $L^p([0,T]; M_+)$ and $C([0,T]; M_+)$ are similar to its definition in Banach spaces. We shall use the following time-space compactness lemma for piecewise constant functions in the compactness argument in sections 4 and 5.

LEMMA 2.3 (Chen, Jüngel, and Liu [10, Theorem 4.3]). *Let $T > 0, N \in \mathbb{N}, \tau = \frac{T}{N}$, and $u_\tau(t,\cdot) = u_k$, $t \in ((k-1)\tau, k\tau]$, $k = 1, 2, \ldots, N$. Let B, Y be Banach spaces, M_+ be a seminormed nonnegative cone in B, and let either $1 \leq p < \infty$, $r = 1$ or $p = \infty$, $r > 1$. Assume $M_+ \hookrightarrow\hookrightarrow B \hookrightarrow Y$ and*

$$\{u_\tau\} \text{ is a bounded subset of } L^p(0,T; M_+), \text{then} \tag{2.3}$$

$$\tau^{-1}\|\mathfrak{t}_\tau u_\tau - u_\tau\|_{L^r(0,T-\tau;Y)} \leq C \quad \forall \tau > 0, \tag{2.4}$$

where $\mathfrak{t}_\tau u_\tau(t) := u_\tau(t+\tau)$. If $p < \infty$, then $\{u_\tau\}$ is relatively compact in $L^p([0,T]; B)$; if $p = \infty$, there exists a subsequence of $\{u_\tau\}$ which converges in $L^q([0,T]; B)$ with $1 \leq q < \infty$ to a limit which belongs to $C([0,T]; B)$.

Particularly, letting $M_+ = X$ be a Banach space, Lemma 2.3 becomes the linear compact result, Theorem 1 of Dreher and Jüngel [16].

Define $F(s) := s(\ln s - 1) + 1$, $s \in [0,\infty)$ and some cut-off functions below. These cut-off functions will be used in the approximate problem and the entropy estimate in sections 4 and 5.

DEFINITION 2.4. *Let $L > 1$. Define*

$$E^L(s) := \begin{cases} 0, & \text{if } s \leq 0, \\ s, & \text{if } 0 \leq s \leq L, \\ L, & \text{if } s \geq L; \end{cases} \quad F^L(s) := \begin{cases} s(\ln s - 1) + 1, & 0 \leq s \leq L, \\ \frac{s^2 - L^2}{2L} + s(\ln L - 1) + 1, & s \geq L. \end{cases}$$

With some elementary computations, one could verify the following properties (also see Barrett and Süli [5], [6] for some of them).

LEMMA 2.5. *Let $L > 1$. Then*

$$E^L \in C^{0,1}(\mathbb{R});\ F^L \in C^{2,1}(\mathbb{R}^+) \cap C([0,\infty)), \tag{2.5}$$

$$F^L(s) \geq F(s)\ \forall s \in [0,\infty), \tag{2.6}$$

$$(F^L)''(s) = [E^L(s)]^{-1} \geq s^{-1}\ \forall s \in \mathbb{R}^+, \tag{2.7}$$

$$(F^L)''(s+\alpha) \leq \frac{1}{\alpha}\ \forall \alpha \in (0,1), \forall s \in [0,\infty), \tag{2.8}$$

$$\forall s \in [0,\infty), \lim_{L\to\infty} E^L(s) = s, \tag{2.9}$$

$$F^L(E^L(s) + \alpha) \leq \alpha + \frac{\alpha^2}{2} + F(s+\alpha)\ \forall \alpha \in (0,1),\ \forall s \in [0,\infty). \tag{2.10}$$

The global weak solutions with a finite relative entropy to the superlinear Hookean model with $\varepsilon > 0$ are defined as below.

DEFINITION 2.6. *Let $d = 2, 3, 4$ and M be the Maxwellian for the superlinear Hookean model. Suppose $\mathbf{u}_{in} \in \boldsymbol{H}$ and $f_{in} \in L^\infty(\Omega; L^1(\mathbb{R}^d))$ such that*

$$f_{in} \geq 0 \ a.e.\ \text{on}\ \Omega \times \mathbb{R}^d, \int_{\Omega\times\mathbb{R}^d} \left[f_{in} \left(\ln \frac{f_{in}}{M} - 1 \right) + M \right] d\mathbf{n} d\mathbf{x} < \infty. \tag{2.11}$$

A pair of measurable functions $(\mathbf{u}, f)$ is called a global weak entropy solution of (1.1)–(1.4) *with initial/boundary conditions* (1.9)–(1.10) *if*

$$\mathbf{u} \in L^\infty(0,\infty; \boldsymbol{H}) \cap L^2(0,\infty; \boldsymbol{V}),\ \mathbf{u} \in H^1(0,\infty; (\boldsymbol{V}^{2+[d/2]})'), \tag{2.12}$$

$$f \geq 0 \ a.e.\ \text{on}\ (0,\infty) \times \Omega \times \mathbb{R}^d,$$
$$\int_{\Omega\times\mathbb{R}^d} \left[f(t) \left(\ln \frac{f(t)}{M} - 1 \right) + M \right] d\mathbf{n} d\mathbf{x} < \infty \ a.e.\ \text{on}\ (0,\infty), \tag{2.13}$$

$$\nabla_{\mathbf{x}} \sqrt{f} \in L^2(0,\infty; L^2(\Omega\times\mathbb{R}^d)), M^{\frac{1}{2}} \nabla_{\mathbf{n}} \sqrt{\frac{f}{M}} \in L^2(0,\infty; L^2(\Omega\times\mathbb{R}^d)), \tag{2.14}$$

$$f \in L^\infty((0,\infty)\times\Omega; L^1(\mathbb{R}^d)),\ f \in H^1(0,\infty; (H^{2+d}(\Omega\times\mathbb{R}^d))'); \tag{2.15}$$

for any $\mathbf{v} \in \boldsymbol{C}_0^\infty([0,\infty)\times\Omega)$ with $\nabla_{\mathbf{x}} \cdot \mathbf{v} = 0$,

$$- \int_0^\infty \int_\Omega \mathbf{u} \cdot \partial_t \mathbf{v} d\mathbf{x} dt + \int_0^\infty \int_\Omega \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v} d\mathbf{x} dt + \int_0^\infty \int_\Omega (\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} \cdot \mathbf{v} d\mathbf{x} dt$$
$$= - \int_0^\infty \int_{\Omega\times\mathbb{R}^d} \nabla_{\mathbf{n}} U \otimes \mathbf{n} f : \nabla_{\mathbf{x}} \mathbf{v} d\mathbf{n} d\mathbf{x} dt + \int_\Omega \mathbf{u}_{in}(\mathbf{x}) \cdot \mathbf{v}(0,\mathbf{x}) d\mathbf{x}; \tag{2.16}$$

for any $\varphi \in C_0^\infty([0,\infty)\times\overline{\Omega}\times\mathbb{R}^d)$,

$$- \int_0^\infty \int_{\Omega\times\mathbb{R}^d} f \partial_t \varphi d\mathbf{n} d\mathbf{x} dt + \int_0^\infty \int_{\Omega\times\mathbb{R}^d} (\mathbf{u} \cdot \nabla_{\mathbf{x}} f) \varphi d\mathbf{n} d\mathbf{x} dt$$
$$+ \varepsilon \int_0^\infty \int_{\Omega\times\mathbb{R}^d} \nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{x}} \varphi d\mathbf{n} d\mathbf{x} dt + \int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \nabla_{\mathbf{n}} \frac{f}{M} \cdot \nabla_{\mathbf{n}} \varphi d\mathbf{n} d\mathbf{x} dt$$
$$= \int_0^\infty \int_{\Omega\times\mathbb{R}^d} \nabla_{\mathbf{x}} \mathbf{u} \mathbf{n} f \cdot \nabla_{\mathbf{n}} \varphi d\mathbf{n} d\mathbf{x} dt + \int_{\Omega\times\mathbb{R}^d} f_{in}(\mathbf{x},\mathbf{n}) \varphi(0,\mathbf{x},\mathbf{n}) d\mathbf{n} d\mathbf{x}; \tag{2.17}$$

and $(\mathbf{u}, f)$ satisfies the following energy inequality, for a.e. $t \in [0,\infty)$,

$$\|\mathbf{u}(t)\|^2_{L^2(\Omega)} + 2 \int_{\Omega\times\mathbb{R}^d} \left[f(t) \left(\ln \frac{f(t)}{M} - 1 \right) + M \right] d\mathbf{n} d\mathbf{x} + 2 \int_0^t \|\nabla_{\mathbf{x}} \mathbf{u}(s)\|^2_{L^2(\Omega)} ds$$
$$+ 4 \int_0^t \left(\varepsilon \left\| \nabla_{\mathbf{x}} \sqrt{f(s)} \right\|^2_{L^2(\Omega\times\mathbb{R}^d)} + \left\| M^{\frac{1}{2}} \nabla_{\mathbf{n}} \sqrt{\frac{f(s)}{M}} \right\|^2_{L^2(\Omega\times\mathbb{R}^d)} \right) ds$$
$$\leq \|\mathbf{u}_{in}\|^2_{L^2(\Omega)} + 2 \int_{\Omega\times\mathbb{R}^d} \left[f_{in} \left(\ln \frac{f_{in}}{M} - 1 \right) + M \right] d\mathbf{n} d\mathbf{x}. \tag{2.18}$$

3. Compact embedding theorems for weighted spaces. Arising from the relative entropy estimate of the FENE and the Hookean model, in this section, we will systematically study the continuous embedding and (non)compact embedding theorems for some weighted spaces on the unit ball B and on $\mathbb{R}^d$, for all dimensions $d \in \mathbb{N}$. As mentioned in section 1.3 of the introduction, we note that both the key idea for checking the uniform integrability condition and the results are different from that of Lemma 5.2 in Antoci [1] and Theorem 3.1 in Hooton [19] as well as Theorem Appendix B.1 in Barrett and Süli [6]. Compared with their compactness embedding results, our results were applied to more general weight functions. Indeed, the condition on the weight functions in most of our results was sharp.

3.1. Compactness theorem for weighted spaces on the unit ball. Define $D_r = \{\mathbf{n} \in \mathbb{R}^d : r \leq |\mathbf{n}| < 1\}$ $(0 < r < 1)$.

LEMMA 3.1 (Hardy-type inequality). *Let* $k > 0$, $1 \leq p < \infty$, $d \in \mathbb{N}$, *and* $r_0 = \frac{k+2p(d-1)}{2k+2p(d-1)}$. *Then* $\frac{1}{2} \leq r_0 < 1$ *and* $\forall u \in C^1(\bar{B})$,

$$\int_{D_{r_0}} (1-|\mathbf{n}|)^{k-1}|u|^p d\mathbf{n}$$
$$\leq 2\left(\frac{p}{k}\right)^p \int_{D_{r_0}} (1-|\mathbf{n}|)^{k+p-1}|\nabla_{\mathbf{n}} u|^p d\mathbf{n} + \frac{2p(1-r_0)^k}{k}\int_{|\mathbf{n}|=r_0} |u|^p dS. \tag{3.1}$$

Proof. Let $z(\mathbf{n}) = 1 - |\mathbf{n}|$ in D_{r_0}. Then $|\nabla_{\mathbf{n}} z| = 1$, $\Delta_{\mathbf{n}} z = \frac{1-d}{|\mathbf{n}|}$, and hence

$$\nabla_{\mathbf{n}} \cdot \left(z^k \nabla_{\mathbf{n}} z\right) = kz^{k-1} + z^k \frac{1-d}{|\mathbf{n}|} \tag{3.2}$$

in D_{r_0}. Multiplying (3.2) with $|u|^p$ and integrating over D_{r_0}, we have that

$$\int_{D_{r_0}} |u|^p \nabla_{\mathbf{n}} \cdot \left(z^k \nabla_{\mathbf{n}} z\right) d\mathbf{n}$$
$$= k\int_{D_{r_0}} z^{k-1}|u|^p d\mathbf{n} + (1-d)\int_{D_{r_0}} z^k |u|^p \frac{1}{|\mathbf{n}|} d\mathbf{n}$$
$$= [k+(d-1)]\int_{D_{r_0}} z^{k-1}|u|^p d\mathbf{n} + (1-d)\int_{D_{r_0}} z^{k-1}|u|^p \frac{1}{|\mathbf{n}|} d\mathbf{n}$$
$$\geq \left[k+(d-1)-\frac{d-1}{r_0}\right]\int_{D_{r_0}} z^{k-1}|u|^p d\mathbf{n}. \tag{3.3}$$

Case 1. For $1 < p < \infty$, it follows from integration by parts and the Young inequality

$$ab \leq \eta a^p + (\eta p)^{-\frac{1}{p-1}} \frac{p-1}{p} b^{\frac{p}{p-1}} \ (a, b \geq 0, \eta > 0)$$

with $\eta = \frac{1}{p}\left(\frac{p}{k}\right)^{p-1}$ that

$$\int_{D_{r_0}} |u|^p \nabla_{\mathbf{n}} \cdot \left(z^k \nabla_{\mathbf{n}} z\right) d\mathbf{n}$$
$$= -p\int_{D_{r_0}} |u|^{p-2} u \nabla_{\mathbf{n}} u \cdot (z^k \nabla_{\mathbf{n}} z) d\mathbf{n} + \int_{|\mathbf{n}|=r_0} |u|^p \left(z^k \nabla_{\mathbf{n}} z\right) \cdot \nu dS$$
$$\leq p\int_{D_{r_0}} \left(z^{\frac{k+p-1}{p}}|\nabla_{\mathbf{n}} u|\right)\left(z^{(k-1)\frac{p-1}{p}}|u|^{p-1}\right) d\mathbf{n} + \int_{|\mathbf{n}|=r_0} |u|^p z^k dS$$
$$\leq \left(\frac{p}{k}\right)^{p-1}\int_{D_{r_0}} z^{k+p-1}|\nabla_{\mathbf{n}} u|^p d\mathbf{n} + \frac{p-1}{p}k\int_{D_{r_0}} z^{k-1}|u|^p d\mathbf{n}$$
$$+ (1-r_0)^k \int_{|\mathbf{n}|=r_0} |u|^2 dS. \tag{3.4}$$

Case 2. For $p = 1$, one has

$$\begin{aligned}
&\int_{D_{r_0}} |u|\nabla_{\mathbf{n}} \cdot \left(z^k \nabla_{\mathbf{n}} z\right) d\mathbf{n} \\
&= -\int_{D_{r_0}} |u|^{-1} u \nabla_{\mathbf{n}} u \cdot (z^k \nabla_{\mathbf{n}} z) d\mathbf{n} + \int_{|\mathbf{n}|=r_0} |u| \left(z^k \nabla_{\mathbf{n}} z\right) \cdot \nu dS \\
&\leq \int_{D_{r_0}} z^k |\nabla_{\mathbf{n}} u| d\mathbf{n} + (1-r_0)^k \int_{|\mathbf{n}|=r_0} |u| dS.
\end{aligned}$$

Hence the result of (3.4) also holds for $p = 1$.

We deduce from (3.3) and (3.4) that for $1 \leq p < \infty$, one has

$$\begin{aligned}
&\left[\frac{k + p(d-1)}{p} - \frac{d-1}{r_0}\right] \int_{D_{r_0}} z^{k-1} |u|^2 d\mathbf{n} \\
&\leq \left(\frac{p}{k}\right)^{p-1} \int_{D_{r_0}} z^{k+p-1} |\nabla_{\mathbf{n}} u|^p d\mathbf{n} + (1-r_0)^k \int_{|\mathbf{n}|=r_0} |u|^p dS. \qquad (3.5)
\end{aligned}$$

Since

$$\frac{k + p(d-1)}{p} - \frac{d-1}{r_0} = \frac{k + p(d-1)}{p} \frac{k}{k + 2p(d-1)} > \frac{k}{2p},$$

we have from (3.5) that

$$\frac{k}{2p} \int_{D_{r_0}} z^{k-1} |u|^p d\mathbf{n} \leq \left(\frac{p}{k}\right)^{p-1} \int_{D_{r_0}} z^{k+p-1} |\nabla_{\mathbf{n}} u|^p d\mathbf{n} + (1-r_0)^k \int_{|\mathbf{n}|=r_0} |u|^p dS. \qquad (3.6)$$

This ends the proof of (3.1). ◻

THEOREM 3.2. *Let* $k > 0$, $1 \leq p < \infty$. *Then*

$$W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B) \hookrightarrow L^p_{(1-|\mathbf{n}|)^{k-1}}(B) \quad (\forall d \in \mathbb{N}). \qquad (3.7)$$

Proof. We deduce from Lemma 3.1 and a trace theorem that

$$\int_{D_{r_0}} (1-|\mathbf{n}|)^{k-1} |u|^p d\mathbf{n} \leq C(r_0, p, k) \|u\|^p_{W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B)} \ \forall u \in C^1(\bar{B}), \qquad (3.8)$$

where $r_0 = \frac{k+2p(d-1)}{2k+2p(d-1)}$. Then it follows from

$$\int_{B \setminus D_{r_0}} (1-|\mathbf{n}|)^{k-1} |u|^p d\mathbf{n} \leq (1-r_0)^{-p} \|u\|^p_{L^p_{(1-|\mathbf{n}|)^{k+p-1}}(B)} \ \forall u \in C^1(\bar{B}), \qquad (3.9)$$

that

$$\int_{B} (1-|\mathbf{n}|)^{k-1} |u|^p d\mathbf{n} \leq C(r_0, p, k) \|u\|^p_{W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B)} \ \forall u \in C^1(\bar{B}). \qquad (3.10)$$

Applying the density of $C^1(\bar{B})$ in $W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B)$, we finish the proof of (3.7). ◻

THEOREM 3.3. *Let* $k > 0$, $1 \leq p < \infty$. *Then for any* $\epsilon \in (0, 1)$,

$$W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B) \hookrightarrow\hookrightarrow L^p_{(1-|\mathbf{n}|)^{k-1+\epsilon}}(B) \quad (\forall d \in \mathbb{N}). \qquad (3.11)$$

Proof. For any bounded sequence $\{u_i\}$ in $W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B)$, one has from Theorem 3.2 that for any $r_0 = \frac{k+2p(d-1)}{2k+2p(d-1)} < r < 1$,

$$\begin{aligned}\int_{D_r}(1-|\mathbf{n}|)^{k-1+\epsilon}|u_i|^p d\mathbf{n} &\le (1-r)^\epsilon \int_{D_{r_0}}(1-|\mathbf{n}|)^{k-1}|u_i|^p d\mathbf{n}\\ &\le C(1-r)^\epsilon \|u_i\|^p_{W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B)} \le C(1-r)^\epsilon. \end{aligned} \tag{3.12}$$

One has from the Rellich–Kondrachov theorem that

$$\begin{aligned}&W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B\setminus D_r)\\ &\quad \equiv W^{1,p}(B\setminus D_r) \hookrightarrow\hookrightarrow L^p(B\setminus D_r) \equiv L^p_{(1-|\mathbf{n}|)^{k-1+\epsilon}}(B\setminus D_r).\end{aligned} \tag{3.13}$$

We deduce from the uniform integrability (3.12), (3.13) and the standard diagonal argument that there exists a Cauchy subsequence of $\{u_i\}$ in $L^2_{(1-|\mathbf{n}|)^{k-1+\epsilon}}(B)$ and hence converges there. This ends the proof. (Indeed, (3.12) and (3.13) are enough for us to conclude the proof by applying Theorem 2.4 in Opic [27] directly instead of mentioning the diagonal argument). ∎

Remark 3.4. With a similar proof, we know that Theorems 3.2 and 3.3 hold for any ball centered at the origin. Let $u_\lambda(\mathbf{n}) = \lambda^{\frac{k-1+d}{p}} u(\lambda\mathbf{n})$ $(\lambda > 0)$ and $B_\lambda = \{\mathbf{n} \in \mathbb{R}^d : |\mathbf{n}| < \lambda\}$. Then $\|u_\lambda\|_{L^p_{(1-|\mathbf{n}|)^{k-1}}(B)} = \|u\|_{L^p_{(\lambda-|\mathbf{n}|)^{k-1}}(B_\lambda)}$ and $\|\nabla_{\mathbf{n}} u_\lambda\|_{L^p_{(1-|\mathbf{n}|)^{k+p-1}}(B)} = \|\nabla_{\mathbf{n}} u\|_{L^p_{(\lambda-|\mathbf{n}|)^{k+p-1}}(B_\lambda)}$. These reveal that the (compact) embeddings in Theorems 3.2 and 3.3 may be sharp, which can be proved strictly as below.

Remark 3.5. The compact embedding in Theorem 3.3 is sharp. Indeed, let

$$\varphi_i(\mathbf{n}) = 3^{\frac{(k+p)i}{p}}\left(\frac{1}{3^i} - \left||\mathbf{n}| - 1 + \frac{2}{3^i}\right|\right)\chi_{\left(1-\frac{1}{3^{i-1}},\,1-\frac{1}{3^i}\right)}(|\mathbf{n}|) \ \text{ in } \ B. \tag{3.14}$$

Then $\|\varphi_i\|_{L^p_{(1-|\mathbf{n}|)^{k-1}}(B)} = O(1)$ and it follows from $\forall i \ne j$,

$$\left(1-\frac{1}{3^{i-1}},\,1-\frac{1}{3^i}\right)\bigcap\left(1-\frac{1}{3^{j-1}},\,1-\frac{1}{3^j}\right) = \emptyset$$

that $|\varphi_i - \varphi_j|^p = |\varphi_i|^p + |\varphi_j|^p$ in B and

$$\|\varphi_i - \varphi_j\|^p_{L^p_{(1-|\mathbf{n}|)^{k-1}}(B)} = \|\varphi_i\|^p_{L^p_{(1-|\mathbf{n}|)^{k-1}}(B)} + \|\varphi_j\|^p_{L^p_{(1-|\mathbf{n}|)^{k-1}}(B)} = O(1).$$

Hence $\{\varphi_i\}$ has no convergent subsequence in $L^p_{(1-|\mathbf{n}|)^{k-1}}(B)$. Since

$$\begin{aligned}&|\nabla_{\mathbf{n}}\varphi_i(\mathbf{n})| = 3^{\frac{(k+p)i}{p}}\chi_{\left[1-\frac{1}{3^{i-1}},\,1-\frac{1}{3^i}\right]}(|\mathbf{n}|) \text{ in } B\\ &\text{and } \|\nabla_{\mathbf{n}}\varphi_i\|_{L^p_{(1-|\mathbf{n}|)^{k+p-1}}(B)} = O(1),\end{aligned} \tag{3.15}$$

we have $\|\varphi_i\|_{W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B)} = O(1)$, and hence $\{\varphi_i\}$ is bounded in $W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B)$. So (3.11) does not hold for $\epsilon = 0$. That is, the continuous embedding (3.7) is not compact.

Remark 3.6. The continuous embedding in Theorem 3.2 is sharp. In fact, define $\{\varphi_i\}$ by (3.14). Then $\|\varphi_i\|_{L^p_{(1-|\mathbf{n}|)^{k-1-\epsilon}}(B)} = O(3^{\epsilon i}), \epsilon \in (0,k)$ and

$$\|\varphi_i\|_{W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B)} = O(1).$$

Therefore,

$$W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(B) \not\hookrightarrow L^p_{(1-|\mathbf{n}|)^{k-1-\epsilon}}(B)\ \forall \epsilon \in (0,k). \tag{3.16}$$

The following compact embedding result will be used in the discussions of the FENE model in section 5.

PROPOSITION 3.7. *Let $k > 0$, $1 \le p < \infty$. Then for any $\epsilon \in (0,1)$,*

$$W^{1,p}_{(1-|\mathbf{n}|)^{k+p-1}}(\Omega \times B) \hookrightarrow\hookrightarrow L^p_{(1-|\mathbf{n}|)^{k-1+\epsilon}}(\Omega \times B)\ \ (\forall d \in \mathbb{N}). \tag{3.17}$$

Proof. The proof is similar to that of Theorem 3.3. The only difference is to replace B by $\Omega \times B$ in the proof. ∎

3.2. Compactness theorems for weighted spaces on $\mathbb{R}^d$.

3.2.1. Continuous embedding for both superlinear and linear Hookean Maxwellian weights.

THEOREM 3.8. *Let $2 \le p < \infty$, M be the Maxwellian for both the superlinear and linear Hookean model. Then*

$$W^{1,p}_M(\mathbb{R}^d) \hookrightarrow L^p_{M(1+U(\mathbf{n}))}(\mathbb{R}^d)\ \ (\forall d \in \mathbb{N}). \tag{3.18}$$

Proof. One has $U \in W^{2,\infty}_{loc}(\mathbb{R}^d; \mathbb{R}_{\ge 0})$ and

$$D^2U(\mathbf{n}) = V''\left(\frac{1}{2}|\mathbf{n}|^2\right)\mathbf{n}\otimes\mathbf{n} + V'\mathrm{Id} \ge \mathrm{Id},\ \forall \mathbf{n} \in \mathbb{R}^d.$$

That is, $U(\mathbf{n})$ satisfies a special case of the well-known Bakry–Emery condition (see p. 64, [3]) which implies the following logarithmic Sobolev inequality (see Theorem 5.2, [20] or pp. 76–77, [3])

$$\begin{aligned} &\int_{\mathbb{R}^d} M|\psi|^2 \ln\left(|\psi|^2\right) d\mathbf{n} \\ &\le 2\|\nabla_{\mathbf{n}}\psi\|^2_{L^2_M(\mathbb{R}^d)} + \|\psi\|^2_{L^2_M(\mathbb{R}^d)} \ln\left(\|\psi\|^2_{L^2_M(\mathbb{R}^d)}\right)\ \forall \psi \in H^1_M(\mathbb{R}^d). \end{aligned} \tag{3.19}$$

It follows from (3.19) and the Fenchel–Young inequality,

$$rs \le r(\ln r - 1) + e^s\ \forall r, s \in [0,\infty) \tag{3.20}$$

that for any $\psi \in H^1_M(\mathbb{R}^d)$,

$$\begin{aligned} \int_{\mathbb{R}^d} MU(\mathbf{n})|\psi|^2 d\mathbf{n} &= 2\int_{\mathbb{R}^d} M\frac{U(\mathbf{n})}{2}|\psi|^2 d\mathbf{n} \\ &\le 2\int_{\mathbb{R}^d} M|\psi|^2 \ln\left(|\psi|^2\right) d\mathbf{n} + 2\int_{\mathbb{R}^d} Me^{\frac{U(\mathbf{n})}{2}} d\mathbf{n} \\ &\le C\left[\|\nabla_{\mathbf{n}}\psi\|^2_{L^2_M(\mathbb{R}^d)} + \|\psi\|^2_{L^2_M(\mathbb{R}^d)} \ln\left(\|\psi\|^2_{L^2_M(\mathbb{R}^d)}\right) + 1\right]. \end{aligned} \tag{3.21}$$

For any $\varphi \in W^{1,p}_M(\mathbb{R}^d)$ and $2 < p < \infty$, $\left\||\varphi|^{\frac{p}{2}}\right\|^2_{L^2_M(\mathbb{R}^d)} = \|\varphi\|^p_{L^p_M(\mathbb{R}^d)}$, it follows from the Hölder inequality that

$$\left\|\nabla_{\mathbf{n}}\left(|\varphi|^{\frac{p}{2}}\right)\right\|^2_{L^2_M(\mathbb{R}^d)} = \frac{p}{2}\int_{\mathbb{R}^d} M|\varphi|^{p-2}|\nabla_{\mathbf{n}}\varphi|^2 d\mathbf{n} \le \frac{p}{2}\|\varphi\|^{p-2}_{L^p_M(\mathbb{R}^d)}\|\nabla_{\mathbf{n}}\varphi\|^2_{L^p_M(\mathbb{R}^d)}.$$

Hence $|\varphi|^{\frac{p}{2}} \in H_M^1(\mathbb{R}^d)$ and we take $\psi = |\varphi|^{\frac{p}{2}}$ in (3.21) so that

$$\int_{\mathbb{R}^d} MU(\mathbf{n})|\varphi|^p d\mathbf{n}$$
$$\le C\left[\|\varphi\|^{p-2}_{L^p_M(\mathbb{R}^d)}\|\nabla_{\mathbf{n}}\varphi\|^2_{L^p_M(\mathbb{R}^d)} + \|\varphi\|^p_{L^p_M(\mathbb{R}^d)} \ln\left(\|\varphi\|^p_{L^p_M(\mathbb{R}^d)}\right) + 1\right]. \tag{3.22}$$

Combining (3.21) and (3.22), we deduce that for any $2 \le p < \infty$, $W_M^{1,p}(\mathbb{R}^d)$ is a subset of $L^p_{M(1+U(\mathbf{n}))}(\mathbb{R}^d)$ and

$$\|\varphi\|_{W_M^{1,p}(\mathbb{R}^d)} \le 1 \ \text{ implies } \|\varphi\|_{L^p_{M(1+U(\mathbf{n}))}(\mathbb{R}^d)} \le C. \tag{3.23}$$

Therefore

$$\left\|\frac{\psi}{\|\psi\|_{W_M^{1,p}(\mathbb{R}^d)}}\right\|_{L^p_{M(1+U(\mathbf{n}))}(\mathbb{R}^d)} \le C \ \forall\, \psi \in W_M^{1,p}(\mathbb{R}^d)\setminus\{0\}$$

and hence we have

$$\|\psi\|_{L^p_{M(1+U(\mathbf{n}))}(\mathbb{R}^d)} \le C\|\psi\|_{W_M^{1,p}(\mathbb{R}^d)} \ \forall \varphi \in W_M^{1,p}(\mathbb{R}^d).$$

This finishes the proof of Lemma 3.8. ∎

3.2.2. Compact embedding theorem for the superlinear Hookean Maxwellian weight.

THEOREM 3.9. *Let $2 \le p < \infty$, M be the Maxwellian for the superlinear Hookean model. Then*

$$W_M^{1,p}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p_{M(1+|\mathbf{n}|^2)}(\mathbb{R}^d) \ \ (\forall d \in \mathbb{N}). \tag{3.24}$$

Proof. It follows from Theorem 3.8 and $L^p_{M(1+U(\mathbf{n}))}(\mathbb{R}^d) \hookrightarrow L^p_{M(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$ that

$$W_M^{1,p}(\mathbb{R}^d) \hookrightarrow L^p_{M(1+|\mathbf{n}|^2)}(\mathbb{R}^d). \tag{3.25}$$

Next we show the compactness of this embedding (3.25). Suppose that $\{\varphi_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $W_M^{1,p}(\mathbb{R}^d)$ and hence bounded in $L^p_{M(1+U(\mathbf{n}))}(\mathbb{R}^d)$ in view of (3.18). Let $R \gg 1$, and define $D_R := \{\mathbf{n} \in \mathbb{R}^d : |\mathbf{n}| \ge R\}$. Then we have from (1.22) that

$$\begin{aligned}\int_{D_R} M(1+|\mathbf{n}|^2)|\varphi_i|^p d\mathbf{n} &\le \sup_{|\mathbf{n}|\ge R}\frac{1+|\mathbf{n}|^2}{1+U(\mathbf{n})}\int_{D_R} M(1+U(\mathbf{n}))|\varphi_i|^p d\mathbf{n}\\ &\le 4\frac{(\frac{1}{2}R^2)}{V(\frac{1}{2}R^2)}\int_{\mathbb{R}^d} M(1+U(\mathbf{n}))|\varphi_i|^p d\mathbf{n} \le C\delta(R),\end{aligned}$$

where $\delta(R) = \frac{(\frac{1}{2}R^2)}{V(\frac{1}{2}R^2)}$ and in view of (1.6), $\lim_{R\to\infty}\delta(R) = 0$. Then using the diagonal argument similar to the proof of Theorem 3.3, we obtain (3.24). This finishes the proof. ∎

As we mentioned in the introduction, the key to obtaining the weak compactness for the approximate stress tensors $\{\sigma_k\}$ in $L^1((0,T)\times\Omega)$ for the superlinear Hookean model is the following compact embedding result.

PROPOSITION 3.10. *Let $2 \le p < \infty$, M be the Maxwellian for the superlinear Hookean model. Then*

$$W_M^{1,p}(\Omega\times\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d) \ \ (\forall d \in \mathbb{N}). \tag{3.26}$$

Proof. The proof is similar to that of Theorem 3.9. The only difference is to replace $\mathbb{R}^d$ by $\Omega\times\mathbb{R}^d$ in the proof. ∎

3.2.3. Noncompact embedding theorem in $L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$.

LEMMA 3.11 (Parseval-type identity).

$$\|\psi\|_{H^1(\mathbb{R}^d)\cap L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = \|\mathscr{F}\psi\|_{H^1(\mathbb{R}^d)\cap L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} \quad (\forall d \in \mathbb{N}). \tag{3.27}$$

Here $H^1(\mathbb{R}^d) \cap L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$ *is the intersection space of* $H^1(\mathbb{R}^d)$ *and* $L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$ *with the maximal norm.*

Proof. We have from Plancherel's theorem that

$$\begin{aligned}\|\psi\|_{L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} &= \left\|(1+|\cdot|^2)^{\frac{1}{2}}\psi\right\|_{L^2(\mathbb{R}^d)} \\ &= \left\|\mathscr{F}^{-1}\left\{(1+|\cdot|^2)^{\frac{1}{2}}\mathscr{F}\left(\mathscr{F}^{-1}\psi\right)\right\}\right\|_{L^2(\mathbb{R}^d)} = \|\mathscr{F}^{-1}\psi\|_{H^1(\mathbb{R}^d)}.\end{aligned}$$

It follows from this and $\mathscr{F}^{-1}\psi(\cdot) = \mathscr{F}\psi(-\cdot)$ that

$$\|\psi\|_{H^1(\mathbb{R}^d)} = \|\mathscr{F}\psi\|_{L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = \|\mathscr{F}^{-1}\psi\|_{L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)}$$

and hence

$$\|\psi\|_{L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = \|\mathscr{F}\psi\|_{H^1(\mathbb{R}^d)}. \tag{3.28}$$

Consequently, (3.27) holds. □

THEOREM 3.12.

$$H^1(\mathbb{R}^d) \cap L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d) \hookrightarrow\not\hookrightarrow L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d) \quad (\forall d \in \mathbb{N}). \tag{3.29}$$

Proof. We use the method of contradiction. Suppose that

$$H^1(\mathbb{R}^d) \cap L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d). \tag{3.30}$$

Then we have from (3.27) and (3.28) that

$$H^1(\mathbb{R}^d) \cap L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d) \hookrightarrow\hookrightarrow H^1(\mathbb{R}^d). \tag{3.31}$$

We know that a sequence bounded in $H^1(\mathbb{R}^d)$ with compact support does not necessarily have a convergent subsequence in $H^1(\mathbb{R}^d)$. Therefore (3.31) does not hold. Then the assumption (3.30) is not correct. This ends the proof. □

Remark 3.13. There is also a constructive proof for Theorem 3.12. Indeed, let

$$\varphi_i(\mathbf{n}) = i^{-\frac{d+1}{2}}\left(\frac{1}{2} - \big||\mathbf{n}| - i\big|\right)\chi_{\left(i-\frac{1}{2},\, i+\frac{1}{2}\right)}(|\mathbf{n}|) \text{ in } \mathbb{R}^d. \tag{3.32}$$

Then $\|\varphi_i\|_{L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = O(1)$ and it follows from $\forall i \neq j$,

$$\left(i - \frac{1}{2},\, i + \frac{1}{2}\right) \bigcap \left(j - \frac{1}{2},\, j + \frac{1}{2}\right) = \emptyset$$

that $|\varphi_i - \varphi_j|^2 = |\varphi_i|^2 + |\varphi_j|^2$ in $\mathbb{R}^d$ and

$$\|\varphi_i - \varphi_j\|^2_{L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = \|\varphi_i\|^2_{L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} + \|\varphi_j\|^2_{L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = O(1).$$

Hence $\{\varphi_i\}$ has no convergent subsequence in $L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$. Since

$$|\nabla_{\mathbf{n}}\varphi_i(\mathbf{n})| = i^{-\frac{d+1}{2}}\chi_{\left(i-\frac{1}{2},\, i+\frac{1}{2}\right)}(|\mathbf{n}|) \text{ in } \mathbb{R}^d \text{ and } \|\varphi_i\|_{H^1(\mathbb{R}^d)} = O(i^{-1}), \tag{3.33}$$

we have $\|\varphi_i\|_{H^1(\mathbb{R}^d)\cap L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = O(1)$. Hence $\{\varphi_i\}$ is bounded in $H^1(\mathbb{R}^d) \cap L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$ which has no convergent subsequence in $L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$. So (3.29) holds.

3.2.4. (Non)compact embedding theorem for the linear Hookean Maxwellian weight.

THEOREM 3.14. *Let $2 \le p < \infty$, $\tilde{M}(\mathbf{n}) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|\mathbf{n}|^2}{2}}$, i.e., the Maxwellian for the linear Hookean model. Then for any $\epsilon \in (0,1)$,*

$$W^{1,p}_{\tilde{M}}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^p_{\tilde{M}(1+|\mathbf{n}|^2)^{1-\epsilon}}(\mathbb{R}^d) \ \ (\forall d \in \mathbb{N}). \tag{3.34}$$

Proof. It follows from Theorem 3.8 and $L^p_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d) \hookrightarrow L^p_{\tilde{M}(1+|\mathbf{n}|^2)^{1-\epsilon}}(\mathbb{R}^d)$ that

$$W^{1,p}_{\tilde{M}}(\mathbb{R}^d) \hookrightarrow L^p_{\tilde{M}(1+|\mathbf{n}|^2)^{1-\epsilon}}(\mathbb{R}^d). \tag{3.35}$$

For the bounded sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ in $W^{1,p}_{\tilde{M}}(\mathbb{R}^d)$, we have

$$\int_{D_R} \tilde{M}(1+|\mathbf{n}|^2)^{1-\epsilon}|\varphi_i|^p d\mathbf{n}d\mathbf{x} \le \frac{1}{(1+R^2)^\epsilon}\int_{\mathbb{R}^d} \tilde{M}(1+|\mathbf{n}|^2)|\varphi_i|^p d\mathbf{n}d\mathbf{x} \le C\frac{1}{(1+R^2)^\epsilon}.$$

Then by applying a similar discussion to that in Theorem 3.3, we finish the proof. ◻

THEOREM 3.15. *Let $\tilde{M}(\mathbf{n}) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|\mathbf{n}|^2}{2}}$, i.e. the Maxwellian for the linear Hookean model. Then*

$$H^1_{\tilde{M}}(\mathbb{R}^d) \hookrightarrow\not\hookrightarrow L^2_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d) \ \ (\forall d \in \mathbb{N}).$$

Proof. We have from Theorem 3.8 that

$$H^1_{\tilde{M}}(\mathbb{R}^d) \hookrightarrow L^2_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d) \tag{3.36}$$

and hence the space equivalence

$$H^1_{\tilde{M}}(\mathbb{R}^d) \equiv H^1_{\tilde{M}}(\mathbb{R}^d) \cap L^2_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d), \tag{3.37}$$

where $H^1_{\tilde{M}}(\mathbb{R}^d) \cap L^2_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$ denotes the intersection space of $H^1_{\tilde{M}}(\mathbb{R}^d)$ and $L^2_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$ with maximal norm.

Next we prove that the embedding (3.36) is not compact by contradiction. Assume that

$$H^1_{\tilde{M}}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^2_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d), \tag{3.38}$$

then it follows from (3.37) that

$$H^1_{\tilde{M}}(\mathbb{R}^d) \cap L^2_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^2_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d). \tag{3.39}$$

Letting $\psi := \sqrt{\tilde{M}}\varphi$ and noting

$$\sqrt{\tilde{M}}\nabla_{\mathbf{n}}\varphi = \nabla_{\mathbf{n}}\left(\sqrt{\tilde{M}}\varphi\right) + \frac{\mathbf{n}}{2}\left(\sqrt{\tilde{M}}\varphi\right) = \nabla_{\mathbf{n}}\psi + \frac{\mathbf{n}}{2}\psi, \tag{3.40}$$

we deduce that

$$\|\varphi\|_{L^2_{\tilde{M}(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = \|\psi\|_{L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)}$$

and

$$\|\varphi\|_{H^1_{\tilde M}(\mathbb{R}^d)\cap L^2_{\tilde M(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} \le C \ \text{ if and only if } \ \|\psi\|_{H^1(\mathbb{R}^d)\cap L^2_{(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} \le C.$$

Hence (3.39) implies (3.30). This contradicts (3.29). This ends the proof of Theorem 3.15. □

Remark 3.16. Theorem 3.15 shows that Theorem 3.9 with $p = 2$ holds only for M with the superlinear assumption (1.6) at far field, while for the Maxwellian corresponding to the linear Hookean model, Theorem 3.9 with $p = 2$ does not hold. Moreover, Theorem 3.14 with $p = 2$ does not hold for $\epsilon = 0$. Therefore, both the compactness results in Theorem 3.9 and Theorem 3.14 are sharp in the case $p = 2$.

Remark 3.17. There is a constructive proof of Theorem 3.15 as well. We only need to find a bounded sequence in $H^1_{\tilde M}(\mathbb{R}^d)$ which has no convergent subsequence in $L^2_{\tilde M(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$. More precisely, let

$$\varphi_i(\mathbf{n}) = i^{-\frac{d+1}{2}} e^{\frac{|\mathbf{n}|^2}{4}} \left(\frac{1}{2} - \big||\mathbf{n}| - i\big|\right) \chi_{\left(i-\frac{1}{2},\, i+\frac{1}{2}\right)}(|\mathbf{n}|) \ \text{ in } \ \mathbb{R}^d. \tag{3.41}$$

Then $\|\varphi_i\|_{L^2_{\tilde M(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = O(1)$ and $\|\varphi_i - \varphi_j\|_{L^2_{\tilde M(1+|\mathbf{n}|^2)}(\mathbb{R}^d)} = O(1)$. Hence $\{\varphi_i\}$ has no convergent subsequence in $L^2_{\tilde M(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$. Since

$$\begin{aligned} &|\nabla_{\mathbf{n}}\varphi_i(\mathbf{n})| \\ &= i^{-\frac{d+1}{2}} e^{\frac{|\mathbf{n}|^2}{4}} \left|\operatorname{sign}\{i - |\mathbf{n}|\} + \frac{|\mathbf{n}|}{2}\left(\frac{1}{2} - \big||\mathbf{n}| - i\big|\right)\right| \chi_{\left(i-\frac{1}{2},\, i+\frac{1}{2}\right)}(|\mathbf{n}|) \text{ in } \mathbb{R}^d \end{aligned} \tag{3.42}$$

and $\|\nabla_{\mathbf{n}}\varphi_i\|_{L^2_{\tilde M}(\mathbb{R}^d)} = O(1)$, we have $\{\varphi_i\}$ is bounded in $H^1_{\tilde M}(\mathbb{R}^d)$. So the sequence $\{\varphi_i\}$ is the example needed to show that there is a bounded sequence in $H^1_{\tilde M}(\mathbb{R}^d)$ which has no convergent subsequence in $L^2_{\tilde M(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$.

Remark 3.18. The continuous embedding $H^1_{\tilde M}(\mathbb{R}^d) \hookrightarrow L^2_{\tilde M(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$ in Theorem 3.15 is sharp. In fact, define $\{\varphi_i\}$ by (3.41), then $\|\varphi_i\|_{L^2_{\tilde M(1+|\mathbf{n}|^2)^{1+\epsilon}}(\mathbb{R}^d)} = O(i^\epsilon)$, $\epsilon \in (0, 1)$, and $\|\varphi_i\|_{H^1_{\tilde M}(\mathbb{R}^d)} = O(1)$. Hence

$$H^1_{\tilde M}(\mathbb{R}^d) \not\hookrightarrow L^2_{\tilde M(1+|\mathbf{n}|^2)^{1+\epsilon}}(\mathbb{R}^d) \ \forall \epsilon \in (0, 1).$$

4. Global existence of weak entropy solutions for the superlinear Hookean model. In this section, following the method of Barrett and Süli [5], [6] with some improvements, we establish the global existence of weak solutions for the general superlinear Hookean model with $\varepsilon > 0$. We refer the reader to section 1.3 of the introduction for a summary of our contributions. Throughout this section, let M be the Maxwellian for the superlinear Hookean model.

First, we use a semi-implicit scheme to construct a sequence of approximate solutions. In this construction, we apply the Leray–Schauder fixed point theorem and cut-off techniques to prove the existence of the solution to the discrete problem. Then we use compactness to show that these constructed approximate solutions have a subsequence which converges to a weak solution.

Now we state our main result.

THEOREM 4.1. *Let* $d = 2, 3, 4$ *and* M *be the Maxwellian for the superlinear Hookean model. Suppose* $\mathbf{u}_{in} \in \boldsymbol{H}$ *and* $f_{in} \in L^\infty(\Omega; L^1(\mathbb{R}^d))$ *such that* $f_{in} \ge 0$ *a.e. on* $\Omega \times \mathbb{R}^d$, $\int_{\Omega\times\mathbb{R}^d}[f_{in}(\ln \frac{f_{in}}{M} - 1) + M] d\mathbf{n} d\mathbf{x} < \infty$. *Then there exists a global weak entropy solution* $(\mathbf{u}, f)$ *of* (1.1)–(1.4) *with initial/boundary conditions* (1.9)–(1.10).

4.1. Approximate problem. In the construction of the approximate problem, a cut-off function chopping off above by some $L > 1$ and chopping off below by 0 is used to ensure the boundedness of the linear functional (4.5) and (4.7) for the discrete Fokker–Planck equation required by the Lax–Milgram theorem, and the boundedness estimates for the existence of fixed point solutions needed by the Leray–Schauder fixed point theorem. Using this effective cut-off, we obtain the existence of weak solutions in $\boldsymbol{V} \times H^1_M$ for the approximate problem; then by applying the standard method for the resulting elliptic equation we get the nonnegativity of approximate distribution functions.

For any fixed $0 < \tau \ll 1$ and for any $k \in \mathbb{N}$, given $(\mathbf{u}_{k-1}, \hat{f}_{k-1})$, the approximate problem with cut-off reads

$$\begin{aligned}
&\int_\Omega \frac{\mathbf{u}_k - \mathbf{u}_{k-1}}{\tau} \cdot \mathbf{v} d\mathbf{x} + \int_\Omega \nabla_\mathbf{x} \mathbf{u}_k : \nabla_\mathbf{x} \mathbf{v} d\mathbf{x} + \int_\Omega (\mathbf{u}_{k-1} \cdot \nabla_\mathbf{x}) \mathbf{u}_k \cdot \mathbf{v} d\mathbf{x} \\
&\quad = - \int_{\Omega \times \mathbb{R}^d} M \nabla_\mathbf{n} U \otimes \mathbf{n} \hat{f}_k : \nabla_\mathbf{x} \mathbf{v} d\mathbf{n} d\mathbf{x} \ \forall \mathbf{v} \in \boldsymbol{V};
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\int_{\Omega \times \mathbb{R}^d} M \frac{\hat{f}_k - \hat{f}_{k-1}}{\tau} \varphi d\mathbf{n} d\mathbf{x} + \int_{\Omega \times \mathbb{R}^d} M (\mathbf{u}_{k-1} \cdot \nabla_\mathbf{x} \hat{f}_k) \varphi d\mathbf{n} d\mathbf{x} \\
&+ \varepsilon \int_{\Omega \times \mathbb{R}^d} M \nabla_\mathbf{x} \hat{f}_k \cdot \nabla_\mathbf{x} \varphi d\mathbf{n} d\mathbf{x} + \int_{\Omega \times \mathbb{R}^d} M \nabla_\mathbf{n} \hat{f}_k \cdot \nabla_\mathbf{n} \varphi d\mathbf{n} d\mathbf{x} \\
&= \int_{\Omega \times \mathbb{R}^d} M \nabla_\mathbf{x} \mathbf{u}_k \mathbf{n} E^{\tau^{-\frac{1}{4}}} (\hat{f}_k) \cdot \nabla_\mathbf{n} \varphi d\mathbf{n} d\mathbf{x} \ \forall \varphi \in H^1_M(\Omega \times \mathbb{R}^d).
\end{aligned} \tag{4.2}$$

Remark 4.2. We note that (4.2) implies a weak formulation of the discrete (1.20), saying for any $\varrho \in H^1(\Omega)$,

$$\int_\Omega \frac{\rho_k - \rho_{k-1}}{\tau} \varrho d\mathbf{x} + \int_\Omega (\mathbf{u}_{k-1} \cdot \nabla_\mathbf{x} \rho_k) \varrho d\mathbf{x} + \varepsilon \int_\Omega \nabla_\mathbf{x} \rho_k \cdot \nabla_\mathbf{x} \varrho d\mathbf{x} = 0, \tag{4.3}$$

where $\rho_k = \int_{\mathbb{R}^d} M \hat{f}_k d\mathbf{n}$, $k = 0, 1, 2, \ldots$. Our uniform estimates below are based on (4.3).

DEFINITION 4.3.

$$Z := \Big\{ \hat{f} \in L^2_M(\Omega \times \mathbb{R}^d) : \hat{f} \geq 0 \ a.e. \ on \ \Omega \times \mathbb{R}^d \Big\}. \tag{4.4}$$

PROPOSITION 4.4. *Let* $(\mathbf{u}_{k-1}, \hat{f}_{k-1}) \in \boldsymbol{V} \times Z$. *Then there exists* $(\mathbf{u}_k, \hat{f}_k) \in \boldsymbol{V} \times (Z \cap H^1_M(\Omega \times \mathbb{R}^d))$ *which solves* (4.1)–(4.2).

Proof. Step 1. Let $\hat{f}^* \in L^2_M(\Omega \times \mathbb{R}^d)$. We claim that there exists a unique element $\mathbf{u} \in \boldsymbol{V}$ such that

$$a(\mathbf{u}, \mathbf{v}) = A(\hat{f}^*)(\mathbf{v}) \ \forall \mathbf{v} \in \boldsymbol{V}, \tag{4.5}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v} d\mathbf{x} + \tau \int_\Omega \nabla_\mathbf{x} \mathbf{u} : \nabla_\mathbf{x} \mathbf{v} d\mathbf{x} + \tau \int_\Omega (\mathbf{u}_{k-1} \cdot \nabla_\mathbf{x}) \mathbf{u} \cdot \mathbf{v} d\mathbf{x} \ \forall \mathbf{u}, \mathbf{v} \in \boldsymbol{V};$$

$$A(\hat{f}^*)(\mathbf{v}) = \int_\Omega \mathbf{u}_{k-1} \cdot \mathbf{v} d\mathbf{x} - \tau \int_{\Omega \times \mathbb{R}^d} M \nabla_\mathbf{n} U \otimes \mathbf{n} \hat{f}^* : \nabla_\mathbf{x} \mathbf{v} d\mathbf{n} d\mathbf{x} \ \forall \mathbf{v} \in \boldsymbol{V}.$$

In fact, noting that $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $\nabla_\mathbf{x} \cdot \mathbf{u}_{k-1} = 0$, we have

$$\left| \int_\Omega (\mathbf{u}_{k-1} \cdot \nabla_\mathbf{x}) \mathbf{u} \cdot \mathbf{v} d\mathbf{x} \right| \leq \|\mathbf{u}_{k-1}\|_{L^4(\Omega)} \|\nabla_\mathbf{x} \mathbf{u}\|_{L^2(\Omega)} \|\mathbf{v}\|_{L^4(\Omega)} \leq C \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)}$$

and $\int_\Omega (\mathbf{u}_{k-1}\cdot\nabla_\mathbf{x})\mathbf{u}\cdot\mathbf{u}d\mathbf{x}=0$. Then $a(\cdot,\cdot)$ is a bounded, coercive bilinear functional on $\boldsymbol{V}$. It follows from Lemma 2.1 and the Hölder inequality that

$$\begin{aligned}&\left\|\int_{\mathbb{R}^d} M\nabla_\mathbf{n}U\otimes\mathbf{n}\hat{f}^*d\mathbf{n}\right\|_{L^2(\Omega)}\\ &\le C\left\|\||\mathbf{n}||\nabla_\mathbf{n}U|\|_{L^2_M(\mathbb{R}^d)}\left\|\hat{f}^*\right\|_{L^2_M(\mathbb{R}^d)}\right\|_{L^2(\Omega)}\le C\left\|\hat{f}^*\right\|_{L^2_M(\Omega\times\mathbb{R}^d)}.\end{aligned}\tag{4.6}$$

Thus $A(\hat{f}^*)\in\boldsymbol{V}'$. Therefore, by the Lax–Milgram theorem, we finish the proof of Step 1.

Step 2. We prove that for such $\hat{f}^*\in L^2_M(\Omega\times\mathbb{R}^d)$ and solution $\mathbf{u}\in\boldsymbol{V}$ in (4.5), there exists a unique element $\hat{f}\in H^1_M(\Omega\times\mathbb{R}^d)$ such that

$$b(\hat{f},\varphi)=B(\hat{f}^*,\mathbf{u})(\varphi)\ \forall\varphi\in H^1_M(\Omega\times\mathbb{R}^d),\tag{4.7}$$

where

$$\begin{aligned}b(\hat{f},\varphi)=&\int_{\Omega\times\mathbb{R}^d}M\hat{f}\varphi d\mathbf{n}d\mathbf{x}+\tau\int_{\Omega\times\mathbb{R}^d}M\left(\varepsilon\nabla_\mathbf{x}\hat{f}\cdot\nabla_\mathbf{x}\varphi+\nabla_\mathbf{n}\hat{f}\cdot\nabla_\mathbf{n}\varphi\right)d\mathbf{n}d\mathbf{x}\\ &+\tau\int_{\Omega\times\mathbb{R}^d}M\left(\mathbf{u}_{k-1}\cdot\nabla_\mathbf{x}\hat{f}\right)\varphi d\mathbf{n}d\mathbf{x}\ \forall\hat{f},\varphi\in H^1_M(\Omega\times\mathbb{R}^d),\\ B(\hat{f}^*,\mathbf{u})(\varphi)=&\int_{\Omega\times\mathbb{R}^d}M\hat{f}_{k-1}\varphi d\mathbf{n}d\mathbf{x}\\ &+\tau\int_{\Omega\times\mathbb{R}^d}M\nabla_\mathbf{x}\mathbf{u}\mathbf{n}E^{\tau^{-\frac14}}(\hat{f}^*)\cdot\nabla_\mathbf{n}\varphi d\mathbf{n}d\mathbf{x},\ \forall\varphi\in H^1_M(\Omega\times\mathbb{R}^d)\end{aligned}$$

in which $E^{\tau^{-\frac14}}$ is the cut-off function given by Definition 2.4.

Indeed, since $H^1_M(\Omega\times\mathbb{R}^d)\hookrightarrow H^1(\Omega;L^2_M(\mathbb{R}^d))\hookrightarrow L^4(\Omega;L^2_M(\mathbb{R}^d))$, we have

$$\begin{aligned}\left|\int_{\Omega\times\mathbb{R}^d}M(\mathbf{u}_{k-1}\cdot\nabla_\mathbf{x}\hat{f})\varphi d\mathbf{n}d\mathbf{x}\right|&\le\|\mathbf{u}_{k-1}\|_{L^4(\Omega)}\|\nabla_\mathbf{x}\hat{f}\|_{L^2(\Omega;L^2_M(\mathbb{R}^d))}\|\varphi\|_{L^4(\Omega;L^2_M(\mathbb{R}^d))}\\ &\le C\|\hat{f}\|_{H^1_M(\Omega\times\mathbb{R}^d)}\|\varphi\|_{H^1_M(\Omega\times\mathbb{R}^d)}\end{aligned}$$

and by noting that $\nabla_\mathbf{x}\cdot\mathbf{u}_{k-1}=0$ gives

$$\int_{\Omega\times\mathbb{R}^d}M(\mathbf{u}_{k-1}\cdot\nabla_\mathbf{x}\hat{f})\hat{f}d\mathbf{n}d\mathbf{x}=\frac12\int_\Omega\mathbf{u}_{k-1}\cdot\nabla_\mathbf{x}\left[\int_{\mathbb{R}^d}M\hat{f}^2d\mathbf{n}\right]d\mathbf{x}=0.$$

Therefore $b(\cdot,\cdot)$ is a bounded and coercive bilinear functional on $H^1_M(\Omega\times\mathbb{R}^d)$. It follows from Definition 2.4 that $0\le E^{\tau^{-\frac14}}(s)\le\tau^{-\frac14}\,(\forall s\in\mathbb{R})$ and from a similar discussion as (4.6) that

$$\begin{aligned}\left|\int_{\Omega\times\mathbb{R}^d}M\nabla_\mathbf{x}\mathbf{u}\mathbf{n}E^{\tau^{-\frac14}}(\hat{f}^*)\cdot\nabla_\mathbf{n}\varphi d\mathbf{n}d\mathbf{x}\right|&\le\tau^{-\frac14}\|\nabla_\mathbf{x}\mathbf{u}\|_{L^2(\Omega)}\left\|\int_{\mathbb{R}^d}M|\mathbf{n}||\nabla_\mathbf{n}\varphi|d\mathbf{n}\right\|_{L^2(\Omega)}\\ &\le C(\tau)\|\varphi\|_{H^1_M(\Omega\times\mathbb{R}^d)}.\end{aligned}\tag{4.8}$$

Therefore $B(\hat{f}^*,\mathbf{u})\in(H^1_M(\Omega\times\mathbb{R}^d))'$. We thus finish the proof of Step 2 by the Lax–Milgram theorem.

Step 3. The solution $\hat{f}$ from a given function $\hat{f}^*$ in the procedure (4.5) and (4.7) defines a mapping $\Phi : L^2_M(\Omega \times \mathbb{R}^d) \to L^2_M(\Omega \times \mathbb{R}^d)$, $\hat{f}^* \mapsto \hat{f} := \Phi(\hat{f}^*) \in H^1_M(\Omega \times \mathbb{R}^d)$. By the Leray–Schauder fixed point theorem (see Theorem 11.3, [18]), we obtain a fixed point solution $\hat{f}$ to $\Phi(\hat{f}) = \hat{f}$, and hence a solution $(\mathbf{u}, \hat{f}) \in \boldsymbol{V} \times H^1_M(\Omega \times \mathbb{R}^d)$ to (4.1) and (4.2). For explicitness, we relabel $(\mathbf{u}, \hat{f})$ as $(\mathbf{u}_k, \hat{f}_k)$.

To prove this, we only need to show the following three claims:

Claim 1. $\Phi : L^2_M(\Omega \times \mathbb{R}^d) \to L^2_M(\Omega \times \mathbb{R}^d)$ is continuous.

Claim 2. Φ is compact.

Claim 3. $\Lambda := \{\hat{f} \in L^2_M(\Omega \times \mathbb{R}^d) : \hat{f} = \sigma\Phi(\hat{f}) \text{ for some } \sigma \in (0, 1]\}$ is bounded in $L^2_M(\Omega \times \mathbb{R}^d)$.

Proof of Claim 1. Set $\hat{f} := \Phi(\hat{f}^*)$ and $\hat{f}_m := \Phi(\hat{f}^*_m)$, $m \in \mathbb{N}$. If

$$\hat{f}^*_m \to \hat{f}^* \text{ in } L^2_M(\Omega \times \mathbb{R}^d), \text{ as } m \to \infty, \tag{4.9}$$

we need to show

$$\hat{f}_m \to \hat{f} \text{ in } L^2_M(\Omega \times \mathbb{R}^d), \text{ as } m \to \infty. \tag{4.10}$$

Indeed, for $\hat{f}^*$ and $\hat{f}^*_m$, in view of the definition of Φ, there exist a unique $\mathbf{u} \in \boldsymbol{V}$ and $\mathbf{u}_m \in \boldsymbol{V}$ such that

$$a(\mathbf{u}, \mathbf{v}) = A(\hat{f}^*)(\mathbf{v}), \qquad a(\mathbf{u}_m, \mathbf{v}) = A(\hat{f}^*_m)(\mathbf{v}) \ \forall \mathbf{v} \in \boldsymbol{V}, \tag{4.11}$$

$$b(\hat{f}, \varphi) = B(\hat{f}^*, \mathbf{u})(\varphi), \qquad b(\hat{f}_m, \varphi) = B(\hat{f}^*_m, \mathbf{u}_m)(\varphi) \ \forall \varphi \in H^1_M(\Omega \times \mathbb{R}^d). \tag{4.12}$$

By subtracting the terms in (4.11), we obtain

$$a(\mathbf{u}_m, \mathbf{v}) - a(\mathbf{u}, \mathbf{v}) = A(\hat{f}^*_m)(\mathbf{v}) - A(\hat{f}^*)(\mathbf{v})$$

and by taking $\mathbf{v} = \mathbf{u}_m - \mathbf{u}$, and using $\int_\Omega (\mathbf{u}_{k-1} \cdot \nabla_{\mathbf{x}})(\mathbf{u}_m - \mathbf{u}) \cdot (\mathbf{u}_m - \mathbf{u}) d\mathbf{x} = 0$ and in view of (4.6) we have that

$$\begin{aligned}
&\int_\Omega |\mathbf{u}_m - \mathbf{u}|^2 d\mathbf{x} + \tau \int_\Omega |\nabla_{\mathbf{x}} \mathbf{u}_m - \nabla_{\mathbf{x}} \mathbf{u}|^2 d\mathbf{x} \\
&\quad = -\tau \int_{\Omega \times \mathbb{R}^d} M \nabla_{\mathbf{n}} U \otimes \mathbf{n} (\hat{f}^*_m - \hat{f}^*) : (\nabla_{\mathbf{x}} \mathbf{u}_m - \nabla_{\mathbf{x}} \mathbf{u}) d\mathbf{n} d\mathbf{x} \\
&\quad \le \tau \left\| \int_{\mathbb{R}^d} M \nabla_{\mathbf{n}} U \otimes \mathbf{n} (\hat{f}^*_m - \hat{f}^*) d\mathbf{n} \right\|_{L^2(\Omega)} \|\nabla_{\mathbf{x}} \mathbf{u}_m - \nabla_{\mathbf{x}} \mathbf{u}\|_{L^2(\Omega)} \\
&\quad \le C\tau \big\| \hat{f}^*_m - \hat{f}^* \big\|_{L^2_M(\Omega \times \mathbb{R}^d)} \|\nabla_{\mathbf{x}} \mathbf{u}_m - \nabla_{\mathbf{x}} \mathbf{u}\|_{L^2(\Omega)}.
\end{aligned} \tag{4.13}$$

Then from the Cauchy–Schwarz inequality one has that

$$\big\| \mathbf{u}_m - \mathbf{u} \big\|^2_{H^1(\Omega)} \le C(\tau) \big\| \hat{f}^*_m - \hat{f}^* \big\|^2_{L^2_M(\Omega \times \mathbb{R}^d)}.$$

Thus (4.9) yields

$$\mathbf{u}_m \to \mathbf{u} \text{ in } \boldsymbol{H}^1(\Omega) \text{ as } m \to \infty. \tag{4.14}$$

By (4.12), taking the same procedure as above, and noting that

$$\int_{\Omega \times \mathbb{R}^d} M[\mathbf{u}_{k-1} \cdot \nabla_{\mathbf{x}} (\hat{f}^*_m - \hat{f}^*)](\hat{f}^*_m - \hat{f}^*) d\mathbf{n} d\mathbf{x} = 0,$$

one has

$$\begin{aligned}\sqrt{\varepsilon}\big\|\hat{f}_m-\hat{f}\big\|_{H^1_M(\Omega\times\mathbb{R}^d)} &\le C\big\|\nabla_{\mathbf{x}}\mathbf{u}_m\mathbf{n}E^{\tau^{-\frac14}}(\hat{f}^*_m)-\nabla_{\mathbf{x}}\mathbf{u}\mathbf{n}E^{\tau^{-\frac14}}(\hat{f}^*)\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}\\ &\le C\Big(\big\|(\nabla_{\mathbf{x}}\mathbf{u}_m-\nabla_{\mathbf{x}}\mathbf{u})\mathbf{n}E^{\tau^{-\frac14}}(\hat{f}^*_m)\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}\\ &\qquad+\big\|\nabla_{\mathbf{x}}\mathbf{u}\mathbf{n}[E^{\tau^{-\frac14}}(\hat{f}^*_m)-E^{\tau^{-\frac14}}(\hat{f}^*)]\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}\Big)\\ &=:I_1+I_2.\end{aligned}$$

It follows from (4.14) that

$$\begin{aligned}I_1 &\le C\tau^{-\frac14}\|\mathbf{n}\|_{L^2_M(\mathbb{R}^d)}\big\|\nabla_{\mathbf{x}}\mathbf{u}_m-\nabla_{\mathbf{x}}\mathbf{u}\big\|_{L^2(\Omega)}\\ &\le C(\tau)\big\|\nabla_{\mathbf{x}}\mathbf{u}_m-\nabla_{\mathbf{x}}\mathbf{u}\big\|_{L^2(\Omega)}\to 0 \text{ as } m\to\infty.\end{aligned}$$

Next we estimate I_2. Since $\boldsymbol{C}^\infty(\overline{\Omega};\boldsymbol{C}_0^\infty(\mathbb{R}^d))$ is dense in $\boldsymbol{L}^2_M(\Omega\times\mathbb{R}^d)$ and $\nabla_{\mathbf{x}}\mathbf{u}\mathbf{n}\in\boldsymbol{L}^2_M(\Omega\times\mathbb{R}^d)$, we have that

$$\forall\epsilon>0,\exists\,\boldsymbol{w}\in\boldsymbol{C}^\infty(\overline{\Omega};\boldsymbol{C}_0^\infty(\mathbb{R}^d))\text{ such that }\|\nabla_{\mathbf{x}}\mathbf{u}\mathbf{n}-\boldsymbol{w}\|_{\boldsymbol{L}^2_M(\Omega\times\mathbb{R}^d)}<\epsilon.$$

Moreover, we have from $E^{\tau^{-\frac14}}\in C^{0,1}(\mathbb{R})$ with Lipschitz coefficient 1 and (4.9) that

$$\begin{aligned}\exists\, m_0\in\mathbb{N},\forall m>m_0,\quad &\left\|\boldsymbol{w}\left[E^{\tau^{-\frac14}}(\hat{f}^*_m)-E^{\tau^{-\frac14}}(\hat{f}^*)\right]\right\|_{L^2_M(\Omega\times\mathbb{R}^d)}\\ &\le\|\boldsymbol{w}\|_{L^\infty(\Omega\times\mathbb{R}^d)}\big\|\hat{f}^*_m-\hat{f}^*\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}<\epsilon.\end{aligned}$$

Therefore

$$\begin{aligned}I_2 &\le C\Big(\big\|\big(\nabla_{\mathbf{x}}\mathbf{u}\mathbf{n}-\boldsymbol{w}\big)\big[E^{\tau^{-\frac14}}(\hat{f}^*_m)-E^{\tau^{-\frac14}}(\hat{f}^*)\big]\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}\\ &\qquad+\big\|\boldsymbol{w}\big[E^{\tau^{-\frac14}}(\hat{f}^*_m)-E^{\tau^{-\frac14}}(\hat{f}^*)\big]\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}\Big)\\ &\le C\Big(\tau^{-\frac14}\big\|\nabla_{\mathbf{x}}\mathbf{u}\mathbf{n}-\boldsymbol{w}\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}+\big\|\boldsymbol{w}\big[E^{\tau^{-\frac14}}(\hat{f}^*_m)-E^{\tau^{-\frac14}}(\hat{f}^*)\big]\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}\Big)\\ &< C(\tau)\epsilon.\end{aligned}$$

Consequently $\hat{f}_m\to\hat{f}$ in $H^1_M(\Omega\times\mathbb{R}^d)$ and hence (4.10) holds. This ends the proof of *Claim* 1.

Proof of Claim 2. It is easy to deduce that

$$\exists\, C(\tau)>0,\ \forall\hat{f}^*\in L^2_M(\Omega\times\mathbb{R}^d),\sqrt{\varepsilon}\big\|\Phi(\hat{f}^*)\big\|_{H^1_M(\Omega\times\mathbb{R}^d)}\le C(\tau)\left(1+\big\|\hat{f}^*\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}\right).$$

Thus we have from Proposition 3.10 that $H^1_M(\Omega\times\mathbb{R}^d)\hookrightarrow\hookrightarrow L^2_M(\Omega\times\mathbb{R}^d)$ and hence *Claim* 2 holds.

Proof of Claim 3. For any $\hat{f}\in\Lambda$, there exists a unique $\mathbf{u}\in\boldsymbol{V}$ such that

$$a(\mathbf{u},\mathbf{v})=A(\hat{f})(\mathbf{v})\ \forall\mathbf{v}\in\boldsymbol{V},\tag{4.15}$$

$$b(\hat{f},\varphi)=\sigma B(\hat{f},\mathbf{u})(\varphi)\ \forall\varphi\in H^1_M(\Omega\times\mathbb{R}^d).\tag{4.16}$$

Taking $\mathbf{v} = \mathbf{u}$ in (4.15) and similarly to that in (4.13), we have from the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\|\mathbf{u}\|^2_{L^2(\Omega)} + \tau\|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega)} \\
&\quad\le \int_\Omega |\mathbf{u}_{k-1}||\mathbf{u}|d\mathbf{x} + C\tau\big\|\hat{f}\big\|_{L^2_M(\Omega\times\mathbb{R}^d)}\|\nabla_{\mathbf{x}}\mathbf{u}\|_{L^2(\Omega)} \\
&\quad\le \frac{1}{2}\|\mathbf{u}_{k-1}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\tau}{2}\|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega)} + C\tau\big\|\hat{f}\big\|^2_{L^2_M(\Omega\times\mathbb{R}^d)}.
\end{aligned} \tag{4.17}$$

Therefore

$$\tau\|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega)} \le C(k-1) + C\tau\|\hat{f}\|^2_{L^2_M(\Omega\times\mathbb{R}^d)}. \tag{4.18}$$

Taking $\varphi = \hat{f}$ in (4.16), we deduce from the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^d} M|\hat{f}|^2 d\mathbf{n}d\mathbf{x} + \tau\int_{\Omega\times D} M\left(\varepsilon|\nabla_{\mathbf{x}}\hat{f}|^2 + |\nabla_{\mathbf{n}}\hat{f}|^2\right) d\mathbf{n}d\mathbf{x} \\
&\quad\le \sigma\int_{\Omega\times\mathbb{R}^d} M|\hat{f}_{k-1}||\hat{f}|d\mathbf{n}d\mathbf{x} + \sigma\tau^{\frac{3}{4}}\int_{\Omega\times\mathbb{R}^d} M|\mathbf{n}||\nabla_{\mathbf{n}}\hat{f}||\nabla_{\mathbf{x}}\mathbf{u}|d\mathbf{n}d\mathbf{x} \\
&\quad\le \frac{1}{2}\int_{\Omega\times\mathbb{R}^d} M|\hat{f}_{k-1}|^2 d\mathbf{n}d\mathbf{x} + \frac{1}{2}\int_{\Omega\times\mathbb{R}^d} M|\hat{f}|^2 d\mathbf{n}d\mathbf{x} + \frac{\tau}{2}\int_{\Omega\times\mathbb{R}^d} M|\nabla_{\mathbf{n}}\hat{f}|^2 d\mathbf{n}d\mathbf{x} \\
&\quad + C\tau^{\frac{1}{2}}\int_{\Omega\times\mathbb{R}^d} M|\mathbf{n}|^2|\nabla_{\mathbf{x}}\mathbf{u}|^2 d\mathbf{n}d\mathbf{x}.
\end{aligned} \tag{4.19}$$

Since

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^d} M|\mathbf{n}|^2|\nabla_{\mathbf{x}}\mathbf{u}|^2 d\mathbf{n}d\mathbf{x} &= \left(\int_{\mathbb{R}^d} M|\mathbf{n}|^2 d\mathbf{n}\right)\left(\int_\Omega |\nabla_{\mathbf{x}}\mathbf{u}|^2 d\mathbf{x}\right) \\
&\le C\int_\Omega |\nabla_{\mathbf{x}}\mathbf{u}|^2 d\mathbf{x},
\end{aligned} \tag{4.20}$$

we have from (4.18) and (4.19) that

$$\|\hat{f}\|^2_{L^2_M(\Omega\times\mathbb{R}^d))} \le \|\hat{f}_{k-1}\|^2_{L^2_M(\Omega\times\mathbb{R}^d))} + C\tau^{\frac{1}{2}}\|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega)} \le C(k-1,\tau) + C\tau^{\frac{1}{2}}\|\hat{f}\|^2_{L^2_M(\Omega\times\mathbb{R}^d))}.$$

Noting that $C\tau^{\frac{1}{2}} < \frac{1}{2}$, we have $\|\hat{f}\|_{L^2_M(\Omega\times\mathbb{R}^d)} \le C(k-1,\tau)$ and then Claim 3 is proven.

Step 4. We prove the nonnegativity for $\hat{f}_k$. In fact, set $[\hat{f}_k]^- := \min\{\hat{f}_k, 0\}$. Then $[\hat{f}_k]^- \in H^1_M(\Omega\times\mathbb{R}^d)$. Choosing $\varphi = [\hat{f}_k]^-$ in (4.2) and noting that $E^{\tau^{-\frac{1}{4}}}(\hat{f}_k)\nabla_{\mathbf{n}}[\hat{f}_k]^- = 0$, we deduce that

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^d} M\left|[\hat{f}_k]^-\right|^2 d\mathbf{n}d\mathbf{x} + \tau\int_{\Omega\times\mathbb{R}^d} M\left(\varepsilon\left|\nabla_{\mathbf{x}}[\hat{f}_k]^-\right|^2 + \left|\nabla_{\mathbf{n}}[\hat{f}_k]^-\right|^2\right) d\mathbf{n}d\mathbf{x} \\
&\quad = \int_{\Omega\times\mathbb{R}^d} M\hat{f}_{k-1}[\hat{f}_k]^- d\mathbf{n}d\mathbf{x} \le 0.
\end{aligned} \tag{4.21}$$

Therefore $[\hat{f}_k]^- = 0$ a.e. on $\Omega\times\mathbb{R}^d$ and hence $\hat{f}_k \ge 0$ a.e. on $\Omega\times\mathbb{R}^d$. Thus $\hat{f}_k \in Z$. This finishes the proof of Proposition 4.4. ▯

For any fixed constant $c > 0$, define $Z_c := \{\varphi \in Z : \|\varphi\|_{L^\infty(\Omega;L^1_M(\mathbb{R}^d))} \le c\}$. By choosing $\varphi \equiv \varrho \in H^1(\Omega)$, (4.2) becomes (4.3). If $\|\rho_{k-1}\|_{L^\infty(\Omega)} = \|\hat{f}_{k-1}\|_{L^\infty(\Omega;L^1_M(\mathbb{R}^d))} \le c$, then (4.3) implies $\|\rho_k\|_{L^\infty(\Omega)} = \|\hat{f}_k\|_{L^\infty(\Omega;L^1_M(\mathbb{R}^d))} \le c$. Therefore one can establish the following result.

Corollary 4.5. *Let* $(\mathbf{u}_{k-1}, \hat{f}_{k-1}) \in \boldsymbol{V}\times Z_c$. *Then there exists* $(\mathbf{u}_k, \hat{f}_k) \in \boldsymbol{V}\times(Z_c\cap H^1_M(\Omega\times\mathbb{R}^d))$ *which solves* (4.1)–(4.2).

4.2. Uniform estimates in τ, ε, and time t. Suppose $\mathbf{u}_{in} \in \boldsymbol{H}$ and $\hat{f}_{in} \in L^\infty(\Omega; L^1_M(\mathbb{R}^d))$ such that $\hat{f}_{in} \geq 0$ a.e. on $\Omega \times \mathbb{R}^d$, $\int_{\Omega\times\mathbb{R}^d} MF(\hat{f}_{in})d\mathbf{n}d\mathbf{x} < \infty$. We regularize $\mathbf{u}_{in}$ by $\mathbf{u}_0 = \mathbf{u}_0(\tau)$ which is the weak solution of $\mathbf{u}_0 - \tau^{\frac{1}{4}}\Delta\mathbf{u}_0 = \mathbf{u}_{in}$, with boundary condition $\mathbf{u}_0\big|_{\partial\Omega} = 0$. Therefore

$$\|\mathbf{u}_0\|^2_{L^2(\Omega)} + \tau^{\frac{1}{4}}\|\nabla_{\mathbf{x}}\mathbf{u}_0\|^2_{L^2(\Omega)} \leq \|\mathbf{u}_{in}\|^2_{L^2(\Omega)} \tag{4.22}$$

and $\mathbf{u}_0 \rightharpoonup \mathbf{u}_{in}$ in $\boldsymbol{H}$ as $\tau \to 0$. Furthermore, let $\hat{f}_0 = \hat{f}_0(\tau) := E^{\tau^{-\frac{1}{4}}}(\hat{f}_{in})$. Then $(\mathbf{u}_0, \hat{f}_0) \in \boldsymbol{V} \times Z_{\|\hat{f}_{in}\|_{L^\infty(\Omega;L^1_M(\mathbb{R}^d))}}$. Using Corollary 4.5, as the time step updates, we obtain a sequence of approximate solutions

$$(\mathbf{u}_k, \hat{f}_k) \in \boldsymbol{V} \times \left(Z_{\|\hat{f}_{in}\|_{L^\infty(\Omega;L^1_M(\mathbb{R}^d))}} \cap H^1_M(\Omega\times\mathbb{R}^d)\right) \ (k \in \mathbb{N}) \tag{4.23}$$

to (4.1)–(4.2). Equation (4.23) implies the following lemma directly.

LEMMA 4.6.

$$\sup_{k\in\mathbb{N}} \|\hat{f}_k\|_{L^\infty(\Omega;L^1_M(\mathbb{R}^d))} \leq \|\hat{f}_{in}\|^2_{L^\infty(\Omega;L^1_M(\mathbb{R}^d))}. \tag{4.24}$$

Based on Lemma 4.6, we establish the following two lemmas for the entropy estimate and the time regularity estimate, respectively.

LEMMA 4.7. *For any $k \in \mathbb{N}$,*

$$\begin{aligned} &\|\mathbf{u}_k\|^2_{L^2(\Omega)} + 2\int_{\Omega\times\mathbb{R}^d} MF(\hat{f}_k)d\mathbf{n}d\mathbf{x} + \sum_{i=1}^k \|\mathbf{u}_i - \mathbf{u}_{i-1}\|^2_{L^2(\Omega)} \\ &\quad + 2\tau\sum_{i=1}^k \|\nabla_{\mathbf{x}}\mathbf{u}_i\|^2_{L^2(\Omega)} + 4\tau\sum_{i=1}^k\left(\varepsilon\left\|\nabla_{\mathbf{x}}\sqrt{\hat{f}_i}\right\|^2_{L^2_M(\Omega\times\mathbb{R}^d)} + \left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_i}\right\|^2_{L^2_M(\Omega\times\mathbb{R}^d)}\right) \\ &\leq \|\mathbf{u}_{in}\|^2_{L^2(\Omega)} + 2\int_{\Omega\times\mathbb{R}^d} MF(\hat{f}_{in})d\mathbf{n}d\mathbf{x}. \end{aligned} \tag{4.25}$$

Proof. Let $\alpha \in (0,1)$ and denote $L := \tau^{-\frac{1}{4}}$. Let F^L be the function defined in Definition 2.4. Taking $\varphi = (F^L)'(\hat{f}_k + \alpha) \in H^1_M(\Omega\times\mathbb{R}^d)$ in (4.2) and noting

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^d} M(\mathbf{u}_{k-1}\cdot\nabla_{\mathbf{x}}\hat{f}_k)(F^L)'(\hat{f}_k+\alpha)d\mathbf{n}d\mathbf{x} \\ &\quad = \int_\Omega \mathbf{u}_{k-1}\cdot\nabla_{\mathbf{x}}\left(\int_{\mathbb{R}^d} MF^L(\hat{f}_k+\alpha)d\mathbf{n}\right)d\mathbf{x} = 0, \end{aligned}$$

we have from the convexity of F^L that

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^d} M\left(F^L(\hat{f}_k+\alpha) - F^L(\hat{f}_{k-1}+\alpha)\right)d\mathbf{n}d\mathbf{x} \\ &\quad + \tau\int_{\Omega\times\mathbb{R}^d} M\left(\varepsilon|\nabla_{\mathbf{x}}\hat{f}_k|^2 + |\nabla_{\mathbf{n}}\hat{f}_k|^2\right)(F^L)''(\hat{f}_k+\alpha)d\mathbf{n}d\mathbf{x} \\ &\leq \tau\int_{\Omega\times\mathbb{R}^d} M\left(E^L(\hat{f}_k)(F^L)''(\hat{f}_k+\alpha)\right)\left(\nabla_{\mathbf{x}}\mathbf{u}_k\mathbf{n}\cdot\nabla_{\mathbf{n}}\hat{f}_k\right)d\mathbf{n}d\mathbf{x} \\ &= \tau\int_{\Omega\times\mathbb{R}^d} M\left(E^L(\hat{f}_k)(F^L)''(\hat{f}_k+\alpha) - 1\right)\left(\nabla_{\mathbf{x}}\mathbf{u}_k\mathbf{n}\cdot\nabla_{\mathbf{n}}\hat{f}_k\right)d\mathbf{n}d\mathbf{x} \\ &\quad + \tau\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{x}}\mathbf{u}_k\mathbf{n}\cdot\nabla_{\mathbf{n}}\hat{f}_k d\mathbf{n}d\mathbf{x} =: J_1 + J_2. \end{aligned} \tag{4.26}$$

The Cauchy–Schwarz inequality, together with $E^L \in C^{0,1}(\mathbb{R})$ with Lipschitz coefficient 1, (2.7)–(2.8), and (4.20) implies

$$\begin{aligned} J_1 &\le \tau \int_{\Omega\times\mathbb{R}^d} M|\mathbf{n}||\nabla_{\mathbf{x}}\mathbf{u}_k||\nabla_{\mathbf{n}}\hat{f}_k||(F^L)''(\hat{f}_k+\alpha)||E^L(\hat{f}_k+\alpha)-E^L(\hat{f}_k)|d\mathbf{n}d\mathbf{x} \\ &\le \sqrt{\alpha}\tau \int_{\Omega\times\mathbb{R}^d} M|\mathbf{n}||\nabla_{\mathbf{x}}\mathbf{u}_k||\nabla_{\mathbf{n}}\hat{f}_k|\sqrt{(F^L)''(\hat{f}_k+\alpha)}d\mathbf{n}d\mathbf{x} \\ &\le \frac{\alpha\tau}{2}\int_{\Omega\times\mathbb{R}^d} M|\mathbf{n}|^2|\nabla_{\mathbf{x}}\mathbf{u}_k|^2 d\mathbf{n}d\mathbf{x} + \frac{\tau}{2}\int_{\Omega\times\mathbb{R}^d} M|\nabla_{\mathbf{n}}\hat{f}_k|^2(F^L)''(\hat{f}_k+\alpha)d\mathbf{n}d\mathbf{x} \\ &\le C\alpha\tau\int_\Omega |\nabla_{\mathbf{x}}\mathbf{u}_k|^2 d\mathbf{x} + \frac{\tau}{2}\int_{\Omega\times\mathbb{R}^d} M|\nabla_{\mathbf{n}}\hat{f}_k|^2(F^L)''(\hat{f}_k+\alpha)d\mathbf{n}d\mathbf{x}. \end{aligned} \tag{4.27}$$

It follows from Lemma 2.2 that

$$J_2 = \tau\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{n}}U\otimes\mathbf{n}\hat{f}_k : \nabla_{\mathbf{x}}\mathbf{u}_k d\mathbf{n}d\mathbf{x}. \tag{4.28}$$

Taking $\mathbf{v}=\mathbf{u}_k$ in (4.1), one has from the identity

$$2(\mathbf{a}-\mathbf{b})\cdot\mathbf{a} = |\mathbf{a}|^2 + |\mathbf{a}-\mathbf{b}|^2 - |\mathbf{b}|^2 \ \forall \mathbf{a},\mathbf{b}\in\mathbb{R}^d \tag{4.29}$$

that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\mathbf{u}_k|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega |\mathbf{u}_k-\mathbf{u}_{k-1}|^2 d\mathbf{x} + \tau\int_\Omega |\nabla_{\mathbf{x}}\mathbf{u}_k|^2 d\mathbf{x} \\ &\quad = \frac{1}{2}\int_\Omega |\mathbf{u}_{k-1}|^2 d\mathbf{x} - \tau\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{n}}U\otimes\mathbf{n}\hat{f}_k : \nabla_{\mathbf{x}}\mathbf{u}_k d\mathbf{n}d\mathbf{x}. \end{aligned} \tag{4.30}$$

Combining (4.26)–(4.30) and summing up, we have by noting $\hat{f}_0 = E^L(\hat{f}_{in})$ and (4.22) that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\mathbf{u}_k|^2 d\mathbf{x} + \int_{\Omega\times\mathbb{R}^d} MF^L(\hat{f}_k+\alpha)d\mathbf{n}d\mathbf{x} \\ &\qquad + \frac{1}{2}\sum_{i=1}^k\int_\Omega |\mathbf{u}_i-\mathbf{u}_{i-1}|^2 d\mathbf{x} + \tau(1-C\alpha)\sum_{i=1}^k\int_\Omega |\nabla_{\mathbf{x}}\mathbf{u}_i|^2 d\mathbf{x} \\ &\qquad + \frac{\tau}{2}\sum_{i=1}^k\int_{\Omega\times\mathbb{R}^d} M\Big(\varepsilon|\nabla_{\mathbf{x}}\hat{f}_i|^2 + |\nabla_{\mathbf{n}}\hat{f}_i|^2\Big)(F^L)''(\hat{f}_i+\alpha)d\mathbf{n}d\mathbf{x} \\ &\quad \le \frac{1}{2}\int_\Omega |\mathbf{u}_{in}|^2 d\mathbf{x} + \int_{\Omega\times\mathbb{R}^d} MF^L(E^L(\hat{f}_{in})+\alpha)d\mathbf{n}d\mathbf{x}. \end{aligned} \tag{4.31}$$

Thus it follows from (2.6), (2.7), and (2.10) that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\mathbf{u}_k|^2 d\mathbf{x} + \int_{\Omega\times\mathbb{R}^d} MF(\hat{f}_k+\alpha)d\mathbf{n}d\mathbf{x} + \frac{1}{2}\sum_{i=1}^k\int_\Omega |\mathbf{u}_i-\mathbf{u}_{i-1}|^2 d\mathbf{x} \\ &\quad + \tau(1-C\alpha)\sum_{i=1}^k\int_\Omega |\nabla_{\mathbf{x}}\mathbf{u}_i|^2 d\mathbf{x} + \frac{\tau}{2}\sum_{i=1}^k\int_{\Omega\times\mathbb{R}^d} M\left(\varepsilon\frac{|\nabla_{\mathbf{x}}\hat{f}_i|^2}{\hat{f}_i+\alpha} + \frac{|\nabla_{\mathbf{n}}\hat{f}_i|^2}{\hat{f}_i+\alpha}\right)d\mathbf{n}d\mathbf{x} \\ &\le \frac{1}{2}\int_\Omega |\mathbf{u}_{in}|^2 d\mathbf{x} + \int_{\Omega\times\mathbb{R}^d} M\left[\alpha + \frac{\alpha^2}{2} + F(\hat{f}_{in}+\alpha)\right]d\mathbf{n}d\mathbf{x}. \end{aligned}$$

By choosing a sufficiently small $\alpha > 0$ and then performing $\alpha \to 0$, one finishes the proof of (4.25) by applying Lebesgue's dominated convergence theorem and Fatou's lemma. ∎

LEMMA 4.8.

$$\tau \sum_{k=1}^{\infty} \left\| \frac{\mathbf{u}_k - \mathbf{u}_{k-1}}{\tau} \right\|^2_{(\boldsymbol{V}^{2+[d/2]})'} + \tau \sum_{k=1}^{\infty} \left\| M \frac{\hat{f}_k - \hat{f}_{k-1}}{\tau} \right\|^2_{(H^{2+d}(\Omega\times\mathbb{R}^d))'} \leq C. \tag{4.32}$$

Proof. It follows from (4.1) that for any $\mathbf{v} \in \boldsymbol{V}^{2+[d/2]}$,

$$\begin{aligned}
&\left| \int_\Omega \frac{\mathbf{u}_k - \mathbf{u}_{k-1}}{\tau} \cdot \mathbf{v} d\mathbf{x} \right| \\
&\quad \leq \|\nabla_{\mathbf{x}} \mathbf{u}_k\|_{L^2(\Omega)} \|\nabla_{\mathbf{x}} \mathbf{v}\|_{L^2(\Omega)} + \|\nabla_{\mathbf{x}} \mathbf{u}_k\|_{L^2(\Omega)} \|\mathbf{u}_{k-1}\|_{L^2(\Omega)} \|\mathbf{v}\|_{L^\infty(\Omega)} \\
&\qquad + \left| \int_{\Omega\times\mathbb{R}^d} M \nabla_{\mathbf{n}} U \otimes \mathbf{n} \hat{f}_k : \nabla_{\mathbf{x}} \mathbf{v} d\mathbf{n} d\mathbf{x} \right|.
\end{aligned}$$

We have from Lemma 2.2 that

$$\begin{aligned}
&\left| \int_{\Omega\times\mathbb{R}^d} M \nabla_{\mathbf{n}} U \otimes \mathbf{n} \hat{f}_k : \nabla_{\mathbf{x}} \mathbf{v} d\mathbf{n} d\mathbf{x} \right| \\
&\quad = \left| \int_{\Omega\times\mathbb{R}^d} M \nabla_{\mathbf{x}} \mathbf{v} \mathbf{n} \cdot \nabla_{\mathbf{n}} \hat{f}_k d\mathbf{n} d\mathbf{x} \right| \\
&\quad \leq 2 \int_{\Omega\times\mathbb{R}^d} M \left| \mathbf{n} \sqrt{\hat{f}_k} \right| \left| \nabla_{\mathbf{n}} \sqrt{\hat{f}_k} \right| d\mathbf{n} d\mathbf{x} \cdot \|\nabla_{\mathbf{x}} \mathbf{v}\|_{L^\infty(\Omega)} \\
&\quad \leq 2 \left\| \mathbf{n} \sqrt{\hat{f}_k} \right\|_{L^2_M(\Omega\times\mathbb{R}^d)} \left\| \nabla_{\mathbf{n}} \sqrt{\hat{f}_k} \right\|_{L^2_M(\Omega\times\mathbb{R}^d)} \|\nabla_{\mathbf{x}} \mathbf{v}\|_{L^\infty(\Omega)}.
\end{aligned}$$

It follows from the Fenchel–Young inequality (3.20) that

$$\begin{aligned}
\left\| |\mathbf{n}| \sqrt{\hat{f}_k} \right\|^2_{L^2_M(\Omega\times\mathbb{R}^d)} &= 4 \int_{\Omega\times\mathbb{R}^d} M \left(\frac{|\mathbf{n}|^2}{4} \right) \hat{f}_k d\mathbf{n} d\mathbf{x} \\
&\leq 4 \int_{\Omega\times\mathbb{R}^d} M F(\hat{f}_k) d\mathbf{n} d\mathbf{x} + 4 \int_{\Omega\times\mathbb{R}^d} M e^{\frac{|\mathbf{n}|^2}{4}} d\mathbf{n} d\mathbf{x} \\
&\leq C \left(\int_{\Omega\times\mathbb{R}^d} M F(\hat{f}_k) d\mathbf{n} d\boldsymbol{x} + 1 \right).
\end{aligned} \tag{4.33}$$

Then one has from $\boldsymbol{V}^{2+[d/2]} \hookrightarrow \boldsymbol{W}^{1,\infty}(\Omega)$ that

$$\begin{aligned}
&\left| \int_\Omega \frac{\mathbf{u}_k - \mathbf{u}_{k-1}}{\tau} \cdot \mathbf{v} d\mathbf{x} \right| \\
&\quad \leq C \Bigg[\|\nabla_{\mathbf{x}} \mathbf{u}_k\|_{L^2(\Omega)} \left(1 + \|\mathbf{u}_{k-1}\|_{L^2(\Omega)}\right) \\
&\qquad + \left\| \nabla_{\mathbf{n}} \sqrt{\hat{f}_k} \right\|_{L^2_M(\Omega\times\mathbb{R}^d)} \left(\int_{\Omega\times\mathbb{R}^d} M F(\hat{f}_k) d\mathbf{n} d\mathbf{x} + 1 \right)^{1/2} \Bigg] \|\mathbf{v}\|_{\boldsymbol{V}^{2+[d/2]}}
\end{aligned}$$

and hence from (4.25) that

$$\begin{aligned}
&\tau\sum_{k=1}^{\infty}\left\|\frac{\mathbf{u}_k-\mathbf{u}_{k-1}}{\tau}\right\|^2_{\left(\boldsymbol{V}^{2+[d/2]}\right)'}\\
&\quad\le C\tau\sum_{k=1}^{\infty}\|\nabla_{\mathbf{x}}\mathbf{u}_k\|^2_{L^2(\Omega)}\left(1+\sup_{k\in\mathbb{N}}\|\mathbf{u}_{k-1}\|^2_{L^2(\Omega)}\right)\\
&\qquad+C\tau\sum_{k=1}^{\infty}\left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_k}\right\|^2_{L^2_M(\Omega\times\mathbb{R}^d)}\left(\sup_{k\in\mathbb{N}}\int_{\Omega\times\mathbb{R}^d}MF(\hat{f}_k)d\mathbf{n}d\mathbf{x}+1\right)\le C.
\end{aligned}$$

For any $\varphi\in H^{2+d}(\Omega\times\mathbb{R}^d)$, and by noting

$$H^{2+d}(\Omega\times\mathbb{R}^d)\hookrightarrow W^{1,\infty}(\Omega\times\mathbb{R}^d)\hookrightarrow H^1_M(\Omega\times\mathbb{R}^d)$$

and

$$\int_{\Omega\times\mathbb{R}^d}M(\mathbf{u}_{k-1}\cdot\nabla_{\mathbf{x}}\hat{f}_k)\varphi d\mathbf{n}d\mathbf{x}=-\int_{\Omega\times\mathbb{R}^d}M(\mathbf{u}_{k-1}\cdot\nabla_{\mathbf{x}}\varphi)\hat{f}_k d\mathbf{n}d\mathbf{x},$$

we deduce from (4.2) that

$$\begin{aligned}
&\left|\int_{\Omega\times\mathbb{R}^d}M\frac{\hat{f}_k-\hat{f}_{k-1}}{\tau}\varphi d\mathbf{n}d\mathbf{x}\right|\\
&\quad\le C\left(\int_{\Omega\times\mathbb{R}^d}M|\mathbf{u}_{k-1}||\hat{f}_k||\nabla_{\mathbf{x}}\varphi|d\mathbf{n}d\mathbf{x}+\varepsilon\int_{\Omega\times\mathbb{R}^d}M|\nabla_{\mathbf{x}}\hat{f}_k||\nabla_{\mathbf{x}}\varphi|d\mathbf{n}d\mathbf{x}\right)\\
&\qquad+C\left(\int_{\Omega\times\mathbb{R}^d}M|\nabla_{\mathbf{n}}\hat{f}_k||\nabla_{\mathbf{n}}\varphi|d\mathbf{n}d\mathbf{x}+\int_{\Omega\times\mathbb{R}^d}M|\mathbf{n}||\nabla_{\mathbf{x}}\mathbf{u}_k||\hat{f}_k||\nabla_{\mathbf{n}}\varphi|d\mathbf{n}d\mathbf{x}\right)\\
&\quad=:P_1+P_2+P_3+P_4.
\end{aligned}$$

It follows from (4.24) and the Hölder inequality that

$$P_1\le C\|\mathbf{u}_{k-1}\|_{L^2(\Omega)}\|\hat{f}_k\|_{L^2(\Omega;L^1_M(\mathbb{R}^d))}\|\nabla_{\mathbf{x}}\varphi\|_{L^\infty(\Omega\times\mathbb{R}^d)}\le C\|\mathbf{u}_{k-1}\|_{L^2(\Omega)}\|\varphi\|_{H^{2+d}(\Omega\times\mathbb{R}^d)};$$

$$\begin{aligned}
P_2&\le C\varepsilon\left\|\nabla_{\mathbf{x}}\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times\mathbb{R}^d)}\left\|\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times\mathbb{R}^d)}\|\nabla_{\mathbf{x}}\varphi\|_{L^\infty(\Omega\times\mathbb{R}^d)}\\
&\le C\varepsilon\left\|\nabla_{\mathbf{x}}\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times\mathbb{R}^d)}\|\varphi\|_{H^{2+d}(\Omega\times\mathbb{R}^d)}.
\end{aligned}$$

Similarly,

$$P_3\le C\left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times\mathbb{R}^d)}\|\varphi\|_{H^{2+d}(\Omega\times\mathbb{R}^d)}.$$

We have from the Hölder inequality, (4.24), (4.33), and (4.25) that

$$\begin{aligned}
P_4&\le C\|\nabla_{\mathbf{x}}\mathbf{u}_k\|_{L^2(\Omega)}\left\||\mathbf{n}|\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times\mathbb{R}^d)}\left\|\sqrt{\hat{f}_k}\right\|_{L^\infty(\Omega;L^2_M(\mathbb{R}^d))}\|\nabla_{\mathbf{n}}\varphi\|_{L^\infty(\Omega\times\mathbb{R}^d)}\\
&\le C\|\nabla_{\mathbf{x}}\mathbf{u}_k\|_{L^2(\Omega)}\left(\int_{\Omega\times\mathbb{R}^d}MF(\hat{f}_k)d\mathbf{n}d\mathbf{x}+1\right)^{1/2}\|\varphi\|_{H^{2+d}(\Omega\times\mathbb{R}^d)}\\
&\le C\|\nabla_{\mathbf{x}}\mathbf{u}_k\|_{L^2(\Omega)}\|\varphi\|_{H^{2+d}(\Omega\times\mathbb{R}^d)}.
\end{aligned}$$

Therefore

$$\left|\int_{\Omega\times\mathbb{R}^d} M\frac{\hat{f}_k-\hat{f}_{k-1}}{\tau}\varphi d\mathbf{n}d\mathbf{x}\right|$$
$$\le C\Bigg(\|\mathbf{u}_{k-1}\|_{L^2(\Omega)}+\varepsilon\left\|\nabla_{\mathbf{x}}\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times\mathbb{R}^d)}$$
$$+\left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times\mathbb{R}^d)}+\|\nabla_{\mathbf{x}}\mathbf{u}_k\|_{L^2(\Omega)}\Bigg)\|\varphi\|_{H^{2+d}(\Omega\times\mathbb{R}^d)}.$$

This, the Poincaré inequality, (4.22), and (4.25) imply

$$\tau\sum_{k=1}^{\infty}\left\|M\frac{\hat{f}_k-\hat{f}_{k-1}}{\tau}\right\|^2_{(H^{2+d}(\Omega\times\mathbb{R}^d))'}$$
$$\le C\tau\|\mathbf{u}_0\|^2_{L^2(\Omega)}+C\tau\sum_{k=1}^{\infty}\Bigg(\varepsilon^2\left\|\nabla_{\mathbf{x}}\sqrt{\hat{f}_k}\right\|^2_{L^2(\Omega\times\mathbb{R}^d)}+\left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_k}\right\|^2_{L^2_M(\Omega\times\mathbb{R}^d)}$$
$$+\|\nabla_{\mathbf{x}}\mathbf{u}_k\|^2_{L^2(\Omega)}\Bigg)\le C.$$

This finishes the proof of Lemma 4.8. ☐

4.3. Convergence and proof of Theorem 4.1.

DEFINITION 4.9. *Define the piecewise function in t by*

$$\mathbf{u}_\tau(t,\cdot):=\mathbf{u}_k(\cdot),\ \pi_\tau\mathbf{u}_\tau(t,\cdot):=\mathbf{u}_{k-1}(\cdot),\ t\in((k-1)\tau,k\tau],\ k\in\mathbb{N}$$

and the difference quotient of size τ by

$$\partial_t^\tau\mathbf{u}_\tau(t,\cdot):=\frac{\mathbf{u}_k(\cdot)-\mathbf{u}_{k-1}(\cdot)}{\tau},\ t\in((k-1)\tau,k\tau],\ k\in\mathbb{N}.$$

Likewise, define ρ_τ, $\hat{f}_\tau$, and $\partial_t^\tau\hat{f}_\tau$.

4.3.1. Convergence. The compactness discussion is crucial to getting strong convergence. Using time-space compactness theorems with an hypothesis on derivatives (such as the Aubin–Lions–Simon lemma, see [29, Theorem 5]; the Dubinskiĭ lemma, see [7, Theorem 2.1] and [17, Theorem 1]), requires the traditional Rothe method in evolutionary PDEs (see [28] and [22]) which needs the construction of linear interpolation functions (also known as Rothe functions). We refer the reader to section 1.3 for a brief discussion on some of the difficulties that arise from using Rothe methods. Here, we shall apply Lemma 2.3 (i.e., Theorem 4.3 in Chen, Jüngel, and Liu [10]) and Theorem 1 in [16], a nonlinear and a linear time-space compactness theorem with the simple time criterion (2.4) for piecewise constant functions directly to avoid using these complicated Rothe functions.

PROPOSITION 4.10. *As $\tau\to 0$, there exists a subsequence of $\{(\mathbf{u}_\tau,\hat{f}_\tau)\}_{0<\tau\ll 1}$, not relabeled, and a pair of functions $(\mathbf{u},\hat{f})$ with regularity*

$$\mathbf{u}\in L^\infty(0,\infty;\boldsymbol{H})\cap L^2(0,\infty;\boldsymbol{V}),\ \hat{f}\ge 0\ \textit{a.e. on}\ [0,\infty)\times\Omega\times\mathbb{R}^d, \tag{4.34}$$

$$\hat{f}\in L^\infty((0,\infty)\times\Omega;L^1_M(\mathbb{R}^d)),\ \sqrt{\hat{f}}\in L^2(0,\infty;H^1_M(\Omega\times\mathbb{R}^d)), \tag{4.35}$$

which also satisfies (2.18) *with* $f = M\hat{f}$ *such that for any* $T > 0$*,*

$$\mathbf{u}_\tau \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(0,\infty;\boldsymbol{H}), \tag{4.36}$$

$$\mathbf{u}_\tau \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,\infty;\boldsymbol{V}), \tag{4.37}$$

$$\mathbf{u}_\tau \to \mathbf{u} \quad \text{in } L^2(0,T;L^p(\Omega))\ (\forall 2 \le p < 4), \tag{4.38}$$

$$\pi_\tau \mathbf{u}_\tau \to \mathbf{u} \quad \text{in } L^2((0,T)\times\Omega), \tag{4.39}$$

$$\sqrt{\hat{f}_\tau} \overset{*}{\rightharpoonup} \sqrt{\hat{f}} \quad \text{in } L^\infty((0,\infty)\times\Omega; L^2_M(\mathbb{R}^d)), \tag{4.40}$$

$$\sqrt{\hat{f}_\tau} \rightharpoonup \sqrt{\hat{f}} \quad \text{in } L^2(0,\infty; H^1_M(\Omega\times\mathbb{R}^d)), \tag{4.41}$$

$$\sqrt{\hat{f}_\tau} \to \sqrt{\hat{f}} \quad \text{in } L^2(0,T; L^2_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)), \tag{4.42}$$

$$\hat{f}_\tau \to \hat{f} \quad \text{in } L^p((0,T)\times\Omega; L^1_M(\mathbb{R}^d))\ (\forall 2 \le p < \infty), \tag{4.43}$$

$$E^{\tau^{-\frac{1}{4}}}(\hat{f}_\tau) \to \hat{f} \quad \text{in } L^p((0,T)\times\Omega; L^1_M(\mathbb{R}^d))\ (\forall 2 \le p < \infty). \tag{4.44}$$

Proof. Applying (4.24)–(4.25), we deduce that there exists a subsequence of $\{(\mathbf{u}_\tau, \hat{f}_\tau)\}_{0<\tau\ll 1}$, not relabeled, and a pair of functions $(\mathbf{u}, \hat{f})$, such that (4.34)–(4.37) and (4.40)–(4.41) hold.

For any fixed $T > 0$, we let $N = \frac{T}{\tau}$ (otherwise let $N = [T/\tau] + 1$). For any $0 < \tau \ll 1$, we have from (4.32) that

$$\|\mathbb{t}_\tau \mathbf{u}_\tau - \mathbf{u}_\tau\|^2_{L^2\left(0,T-\tau;\left(\mathbf{V}^{2+[d/2]}\right)'\right)} = \tau \sum_{k=1}^{N-1} \|\mathbf{u}_{k+1} - \mathbf{u}_k\|^2_{\left(\mathbf{V}^{2+[d/2]}\right)'} \le C\tau^2, \tag{4.45}$$

where $\mathbb{t}_\tau u_\tau(t) := u_\tau(t+\tau)$. Employing Theorem 1 in [16], we obtain (4.38) from (4.25), (4.45), and $\boldsymbol{V} \hookrightarrow\hookrightarrow \boldsymbol{L}^p(\Omega)\cap\boldsymbol{H} \hookrightarrow (\mathbf{V}^{2+[d/2]})'$ ($\forall 2 \le p < 4$). It follows from (4.25) that

$$\|\pi_\tau \mathbf{u}_\tau - \mathbf{u}_\tau\|^2_{L^2((0,T)\times\Omega)} = \tau \sum_{k=1}^{N} \|\mathbf{u}_k - \mathbf{u}_{k-1}\|^2_{L^2(\Omega)} \le C\tau.$$

This and (4.38) yield (4.39).

Following the idea of section 5 in Barrett and Süli [6], we define

$$M_+ := \left\{ f \ge 0 : \sqrt{f} \in H^1_M(\Omega\times\mathbb{R}^d) \right\} \quad \text{with} \quad [f]_{M_+} := \left\| \sqrt{f} \right\|^2_{H^1_M(\Omega\times\mathbb{R}^d)}$$

and

$$Y := \left\{ f :\ Mf \in \left(H^{2+d}(\Omega\times\mathbb{R}^d)\right)' \right\} \quad \text{with} \quad \|f\|_Y := \|Mf\|_{(H^{2+d}(\Omega\times\mathbb{R}^d))'}\,.$$

Then M_+ is a seminormed nonnegative cone in $L^1_M(\Omega\times\mathbb{R}^d)$ and Y is a Banach space. It follows from Proposition 3.10 with $p = 2$, i.e., $H^1_M(\Omega\times\mathbb{R}^d) \hookrightarrow\hookrightarrow L^2_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)$ that

$$M_+ \hookrightarrow\hookrightarrow L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d) \hookrightarrow Y. \tag{4.46}$$

Indeed, we have from

$$\|f\|_{L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)} = \left\|\sqrt{f}\right\|^2_{L^2_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)} \le C \left\|\sqrt{f}\right\|^2_{H^1_M(\Omega\times\mathbb{R}^d)} \le C[f]_{M_+}$$

that $M_+ \hookrightarrow L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)$ and for the bounded sequence $\{f_j\}$ in M_+, $\{\sqrt{f_j}\}$ is bounded in $H^1_M(\Omega\times\mathbb{R}^d)$. Therefore we deduce from $H^1_M(\Omega\times\mathbb{R}^d)\hookrightarrow\hookrightarrow L^2_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)$ that $\{\sqrt{f_j}\}$ has a convergent subsequence in $L^2_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)$ and hence $\{f_j\}$ has a convergent subsequence in $L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)$. Consequently, $M_+\hookrightarrow\hookrightarrow L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)$. Since $H^{2+d}(\Omega\times\mathbb{R}^d)\hookrightarrow L^\infty(\Omega\times\mathbb{R}^d)$, we have $\forall f\in L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)$,

$$\begin{aligned}|\langle Mf,g\rangle| &\le \|f\|_{L^1_M(\Omega\times\mathbb{R}^d)}\|g\|_{L^\infty(\Omega\times\mathbb{R}^d)}\\ &\le C\|f\|_{L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)}\|g\|_{H^{2+d}(\Omega\times\mathbb{R}^d)}\ \forall g\in H^{2+d}(\Omega\times\mathbb{R}^d).\end{aligned}$$

Then $f\in Y$ and $\|f\|_Y\le C\|f\|_{L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)}$ which finishes the proof of (4.46).

We have from (4.32) that

$$\left\|\mathbb{t}_\tau\hat f_\tau-\hat f_\tau\right\|^2_{L^2(0,T-h;Y)}=\tau\sum_{k=1}^{N-1}\left\|\hat f_{k+1}-\hat f_k\right\|^2_Y\le C\tau^2, \tag{4.47}$$

where $\mathbb{t}_\tau\hat f_\tau(t):=\hat f_\tau(t+\tau)$. Clearly, (4.25) yields

$$\|\hat f_\tau\|_{L^1(0,T;M_+)}=\left\|\sqrt{\hat f_\tau}\right\|^2_{L^2\left(0,T;H^1_M(\Omega\times\mathbb{R}^d)\right)}\le C(\varepsilon). \tag{4.48}$$

By applying Lemma 2.3, we deduce from (4.46)–(4.48) that there exists a subsequence of $\{\hat f_\tau\}_{0<\tau\ll 1}$, not relabeled, and an element $\hat f\in L^1(0,T;L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d))$ such that

$$\hat f_\tau\to\hat f \text{ in } L^1(0,T;L^1_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d)) \tag{4.49}$$

and hence by considering the inequality $|\sqrt{a}-\sqrt{b}|^2\le|a-b|\,(a,b\ge 0)$, (4.42) holds. Employing the interpolation inequality for L^p-norms, we get from (4.24) and (4.35) that for any $p\in[2,\infty)$,

$$\begin{aligned}\|\hat f_\tau-\hat f\|_{L^p((0,T)\times\Omega;L^1_M(\mathbb{R}^d))} &\le \|\hat f_\tau-\hat f\|^{1-\frac1p}_{L^\infty((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}\|\hat f_\tau-\hat f\|^{\frac1p}_{L^1((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}\\ &\le C\|\hat f_\tau-\hat f\|^{\frac1p}_{L^1((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}.\end{aligned}$$

This and (4.49) imply (4.43).

In light of the weakly lower semicontinuity of norm, we obtain the energy inequality (2.18) with $f=M\hat f$ directly from (4.25) and the convergent results (4.36)–(4.38) and (4.41)–(4.43).

Finally, we prove (4.44). Indeed, we have from $E^{\tau^{-\frac14}}\in C^{0,1}(\mathbb{R})$ with Lipschitz coefficient 1 that for any $p\in[2,\infty)$,

$$\begin{aligned}&\|E^{\tau^{-\frac14}}(\hat f_\tau)-\hat f\|_{L^p((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}\\ &\quad\le\|E^{\tau^{-\frac14}}(\hat f_\tau)-E^{\tau^{-\frac14}}(\hat f)\|_{L^p((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}\\ &\qquad+\|E^{\tau^{-\frac14}}(\hat f)-\hat f\|_{L^p((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}\\ &\quad\le\|\hat f_\tau-\hat f\|_{L^p((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}+\|E^{\tau^{-\frac14}}(\hat f)-\hat f\|_{L^p((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}.\end{aligned} \tag{4.50}$$

Moreover, employing Lebesgue's dominated convergence theorem, one deduces from (2.9) and $0 \leq E^{\tau^{-\frac{1}{4}}}(\hat{f}) \leq \hat{f}$ that

$$\|E^{\tau^{-\frac{1}{4}}}(\hat{f}) - \hat{f}\|_{L^p((0,T)\times\Omega;L^1_M(\mathbb{R}^d))} \to 0 \text{ as } \tau \to 0. \tag{4.51}$$

Thus, (4.51), (4.43), and (4.50) imply (4.44). This ends the proof of Proposition 4.10. ◻

4.3.2. Proof of Theorem 4.1. We need to establish the convergence of discrete derivatives $\partial_t^\tau \mathbf{u}_\tau$ and $\partial_t^\tau f_\tau$ as well as their weak integrals. These follow from the time regularity estimates (Lemma 4.8) of $\partial_t^\tau \mathbf{u}_\tau$, $\partial_t^\tau f_\tau$ and their convergence to $\partial_t \mathbf{u}$, $\partial_t f$ in the sense of distributions.

Proof of Theorem 4.1. In view of Definition 4.9, the weak approximate form reads, for any $\mathbf{v} \in \boldsymbol{C}_0^\infty([0,\infty)\times\Omega)$ with $\nabla_{\mathbf{x}} \cdot \mathbf{v} = 0$,

$$\begin{aligned}&\int_0^\infty \int_\Omega \partial_t^\tau \mathbf{u}_\tau \cdot \mathbf{v} d\mathbf{x} dt + \int_0^\infty \int_\Omega \nabla_{\mathbf{x}} \mathbf{u}_\tau : \nabla_{\mathbf{x}} \mathbf{v} d\mathbf{x} dt + \int_0^\infty \int_\Omega (\pi_\tau \mathbf{u}_\tau \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\tau \cdot \mathbf{v} d\mathbf{x} dt \\ &= -\int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \nabla_{\mathbf{n}} U \otimes \mathbf{n} \hat{f}_\tau : \nabla_{\mathbf{x}} \mathbf{v} d\mathbf{n} d\mathbf{x} dt;\end{aligned} \tag{4.52}$$

and for any $\varphi \in C_0^\infty([0,\infty)\times\overline{\Omega}\times\mathbb{R}^d)$,

$$\begin{aligned}&\int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \partial_t^\tau \hat{f}_\tau \varphi d\mathbf{n} d\mathbf{x} dt + \int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \big(\pi_\tau \mathbf{u}_\tau \cdot \nabla_{\mathbf{x}} \hat{f}_\tau\big) \varphi d\mathbf{n} d\mathbf{x} dt \\ &\quad + \varepsilon \int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \nabla_{\mathbf{x}} \hat{f}_\tau \cdot \nabla_{\mathbf{x}} \varphi d\mathbf{n} d\mathbf{x} dt + \int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \nabla_{\mathbf{n}} \hat{f}_\tau \cdot \nabla_{\mathbf{n}} \varphi d\mathbf{n} d\mathbf{x} dt \\ &= \int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \nabla_{\mathbf{x}} \mathbf{u}_\tau \mathbf{n} E^{\tau^{-\frac{1}{4}}}(\hat{f}_\tau) \cdot \nabla_{\mathbf{n}} \varphi d\mathbf{n} d\mathbf{x} dt.\end{aligned} \tag{4.53}$$

We first claim that as $\tau \to 0$,

$$\begin{aligned}&\int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \partial_t^\tau \hat{f}_\tau \varphi d\mathbf{n} d\mathbf{x} dt \\ &\to -\int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \hat{f} \partial_t \varphi d\mathbf{n} d\mathbf{x} dt - \int_{\Omega\times\mathbb{R}^d} M \hat{f}_{in}(\mathbf{x},\mathbf{n}) \varphi(0,\mathbf{x},\mathbf{n}) d\mathbf{n} d\mathbf{x},\end{aligned} \tag{4.54}$$

$$M \partial_t^\tau \hat{f}_\tau \rightharpoonup M \partial_t \hat{f} \ \text{ in } \ L^2(0,\infty; (H^{2+d}(\Omega\times\mathbb{R}^d))'). \tag{4.55}$$

Indeed,

$$\begin{aligned}&\int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \partial_t^\tau \hat{f}_\tau \varphi d\mathbf{n} d\mathbf{x} dt \\ &= \int_\tau^\infty \int_{\Omega\times\mathbb{R}^d} M \frac{\hat{f}_\tau(t) - \hat{f}_\tau(t-\tau)}{\tau} \varphi d\mathbf{n} d\mathbf{x} dt + \int_0^\tau \int_{\Omega\times\mathbb{R}^d} M \frac{\hat{f}_\tau(t) - \hat{f}_0}{\tau} \varphi d\mathbf{n} d\mathbf{x} dt \\ &= \int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \frac{\hat{f}_\tau(t)}{\tau} \varphi d\mathbf{n} d\mathbf{x} dt - \int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \frac{\hat{f}_\tau(t)}{\tau} \varphi(t+\tau) d\mathbf{n} d\mathbf{x} dt \\ &\quad - \int_0^\tau \int_{\Omega\times\mathbb{R}^d} M \frac{E^{\tau^{-\frac{1}{4}}}(\hat{f}_{in})}{\tau} \varphi d\mathbf{n} d\mathbf{x} dt \\ &= -\int_0^\infty \int_{\Omega\times\mathbb{R}^d} M \hat{f}_\tau(t) \frac{\varphi(t+\tau) - \varphi(t)}{\tau} d\mathbf{n} d\mathbf{x} dt - \int_0^\tau \int_{\Omega\times\mathbb{R}^d} M E^{\tau^{-\frac{1}{4}}}(\hat{f}_{in}) \frac{\varphi}{\tau} d\mathbf{n} d\mathbf{x} dt.\end{aligned}$$

Assume that the compact support of φ is a subset of $[0,T)\times\overline{\Omega}\times\mathbb{R}^d$, then

$$\begin{aligned}
&\bigg|\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\partial_t^\tau \hat f_\tau\varphi d\mathbf{n}d\mathbf{x}dt+\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\hat f\partial_t\varphi d\mathbf{n}d\mathbf{x}dt\\
&+\int_{\Omega\times\mathbb{R}^d} M\hat f_{in}(\mathbf{x},\mathbf{n})\varphi(0,\mathbf{x},\mathbf{n})d\mathbf{n}d\mathbf{x}\bigg|\\
&\le\left|\int_{T-\tau}^T\int_{\Omega\times\mathbb{R}^d} M\left(\hat f_\tau\frac{\varphi}{\tau}+\hat f\partial_t\varphi\right)d\mathbf{n}d\mathbf{x}dt\right|\\
&\quad+\left|\int_0^{T-\tau}\int_{\Omega\times\mathbb{R}^d} M\left(\hat f\partial_t\varphi-\hat f_\tau\frac{\varphi(t+\tau)-\varphi(t)}{\tau}\right)d\mathbf{n}d\mathbf{x}dt\right|\\
&\quad+\left|\int_{\Omega\times\mathbb{R}^d} M\left(\hat f_{in}\varphi(0)-E^{\tau^{-\frac14}}(\hat f_{in})\int_0^\tau\frac{\varphi}{\tau}dt\right)d\mathbf{n}d\mathbf{x}\right|\\
&=: I_1+I_2+I_3.
\end{aligned}$$

Thanks to $\varphi(T)=0$, we have from the mean value theorem of differentials and (4.24) that

$$\begin{aligned}
I_1&\le\tau\|\partial_t\varphi\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^d)}(\|\hat f_\tau\|_{L^\infty(0,T;L^1_M(\Omega\times\mathbb{R}^d))}+\|\hat f\|_{L^\infty(0,T;L^1_M(\Omega\times\mathbb{R}^d))})\le C\tau,\\
I_2&\le\left|\int_0^{T-\tau}\int_{\Omega\times\mathbb{R}^d} M\hat f\left(\partial_t\varphi-\frac{\varphi(t+\tau)-\varphi(t)}{\tau}\right)d\mathbf{n}d\mathbf{x}\right|\\
&\quad+\left|\int_0^{T-\tau}\int_{\Omega\times\mathbb{R}^d} M\left(\hat f-\hat f_\tau\right)\frac{\varphi(t+\tau)-\varphi(t)}{\tau}d\mathbf{n}d\mathbf{x}\right|\\
&\le\tau\|\partial_{tt}\varphi\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^d)}\|\hat f\|_{L^\infty(0,T;L^1_M(\Omega\times\mathbb{R}^d))}\\
&\quad+\|\hat f-\hat f_\tau\|_{L^1((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}\|\partial_t\varphi\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^d)}\\
&\le C\left(\tau+\|\hat f-\hat f_\tau\|_{L^1((0,T)\times\Omega;L^1_M(\mathbb{R}^d))}\right).
\end{aligned}$$

It follows from the proof of (4.51) and the mean value theorem that

$$\begin{aligned}
I_3\le&\int_{\Omega\times\mathbb{R}^d} M\left|\hat f_{in}-E^{\tau^{-\frac14}}(\hat f_{in})\right||\varphi(0)|d\mathbf{n}d\mathbf{x}\\
&+\int_{\Omega\times\mathbb{R}^d} ME^{\tau^{-\frac14}}(\hat f_{in})\left|\varphi(0)-\frac{1}{\tau}\int_0^\tau\varphi(t)dt\right|d\mathbf{n}d\mathbf{x}\to 0.
\end{aligned}$$

Therefore (4.54) is proved. Moreover, if we take $\varphi\in C_0^\infty((0,\infty)\times\overline{\Omega}\times\mathbb{R}^d)$, then (4.54) implies

$$M\partial_t^\tau\hat f_\tau\rightharpoonup M\partial_t\hat f\ \text{ in }\ \mathscr{D}'((0,\infty);(C^{2+d}(\overline{\Omega};\mathscr{D}(\mathbb{R}^d)))'). \tag{4.56}$$

We have from (4.32) that $\|M\partial_t^\tau\hat f_\tau\|_{L^2(0,\infty;(H^{2+d}(\Omega\times\mathbb{R}^d))')}\le C$. This and (4.56) yield (4.55). Likewise, we could deduce from (4.32) that

$$\int_0^\infty\int_\Omega\partial_t^\tau\mathbf{u}_\tau^L\cdot\mathbf{v}d\mathbf{x}dt\to-\int_0^\infty\int_\Omega\mathbf{u}\partial_t\mathbf{v}d\mathbf{n}d\mathbf{x}dt-\int_\Omega\mathbf{u}_{in}(\mathbf{x})\cdot\mathbf{v}(0,\mathbf{x})d\mathbf{x},$$
$$\partial_t^\tau\mathbf{u}_\tau\rightharpoonup\partial_t\mathbf{u}\ \text{ in }\ L^2\left(0,\infty;\left(\boldsymbol{V}^{2+[d/2]}\right)'\right).$$

Next, we prove

$$\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{n}}U\otimes\mathbf{n}\hat{f}_\tau:\nabla_{\mathbf{x}}\mathbf{v}d\mathbf{n}d\mathbf{x}dt\to\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{n}}U\otimes\mathbf{n}\hat{f}:\nabla_{\mathbf{x}}\mathbf{v}d\mathbf{n}d\mathbf{x}dt, \tag{4.57}$$

$$\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{x}}\hat{f}_\tau\cdot\nabla_{\mathbf{x}}\varphi d\mathbf{n}d\mathbf{x}dt\to\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{x}}\hat{f}\cdot\nabla_{\mathbf{x}}\varphi d\mathbf{n}d\mathbf{x}dt, \tag{4.58}$$

$$\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{n}}\hat{f}_\tau\cdot\nabla_{\mathbf{n}}\varphi d\mathbf{n}d\mathbf{x}dt\to\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{n}}\hat{f}\cdot\nabla_{\mathbf{n}}\varphi d\mathbf{n}d\mathbf{x}dt, \tag{4.59}$$

$$\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\big(\pi_\tau\mathbf{u}_\tau\cdot\nabla_{\mathbf{x}}\hat{f}_\tau\big)\varphi d\mathbf{n}d\mathbf{x}dt\to\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\big(\mathbf{u}\cdot\nabla_{\mathbf{x}}\hat{f}\big)\varphi d\mathbf{n}d\mathbf{x}dt, \tag{4.60}$$

$$\int_0^\infty\int_{\Omega}(\pi_\tau\mathbf{u}_\tau\cdot\nabla_{\mathbf{x}})\mathbf{u}_\tau\cdot\mathbf{v}d\mathbf{x}dt\to\int_0^\infty\int_{\Omega}(\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u}\cdot\mathbf{v}d\mathbf{x}dt, \tag{4.61}$$

$$\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{x}}\mathbf{u}_\tau\mathbf{n}E^{\tau^{-\frac{1}{4}}}(\hat{f}_\tau)\cdot\nabla_{\mathbf{n}}\varphi d\mathbf{n}d\mathbf{x}dt\to\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{x}}\mathbf{u}\mathbf{n}\hat{f}\cdot\nabla_{\mathbf{n}}\varphi d\mathbf{n}d\mathbf{x}dt. \tag{4.62}$$

In fact, it follows from Lemma 2.2, (4.42), and (4.41) that

$$\begin{aligned}
&\left|\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{n}}U\otimes\mathbf{n}(\hat{f}_\tau-\hat{f}):\nabla_{\mathbf{x}}\mathbf{v}d\mathbf{n}d\mathbf{x}dt\right|\\
&\quad=\left|\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{x}}\mathbf{v}\mathbf{n}\cdot\Big(\nabla_{\mathbf{n}}\hat{f}_\tau-\nabla_{\mathbf{n}}\hat{f}\Big)d\mathbf{n}d\mathbf{x}dt\right|\\
&\quad\le 2\left|\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{x}}\mathbf{v}\mathbf{n}\cdot\Big(\sqrt{\hat{f}_\tau}-\sqrt{\hat{f}}\Big)\nabla_{\mathbf{n}}\sqrt{\hat{f}_\tau}d\mathbf{n}d\mathbf{x}dt\right|\\
&\qquad+2\left|\int_0^\infty\int_{\Omega\times\mathbb{R}^d} M\nabla_{\mathbf{x}}\mathbf{v}\mathbf{n}\cdot\sqrt{\hat{f}}\Big(\nabla_{\mathbf{n}}\sqrt{\hat{f}_\tau}-\nabla_{\mathbf{n}}\sqrt{\hat{f}}\Big)d\mathbf{n}d\mathbf{x}dt\right|\to 0
\end{aligned}$$

and hence (4.57) holds. Similarly, one has (4.58) and (4.59).

Noting $\pi_\tau\mathbf{u}_\tau,\mathbf{u}\in\boldsymbol{V}$ and integrating by parts, we then establish (4.60) directly from (4.39) and (4.43). Clearly, (4.37) and (4.39) imply (4.61); (4.37) and (4.44) imply (4.62). Therefore, we have from Proposition 4.10 that Theorem 4.1 holds by setting $f:=M\hat{f}$. ◻

5. A remark on global weak entropy solutions to the FENE model. Following the argument of section 4, we can tackle the FENE model with $\varepsilon>0$, with initial data $f_{in}\in L^2(\Omega;L^1(B))$ instead of $f_{in}\in L^\infty(\Omega;L^1(B))$ in [5] by utilizing the estimate on the density equation (1.20).

Theorem 5.1. *Let $d=2,3$ and $k>1$ in the potential U (see* (1.23)*). Suppose $\mathbf{u}_{in}\in\boldsymbol{H}$ and $f_{in}\in L^2(\Omega;L^1(B))$ such that $f_{in}\ge 0$ a.e. on $\Omega\times B$, $\int_{\Omega\times B}[f_{in}(\ln\frac{f_{in}}{M}-1)+M]d\mathbf{n}d\mathbf{x}<\infty$. Then there exists a global weak entropy solution $(\mathbf{u},f)$ to the FENE model* (1.1)–(1.4) *with initial/boundary conditions* (1.9)–(1.11), *and it has regularity $\mathbf{u}\in H^1(0,\infty;(\boldsymbol{V}^2)')$, $f\in L^\infty(0,\infty;L^2(\Omega;L^1(B)))$.*

Proof. For the readers' convenience, we show the details in the appendix. ◻

Remark 5.2. Using a nearly identical argument with section 4, we can obtain the existence of global weak solutions to the FENE model with initial data $f_{in}\in L^\infty(\Omega;L^1(B))$ for $d=2,3,4$ and $k>1$.

Appendix A. We shall show the proof of Theorem 5.1, which is similar to that of section 4, and we follow the method of Barrett and Süli [5], [6] with some improvements as listed in the introduction. We know that U and M defined by (1.23) with $k > 1$ satisfy the same properties as Lemma 2.1 and Lemma 2.2 except that $\mathbb{R}^d$ is replaced by B. We label the corresponding results as Lemma 2.1_A and Lemma 2.2_A (a special case of Lemma 3.1 in [5]), respectively. Moreover, we have from $H^1_M(\Omega \times B) \hookrightarrow H^1_{M(1-|\mathbf{n}|)}(\Omega \times B)$ and Proposition 3.7 with $\epsilon = 1$, $p = 2$ that $H^1_M(\Omega \times B) \hookrightarrow\hookrightarrow L^2_M(\Omega \times B)$.

A.1. Approximate problem. For any fixed $0 < \tau \ll 1$ and for any $k \in \mathbb{N}$, given $(\mathbf{u}_{k-1}, \hat{f}_{k-1})$, the approximate problem of the FENE model (1.13)–(1.17) is exactly the same as (4.1)–(4.2) except that $\mathbb{R}^d$ is replaced by B, and we label these corresponding formulas as $(4.1)_A$–$(4.2)_A$. Moreover $(4.2)_A$ also implies a weak formulation of the discrete problem (1.20). We also denote $(4.3)_A$ by (4.3) with $\mathbb{R}^d$ replaced by B. The uniform convergence for $\hat{f}_k$ is based on $(4.3)_A$. Define $Z := \{\hat{f} \in L^2_M(\Omega \times B) : B) : \hat{f} \geq 0$ a.e. on $\Omega \times B\}$.

PROPOSITION A.1. *Let* $(\mathbf{u}_{k-1}, \hat{f}_{k-1}) \in \boldsymbol{V} \times Z$. *Then there exists* $(\mathbf{u}_k, \hat{f}_k) \in \boldsymbol{V} \times (Z \cap H^1_M(\Omega \times B))$ *which solves* $(4.1)_A$–$(4.2)_A$.

Proof. With $\mathbb{R}^d$ replaced by B, the proof is identical to that of Proposition 4.4 in view of Lemma 2.1_A and $H^1_M(\Omega \times B) \hookrightarrow\hookrightarrow L^2_M(\Omega \times B)$. ∎

A.2. Uniform estimates in $\boldsymbol{\tau}$, $\boldsymbol{\varepsilon}$, and time $\boldsymbol{t}$. Suppose $\mathbf{u}_{in} \in \boldsymbol{H}$ and $\hat{f}_{in} \in L^2(\Omega; L^1_M(B))$ such that $\hat{f}_{in} \geq 0$ a.e. on $\Omega \times B$, $\int_{\Omega\times B} MF(\hat{f}_{in}) d\mathbf{n} d\mathbf{x} < \infty$. We regularize $\mathbf{u}_{in}$ by $\mathbf{u}_0 = \mathbf{u}_0(\tau)$ which is the weak solution of $\mathbf{u}_0 - \tau^{\frac{1}{4}} \Delta \mathbf{u}_0 = \mathbf{u}_{in}$ with boundary condition $\mathbf{u}_0\big|_{\partial\Omega} = 0$ and it satisfies $\|\mathbf{u}_0\|^2_{L^2(\Omega)} + \tau^{\frac{1}{4}} \|\nabla_{\mathbf{x}} \mathbf{u}_0\|_{L^2(\Omega)} \leq \|\mathbf{u}_{in}\|^2_{L^2(\Omega)}$. Let $\hat{f}_0 = \hat{f}_0(\tau) := E^{\tau^{-\frac{1}{4}}}(\hat{f}_{in})$. Then $(\mathbf{u}_0, \hat{f}_0) \in \boldsymbol{V} \times Z$. Using Proposition A.1, as the time-step updates, we obtain a sequence of approximate solutions $(\mathbf{u}_k, \hat{f}_k) \in \boldsymbol{V} \times (Z \cap H^1_M(\Omega \times B))$ $(k \in \mathbb{N})$ to $(4.1)_A$–$(4.2)_A$. Define $\rho_{in} = \int_B M \hat{f}_{in} d\mathbf{n}$ and recall that $\rho_k = \int_B M \hat{f}_k d\mathbf{n}\, (k = 0, 1, 2, \ldots)$ in $(4.3)_A$.

LEMMA A.2.

$$\sup_{k \leq \mathbb{N}} \|\rho_k\|^2_{L^2(\Omega)} + \sum_{k=1}^{\infty} \|\rho_k - \rho_{k-1}\|^2_{L^2(\Omega)} + 2\varepsilon\tau \sum_{k=1}^{\infty} \|\nabla_{\mathbf{x}} \rho_k\|^2_{L^2(\Omega)}$$

$$\leq \|\rho_{in}\|^2_{L^2(\Omega)} = \|\hat{f}_{in}\|^2_{L^2(\Omega; L^1_M(D))}. \tag{A.1}$$

Proof. Noting $\rho_k \in H^1(\Omega)$ and taking $\psi = \rho_k$ in $(4.3)_A$, we have

$$\int_\Omega \frac{\rho_k - \rho_{k-1}}{\tau} \rho_k d\mathbf{x} + \int_\Omega (\mathbf{u}_{k-1} \cdot \nabla_{\mathbf{x}} \rho_k) \rho_k d\mathbf{x} + \varepsilon \int_\Omega |\nabla_{\mathbf{x}} \rho_k|^2 d\mathbf{x} = 0.$$

Using $\nabla_{\mathbf{x}} \cdot \mathbf{u}^L_k = 0$ and the identity (4.29), one has

$$\frac{1}{2} \int_\Omega |\rho_k|^2 d\mathbf{x} + \frac{1}{2} \int_\Omega |\rho_k - \rho_{k-1}|^2 d\mathbf{x} + \varepsilon\tau \int_\Omega |\nabla_{\mathbf{x}} \rho_k|^2 d\mathbf{x} = \frac{1}{2} \int_\Omega |\rho_{k-1}|^2 d\mathbf{x}. \tag{A.2}$$

Sum up (A.2) to obtain

$$\frac{1}{2} \int_\Omega |\rho_k|^2 d\mathbf{x} + \frac{1}{2} \sum_{i=1}^{k} \int_\Omega |\rho_i - \rho_{i-1}|^2 d\mathbf{x} + \varepsilon\tau \sum_{i=1}^{k} \int_\Omega |\nabla_{\mathbf{x}} \rho_i|^2 d\mathbf{x}$$

$$= \frac{1}{2} \int_\Omega |\rho_0|^2 d\mathbf{x} \leq \frac{1}{2} \int_\Omega |\rho_{in}|^2 d\mathbf{x}.$$

This ends the proof of Lemma A.2. ∎

LEMMA A.3. *For any $k \in \mathbb{N}$,*

$$\|\mathbf{u}_k\|^2_{L^2(\Omega)} + 2\int_{\Omega\times\mathbb{R}^d} MF(\hat{f}_k)d\mathbf{n}d\mathbf{x} + \sum_{i=1}^k \|\mathbf{u}_i - \mathbf{u}_{i-1}\|^2_{L^2(\Omega)}$$
$$+ 2\tau\sum_{i=1}^k \|\nabla_{\mathbf{x}}\mathbf{u}_i\|^2_{L^2(\Omega)} + 4\tau\sum_{i=1}^k\left(\varepsilon\left\|\nabla_{\mathbf{x}}\sqrt{\hat{f}_i}\right\|^2_{L^2_M(\Omega\times B)} + \left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_i}\right\|^2_{L^2_M(\Omega\times B)}\right)$$
$$\leq \|\mathbf{u}_{in}\|^2_{L^2(\Omega)} + 2\int_{\Omega\times B} MF(\hat{f}_{in})d\mathbf{n}d\mathbf{x}. \tag{A.3}$$

Proof. With $\mathbb{R}^d$ replaced by B, the proof is identical with that of Lemma 4.7 in view of Lemma 2.2_A. ∎

LEMMA A.4.

$$\tau\sum_{k=1}^\infty\left\|\frac{\mathbf{u}_k - \mathbf{u}_{k-1}}{\tau}\right\|^2_{(\boldsymbol{V}^2)'} + \tau\sum_{k=1}^\infty\left\|M\frac{\hat{f}_k - \hat{f}_{k-1}}{\tau}\right\|^2_{(H^{2+d}(\Omega\times B))'} \leq C. \tag{A.4}$$

Proof. It follows from $(4.1)_A$ that for any $\mathbf{v} \in \boldsymbol{V}^2$,

$$\left|\int_\Omega \frac{\mathbf{u}_k - \mathbf{u}_{k-1}}{\tau}\cdot\mathbf{v}d\mathbf{x}\right|$$
$$\leq \|\nabla_{\mathbf{x}}\mathbf{u}_k\|_{L^2(\Omega)}\|\nabla_{\mathbf{x}}\mathbf{v}\|_{L^2(\Omega)} + \|\nabla_{\mathbf{x}}\mathbf{u}_k\|_{L^2(\Omega)}\|\mathbf{u}_{k-1}\|_{L^2(\Omega)}\|\mathbf{v}\|_{L^\infty(\Omega)}$$
$$+ \left|\int_{\Omega\times B} M\nabla_{\mathbf{n}}U\otimes\mathbf{n}\hat{f}_k : \nabla_{\mathbf{x}}\mathbf{v}d\mathbf{n}d\mathbf{x}\right|.$$

We have from Lemma 2.2_A that

$$\left|\int_{\Omega\times B} M\nabla_{\mathbf{n}}U\otimes\mathbf{n}\hat{f}_k : \nabla_{\mathbf{x}}\mathbf{v}d\mathbf{n}d\mathbf{x}\right|$$
$$= \left|\int_{\Omega\times B} M\nabla_{\mathbf{x}}\mathbf{v}\mathbf{n}\cdot\nabla_{\mathbf{n}}\hat{f}_k d\mathbf{n}d\mathbf{x}\right|$$
$$\leq 2\|\nabla_{\mathbf{x}}\mathbf{v}\|_{L^4(\Omega)}\left\|\sqrt{\hat{f}_k}\right\|_{L^4(\Omega;L^2_M(B))}\left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times B)}$$
$$= 2\|\nabla_{\mathbf{x}}\mathbf{v}\|_{L^4(\Omega)}\|\rho_k\|^{\frac{1}{2}}_{L^2(\Omega)}\left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times B)}.$$

Then it follows from $\boldsymbol{V}^2 \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and $H^1(\Omega)\hookrightarrow L^4(\Omega)$ that

$$\left|\int_\Omega \frac{\mathbf{u}_k - \mathbf{u}_{k-1}}{\tau}\cdot\mathbf{v}d\mathbf{x}\right|$$
$$\leq 2\left[\|\nabla_{\mathbf{x}}\mathbf{u}_k\|_{L^2(\Omega)}\left(1 + \|\mathbf{u}_{k-1}\|_{L^2(\Omega)}\right) + \left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_k}\right\|_{L^2_M(\Omega\times B)}\|\rho_k\|^{\frac{1}{2}}_{L^2(\Omega)}\right]\|\mathbf{v}\|_{\boldsymbol{V}^2}$$

and hence from (A.1) and (A.3) we have that

$$\tau\sum_{k=1}^\infty\left\|\frac{\mathbf{u}_k - \mathbf{u}_{k-1}}{\tau}\right\|^2_{(\boldsymbol{V}^2)'} \leq C\tau\sum_{k=1}^\infty\|\nabla_{\mathbf{x}}\mathbf{u}_k\|^2_{L^2(\Omega)}\left(1 + \sup_{k\in\mathbb{N}}\|\mathbf{u}_{k-1}\|^2_{L^2(\Omega)}\right)$$
$$+ C\tau\sum_{k=1}^\infty\left\|\nabla_{\mathbf{n}}\sqrt{\hat{f}_k}\right\|^2_{L^2_M(\Omega\times B)}\sup_{k\in\mathbb{N}}\|\rho_k\|_{L^2(\Omega)} \leq C.$$

The proof of the latter estimate in (A.4) is nearly identical with that of (4.32). The only difference is that the estimate for P_4 in the proof of (4.32) is replaced by

$$\begin{aligned} P_4 &\leq C\|\nabla_{\mathbf{x}}\mathbf{u}_k\|_{L^2(\Omega)}\|\hat{f}_k\|_{L^2(\Omega;L^1_M(B))}\|\nabla_{\mathbf{n}}\varphi\|_{L^\infty(\Omega\times B)} \\ &\leq C\|\nabla_{\mathbf{x}}\mathbf{u}_k\|_{L^2(\Omega)}\|\varphi\|_{H^{2+d}(\Omega\times B)}. \qquad \square \end{aligned}$$

A.3. Convergence.

PROPOSITION A.5. *As* $\tau \to 0$, *there exists a subsequence of* $\{(\mathbf{u}_\tau, \hat{f}_\tau)\}_{0<\tau\ll 1}$, *not relabeled, and a pair of functions* $(\mathbf{u}, \hat{f})$ *satisfying* $\hat{f} \in L^\infty(0,\infty; L^2(\Omega; L^1_M(B)))$ *and other similar properties in* (4.34)–(4.35) *such that for any* $T > 0$, (4.36)–(4.37), (4.39), (4.41) *with* $\mathbb{R}^d$ *replaced by* B *hold and*

$$\mathbf{u}_\tau \to \mathbf{u} \quad \text{in } L^2(0,T;L^p(\Omega))\ (\forall 2 \leq p < 6), \tag{A.5}$$

$$\sqrt{\hat{f}_\tau} \overset{*}{\rightharpoonup} \sqrt{\hat{f}} \quad \text{in } L^\infty(0,\infty;L^4(\Omega;L^2_M(B))), \tag{A.6}$$

$$\hat{f}_\tau \to \hat{f} \quad \text{in } L^2((0,T)\times\Omega;L^1_M(B)), \tag{A.7}$$

$$\sqrt{\hat{f}_\tau} \to \sqrt{\hat{f}} \quad \text{in } L^4((0,T)\times\Omega;L^2_M(B)), \tag{A.8}$$

$$E^{\tau^{-\frac{1}{4}}}(\hat{f}_\tau) \to \hat{f} \quad \text{in } L^2((0,T)\times\Omega;L^1_M(B)). \tag{A.9}$$

Proof. Similarly to the proof of Proposition 4.10, we deduce that there exists a subsequence of $\{(\mathbf{u}_\tau, \hat{f}_\tau)\}_{0<\tau\ll 1}$, not relabeled, and a pair of functions $(\mathbf{u}, \hat{f})$, such that all the formulas except (A.5)–(A.9) hold and

$$\hat{f}_\tau \to \hat{f} \quad \text{in } L^1(0,T;L^1_M(\Omega\times B)), \tag{A.10}$$

$$\rho_\tau \to \rho \quad \text{in } L^2(0,T;L^p(\Omega))\ (\forall 2 \leq p < 6). \tag{A.11}$$

Then by noting $\rho_\tau = \int_B M\hat{f}_\tau d\mathbf{n}$, one has from (A.10) and (A.11) that

$$\rho = \int_B M\hat{f}d\mathbf{n} \ \text{ and } \int_B M\hat{f}_\tau d\mathbf{n} \to \int_B M\hat{f}d\mathbf{n} \text{ in } L^2(0,T;L^p(\Omega))\ (\forall 2 \leq p < 6). \tag{A.12}$$

It follows from the Gagliardo–Nirenberg inequality that

$$\|\rho_\tau\|_{L^4(\Omega)} \leq \|\rho_\tau\|^{\frac{d}{4}}_{H^1(\Omega)}\|\rho_\tau\|^{1-\frac{d}{4}}_{L^2(\Omega)}$$

and then from (A.1) and the Hölder inequality that

$$\begin{aligned} \|\hat{f}_\tau\|_{L^{\frac{8}{d}}(0,T;L^4(\Omega;L^1_M(B)))} &= \|\rho_\tau\|_{L^{\frac{8}{d}}(0,T;L^4(\Omega))} \\ &\leq \|\rho_\tau\|^{\frac{d}{4}}_{L^2(0,T;H^1(\Omega))}\|\rho_\tau\|^{1-\frac{d}{4}}_{L^\infty(0,T;L^2(\Omega))} \leq C(\varepsilon). \end{aligned}$$

Moreover, (A.1) also yields

$$\rho_\tau \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{A.13}$$

$$\rho_\tau \rightharpoonup \rho \quad \text{in } L^2(0,T;H^1(\Omega)). \tag{A.14}$$

Therefore $\hat{f} \in L^{\frac{8}{d}}(0,T;L^4(\Omega;L^1_M(B)))$. Consequently, by the interpolation inequality for L^p-norms, we have

$$\begin{aligned} \|\hat{f}_\tau - \hat{f}\|_{L^2((0,T)\times\Omega;L^1_M(B))} &\leq \|\hat{f}_\tau - \hat{f}\|^{\frac{4}{8-d}}_{L^{\frac{8}{d}}((0,T)\times\Omega;L^1_M(B))}\|\hat{f}_\tau - \hat{f}\|^{\frac{4-d}{8-d}}_{L^1((0,T)\times\Omega;L^1_M(B))} \\ &\leq C(\varepsilon)\|\hat{f}_\tau - \hat{f}\|^{\frac{4-d}{8-d}}_{L^1((0,T)\times\Omega;L^1_M(B))}. \end{aligned} \tag{A.15}$$

This and (A.10) imply (A.7). Based on (A.7), we can prove (A.8)–(A.9) with a discussion similar to that of Proposition 4.10. ◻

With Proposition A.5 and the uniform estimates Lemmas A.2–A.4 at hand, we can finish the proof of Theorem 5.1 by following the exact same arguments as those in section 4.3.2.

Remark A.6. (A.15) only holds for $d = 2, 3$. That is why we cannot deal with the four dimensional FENE model with initial data $f_{in} \in L^2(\Omega; L^1(B))$.

Acknowledgments. We thank Laurent Desvillettes for his proposal about Remark 3.13 and we thank Terrance Pendleton and Wei-Cheng Wang for some helpful comments and suggestions.

REFERENCES

[1] F. Antoci, *Some necessary and some sufficient conditions for the compactness of the embedding of weighted Sobolev spaces*, Ricerche Mat., 52 (2003), pp. 55–71.

[2] A. Arnold, J. A. Carrillo, and C. Manzini, *Refined long-time asymptotics for some polymeric fluid flow models*, Comm. Math. Sci., 8 (2010), pp. 763–782.

[3] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter, *On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations*, Comm. Partial Differential Equations, 26 (2001), pp. 43–100.

[4] H. Bae and K. Trivisa, *On the Doi model for the suspensions of rod-like molecules: Global-in-time existence*, Comm. Math. Sci., 2012, to appear.

[5] J. W. Barrett and E. Süli, *Existence and equilibration of global weak solutions to kinetic models for dilute polymers* I: *Finitely extensible nonlinear bead-spring chains*, Math. Models Methods Appl. Sci., 21 (2011), pp. 1211–1289.

[6] J. W. Barrett and E. Süli, *Existence and equilibration of global weak solutions to kinetic models for dilute polymers* II: *Hookean-type bead-spring chains*, Math. Models Methods Appl. Sci., 22 (2012), 1150024.

[7] J. W. Barrett and E. Süli, *Reflections on Dubinskiĭ's nonlinear compact embedding theorem*, Publ. Inst. Math. (Beograd) (N.S.), 91 (2012), pp. 95–110.

[8] A. Bonito, P. Clément, and M. Picasso, *Mathematical analysis of a simplified Hookean dumbbells model arising from viscoelastic flows*, J. Evol. Equations, 6 (2006), pp. 381–398.

[9] C. L. Bris and T. Lelièvre, *Multiscale modelling of complex fluids: A mathematical initiation*, in Multiscale Modeling and Simulation in Science, Lect. Notes Comput. Sci. Eng. 66, Springer, Berlin, 2009, pp. 49–137.

[10] X. Q. Chen, A. Jüngel, and J.-G. Liu, *Generalized Aubin-Lions Lemmas*, manuscript, 2012.

[11] P. Constantin, *Nonlinear Fokker-Planck Navier-Stokes systems*, Comm. Math. Sci., 3 (2005), pp. 531–544.

[12] P. Constantin, C. Fefferman, E. S. Titi, and A. Zarnescu, *Regularity of coupled two-dimensional nonlinear Fokker-Planck and Navier-Stokes systems*, Comm. Math. Phys., 270 (2007), pp. 789–811.

[13] P. Constantin and N. Masmoudi, *Global well-posedness for a Smoluchowski equation coupled with Navier-Stokes equations in* 2*D*, Comm. Math. Phys., 278 (2008), pp. 179–191.

[14] P. Constantin and G. Seregin, *Global regularity of solutions of coupled Navier-Stokes equations and nonlinear Fokker-Planck equations*, Discrete Contin. Dynam. Systems, 26 (2010), pp. 1185–1196.

[15] M. Doi and S. F. Edwards, *The Theory of Polymer Dynamics*, Oxford University Press, Oxford, 1986.

[16] M. Dreher and A. Jüngel, *Compact families of piecewise constant functions in* $L^p(0,T;B)$, Nonlinear Anal., 75 (2012), pp. 3072–3077.

[17] J. A. Dubinskiĭ, *Weak convergence for nonlinear elliptic and parabolic equations*, Mat. Sb., 67 (1965), pp. 609–642.

[18] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.

[19] J. G. Hooton, *Compact Sobolev imbeddings on finite measure spaces*, J. Math. Anal. Appl., 83 (1981), pp. 570–581.

[20] M. Ledoux, *The Concentration of Measure Phenomenon*, AMS, Providence, RI, 2001.

[21] B. Jourdain, C. L. Bris, T. Lelièvre, and F. Otto, *Long-time asymptotics of a multiscale model for polymeric fluid flows*, Arch. Ration. Mech. Anal., 181 (2006), pp. 97–148.
[22] J. Kačur, *Method of Rothe in Evolutions*, Teubner Texte zur Mathematik 80, B.G. Teubner, Leipzig, 1985.
[23] T. Li and P. Zhang, *Mathematical analysis of multi-scale models of complex fluids*, Comm. Math. Sci., 5 (2007), pp. 1–51.
[24] P. L. Lions and N. Masmoudi, *Global solutions of weak solutions to some micromacro models*, C. R. Acad. Sci. Paris Sér. I Math., 345 (2007), pp. 15–20.
[25] C. Liu and H. Liu, *Boundary conditions for the microscopic FENE models*, SIAM J. Appl. Math., 68 (2008), pp. 1304–1315.
[26] N. Masmoudi, *Global existence of weak solutions to the FENE dumbbell model of polymeric flows*, Invent. Math., 191 (2013), pp. 427–500.
[27] B. Opic, *Necessary and sufficient conditions for imbeddings in weighted Sobolev spaces*, Časopis pěst. Mat., 114 (1989), pp. 343–355.
[28] E. Rothe, *Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben*, Math. Ann., 102 (1930), pp. 650–670.
[29] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.
[30] Y. Z. Sun and Z. F. Zhang, *Global well-posedness for the 2D micro-macro models in the bounded domain*, Comm. Math. Phys., 303 (2011), pp. 361–383.
[31] L. Zhang, H. Zhang, and P. Zhang, *Global existence of weak solutions to the regularized Hookean dumbbell model*, Comm. Math. Sci., 6 (2008), pp. 85–124.