# Basic Themes and Pretty Problems of Nonlinear Solid Mechanics

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**Abstract.** The first part of this paper describes some important underlying themes in the mathematical theory of continuum mechanics that are distinct from formulating and analyzing governing equations. The main part of this paper is devoted to a survey of some concrete, conceptually simple, pretty problems that help illuminate the underlying themes. The paper concludes with a discussion of the crucial role of invariant constitutive equations in computation.

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# I. Basic Themes

## 1. Introduction

The aim of the science of continuum physics is to describe the mechanical, thermodynamical, electromagnetic, and chemical behavior of matter in bulk. This science consists of a rich collection of general mathematical theories based upon a handful of fundamental principles, which are largely well understood. These principles are supplemented with constitutive equations, which describe the material properties of the physical bodies under study.

The combination of fundamental principles and constitutive equations typically leads to quasilinear systems of partial differential equations or nonlocal variants thereof, which are largely not well understood. In the past 50 years, the governing equations of nonlinear continuum physics have been made to look simple, so that their mathematical structure can be exposed. But they are not simple: They offer severe mathematical challenges at the cutting edge or beyond of modern analysis. This is wonderful, because we are not paid to solve simple problems. Responding to the challenges presented by the initial-boundary-value problems of nonlinear continuum physics should profitably occupy scientists for years to come. But there are other mathematical objectives, besides treating initial-boundary-value problems, that should be met in order to enhance the utility of the theories. The purpose of this paper is to describe some of these objectives in the context of the continuum mechanics of solids and then illustrate some small steps toward meeting these objectives with the solution of some fairly elementary pretty problems.

We first sketch the mathematical structure of the 3-dimensional theory, and then describe the 1-dimensional theory from which we obtain our concrete problems. For simplicity of exposition, we assume that any function that is exhibited is ipso facto continuous. We denote partial derivatives with respect to scalar arguments by subscripts and denote some ordinary derivatives by primes. Occasionally we denote time derivatives by superposed dots.

## 2. Continuum Mechanics

**Geometry of deformation**. We identify the material points of a body with the positions $\boldsymbol{x}$ they occupy in some specific reference configuration. These material points $\boldsymbol{x}$ lie in a region $\mathcal{B}$. Let $\boldsymbol{p}(\boldsymbol{x}, t)$ denote the position of material point $\boldsymbol{x}$ at time $t$. Then the *velocity* and *acceleration* of $\boldsymbol{x}$ at time $t$ are $\boldsymbol{p}_t(\boldsymbol{x}, t)$ and $\boldsymbol{p}_{tt}(\boldsymbol{x}, t)$. The fundamental aim of continuum mechanics is to formulate and analyze equations for $\boldsymbol{p}$ when the mechanical properties of $\mathcal{B}$ and its environment are given.

The function $\boldsymbol{p}(\cdot, t)$ should be one-to-one for all $t$, so that distinct material points cannot simultaneously occupy the same position, and should have a positive Jacobian:

$$\det \boldsymbol{p}_{\boldsymbol{x}}(\boldsymbol{x}, t) > 0 \tag{2.1}$$

for (almost) all $\boldsymbol{x}$ and $t$, so that the local ratio of deformed to reference volume never be reduced to 0, and so that the deformation never changes its orientation. Here $\boldsymbol{p_x}$ is the (Fréchet) derivative of $\boldsymbol{p}(\cdot, t)$. Its Cartesian components form a matrix with entries $\partial p_i/\partial x_j$. (For an incompressible body, $\det \boldsymbol{p_x}(\boldsymbol{x}, t) = 1$, a nonlinear partial differential equation.) Why worry about total compression and change of orientation? Because if they not precluded by the constitutive functions, these phenomena can occur for solutions, both analytical and numerical, especially near places and times where the solutions have singularities (no matter how pretty the graphics for the solutions are).

**Stress and the Equations of Motion.** Let $\boldsymbol{x}$ be a material point in body $\mathcal{B}$ and let $\boldsymbol{n}$ be the unit normal to a planar region $\mathcal{P}$ consisting of $\boldsymbol{x}$ and other material points in $\mathcal{B}$. Then the force per unit reference area of $\mathcal{P}$ exerted by the material of $\{\boldsymbol{y} \in \mathcal{B} : (\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{n} > 0\}$ on the material of $\{\boldsymbol{y} \in \mathcal{B} : (\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{n} \leq 0\}$ at material point $\boldsymbol{x}$ at time $t$ can be shown to have the form $\boldsymbol{T}(\boldsymbol{x}, t)\boldsymbol{n}$ where $\boldsymbol{T}$ is a second-order tensor (linear transformation from Euclidean 3-space into itself) called the *first Piola-Kirchhoff stress tensor*. The requirement that the total force on any part of the body at time $t$ equal the time derivative of the linear momentum of that part of the body (a version of Newton's Law of Motion) leads to a local version of the equation of motion for the body $\mathcal{B}$:

$$\nabla \cdot \boldsymbol{T}^* + \boldsymbol{f} = \rho \boldsymbol{p}_{tt}. \tag{2.2}$$

Here the asterisk denotes the transpose, $\rho(\boldsymbol{x})$ is the given mass density at $\boldsymbol{x}$ in the reference configuration, and $\boldsymbol{f}$ is the given force per unit reference volume exerted by external agencies. The $i$th Cartesian component of $\nabla \cdot \boldsymbol{T}^*$ is $\sum_{j=1}^{3} \partial T_{ij}/\partial x_j$. Equations like (2.2) in which the independent variables $\boldsymbol{x}$ identify material points is said to have a *material* (= *referential* = *Lagrangian*) formulation (due to Euler).

**Constitutive equations**. Now we introduce into these general laws what distinguishes different materials and what ensures that the number of equations equal the number of unknowns. We characterize the mechanical behavior of materials by giving a constitutive function that specifies how $\boldsymbol{T}$ depends on the function $\boldsymbol{p}$. Its substitution into the equation of motion (2.2) converts it into an equation for $\boldsymbol{p}$.

One of the conceptually simplest constitutive equations is that for an *elastic* material, for which the stress at a given time depends only on the state of deformation at that time. It has the form

$$\boldsymbol{T}(\boldsymbol{x},t)=\boldsymbol{p}_{\boldsymbol{x}}(\boldsymbol{x},t)\boldsymbol{S}(\boldsymbol{C}(\boldsymbol{x},t),\boldsymbol{x}),\qquad \boldsymbol{C}:=\boldsymbol{p}_{\boldsymbol{x}}^{*}\boldsymbol{p}_{\boldsymbol{x}} \tag{2.3}$$

where $\boldsymbol{S}$ is a symmetric tensor-valued function. The symmetry of $\boldsymbol{S}$ ensures that the total torque on any part of the body equal the time derivative of the angular momentum of that part of the body. The special way that $\boldsymbol{p}_{\boldsymbol{x}}$ enters this equation, both directly and in $\boldsymbol{C}$, is a consequence of the requirement that material properties be unaffected by rigid motions. (It is the acceleration in the equation of motion that accounts for rigid motions.) We shall demonstrate the importance of this invariance under rigid motions.

This constitutive equation makes no provision for internal mechanical dissipation. One that does is the constitutive equation for a *viscoelastic material of strain-rate type*:

$$\boldsymbol{T}(\boldsymbol{x},t)=\boldsymbol{p}_{\boldsymbol{x}}(\boldsymbol{x},t)\boldsymbol{S}(\boldsymbol{C}(\boldsymbol{x},t),\boldsymbol{C}_t(\boldsymbol{x},t),\boldsymbol{x}). \tag{2.4}$$

More generally, the tensor $\boldsymbol{S}$ at $(\boldsymbol{x},t)$ could depend upon the past history of $\boldsymbol{C}$ at $\boldsymbol{x}$ (and could even depend on $\boldsymbol{C}(\boldsymbol{y}-\boldsymbol{x},\tau)$ for $\boldsymbol{y}$ in $\mathcal{B}$ and for $\tau\le t$). Such a dependence ensures invariance under rigid motions.

The material behavior in the form of the constitutive function for $\boldsymbol{T}$ completely determines the mathematical structure of the equations. For example, for an elastic material, the requirement that $\boldsymbol{T}$ be monotone on lines in $\boldsymbol{p}_{\boldsymbol{x}}$ space has the reasonable physical implication that certain components of $\boldsymbol{T}$ must be increasing functions of corresponding components of $\boldsymbol{p}_x$. An elastic material having this property satisfies the Strong Ellipticity Condition (one of the weakest ellipticity conditions), which ensures that the equations of motion are hyperbolic in a way that allows the richest wave-like behavior. The analysis of the equations of equilibrium and motion under this condition [7, 16] is technically difficult, presenting many unresolved questions. Many other mathematically useful constitutive restrictions are unacceptable on physical grounds. It is natural to require the viscoelastic material of (2.4) to account for mechanical dissipation, e.g., by requiring $\boldsymbol{T}$ to be a monotone function of $\boldsymbol{p}_{\boldsymbol{x}t}$. This gives the equations of motion a parabolic character. The analysis of such systems is just beginning.

For a given material, what is the constitutive function for $\boldsymbol{T}$? For many common fluids, including water and air, it is Newtonian, for which the equations of motion are the Navier-Stokes equations, the most successful model in all of continuum mechanics. (These equations are typically given a *spatial* (= *Eulerian* formulation, due to d'Alembert) in which the independent spatial variable is a fixed position in space rather than a material point.)

But for most other fluids, including such common non-Newtonian liquids as paint, lubricating oil, and sour milk, there is no consensus as to what $\boldsymbol{T}$ should be. The same situation prevails for metals undergoing large or fast deformations.

Experimental results, not only for new materials, but even for such old materials like metals, paint, and muscle, are usually fragmentary. Efforts using statistical-mechanical ideas to construct constitutive models agreeing with experiment have frequently failed to produce a consensus.

As we shall see, the equations for thin bodies, e.g., rods and shells, have a mathematical structure similar to that of (2.2)–(2.4), but with fewer independent spatial variables and more dependent variables.

Among the elementary introductions to the material of this section are [5, 29]. Among advanced presentations are [7, 16, 30, 44, 46].

## 3. The Role of Constitutive Equations in Analysis

The analysis of the resulting evolution partial differential equations for physically reasonable constitutive functions is at its infancy, even for one independent spatial variable. There are still deep open problems for steady-state solutions. The invariance under rigid motions is a source of technical difficulty emanating from geometry.

For example, in elasticity where $\boldsymbol{T} = \hat{\boldsymbol{T}}(\boldsymbol{p}_x, \boldsymbol{x})$, mathematical analysis would be much easier if

$$\boldsymbol{A} : \frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{p}_x} : \boldsymbol{A} > \mathrm{const}|\boldsymbol{A}|^2 \quad \forall\, \boldsymbol{A}, \tag{3.1}$$

but geometry and mechanics show that this condition is not reasonable. A more natural condition is that this inequality hold only for all tensors of rank 1. But the analysis of special cases of this condition, begun 30 years ago by Ball [14], still has deep open problems.

The further weakening of this condition (by allowing potential wells) has led to exciting new analyses in which there are problems in the calculus of variations lacking minimizers, for which infimizing sequences carry detailed information about the microscopic behavior of crystalline solids [24, 26, 38]. Here is a case where the continuum theory gives useful information about the molecular (statistical-mechanical) theory.

The dynamical equations of nonlinear elasticity admit shocks. When shocks occur, general integral formulations of the equations of motion fail to deliver a unique way to continue a solution past the time of a shock. What is needed is an admissibility condition that picks out a unique continuation that can be judged as physically reasonable.

Various admissibility conditions have been propounded on the basis of purely mathematical criteria to ensure uniqueness, on the basis of thermodynamic arguments associated with the concept of entropy to ensure that shocks have a suitable dissipative character, on the basis of limit arguments in which the shock is realized as a discontinuity that arises in the asymptotic limit of a problem with dissipation as the dissipation goes to zero, and on the basis of numerical procedures [23, 42]. While these methods often agree for simple problems with one spatial dimension, in particular for models of gas dynamics, they have not been reconciled for more complicated problems, even with but one spatial dimension, despite a growing body of penetrating research. In particular, as we shall see, many standard numerical methods for equations for solids (inspired by gas dynamics) lack the appropriate invariance, and can give erroneous solutions (which nevertheless have beautiful graphics).

Our lack of detailed information about constitutive equations for many real materials (other than Newtonian fluids like air and water) need not prevent us from mathematically analyzing problems for such materials. We can exploit our agnostic lack of detailed information about constitutive functions by using and developing powerful qualitative methods to treat whole classes of materials at one time. One aim of such an analysis would be to determine thresholds in material response separating materials having qualitatively different behavior in certain environments. We shall give some simple examples of this process.

## 4. Experimentation and Inverse Problems

There are fundamental scientific difficulties in determining constitutive response experimentally: If we are presented with an unknown material that is known to be of a certain class, say, viscoelastic of strain-rate type (2.4), then, in principle, we could formulate a program of experiments to determine $\boldsymbol{S}$ in which $\boldsymbol{p}_{\boldsymbol{x}}$ and $\boldsymbol{p}_{\boldsymbol{x}t}$ are imposed and $\boldsymbol{T}$ is measured. But if we do not know the class of constitutive equations within the manifold possibilities, then it must be guessed, and an experimental program carried out for each such guess.

There are many areas in which we need far more compelling information on constitutive response: large-strain thermoviscoplasticity, non-Newtonian fluids, electro-magneto-mechanical interactions. In each of these areas, memory effects play a central role, and in each of these areas mathematical analysis that confronts the underlying physics is in its infancy.

The needed mathematical tools are more powerful inverse methods that allow one to determine properties of an equation from information about its solutions. A very simple example of such an inverse methods will be given in Section 8. (There is an extensive body of literature on a variety of inverse problems, most related to linear differential equations, which give rise to nonlinear inverse problems; See [32] and the works cited therein.)

## 5. Rationalization of Hierarchies of Theories

There are numerous theories of continuum physics that are related to each other as certain parameters go to limiting values:

1. 3-D solid mechanics $\longrightarrow$ 1-dimensional rod theories as thickness $\to 0$.
2. 3-dimensional solid mechanics $\longrightarrow$ 2-dimensional shell theories as thickness $\to 0$.
3. 3-dimensional solid mechanics $\longrightarrow$ 1-dimensional rod theories or 2-dimensional shell theories as degrees of freedom become small.
4. Rod theories $\longrightarrow$ (real) string theory as bending stiffness $\to 0$.
5. Shell theories $\longrightarrow$ membrane theory as bending stiffness $\to 0$.
6. 3-dimensional fluid mechanics $\longrightarrow$ 2-dimensional lubrication theories as thickness $\to 0$.
7. 3-dimensional gas dynamics $\longrightarrow$ 1-dimensional piston theories as thickness $\to 0$.
8. 3-dimensional electromagnetism $\longrightarrow$ 1-dimensional wire theories as thickness $\to 0$.
9. The equations for compressible media $\longrightarrow$ those for incompressible media as a constitutive function giving the Jacobian as a function of the stress and other kinematical variables approaches 1.
10. Dynamical equations $\longrightarrow$ quasistatic equations as inertias $\to 0$.
11. Dynamical equations $\longrightarrow$ quasistatic equations as forcing rates $\to 0$.
12. Smooth dynamical processes $\longrightarrow$ processes admitting shocks as viscosity or capillarity $\to 0$.
13. Smooth spatial transitions $\longrightarrow$ sharp transitions as capillarity or strain-gradient effects $\to 0$.

14. Statistical physics $\longrightarrow$ continuum physics as the number of particles $\to \infty$.

There has been considerable progress in justifying these processes, and yet there remain deep, interesting, and challenging problems in giving them completely satisfying resolutions. (See [7] for a discussion of the justification of theories for thin bodies.) Here we shall limit our attention to simple examples of processes 10, 11, 12.

There are an immense number of specific applications that form a useful and fascinating complement to the grand open problems of analysis just discussed. Here is a sampling of very elementary versions.

# II. Simple Problems

## 6. The Planar Motion of Rods

In this section we outline the derivation of the partial differential equations for the planar motion of nonlinearly elastic and viscoelastic rods, which are slender solid bodies. The equations we obtain have exactly the same form as those for the plane-strain motion of a cylindrical shell (not necessarily having a circular cross section), in which each section perpendicular to the generators has the same behavior. We shall analyze these equations for a variety of simple problems. These equations are far easier to treat than those discussed in Part I chiefly because they have but one independent spatial variable.

We study the planar motion of a flexible, extensible, shearable nonlinearly elastic rod. The governing equations form the simplest system for a nonlinearly elastic body in which rotation is manifested, in which Lagrange multipliers do not appear (as they would if the rod were inextensible or unshearable or both), and in which the mechanical variables are just resultant forces and couples (in contrast to the equations in a hierarchy of more sophisticated rod theories).

Let $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be a fixed right-handed orthonormal basis for Euclidean 3-space. The *configuration* at time $t$ of a flexible, extensible, shearable rod constrained to move in the $\{\boldsymbol{i}, \boldsymbol{j}\}$-plane is specified by an absolutely continuous vector-valued function $[0, l] \ni s \mapsto \boldsymbol{r}(s, t) \in \operatorname{span}\{\boldsymbol{i}, \boldsymbol{j}\}$ and by a scalar-valued function $[0, l] \ni s \mapsto \theta(s, t)$. The curve $\boldsymbol{r}(\cdot, t)$ may be interpreted as the image at time $t$ of a material curve $s \mapsto \boldsymbol{r}^\circ(s)$ lying within a thin 2-dimensional body in its reference configuration, and the angle $\frac{\pi}{2} + \theta(s,t)$ may be interpreted as characterizing the orientation with respect to $\boldsymbol{i}$ at time $t$ of the material cross-section at the material point with coordinate $s$ in the reference configuration. We take $s$ to be the arc-length parameter of $\boldsymbol{r}^\circ$. See Figure 1.

We introduce the orthonormal basis

$$\boldsymbol{a}(\theta) := \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}, \quad \boldsymbol{b}(\theta) := -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}. \tag{6.1}$$

The vector $\boldsymbol{b}(\theta(s,t))$ gives the orientation of the cross section at $s$ at time $t$, so that we could equally well characterize the configuration at time $t$ by $s \mapsto \boldsymbol{r}(s,t), \boldsymbol{b}(\theta(s,t))$, and we shall do so when convenient. We define the *strain* variables $(\nu, \eta, \mu)$ by

$$\nu := \boldsymbol{r}_s \cdot \boldsymbol{a}(\theta), \quad \eta := \boldsymbol{r}_s \cdot \boldsymbol{b}(\theta), \quad \mu := \theta_s. \tag{6.2}$$

If $t \mapsto (\boldsymbol{r}(\cdot,t), \boldsymbol{b}(\theta(\cdot,t)))$ is a given motion of the rod, then a motion differing from it by a rigid motion has the form

$$t \mapsto \big(\boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{r}(\cdot,t),\ \boldsymbol{Q}(t)\boldsymbol{b}(\theta(\cdot,t))\big) \tag{6.3}$$

where $\boldsymbol{c}$ is a vector-valued function of $t$ only and where $\boldsymbol{Q}$ is an orthogonal transformation of span $\{\boldsymbol{i},\boldsymbol{j}\}$ to itself, depending only on $t$. The matrix of $\boldsymbol{Q}$ with respect to the basis $\{\boldsymbol{i},\boldsymbol{j}\}$ has the form $\begin{bmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{bmatrix}$. It is important to note that the strains for (6.3) are independent of the rigid motion given by $\boldsymbol{c}$ and $\psi$. A necessary condition that deformations preserve orientation is that $\nu$ be everywhere positive.

Let $\boldsymbol{n}(s,t) \equiv N(s,t)\boldsymbol{a}(\theta(s,t)) + H(s,t)\boldsymbol{b}(\theta(s,t))$ be the internal contact force and $M(s,t)$ be the internal contact couple (about $\boldsymbol{k}$) exerted across the material section at $s$. The component $\boldsymbol{n} \cdot (\boldsymbol{r}_s/|\boldsymbol{r}_s|)$ is the tension and the component $H$ is the shear force. Then the classical equations of motion for the rod subject to an external force $\boldsymbol{f}$ per unit reference length and no external couple have the form

$$\rho A \boldsymbol{r}_{tt} = (N\boldsymbol{a} + H\boldsymbol{b})_s + \boldsymbol{f}, \tag{6.4a}$$

$$\rho J \theta_{tt} = M_s + \boldsymbol{k} \cdot [\boldsymbol{r}_s \times (N\boldsymbol{a} + H\boldsymbol{b})], \tag{6.4b}$$

where $\rho A$ and $\rho J$ are positive-valued functions of $s$ [7]. (For a naturally straight rod, $\rho A$ can be interpreted as the mass density per unit reference length and $\rho J$ can be interpreted as the mass moment of inertia of a cross section per unit reference length. For curved rods the interpretations are similar.)

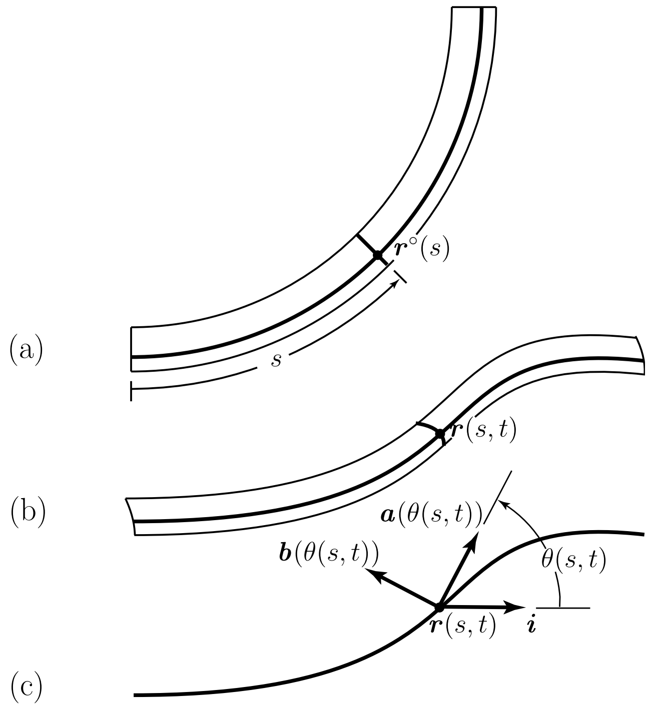



FIGURE 1. (a) A planar rod in its reference configuration. The heavy curve is the base curve $\boldsymbol{r}^\circ$, with arc-length parameter $s$. The short heavy line perpendicular to $\boldsymbol{r}^\circ$ at $s$ is the cross section at $s$. (b) A typical configuration at time $t$ of the rod of (a). The material point $\boldsymbol{r}^\circ(s)$ goes to $\boldsymbol{r}(s,t)$. The long heavy curve is the image at time $t$ of $\boldsymbol{r}^\circ$. The short curve through $\boldsymbol{r}(s,t)$ is the image of the cross section. (c) The rod theory used here models the configuration of (b) by the curve $\boldsymbol{r}(\cdot,t)$ and by the scalar field $\theta(\cdot,t)$ or, equivalently, the unit-vector field $\boldsymbol{b}(\theta(\cdot,t))$, which characterizes the orientation of the cross sections.

The material of the rod is said to be *elastic* if there are constitutive functions $(\nu,\eta,\mu,s)\mapsto N^{\mathrm{E}}(\nu,\eta,\mu,s)$, $H^{\mathrm{E}}(\nu,\eta,\mu,s)$, $M^{\mathrm{E}}(\nu,\eta,\mu,s)$ such that

$$N(s,t)=N^{\mathrm{E}}(\nu(s,t),\eta(s,t),\mu(s,t),s),\quad \text{etc.} \tag{6.5}$$

The material of the rod is said to be *viscoelastic of strain-rate type* if there are constitutive functions $(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}, s) \mapsto \hat{N}(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}, s)$, $\hat{H}(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}, s)$, $\hat{M}(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}, s)$ such that

$$N(s,t) = \hat{N}(\nu(s,t), \eta(s,t), \mu(s,t), \nu_t(s,t), \eta_t(s,t), \mu_t(s,t), s), \quad \text{etc.} \tag{6.6}$$

Note that the fourth and fifth arguments are time derivatives of components of $\boldsymbol{r}_s$ with respect to the moving basis $\{\boldsymbol{a}(\theta(s,t)), \boldsymbol{b}(\theta(s,t))\}$ and not the components of the time derivatives. Since $\hat{N}$, $\hat{H}$, $\hat{M}$ depend only on the indicated arguments, the components of the internal contact force and couple are unaffected by rigid motions.

The material (6.5) is *hyperelastic* if there is a *stored-energy function* $(\nu, \eta, \mu) \mapsto W(\nu, h, \mu, s)$ such that

$$N^{\mathrm{E}} = W_\nu, \quad H^{\mathrm{E}} = W_\eta, \quad M^{\mathrm{E}} = W_\mu. \tag{6.7}$$

We assume that the matrix

$$\begin{bmatrix} W_{\nu\nu} & W_{\nu\eta} & W_{\nu\mu} \\ W_{\eta\nu} & W_{\eta\eta} & W_{\eta\mu} \\ W_{\mu\nu} & W_{\mu\eta} & W_{\mu\mu} \end{bmatrix} \quad \text{is positive-definite.} \tag{6.8}$$

This condition ensures that the motion of hyperelastic rods is governed by a hyperbolic system. In the description of materials that can undergo a phase change, this positive-definiteness is weakened, at least on a compact set of $(\nu, \eta, \mu)$-space [33].

For viscoelastic materials, it is convenient to define the *equilibrium response functions* by

$$N^{\mathrm{E}}(\nu, \eta, \mu, s) := \hat{N}(\nu, \eta, \mu, 0, 0, 0, s), \quad \text{etc.,} \tag{6.9}$$

and to define the *dissipative part* of the constitutive functions by

$$N^{\mathrm{D}}(\nu, \eta, \mu, \nu_t, \eta_t, \mu_t, s) := \hat{N}(\nu, \eta, \mu, \nu_t, \eta_t, \mu_t, s) - N^{\mathrm{E}}(\nu, \eta, \mu, s), \quad \text{etc.,} \tag{6.10}$$

which vanish in equilibrium. It is natural to assume that the *equilibrium response functions* are hyperelastic, in which case (6.6) has the form

$$N = W_\nu(\nu, \eta, \mu, s) + N^{\mathrm{D}}(\nu, \eta, \mu, \nu_t, \eta_t, \mu_t, s), \quad \text{etc.} \tag{6.11}$$

We assume that the matrix

$$\begin{bmatrix} N^{\mathrm{D}}_{\dot{\nu}} & N^{\mathrm{D}}_{\dot{\eta}} & N^{\mathrm{D}}_{\dot{\mu}} \\ H^{\mathrm{D}}_{\dot{\nu}} & H^{\mathrm{D}}_{\dot{\eta}} & H^{\mathrm{D}}_{\dot{\mu}} \\ M^{\mathrm{D}}_{\dot{\nu}} & M^{\mathrm{D}}_{\dot{\eta}} & M^{\mathrm{D}}_{\dot{\mu}} \end{bmatrix} \quad \text{is positive-definite.} \tag{6.12}$$

This positive-definiteness ensures that the motion of viscoelastic rods is governed by a parabolic-hyperbolic system and that the equations have a dissipative character.

It can be shown [7, Sec. 8.7] that a 1-dimensional version of (2.1) has the form

$$\nu > V(\mu, s) \tag{6.13}$$

where $V(\cdot, s)$ is linear and nowhere negative for $\mu > 0$ and for $\mu < 0$, and $V(0, s) = 0$. The limiting condition that $\nu = V(\mu, s)$ corresponds to a total compression. We require that infinite resultants are needed to effect a total compression:

$$|\hat{N}| + |\hat{H}| + |\hat{M}| \to \infty \quad \text{as} \quad \nu - V(\mu, s) \searrow 0. \tag{6.14}$$

Condition (6.13) gives the domain of definition of the constitutive functions $\hat{N}, \hat{H}, \hat{M}$.

We assume that the rod has the natural symmetry property that it is no harder to shear the rod in one sense than in its opposite sense:

$$\begin{aligned} \hat{N}(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}, s) &= \hat{N}(\nu, -\eta, \mu, \dot{\nu}, -\dot{\eta}, \dot{\mu}, s), \\ -\hat{H}(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}, s) &= \hat{H}(\nu, -\eta, \mu, \dot{\nu}, -\dot{\eta}, \dot{\mu}, s), \\ \hat{M}(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}, s) &= \hat{M}(\nu, -\eta, \mu, \dot{\nu}, -\dot{\eta}, \dot{\mu}, s). \end{aligned} \tag{6.15}$$

For a naturally straight rod, $\hat{H}$ and $\hat{M}$ change sign while $\hat{N}$ is unchanged when $(\eta, \mu, \dot{\eta}, \dot{\mu})$ is replaced by $(-\eta, -\mu, -\dot{\eta}, -\dot{\mu})$ [7, Sec. 16.10].

Assuming that the initial-boundary-value problem for (6.4) has a sufficiently regular solution, we take the dot product of (6.4a) with $\boldsymbol{r}_t$, multiply (6.4b) by $\theta_t$, integrate the sum of the resulting products over $[0, l] \times [0, t]$, and use (6.11) to obtain the *energy equation*

$$\int_0^l [\tfrac{1}{2}(\rho A \boldsymbol{r}_t \cdot \boldsymbol{r}_t + \rho J \theta_t^2) + W]\, ds + \int_0^t \int_0^l [N^{\mathrm{D}} \nu_t + H^{\mathrm{D}} \eta_t + M^{\mathrm{D}} \mu_t - \boldsymbol{f} \cdot \boldsymbol{r}_t]\, ds\, d\tau = \text{const.} \tag{6.16}$$

Note that (6.12) implies that $N^{\mathrm{D}} \nu_t + H^{\mathrm{D}} \eta_t + M^{\mathrm{D}} \mu_t \geq 0$.

The variables introduced in this section correspond to those of Section 2: The configuration $\boldsymbol{p}$ of Section 2 corresponds to $\boldsymbol{r}, \theta$; the deformation tensor $\boldsymbol{C}$ to $(\nu, \eta, \mu)$; the stress tensor $\boldsymbol{T}$ to $(N, H, M)$; the condition (2.1) of orientation-preservation to (6.13); the equation of motion (2.2) to (6.4); and the constitutive equation (2.4) to (6.6).

A detailed discussion of the equations of this section, their generalizations, and their relation to the 3-dimensional theory is given in [7]. An existence and regularity theory, global in time, for the generalization of these equations to spatial motions is given by [12].

## 7. Constitutive Thresholds: Radial Motion of a Ring

A uniform circular viscoelastic ring is subject to a spatially uniform hydrostatic pressure of intensity $p(t)$ per unit of *actual* length. $p$ is positive if it is internal, i.e., if it acts outward. Since such a pressure acts normal to $\boldsymbol{r}$, the body force $\boldsymbol{f} = -p\boldsymbol{k} \times \boldsymbol{r}_s$ where we take $s$ to increase in the counterclockwise sense. We take the reference length $l = 2\pi$, assume that the constitutive functions (6.6) are independent of $s$, and that the reference configuration for which $\nu = 1$, $\eta = 0$, $\mu = 1$ is natural, so that all the constitutive functions vanish for these strains. Here we limit our attention to purely radial motions:

$$\theta(s,t) = s, \qquad \boldsymbol{r}(s,t) = -r(t)\boldsymbol{b}(s). \tag{7.1}$$

(We are taking $\boldsymbol{b}(s)$ to point inward.) Thus $\nu(s,t) = r(t)$, $\eta(s,t) = 0$, $\mu(s,t) = 1$. Condition (6.15) then implies that (6.4b) with (6.6) is identically satisfied, and that (6.4a) reduces to

$$\rho A r_{tt} + \hat{N}(r(t), r_t(t)) = p(t)r(t) \tag{7.2}$$

where $\hat{N}(r, \dot{r}) := \hat{N}(r, 0, 1, \dot{r}, 0, 0)$. (The uniformity of the ring means that $\hat{N}$ is independent of $s$.

We first treat the equilibrium problem, for which (7.2) reduces to

$$N^{\mathrm{E}}(r) := W_\nu(r, 0, 1) = pr. \tag{7.3}$$

Condition (6.8) implies that $N^{\mathrm{E}}_\nu(\nu) > 0$, the requirement that the reference configuration be natural implies that $N^{\mathrm{E}}(1) = 0$, and (6.14) implies that $N^{\mathrm{E}}(\nu) \to -\infty$ as $\nu \searrow 0$. For a given $p$ there are as many equilibrium states as there are solutions $r$ of this equation. See Figure 2. For $p \le 0$, (7.3) has exactly one solution. For $p$ positive the situation is much richer: If $N^{\mathrm{E}}$ is an asymptotically strictly superlinear function as its argument approaches $\infty$, i.e., if $N^{\mathrm{E}}(r)/r \to \infty$ as $r \to \infty$, then (7.3) always has at least one solution and may have any algebraically odd number of solutions. If $N^{\mathrm{E}}$ is an asymptotically strictly sublinear function, i.e., if $N^{\mathrm{E}}(r)/r \to 0$ as $r \to \infty$, then (7.3) has at least two solutions for $p$ and always has an algebraically even number of solutions for $p$ less than a critical value and no solutions for $p$ exceeding that value. If $N^{\mathrm{E}}$ is neither asymptotically strictly superlinear nor sublinear, in particular, if $N^{\mathrm{E}}$ is asymptotically linear, i.e., if there is a number $B \in (0, \infty)$ such that $N^{\mathrm{E}}(r)/r \to B$ as $r \to \infty$, then the solvability requires a special analysis. For an asymptotically linear $N^{\mathrm{E}}$, there are no solutions for sufficiently large $p$. The traditional assumption that $N$ is linear in $r - 1$ is thus atypical: There is either exactly one solution or no solution, depending on the ratio of the elastic constant to $p$. A treatment of a large-deformation problem with such a linear constitutive function can therefore be very misleading. These observations indicate that the most important qualitative properties of solutions depend on the asymptotic behavior of $N^{\mathrm{E}}$.

Now let us study the dynamics of the ring under an internal pressure $p$ that is a constant function of time. We multiply $(7.2)_1$ by $r_t$ and use (6.11) to obtain a specialization of the energy equation (6.16):

$$\begin{aligned} &\tfrac{1}{2}\rho A r_t(t)^2 + W(r(t)) + \int_0^t N^{\mathrm{D}}\left(r(\tau), r_t(\tau)\right) r_t(\tau)\, d\tau - \tfrac{p}{2} r(t)^2 \\ &= \tfrac{1}{2}\rho A r_t(0)^2 + W(r(0)) - \tfrac{p}{2} r(0)^2 =: E(0). \end{aligned} \tag{7.4}$$

We make the natural assumption that $W \geq 0$. If $p \leq 0$, i.e., if the pressure is external, then the non-negativity of the integral in (7.4) implies that $\boldsymbol{r}$ and $\boldsymbol{r}_t$ are bounded for all time, and consequently the solution of every initial-value problem for (7.2) exists for all time. Now suppose that $p > 0$. If $W$ is asymptotically strictly superquadratic as its argument goes to $\infty$, then again (7.4) implies $\boldsymbol{r}$ and $\boldsymbol{r}_t$ are bounded for all time and the solution exists for all time. If, furthermore, $W(r) \to \infty$ as $r \searrow 0$, then $\boldsymbol{r}$ has a positive lower bound. Now suppose that $W$ is not asymptotically strictly superquadratic. Then (7.4) implies that there is a positive constant $C$ such that

$$r_t(t)^2 \leq C + C r(t)^2, \tag{7.5}$$

which immediately implies that $r$ and $r_t$ are bounded on any bounded interval of time and therefore exist as solutions of (7.2) for all time. If $W$ is not asymptotically strictly superquadratic, the solution could be unbounded as $t \to \infty$. We now investigate this possibility for elastic rings.

The kinetic energy is $K(t) := \frac{1}{2}\rho A {r_t}^2$. We set

$$\varPhi = \tfrac{1}{2}\rho A r^2. \tag{7.6}$$

We compute $\varPhi_{tt}$, replace the second derivative of $r$ with its expressions from (7.2), and then use the energy equation (7.4) to replace the pressure term

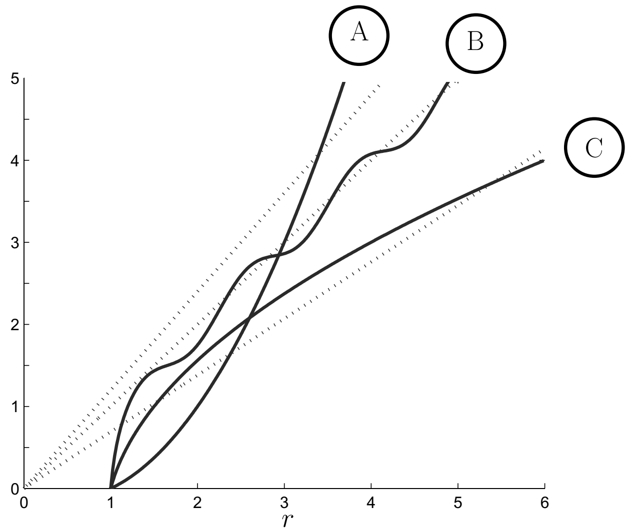



FIGURE 2. Graphs A, B, C of three constitutive functions $N^{\mathrm{E}}$ and graphs, with dotted lines, of the straight lines $r \mapsto pr$ for three different positive $p$'s. Each intersection of a graph of $N^{\mathrm{E}}$ with a straight line with slope $p$ corresponds to a solution of (7.3), which in turn defines an equilibrium state of the ring. The graph A is meant to describe an asymptotically strictly superlinear function. (Of course, this graph cannot exhibit this asymptotic behavior.) For such a function (7.3) has at least one solution for each (positive) $p$. The graph C is meant to describe an asymptotically strictly sublinear function. For such a function (7.3) has at least two solutions for $p$ less than a threshold value and no solutions for $p$ exceeding that value. The graph B is meant to describe a function that has neither of the properties of A and C. Such a graph might describe an asymptotically linear function.

to obtain

$$\begin{aligned}\Phi_{tt} &= 2K - rW_r + pr^2 = 2K - rW_r + 2[K + W - E(0)] \\ &= 4K + 2W - rW_r - 2E(0).\end{aligned} \tag{7.7}$$

We choose $p$ so large that $E(0) < 0$ and assume that $2W \geq rW_r$, so that

$$\Phi_{tt} \geq 4K, \tag{7.8}$$

whence

$$\Phi\Phi_{tt} \geq {\Phi_t}^2. \tag{7.9}$$

Therefore $\Psi := \ln \Phi$ satisfies $\Psi_{tt} \geq 0$, so that $\Psi(t) \geq \Psi_t(0)t + \Psi(0)$, i.e.,

$$\ln \Phi(t) \geq \frac{\Phi_t(0)}{\Phi(0)}\, t + \ln \Phi(0) \quad \Longleftrightarrow \quad \Phi(t) \geq \Phi(0) \exp\left[\frac{\Phi_t(0)}{\Phi(0)}\, t\right]. \tag{7.10}$$

Thus the solution grows exponentially fast if $\Phi_t(0) > 0$. The asymptotically linear behavior of $N^{\mathrm{E}}$ again serves as a threshold separating qualitatively different responses.

The equation of motion for radial motions of an elastic spherical shell under a static pressure differs slightly but significantly from (7.2):

$$\rho A r_{tt} + W_\nu(r) - \tfrac{1}{2} p r^2 = 0 \tag{7.11}$$

(cf. [7]). It has the energy integral

$$\tfrac{1}{2}\rho A {r_t}^2 + W(r) - \tfrac{1}{6} p r^3 = E(0) := \tfrac{1}{2}\rho A r_t(0)^2 + W(r(0)) - \tfrac{1}{6} p r(0)^3. \tag{7.12}$$

Note that $E(0)$ can be negative when $p$ is positive. A study of the equilibrium equation coming from (7.11) shows that if $\nu \mapsto W_\nu(\nu)$ is asymptotically strictly subquadratic as $\nu \to \infty$, i.e., if $\nu^{-2} W_\nu(\nu) \to 0$ as $\nu \to \infty$, then for sufficiently large $p$, there are no equilibrium states. We now show that under an analogous condition, there are initial conditions for which the solution $r$ of (7.11) blows up in finite time (whereas the solution of the corresponding problem for rings exists for all time, although it can grow exponentially fast).

Again we define $\Phi$ by (7.6). Then (7.11) and (7.12) imply that

$$\begin{aligned} \Phi_{tt} &= \rho A {r_t}^2 + \rho A r r_{tt} = \rho A {r_t}^2 - r[W_\nu(r) + \tfrac{1}{2} p r^2] \\ &= \rho A {r_t}^2 - r W_\nu(r) + 3[\tfrac{1}{2}\rho A {r_t}^2 + W(r)] - 3E(0) \\ &= \tfrac{5}{2}\rho A {r_t}^2 + 3W(r) - r W_\nu(r) - 3E(0). \end{aligned} \tag{7.13}$$

Now let us assume that the material is weak in tension in the sense that $3W(\nu) \geq \nu W_\nu(\nu)$ and that $p$ is so large that $E(0) \leq 0$. Then (7.13) implies that

$$\Phi_{tt} \geq \tfrac{5}{2}\rho A {r_t}^2. \tag{7.14}$$

We multiply this inequality by $\Phi$ to get

$$\Phi\Phi_{tt} - \tfrac{5}{4}{\Phi_t}^2 \geq 0, \tag{7.15}$$

so that $\Psi := \Phi^{-1/4}$ satisfies

$$\Psi_{tt} \leq 0, \qquad \Psi(t) \leq \Psi(0) + t\Psi_t(0) \quad \Leftrightarrow \quad \Phi(t) \geq \frac{1}{[\Psi(0) + t\Psi_t(0)]^4}. \tag{7.16}$$

Now $\Psi(0) > 0$. If $r_t(0) > 0$, then $\Psi_t(0) < 0$. Equation (7.16) then implies that $\Phi$ and therefore $r$ blow up in finite time.

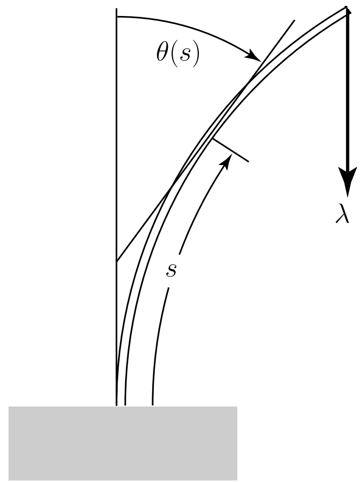
For constant pressures, the behavior of (7.2) and (7.11) can be studied in a phase portrait. The blowup result for the latter presumably could be directly derived from (7.12) by obtaining from it an expression for the time $t$ expended on a trajectory from $r(0)$ to $r(t)$ as an integral from $r(0)$ to $r(t)$ of a certain function, and then exhibiting conditions, like those we have just discussed, ensuring that the improper integral from $r(0)$ to $\infty$ converges.

The treatment culminating in (7.16) is based on Ball [15]; cf. Knops [34]. These methods were used by Calderer [19, 20, 21] to treat the motion of incompressible spherical elastic and viscoelastic shells. For the treatment of problems with time-dependent pressures, see [10]. It is only fair to mention that to maintain a constant internal pressure in any dynamical problem would require a sophisticated control.

Similar threshold effects hold when a ring rotates about its center, in which case there is a conflict between the centrifugal force and the resistance of the material. We look at such a problem below, where it is used to illustrate the effect of invariance on numerical methods for hyperbolic systems. It is interesting to note that in a ring that is simultaneously spinning and radially oscillating, the shear force $H$ cannot be 0 [2]. For a treatment of more complicated effects under the *live* loads due to a barotropic gas and a heavy incompressible fluid, see [11]. Live loads are not necessary for the appearance of threshold effects. E.g., if a wedge given in cylindrical coordinates by $r \leq 1$, $0 \leq \phi \leq \alpha$, $0 \leq z \leq 1$ is subjected to a force in the $z$-direction acting on its edge $r = 0$ and to equilibrating forces on its faces, then the displacement in the $z$-direction is bounded if the energy associated with shear is asymptoticly superquadratic [7, Sec. 14.4].

## 8. An Inverse Problem

We consider the equilibrium of a naturally straight rod of length 1 subject to a scaled terminal thrust $\lambda$ (see Figure 3a, where the invisible unit vector $\boldsymbol{i}$, introduced in Figure 1, is here taken to be vertical, and the unit vector $\boldsymbol{j}$ is taken to be horizontal). We assume that the end $s = 0$ is welded to a rigid wall along the $\boldsymbol{j}$-axis, so that $\boldsymbol{r}(0) = \boldsymbol{o}$ and $\theta(0) = 0$, and assume that the end $s = 1$ is subject to the force $\boldsymbol{n}(1) = \lambda \boldsymbol{i}$ and to zero couple $M(1) = 0$. We assume that there is no body force. Thus (6.4a) implies that $\boldsymbol{n}(s) = -\lambda \boldsymbol{i}$. We further assume that the rod is inextensible and unshearable, so that



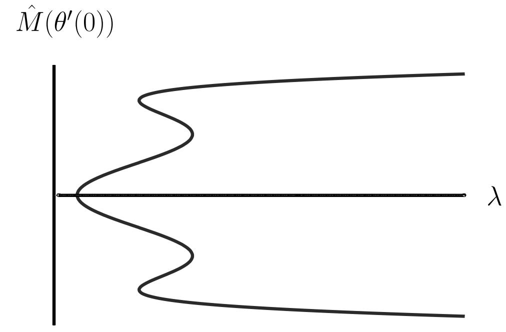




FIGURE 3. (a) The buckling of an inextensible column under force $\lambda$. (b) A bifurcation diagram for this buckling problem, showing the trivial branch and the first bifurcating branch.

$\nu = 1$, $\eta = 0$, $\boldsymbol{r} = \boldsymbol{a}$. In this case, $\boldsymbol{n}$ plays the role of a Lagrange multiplier. We finally assume that the rod is uniform, so that the only constitutive equation has the form $M(s) = M^{\mathrm{E}}(\theta_s(s))$. In this case, $\theta_s$ is the curvature, and the equilibrium version of (6.4) reduces to

$$[M^{\mathrm{E}}(\theta')]' + \lambda \sin\theta = 0, \quad \theta(0) = 0 = M^{\mathrm{E}}(\theta'(1)). \tag{8.1}$$

In this section the prime denotes differentiation with respect to $s$. Conditions (6.8) and (6.15) imply that $M^{\mathrm{E}}$ is strictly increasing and odd. The bifurcation diagram (Figure 3b) can be computed for each function $M^{\mathrm{E}}$. If $M^{\mathrm{E}}$ is a constant multiple of $\theta'$, then on each bifurcating branch, $\lambda$ is a convex function of the amplitude $M^{\mathrm{E}}(\theta'(0))$. When $M^{\mathrm{E}}$ is a nonlinear function, however, this convexity may well be lost (see [39]). We are concerned with the

**Inverse Problem**: Given the lowest nontrivial branch of the bifurcation diagram, find the function $M^{\mathrm{E}}$.

To solve this problem we use the invertibility of $M^{\mathrm{E}}$ to show that it has an inverse $\mu^{\mathrm{E}}$, which has a potential $V$ such that

$$m = M^{\mathrm{E}}(\theta') \quad \Leftrightarrow \quad \theta' = \mu^{\mathrm{E}}(m) =: V_m(m). \tag{8.2}$$

We can then replace (8.1) with an equivalent Hamiltonian system:

$$\theta' = V_m(m), \quad m' = -\lambda \sin\theta. \tag{8.3}$$

$V(m(0)) =: \tau$ serves a magnitude of solution $(\theta, m)$ alternative to $m(0)$ (which is the ordinate in Figure 3b).

Since this system is Hamiltonian, it possesses an energy integral:

$$\begin{aligned} V(m(s)) &= \lambda[\cos\theta(s) - 1] + \tau \\ &= \lambda[\pm\sqrt{1 - (m'(s)/\lambda)^2} - 1] + \tau \end{aligned} \tag{8.4}$$

where we have used $(8.3)_2$. Thus

$$m'(s)^2 = [\tau - V(m)][2\lambda - (\tau - V(m))] \tag{8.5}$$

whence

$$1 = \int_{m(0)}^{0} \frac{ds}{dm}\, dm = \int_0^{m(0)} \frac{dm}{\sqrt{[\tau - V(m)][2\lambda - (\tau - V(m)]}} \,. \tag{8.6}$$

For $m \geq 0$, the function $V$ has an inverse $Q$ so that $V(m) = u$ is equivalent to $m = Q(u)$. We account for the infinite slope of $Q$ at 0 by setting $Q'(u) =: F(u)/\sqrt{u}$. Thus (8.7) is equivalent to

$$1 = \int_0^{\tau} \frac{F(u)\, du}{\sqrt{u[\tau - u][2\lambda - \tau + u]}} \,. \tag{8.7}$$

Now let $\lambda$ be a prescribed function of $\tau$. Then (8.7) for the unknown $F$ becomes a *linear* Volterra integral equation of the first kind with a singular kernel, which is easily solved numerically. A reversal of the steps of this development shows that $F$ determines the constitutive function $M^{\mathrm{E}}$ on an interval corresponding the domain of $\lambda$ as a function of $\tau$. That $M^{\mathrm{E}}$ is increasing must be checked a posteriori.

For this immediate problem, one could contemplate a direct experimental measurement of the bending couple $M$ as a function of the curvature of a rod by bending the shaft into an arc of a circle by equal and opposite bending couples applied at the ends. But it is not easy to measure these couples. In Bell's history [17] there is no indication of experimental work of this sort. Of course, the same objection could ostensibly be raised against an experimental program producing Figure 3b, which requires the measurement of $m(0)$. This objection can be dismissed, however, with the observation that the energy integral (8.8) yields

$$\tau \equiv V(m(0)) = \lambda[1 + \cos\theta(1)] \tag{8.8}$$

provided that $V(0)$, without loss of generality, is taken to vanish. Since $\theta(1)$ can be easily measured (e.g., optically), no direct measurement of $m(0)$ is necessary.

Finally, there is another way to find $M^{\mathrm{E}}$ without measuring it directly, and this can be done for a single value of $\lambda$: The function $\theta$ is measured for this (sufficiently large) $\lambda$. Then the function $m$ is determined from $(8.1)_3$ and $(8.3)_2$ by the formula $m(s) = -\lambda \int_s^1 \sin\theta(\xi)\,d\xi$. The function $\theta'$ is computed from $\theta$, and the graph of the curve $s \mapsto (\theta'(s), m(s))$ in the $(\theta', m)$-plane determines $M$ for the range of $\theta$ corresponding to the given $\lambda$. The disadvantage of this procedure is that is requires monitoring $\theta(s)$ for each $s$ and either computing $\theta'(s)$ from it (which is not numerically convenient) or else monitoring $\theta'(s)$ directly (which is feasible with lasers).

Some of the discussion of this section is based on [9].

A virtue of the procedure leading to (8.7) is that it has applications beyond that of determining the constitutive function $M^{\mathrm{E}}$: To design switches, it might be convenient to make a system snap from one configuration to another at wiggles in the bifurcation diagram Figure 3b. We need only prescribe $\lambda$ as a function of $\tau$ and find $M^{\mathrm{E}}$. But how can one find a material of the prescribed form? One answer is to use a magnetostrictive material, typically an alloy of rare earths, whose mechanical properties are very sensitive to the ambient magnetic field. Such materials are *smart*, which simply means that there is a strong coupling between different kinds of physical processes, allowing opportunities for control. A procedure for doing this using a magnetoelastic feedback is given in [3]. We illustrate such a feedback in the next section.

## 9. Control of Shocks in the Longitudinal Motion of a Magnetostrictive Rod

We examine the purely longitudinal motion of a naturally straight rod, which we suppose is infinitely long. Then $\boldsymbol{r}(s,t) = x(s,t)\boldsymbol{i}$, $\theta(s,t) = 0$, $\nu(s,t) = x_s(s,t)$, $\eta(s,t) = 0$, $\mu(s,t) = 0$. We strengthen the symmetry condition (6.15) so that $H = 0 = M$ for such motions. Let $h(s,t)$ denote the scalar (longitudinal) magnetic field acting at $(s,t)$. We take it to be a control (which can be applied to the rod by surrounding segments of it with solenoids). We assume that the constitutive equation for $N$ has the form $N(s,t) = \tilde{N}(x_s(s,t), h(s,t), s)$. Such functions have been found experimentally for various magnetoelastic materials [48]. Then the specialization of (6.4a) for free longitudinal motion has the form

$$x_{tt} = \tilde{N}(x_s, h(s,t), s)_s. \tag{9.1}$$

If $\tilde{N}(\cdot, h, s)$ is a strictly increasing nonlinear function for fixed function $h$, then for fixed $h$, (9.1) is a quasilinear hyperbolic equation, whose solutions typically exhibit shocks. But if there is a positive-valued function $a$ and a constant $b$ such that in $(\nu, h, z)$-space the plane $z = a(s)\nu + b$ intersects the surface $z = \tilde{N}(\nu, h, s)$ for each fixed $s$, then the solution(s) $\tilde{h}(\nu, s)$ of $a(s)\nu + b = \tilde{N}(\nu, h, s)$ are feedbacks that linearize (9.1):

$$x_{tt} = [a(s)x_s]_s. \tag{9.2}$$

This equation has no shocks, and if $a$ is constant, this equation faithfully transmits signals. Because the rod is thin, it is feasible to localize the magnetic field. This control has the special virtue that it acts on the principal part of the differential operator, and accordingly has a great effect. For rods having a general motion in a plane, one can contrive a triple of magnetic controls that affect the response to flexure, elongation, and shear. Here the constitutive functions $\hat{N}, \hat{H}, \hat{M}$ of (6.6) depend also on a triple $\mathbf{h}$ of such controls. In [4], it is shown how such controls can prevent shocks, and change the dissipation, isotropy, and homogeneity of rods in space.

## 10. Quasistatic Motions under a Slowly Applied Load

Plasticity theory was designed not to study plastics, which are typically viscoelastic, but to study metals. It is meant to account for such effects as permanent plastic deformation, which is readily observed in a paper clip. It must therefore be described by a material with memory. Many studies of bifurcation and stability for plastic bodies assume that the body deforms quasistatically, i.e., so slowly that inertial terms in the equations of motion can be neglected. In this case, time is just a parameter in a steady-state equation.

Buckling problems for elastic bodies can be described mathematically by bifurcation diagrams, like that of Figure 3b, which show the number of equilibrium solutions for each value of the load parameter. The stability and even the dynamic behavior of solutions are often inferred from such a diagram by assuming that the load is applied quasistatically. E.g., in Figure 3b, it is presumed that when the load $\lambda$ is slowly raised past a local maximum on a bifurcating branch, the structure undergoes a sudden dynamic "snap-through" buckling (the antithesis of a quasistatic motion), rapidly coming to rest at another equilibrium state with the same load $\lambda$. The justification of quasistatic behavior for a material with or without memory would require advances in the exciting area of dynamical-systems methods for partial differential equations. Here we look at the simplest possible ordinary differential equation model for quasistatic Euler buckling.

We consider the motion of a simple pendulum of scaled length 1 in the plane. Let $\theta(t)$ be the angle the pendulum makes with a fixed direction $\boldsymbol{j}$, called the vertical. We assume that motion away from the vertical is opposed by a torsional spring applying a (scaled) restoring torque $-\theta$ to the pendulum when it makes angle $\theta$ with $\boldsymbol{j}$. We assume that the motion is also opposed by a frictional torque $-2\nu\dot{\theta}$ with $\nu$ a constant $\geq 0$ when the pendulum moves with angular speed $\dot{\theta}$. Finally, we assume that the bob of the pendulum is subjected to a force $-(\gamma+\varepsilon t)\boldsymbol{j}$ of slowly increasing magnitude where $\gamma$ and $\varepsilon$ are positive constants with $\varepsilon$ small. The motion of $\theta$ is then governed by the initial-value problem

$$\ddot{\theta} + 2\nu\dot{\theta} + \theta = (\gamma + \varepsilon t)\sin\theta, \quad \theta(0) = \alpha, \quad \dot{\theta}(0) = \beta. \tag{10.1}$$

This initial-value problem has a globally defined unique solution. (If $\varepsilon = 0$, this equation has a beautiful family of phase portraits parametrized by $\gamma$.)

We want to determine how solutions behave as $\varepsilon \searrow 0$. To answer this question we have to determine how solutions behave as $t \to \infty$ (for which purpose the smallness of $\varepsilon$ is of no use). Physical intuition suggests that as $t \to \infty$, typically a solution $\theta$ approaches a value in the set $\{(2n+1)\pi\}$, $n$ an integer. The difficulty in demonstrating this conclusion is to control $(\gamma+\varepsilon t)\sin\theta(t)$ as $\gamma+\varepsilon t \to \infty$ while $\sin\theta(t) \to 0$. For this purpose, Lyapunov-type theorems are of limited utility. Difficult bounds relying on the positivity of $\nu$ are needed to show that most solutions are ultimately confined to a large region of attraction about $\theta = (2n+1)\pi$. Then we can set

$$\begin{gathered} \theta = (2n+1)\pi - \psi, \quad \psi = e^{-\nu t}u, \\ \hat{t}(\tau) = \varepsilon^{-1/3}\tau - \varepsilon^{-1}(1-\nu^2+\gamma), \quad v(\tau) = u(\hat{t}(\tau)) \end{gathered} \tag{10.2}$$

to get

$$\begin{aligned} v_{\tau\tau} + \tau v &= \varepsilon^{-2/3}(2n+1)\pi e^{\nu\hat{t}(\tau)} \\ &\quad + (\tau + \varepsilon^{-2/3}(\nu^2-1))e^{-2\nu\hat{t}(\tau)}v^3 m(e^{-\nu\hat{t}(\tau)}v) \end{aligned} \tag{10.3}$$

where $\psi - \sin\psi = \psi^3 m(\psi)$. The homogeneous equation $v_{\tau\tau} + \tau v = 0$ is the (backwards) Airy equation, with independent solutions $\mathrm{Ai}(-\tau)$ and $\mathrm{Bi}(-\tau)$. We use Lagrange's method of variation of parameters to convert the initial-value problem (10.1) to an integral equation with a kernel determined by these Airy functions. The derivation of further bounds from the integral equation employs the properties that

$$\begin{aligned} \mathrm{Ai}(-\tau) &\sim \pi^{-1/2}\tau^{-1/4}\sin(\tfrac{2}{3}\tau^{3/2}+\tfrac{\pi}{4}), \\ \mathrm{Bi}(-\tau) &\sim \pi^{-1/2}\tau^{-1/4}\cos(\tfrac{2}{3}\tau^{3/2}+\tfrac{\pi}{4}), \\ -\mathrm{Ai}'(-\tau) &\sim \pi^{-1/2}\tau^{1/4}\ \cos(\tfrac{2}{3}\tau^{3/2}+\tfrac{\pi}{4}), \\ \mathrm{Bi}'(-\tau) &\sim \pi^{-1/2}\tau^{1/4}\ \sin(\tfrac{2}{3}\tau^{3/2}+\tfrac{\pi}{4}) \end{aligned} \tag{10.4}$$

as $\tau \to \infty$ (cf. [40]). The full bounds support an application of the Contraction Mapping Principle to show that if $\nu > 0$, then for almost every solution $\theta$ there is an integer $n$ such that $\theta(t) \to (2n+1)\pi$ and $\dot{\theta}(t) \to 0$ as $t \to \infty$. For $\nu > 0$ and for $\varepsilon$ sufficiently small, these results support a modification of Hoppensteadt's [31] theory ensuring that a rigorous asymptotics justifies the quasistatic motion.

For $\nu = 0$, let $w$ be the solution fo the linearization of (10.3) about $v = 0$ (obtained by dropping the last term of (10.3)). The properties (10.4) of the Airy functions show that $w(\tau) \to 0$ as $\tau \to \infty$ while $w_\tau$ oscillates unboundedly. This result suggests tha $v$ enjoys the same properties. But this is false: It can be shown that the nonlinear part of (10.3) prevents $v$ from converging to 0, so that the linearization of (10.3) about $v = 0$ is irrelevant for the actual dynamics. $v$ and $\psi$ oscillate about 0 without approaching it, while $v_\tau$ and $\psi_t$ oscillate unboundedly about 0 ! These results are based on [8].

A version of this problem for the dynamical behavior of a naturally straight rod under a slowly increasing terminal thrust consists of the partial differential equations (6.4), (6.6) subject to arbitrary initial conditions and to the boundary conditions

$$\boldsymbol{r}(0,t) = \boldsymbol{o}, \qquad \theta(0,t) = 0, \qquad \boldsymbol{n}(1,t) = -(\gamma + \varepsilon t)\boldsymbol{i}, \qquad M(1,t) = 0. \tag{10.5}$$

## 11. Quasistatic Motions: Springs with Small Mass

In elementary mechanics, we study the motion of a mass point on a massless spring. The spring has no inertia, it merely transmits force. Suppose, instead, that the spring is a viscoelastic rod of scaled length 1 with small mass density $(\rho A)(s) = \varepsilon\sigma(s)$ constrained to execute longitudinal motions. What happens as its mass density goes to 0? The specialization of (6.4a) for free longitudinal motion again has the form

$$\varepsilon\sigma x_{tt} = \partial_s \hat{N}(x_s, x_{st}, s), \qquad 0 < s < 1. \tag{11.1}$$

The requirements that the end $s = 0$ of the spring be fixed at the origin and and that the end $s = 1$ carry a particle of mass $m$ gives the boundary conditions

$$x(0,t) = 0, \tag{11.2}$$

$$m x_{tt}(1,t) = -\hat{N}(x_s(1,t), x_{st}(1,t), 1). \tag{11.3}$$

The second condition is just the equation of motion of the particle. These equations are subject to general initial conditions.

The *reduced problem* is obtained by setting $\varepsilon = 0$, in which case (11.1) yields

$$\hat{N}(x_s, x_{st}, s) = \hat{N}(x_s(1,t), x_{st}(1,t), 1). \tag{11.4}$$

Note that the right-hand side of this equation is given by (11.3). Is this reduced problem governed by the standard ordinary differential equation? I.e., can we "cancel" the $s$-derivatives in (11.3)? What is the asymptotic status of the reduced problem?

If $\hat{N}$ is independent of its second argument, i.e., if the material is elastic, then (11.4) is an algebraic equation for $x_s(s,t)$ in terms of $x_s(1,t)$. If the derivative of $\hat{N}$ with respect to its first argument is positive, in accord with (6.8), and if $\hat{N}$ satisfies the growth conditions described in Section 6, then $x_s(s,t)$ can be uniquely found in terms of $x_s(1,t)$. An integration subject to (11.3) shows that $x(s,t)$ and, in particular, $x(1,t)$ can be expressed in terms of $x_s(1,t)$. The inversion of this last representation yields an expression for $x_s(1,t)$ in terms of $x(1,t)$, which when substituted into (11.3) produces a standard ordinary differential equation. This representation, however, has no asymptotic justification, for which dissipation is needed.

If the derivative of $\hat{N}$ with respect to its second argument has a positive lower bound (so that the material is viscoelastic), then (11.4) is a family of equations, similar to ordinary differential equations, parametrized by $s$ for $x_s(s,t)$ as a function of $t$. These are not ordinary differential equations because they depend on $x_s(1,t), x_{st}(1,t)$. Nevertheless, by suitably combining (11.3) and (11.4), we can obtain bona fide ordinary differential equations for $t \mapsto x_s(\cdot,t), x_s(1,t)$. A combination of the Contraction-Mapping Principle and suitable estimates (reflecting slightly sharpened versions of our constitutive restrictions, needed to ensure that the spring does not suffer a total compression) ensures that this system of ordinary differential equations has a globally defined unique solution. To determine whether this solution is the solution of the standard second-order ordinary differential equation, we suppose that there is a function $g$ such that $x_s(1,t) = g(x(1,t))$, which would convert (11.3) into the requisite form. The exploration of this assumption leads to a contradiction showing that there is no viscoelastic spring for which the reduced problem is governed by a standard second-order ordinary differential equation for all initial conditions. (It is governed by an equation with memory. Only for special initial conditions for special rods is the reduced problem governed by a standard ordinary differential equation.) Thus there are errors in the elementary treatment of massless springs, which are not too serious as we shall shortly point out. For details of this analysis see [1] and for a correction of a proposition there see [7, Sec. 3.11].

For viscoelastic springs, the solution of the reduced problem is the leading term of a regular asymptotic expansion, which is accompanied by an initial-layer expansion. The whole expansion in $\varepsilon$ is rigorous, having appropriate error bounds. The difficulty in demonstrating this lies in showing that the terms of the initial-layer expansion are exponentially decaying in the stretched time variable. The leading term of this expansion is a quasilinear parabolic equation for which such exponential bounds are required for the solution and many of its derivatives. These bounds are obtained by exploiting the Maximum Principle along the lines developed by S. N. Bernstein. The details are given in [49].

If the spring is uniform, then a standard second-order ordinary differential equation provides an attractor for the solution of the reduced problem [47]. This means that the discrepancy between the actual solution of the reduced problem, satisfying an equation with memory, and the attracting ordinary differential equation is transient. What happens for a non-uniform rod is not known.

A related problem is that of the longitudinal motion of a particle attached to two light springs whose ends remote from the particle are fixed. In this case, the equation of motion for the particle, corresponding to (11.3) becomes an internal transmission condition. The analysis of the reduced problem and the demonstration that it has an attractor given by a standard second-order ordinary differential equation for uniform springs is much trickier than that for the problem with a single spring. The techniques are given in [13] for the related problem for the motion of a piston in a closed cylinder in which the springs are replaced by a viscous gas confined to the cylinder.

## 12. Dissipativity and Shock Structure

**1-dimensional gas dynamics.** Let us recall the spatial (= Eulerian) formulation of the equations governing the 1-dimensional longitudinal motion of a compressible, barotropic, viscous gas (in a cylinder) [22]. Let $\varrho(y,t)$ and $v(y,t)$ denote the actual density per unit actual length and the velocity of the material point occupying position $y$ at time $t$. Then the conservation of mass and the balance of linear momentum yield

$$\varrho_t = -(\varrho v)_y, \tag{12.1}$$

$$\varrho(v_t + v v_y) = -p(\varrho)_y + [\kappa(\varrho) v_y]_y \equiv -\varrho P(\varrho)_y + [\kappa(\varrho) v_y]_y \tag{12.2}$$

where $p(\varrho)$ and $\kappa(\varrho)$ are the pressure and viscosity corresponding to the density $\varrho$ and where $P(\varrho) := \int_1^\varrho \sigma^{-1} p'(\sigma)\, d\sigma$. It is convenient to introduce the momentum $m := \varrho v$, which converts (12.1), (12.2) to

$$\varrho_t = -m_y, \tag{12.3}$$

$$m_t = -(\varrho^{-1} m^2)_y - p(\varrho)_y + [\kappa(\varrho)(\varrho^{-1} m)_y]_y. \tag{12.4}$$

We assume that $p(\varrho)$ strictly decreases from $\infty$ to 0 as $\varrho$ increases from 0 to $\infty$, and we take $\kappa$ to be a positive constant. In this case, $v_{yy}$ is a 1-dimensional Laplacian, which gives (12.2) a parabolic character. When $\kappa = 0$, system (12.1), (12.2) or system (12.3), (12.4) forms hyperbolic conservation laws, which have been intensively studied.

All numerical schemes for hyperbolic conservation laws include some sort of numerical regularization [28, 35, 43]. Some schemes, like the Lax-Friedrichs and upwind schemes, when applied to (12.1), (12.2) or to (12.3), (12.4) with $\kappa = 0$ may be regarded as difference schemes for these hyperbolic systems modified by the addition of a variety of viscosity terms to both right-hand sides. A simple such modification of (12.1), (12.2) might have the form

$$\varrho_t = -(\varrho v)_y + \alpha \varrho_{yy}, \tag{12.5}$$

$$v_t = -[\tfrac{1}{2} v^2 + P(\varrho)]_y + \beta v_{yy} \tag{12.6}$$

where, e.g., $\alpha$ and $\beta$ are positive constants. This modification has a viscosity term $\beta v_{yy}$ differing slightly from that of (12.2) and adds viscosity $\alpha \rho_{yy}$

to (12.1). Slemrod [45] observed that this modification (12.5) of the equation for the conservation of mass is equivalent to having the constitutive equations account for capillarity effects as well as viscosity and compression. Numerical schemes of this sort seem very effective for treating shocks in gases, not only for (12.1), (12.2), but also for far more elaborate models. Here we show that the straightforward application of such schemes to certain problems of solid mechanics can lead to serious errors. For this purpose, it is illuminating to convert system (12.1), (12.2) to their material (= Lagrangian) form:

Let $s$ identify a material cross section in a gas contained in a cylindrical tube and let $x(s,t)$ be its position at time $t$. Then its velocity and acceleration are $x_t$ and $x_{tt}$. In consonance with (6.13) we require that $x_s(s,t) > 0$ for all $s, t$, and define the inverse of $x(\cdot,t)$ to be $\hat{s}(\cdot,t)$, so that $y = x(s,t) \Longleftrightarrow s = \hat{s}(y,t)$, and $y = x(\hat{s}(y,t),t)$. Thus $v(y,t) = x_t(\hat{s}(y,t),t)$ and $\varrho(y,t) = (\rho A)(\hat{s}(y,t))$ where $(\rho A)(s)$ is the mass per unit reference length of the gas in the reference configuration. The requirement that mass be conserved is that

$$\int_a^s (\rho A)(\sigma)\,d\sigma = \int_{x(a,t)}^{x(s,t)} \varrho(y,t)\,dy = \int_a^s \varrho(x(\sigma,t),t)x_s(\sigma,t)\,d\sigma \qquad (12.7)$$

for all $a, s$, so that

$$\varrho(x(s,t),t)x_s(s,t) = (\rho A)(s). \qquad (12.8)$$

The time derivative of this equation is equivalent to (12.1) because $v_y(y,t) = x_{st}(\hat{s}(y,t),t)\hat{s}_y(y,t) = x_{st}(\hat{s}(y,t),t)/x_s(\hat{s}(y,t),t)$. The use of (12.7) shows that (12.2) is equivalent to

$$\rho A x_{tt} = -p(\rho A/x_s) + \kappa[x_{st}/x_s]_s. \qquad (12.9)$$

We recognize this equation as having a form just like that for the longitudinal motion of rods (cf. $(10.1)_1$). Note that in the transition from the spatial formulation to the material formulation, the Laplacian form of the dissipative term, with coefficient $\kappa$, has changed. We can write (12.9) as a system of two first-order equations by setting $u = x_s$, $w = x_t$:

$$u_t = w_s, \qquad \rho A w_t = -p(\rho A/u) + \kappa[w_s/u]_s. \qquad (12.10)$$

If we were to modify the version of this system with zero viscosity (i.e., with $\kappa = 0$) in the same mathematical manner by which we obtained (12.5), (12.6) from (12.1), (12.2) with $\kappa = 0$, then we would obtain

$$u_t = w_s + \gamma u_{ss}, \qquad \rho A w_t = -p(\rho A/u) + \delta w_{ss} \qquad (12.11)$$

where $\gamma$ and $\delta$ are positive numbers, which is not equivalent to (12.5), (12.6), and which lacks the structure of (12.10). We refer to the $\alpha, \beta, \gamma, \delta$ of (12.5), (12.6), (12.11) as *artificial viscosities*. We shall show that the analogous introduction of artificial viscosity into the vectorial equations of motion (6.4), (6.5) for elastic rods produces equations with material responses that are not invariant under rigid motions. The consequence is that the discrepancy between the equations of motion (6.4), (6.6) for viscoelastic rods and such a modification of (6.4), (6.5) with artificial viscosity is far more serious than the discrepancy between (12.10) and (12.11). We shall show that the use of the modifications of (6.4), (6.5) with artificial viscosity may lead to serious numerical errors, and we then show how to correct them.

**Modifications of the hyperbolic equations for an elastic rod.** Let us write the equations (6.4), (6.5) for an elastic rod as a first-order system. For simplicity of exposition we assume that $\rho A$ and $\rho J$ are constant, that the constitutive functions $N^{\mathrm{E}}$, etc., do not depend explicitly on $s$, and that $\boldsymbol{f} = \boldsymbol{o}$. We introduce new variables:

$$\boldsymbol{v} := \boldsymbol{r}_t, \quad \omega := \theta_t. \tag{12.12}$$

Then the compatibility equations, expressing the equality of mixed partial derivatives of $\theta$ and $\boldsymbol{r}$, are

$$\theta_t = \omega, \qquad \mu_t = \omega_s, \tag{12.13a}$$

$$[\nu \boldsymbol{a}(\theta) + \eta \boldsymbol{b}(\theta)]_t = \boldsymbol{v}_s. \tag{12.13b}$$

Now we can write (6.4)–(6.7) as

$$\rho A \boldsymbol{v}_t = (N^{\mathrm{E}} \boldsymbol{a} + H^{\mathrm{E}} \boldsymbol{b})_s, \tag{12.13c}$$

$$\rho J \omega_t = \partial_s M^{\mathrm{E}} + \nu H^{\mathrm{E}} - \eta N^{\mathrm{E}}, \tag{12.13d}$$

where the arguments of $N^{\mathrm{E}}, H^{\mathrm{E}}, M^{\mathrm{E}}$ are $\nu, \eta, \mu$. System (12.13) corresponds to seven scalar equations for the seven scalar unknowns $\nu, \eta, \theta, \mu, \omega$ and any two components of $\boldsymbol{v}$. (Once these are found, we can find $\boldsymbol{r}$ by integration.) Depending on how the components of this system are chosen (with respect to $\{\boldsymbol{i}, \boldsymbol{j}\}$ or $\{\boldsymbol{a}, \boldsymbol{b}\}$ or $\{\boldsymbol{r}_s/|\boldsymbol{r}_s|, \boldsymbol{k} \times \boldsymbol{r}_s/|\boldsymbol{r}_s|\}$), these scalar equations have different forms, each of which can be put into the abstract form

$$\mathsf{f}(\mathsf{u})_t = \mathsf{g}(\mathsf{u})_s + \mathsf{h}(\mathsf{u}). \tag{12.14}$$

(Note that (12.1), (12.2) and (12.3), (12.4) each have this form.) A class of modifications of this system by the introduction of a simple version of artificial viscosity has the form

$$\mathbf{f}(\mathbf{u})_t = \mathbf{g}(\mathbf{u})_s + \mathbf{h}(\mathbf{u}) + \mathbf{D} \cdot \mathbf{f}(\mathbf{u})_{ss} \tag{12.15}$$

where $\mathbf{D}$ is a small constant positive-definite diagonal matrix. (Difference equations for (12.15) correspond to a Lax-Friedrichs scheme.)

We construct a vectorial version of the system (12.15) for (12.13):

$$\begin{aligned} \theta_t &= \omega + \alpha_1 \theta_{ss}, \\ \mu_t &= \omega_s + \alpha_1 \mu_{ss}, \\ (\nu \boldsymbol{a} + \eta \boldsymbol{b})_t &= \boldsymbol{v}_s + \alpha_2 (\nu \boldsymbol{a} + \eta \boldsymbol{b})_{ss}, \\ \rho A \boldsymbol{v}_t &= (N^{\mathrm{E}} \boldsymbol{a} + H^{\mathrm{E}} \boldsymbol{b})_s + \alpha_3 \boldsymbol{v}_{ss}, \\ \rho J \omega_t &= M^{\mathrm{E}}_s + \nu H^{\mathrm{E}} - \eta N^{\mathrm{E}} + \alpha_4 \omega_{ss}, \end{aligned} \tag{12.16}$$

where the $\alpha$'s are small positive parameters.

It would be closer to the spirit of the Lax-Friedrichs and upwind schemes if the $\alpha$-terms were introduced into a corresponding system of scalar equations. A portent of some of the difficulties we must overcome is that such modifications of equivalent systems of scalar equations are not equivalent!

The scalar system corresponding to (12.13) that seems most elegant is that in which $\boldsymbol{v} = u\boldsymbol{a} + v\boldsymbol{b}$ and all the equations of (12.13) are decomposed into their $\boldsymbol{a}$ and $\boldsymbol{b}$ components. In this case, $\theta$ does not appear in the componential versions of (12.13b)–(12.13d), so that $(12.13\mathrm{a})_1$ is uncoupled from the rest of the equations (just as is $\boldsymbol{r}$). We accordingly ignore it. The modification in the form of (12.15) of this componential version is

$$\begin{aligned} \nu_t &= u_s - \mu v + \omega \eta + \alpha_1 \nu_{ss}, \\ \eta_t &= v_s + \mu u - \omega \nu + \alpha_2 \eta_{ss}, \\ \mu_t &= \omega_s + \alpha_3 \mu_{ss}, \\ \rho A u_t &= N^{\mathrm{E}}_s - \mu H^{\mathrm{E}} + \rho A \omega v + \alpha_4 \rho A u_{ss}, \\ \rho A v_t &= H^{\mathrm{E}}_s + \mu N^{\mathrm{E}} - \rho A \omega u + \alpha_5 \rho A v_{ss}, \\ \rho J \omega_t &= M^{\mathrm{E}}_s + \nu H^{\mathrm{E}} - \eta N^{\mathrm{E}} + \alpha_6 \rho J \omega_{ss} \end{aligned} \tag{12.17}$$

where the $\alpha$'s are small positive parameters.

**The steadily rotating ring.** Now let us specialize the equations of Section 6 to those for a circular elastic ring of natural radius 1. Since $H^{\mathrm{E}}(\nu, 0, \mu) = 0$ by (6.15), it follows that system (12.13), its vectorial modification (12.16), the hyperbolic system (obtained from (12.17) by setting the $\alpha$'s equal to 0), and the full system (12.17) each admit the steady solution corresponding to the ring rotating with constant angular velocity $\omega_0$ with constant radius $\nu_0$:

$$\nu = \nu_0, \quad \eta = 0, \quad u = \nu_0\omega_0, \quad v = 0, \quad \mu = 1, \quad \omega = \omega_0 \tag{12.18}$$

provided that

$$\rho A\omega_0^2\nu_0 = N^{\mathrm{E}}(\nu_0, 0, 1) \tag{12.19}$$

(which is an equation just like (7.3)).

Now we decompose $\boldsymbol{v}$ with respect to the basis $\{\boldsymbol{i}, \boldsymbol{j}\}$ by setting

$$\boldsymbol{v} = U\boldsymbol{i} + V\boldsymbol{j}, \tag{12.20}$$

obtaining the modified equations

$$\begin{aligned}
\nu_t &= \eta\omega + U_s\cos\theta + V_s\sin\theta + \gamma_3\nu_{ss},\\
\eta_t &= -\nu\omega - U_s\sin\theta + V_s\cos\theta + \gamma_4\eta_{ss},\\
\theta_t &= \omega + \gamma_1\theta_{ss},\\
\mu_t &= \omega_s + \gamma_1\mu_{ss},\\
\rho AU_t &= N_s^{\mathrm{E}}\cos\theta - H_s^{\mathrm{E}}\sin\theta - \mu(N^{\mathrm{E}}\sin\theta + H^{\mathrm{E}}\cos\theta) + \gamma_5\rho AU_{ss},\\
\rho AV_t &= N_s^{\mathrm{E}}\sin\theta + H_s^{\mathrm{E}}\cos\theta + \mu(N^{\mathrm{E}}\cos\theta - H^{\mathrm{E}}\sin\theta) + \gamma_6\rho AV_{ss},\\
\rho J\omega_t &= M_s^{\mathrm{E}} + \nu\hat{H}^{\mathrm{E}} - \eta N^{\mathrm{E}} + \gamma_2\rho J\omega_{ss}.
\end{aligned} \tag{12.21}$$

Let us check whether (12.21) admits the same trivial solution, here with $\theta_0(s,t) = s + \omega_0 t$. Then $(12.21)_{1-4,7}$ yield corresponding trivial solutions $U_0 = \nu_0\omega_0\cos\theta_0$, $V_0 = \nu_0\omega_0\sin\theta_0$ (to within a rigid motion). The remaining two equations yield

$$\begin{aligned}
\nu_0\omega_0^2\rho A\sin\theta_0 &= N_0\sin\theta_0 + \gamma_5\rho A\nu_0\omega_0\cos\theta_0,\\
\nu_0\omega_0^2\rho A\cos\theta_0 &= N_0\cos\theta_0 - \gamma_6\rho A\nu_0\omega_0\sin\theta_0.
\end{aligned} \tag{12.22}$$

These equations are inconsistent unless $\gamma_5\omega_0\cos^2\theta_0 = \gamma_6\omega_0\sin^2\theta_0$. Thus this modification does not admit the trivial solution (12.18) (unless $\omega_0 = 0$)!

**Invariant constitutive equations.** It can be shown that the most general constitutive equations for our rod theory that are invariant under rigid motions are those in which $N, H, M$ depend possibly nonlocally in space and time on the strains $\nu, \eta, \mu$ [7, Chap. 8]. In particular, these resultants can depend on $\nu, \eta, \mu$ and some of their derivatives with respect to $s, t$, but they cannot depend on $\boldsymbol{v}_s$, e.g. We now show that our modifications correspond to constitutive equations that are not invariant, show that their use in numerical treatments can lead to severe errors, and then show how to construct invariant modifications.

Let us write the right-hand side of $(12.16)_4$ as $(N^{\mathrm{E}}\boldsymbol{a}+H^{\mathrm{E}}\boldsymbol{b}+\alpha_3\boldsymbol{v}_s)_s$ and interpret the term in parentheses as an expression of the form $N^+\boldsymbol{a}+H^+\boldsymbol{b}$ where $N^+$ and $H^+$ are modified constitutive functions. Since $\boldsymbol{v}_s$ lacks the form $P\boldsymbol{a}+Q\boldsymbol{b}$ where $P$ and $Q$ depend (nonlocally) on $\nu,\eta,\mu$, in particular, since $\boldsymbol{v}_s = \boldsymbol{r}_{ts} = (\nu\boldsymbol{a}+\eta\boldsymbol{b})_t$, which involves the prohibited $\theta_t$, it follows that these modified functions are not invariant.

It is clear how to modify $(12.16)_{4,5}$ to produce simple dissipative versions that are invariant under rigid motions:

$$\begin{aligned}\rho A\boldsymbol{v}_t &= [(N^{\mathrm{E}}+\beta_3\nu_t)\boldsymbol{a}+(H^{\mathrm{E}}+\beta_4\eta_t)\boldsymbol{b}]_s,\\ \rho J\omega_t &= (M^{\mathrm{E}}+\alpha_4\omega_s)_s+(H^{\mathrm{E}}+\beta_4\eta_t)\nu-(N^{\mathrm{E}}+\beta_3\nu_t)\eta\end{aligned} \tag{12.23}$$

where the $\beta$'s are positive numbers.

We shall tacitly show that the compatibility equation $(12.16)_3$ is not invariant under rigid motions, in the process of adjusting it to make it so. Since the addition of dissipative mechanisms to the compatibility equations (12.13a), (12.13b) plays a critical role in numerical (and many analytic) methods, we cannot avoid this problem by considering difference equations for the system (12.13a), (12.13b), (12.23). An *invariant modification corresponding to the Lax-Friedrichs scheme* is one in which there is a dissipation term in each equation and the system is equivalent to (6.4) with invariant constitutive equations for $N,H,M$.

Let us first show how to reconstitute a version of (6.4b) from $(12.16)_2$ and $(12.23)_2$. (Equation $(12.16)_1$ is essentially equivalent to $(12.16)_2$.) For this reconstituted (6.4b), the issue of invariance does not intervene. Since $(12.16)_2$ expresses the equality of a $t$-derivative and an $s$-derivative on a simply-connected domain of $(s,t)$-space, there is a potential (which must be our variable) $\theta$ such that

$$\theta_s=\mu,\qquad \theta_t=\omega+\alpha_1\mu_s=\omega+\alpha_1\theta_{ss}. \tag{12.24}$$

We replace the $\omega$'s in $(12.23)_2$ with that given by $(12.24)_2$ to obtain

$$\rho J\theta_{tt}=[\hat{M}+(\rho J\alpha_1+\alpha_4)\theta_{st}-\alpha_1\alpha_4\theta_{sss}]_s+(H^{\mathrm{E}}+\beta_4\eta_t)\nu-(N^{\mathrm{E}}+\beta_3\nu_t)\eta, \tag{12.25}$$

which clearly has the requisite invariance. The terms with $\theta_{sst}$ describe viscosity, and, the $\theta_{ssss}$ describes strain-gradient effects. (The role of this derivative is reminiscent of its role in the simplest linear equation for the flexure of a rod: $w_{tt}+\gamma w_{ssss}=0$.)

We now turn to the treatment of $(12.16)_3$ and $(12.23)_1$, which is far trickier because their vectorial character causes the full force of rotational effects to intervene. We modify (12.13b) thus:

$$[\nu \boldsymbol{a}(\theta) + \eta \boldsymbol{b}(\theta)]_t = \boldsymbol{v}_s + \alpha_2[\nu \boldsymbol{a}(\theta) + \eta \boldsymbol{b}(\theta)]_{ss} + \alpha_2 \boldsymbol{g}_s, \tag{12.26}$$

where $\boldsymbol{g}$ is a function at our disposal. This equation implies that there is a vector-valued function $\boldsymbol{r}$ such that

$$\boldsymbol{r}_s = \nu \boldsymbol{a}(\theta) + \eta \boldsymbol{b}(\theta), \qquad \boldsymbol{r}_t = \boldsymbol{v} + \alpha_2[\nu \boldsymbol{a}(\theta) + \eta \boldsymbol{b}(\theta)]_s + \alpha_2 \boldsymbol{g}. \tag{12.27}$$

We replace $\boldsymbol{v}$ in $(12.23)_1$ with its expressions coming from $(12.27)_2$ to obtain

$$\rho A \boldsymbol{r}_{tt} = \rho A \alpha_2[\nu \boldsymbol{a}(\theta) + \eta \boldsymbol{b}(\theta)]_{st} + \rho A \alpha_2 \boldsymbol{g}_t + [(N^{\mathrm{E}} + \beta_3 \nu_t)\boldsymbol{a} + (H^{\mathrm{E}} + \beta_4 \eta_t)\boldsymbol{b})]_s. \tag{12.28}$$

For this equation to have the requisite invariance, the first two terms on the right-hand side must have the form $(P\boldsymbol{a} + Q\boldsymbol{b})_s$ where $P$ and $Q$ depend (possibly nonlocally) only on $\nu, \eta, \mu$; e.g., by depending on these arguments and their derivatives. Now the first term on the right-hand side does not have this form because its coefficients of $\boldsymbol{a}$ and $\boldsymbol{b}$ depend on $\theta_t$. We accordingly choose $\boldsymbol{g}$ to make the sum of the first two terms equal $\rho A \alpha_2[\nu_t \boldsymbol{a}(\theta) + \eta_t \boldsymbol{b}(\theta)]_s$, which gives (12.28) the requisite form

$$\rho A \boldsymbol{r}_{tt} = \{[N^{\mathrm{E}} + (2\rho A \alpha_2 + \beta_3)\nu_t]\boldsymbol{a} + [H^{\mathrm{E}} + (2\rho A \alpha_2 + \beta_4)\eta_t]\boldsymbol{b}\}_s, \tag{12.29}$$

by taking

$$\boldsymbol{g}(s,t) = \boldsymbol{g}(s,0) - \int_0^t [\theta_t(\nu \boldsymbol{b} - \eta \boldsymbol{a})]_s \, d\tau. \tag{12.30}$$

In this case, (12.26) contains an integral, which does not look pretty, but the time derivative of it reduces to

$$[\nu \boldsymbol{a}(\theta) + \eta \boldsymbol{b}(\theta)]_{tt} = \boldsymbol{v}_{st} + \alpha_2[\nu_t \boldsymbol{a}(\theta) + \eta_t \boldsymbol{b}(\theta)]_{ss}. \tag{12.31}$$

Thus this dissipative version of (12.13b) has a parabolic-hyperbolic character, instead of the purely parabolic character of the corresponding equations in (12.16). In particular, the principal parts of the $\boldsymbol{a}$ and $\boldsymbol{b}$ components of (12.31) are $\nu_{tt} - \nu_{sst}$ and $\eta_{tt} - \eta_{sst}$.

Note that (12.29) has a character quite different from that of (12.25) because (12.29) lacks the high space derivatives present in (12.25). The explanation for this is simple: In going from $(12.16)_5$ to $(12.23)_2$, we retained the term $\alpha_4 \omega_{ss}$ because its presence did not interfere with the treatment of invariance. On the other hand, we dropped $\alpha_3 \boldsymbol{v}_{ss}$ in going from $(12.16)_4$ to $(12.23)_1$ because it was likely to cause some trouble. By retaining this term, we could give (12.29) high space derivatives. We omit the details, which would emphasize that the invariant versions that we obtain are not unique.

**Numerical treatment of the rapidly rotating elastic ring.** We have found that if a uniform circular elastic ring is sufficiently strong in resisting extension, then it admits free steady rotations about its center in which the deformed constant radius responds to the centrifugal force. We now treat this problem numerically as an initial-value problem for system (12.21) in which the initial position and velocity fields are exactly those for the steady rotation. We want to see what happens when this problem is treated by standard methods that are not invariant.

We take the natural radius of the ring to be 1, and take $s \in [0, 2\pi]$ to be the arc-length parameter of $\boldsymbol{r}$ in the reference configuration. All our variables save $\theta$ are taken to have period $2\pi$ in $s$, while $\theta(s + 2\pi, t) = \theta(s, t) + 2\pi$. We take initial conditions to be those for the steadily rotating ring of radius 2, so that (12.18) and (12.20) hold with $\nu_0 = 2$. We take the stored-energy function to have the form

$$\begin{aligned} &Gh^{-1}W \\ &= \tfrac{1}{4}\nu^2 + \tfrac{1}{4}\nu^4 + \tfrac{1}{4}\nu^{-2} + \tfrac{1}{2}\nu^2\eta^2 - \nu + \tfrac{1}{2}\eta^2 + \tfrac{1}{4}\eta^4 + \tfrac{10}{3}h^2\nu \\ &\quad + h^2\mu[\tfrac{1}{3}\mu^2 + \tfrac{1}{3}\mu - \tfrac{4}{3}\nu^3 + \tfrac{1}{2}\nu^2\mu - \tfrac{4}{3}\nu\eta^2 - \tfrac{1}{3}\nu + \tfrac{1}{4}\mu\nu^{-4} - \tfrac{1}{3}\nu^{-3} + \tfrac{1}{6}\eta^2\mu] \end{aligned} \tag{12.32}$$

(cf. (6.7)) where $G$ represents the shear modulus (here taken to equal $\frac{4}{10}$ of the elastic modulus) and $2h$ represents the thickness of the ring. This energy, inspired by 2-dimensional considerations, penalizes the vanishing of $\nu$ but does not penalize a violation of (6.13). We scale time so that the shear wave speed is $1/\sqrt{2}$.

To promote the possible appearance of instabilities we use a slightly non-uniform mesh with 200 grid points. We now numerically solve our initial-value problem for the unmodified version first with a standard Lax-Friedrichs scheme, then with a van Leer MUSCL scheme [36], and finally with a difference scheme corresponding to invariant system (12.23), $(12.24)_2$, (12.31).

In Figure 4 we plot $\nu, \eta, \mu$ at a fixed material point as a function of $t$ for each of the two numerical solutions. We see that these variables for the standard Lax-Friedrichs scheme stay close to those of the exact solution, whereas $\nu$ and $\mu$ for the MUSCL scheme suffer sharp jumps before seeming to approach their values for the reference configuration. It is likely that the variation in $\mu$ is so large as to be inconsistent with (6.13), a difficulty that could be avoided by using more refined constitutive functions. In this and the next two figures, $\eta$ is very close to the exact solution $\eta = 0$, suggesting

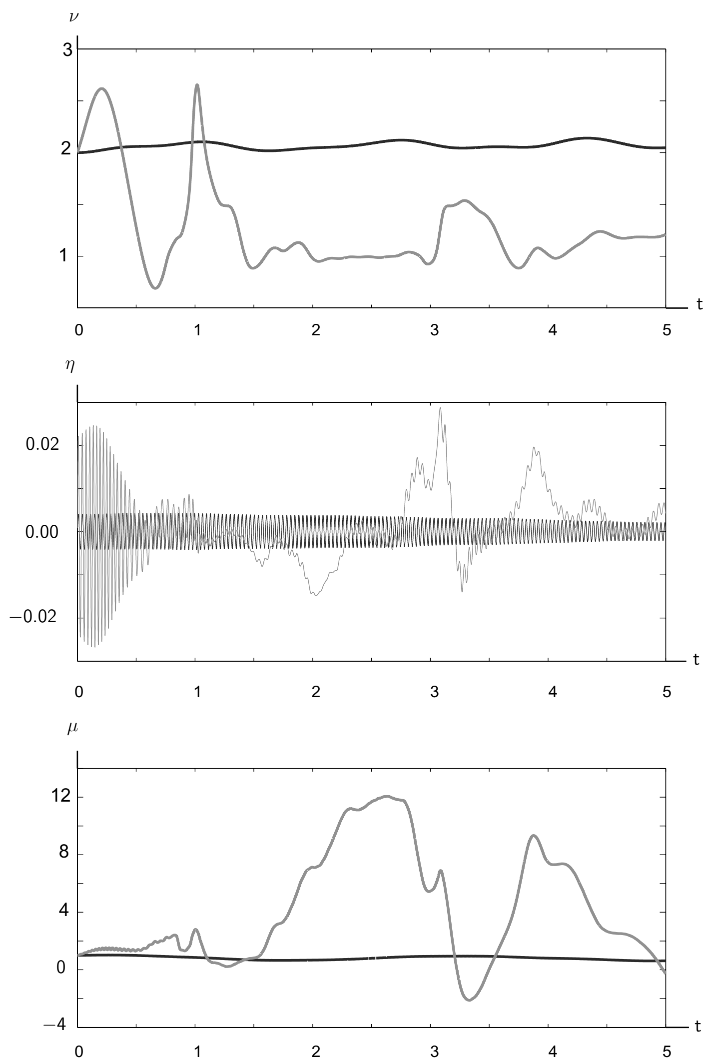



FIGURE 4. Time evolutions of $\nu$, $\eta$, $\mu$ at a fixed material point as computed by the Lax-Friedrichs scheme and the MUSCL scheme. The graphs with the large variation correspond to the latter. The exact solutions, given by the invariant Lax-Friedrichs scheme, are not shown. Note that the scales of each figure are vastly different.

that shear effects are negligible for this problem. (Such effects would be substantial for a motion combining this rotation with spinning [2].) The rapid and small oscillations of $\eta$ in Figure 4 are no doubt due to roundoff error. In Figures 5 and 6 we show numerical solutions for the functions $\nu, \eta, \mu$ at the fixed time $t = 5$. Here these functions computed by the standard Lax-Friedrichs scheme are relatively smooth. The bending strain differs most from the exact solution. The functions computed by the MUSCL scheme are more jagged and differ markedly from the exact solution. Indeed, they exhibit singular behavior near the middle of the range of $s$. We find similar behavior for the velocities.

What is the source of the discrepancies? The invariant dissipation penalizes relative motions of the ring, but not rigid motions. The dissipation for the standard Lax-Friedrichs scheme includes a mechanically curious viscous mechanism akin to air resistance, which would tend to slow down the rotation. Our results indicate that the MUSCL scheme, which adaptively supplies nonlinear viscosity, has a similar mechanism. Such mechanisms change our very special initial conditions at time 0 to typical initial conditions at a later time, and typical conditions for hyperbolic conservation laws typically give rise to shocks.

Why does the standard Lax-Friedrichs scheme not exhibit shock-like behavior, and accordingly appear to be more effective than the MUSCL scheme, which does exhibit such behavior? Since the numerical values for the invariant Lax-Friedrichs scheme are indistinguishable from the exact solution for the time interval treated, we cannot judge the results for the standard Lax-Friedrichs scheme to be good. In our numerical experiments, we found that enlarging the nonuniformity of the mesh, reducing the dissipation, changing the thickness $h$, extending the time interval, and changing the constitutive functions could each enlarge the discrepancy between the solution computed by the standard Lax-Friedrichs scheme and the exact solution, with the computed solutions exhibiting some shock-like behavior and ultimately approaching the natural reference state. The MUSCL scheme, on the other hand, apparently because of its sophisticated dissipative mechanism, is designed to capture shocks in a way superior to that of the Lax-Friedrichs scheme. Our computations suggest that the dissipative mechanism of the MUSCL scheme is not invariant, that its lack of invariance quickly moves the solution away from the initial conditions for the exact solution, and that the scheme with its small dissipation effectively

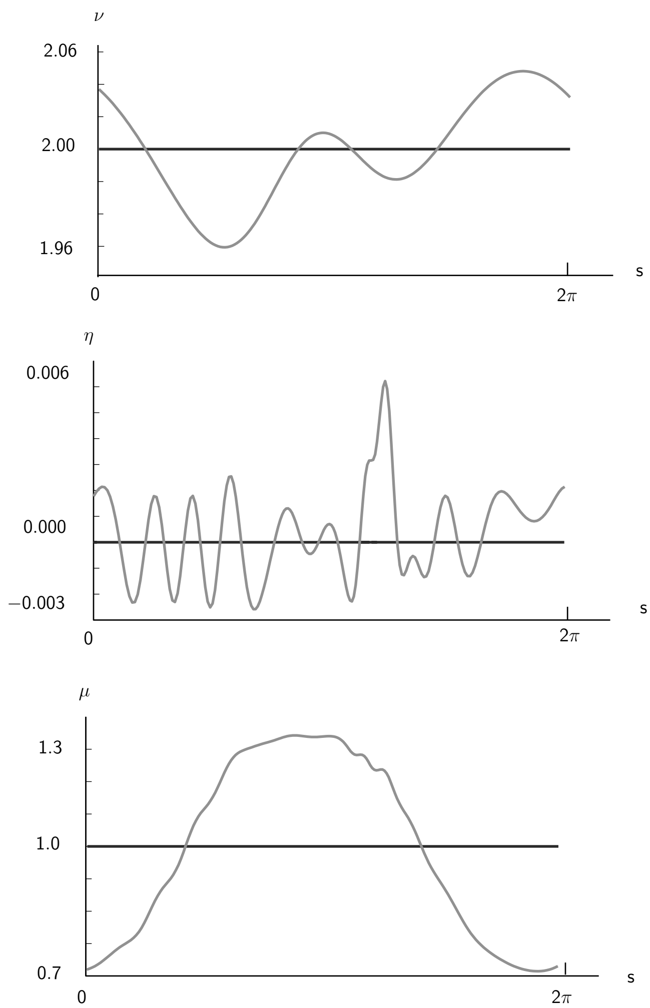



FIGURE 5. The forms of $\nu$, $\eta$, $\mu$ at a fixed time $t = 5$ as computed by a standard Lax-Friedrichs scheme and an invariant Lax-Friedrichs scheme. The graphs of the latter, which are the exact solutions, are the straight lines. Note that the scales of each of these graphs are very different.

captures the resulting shocks. (It would be valuable to modify the MUSCL scheme to make it invariant for problems of solid mechanics.)

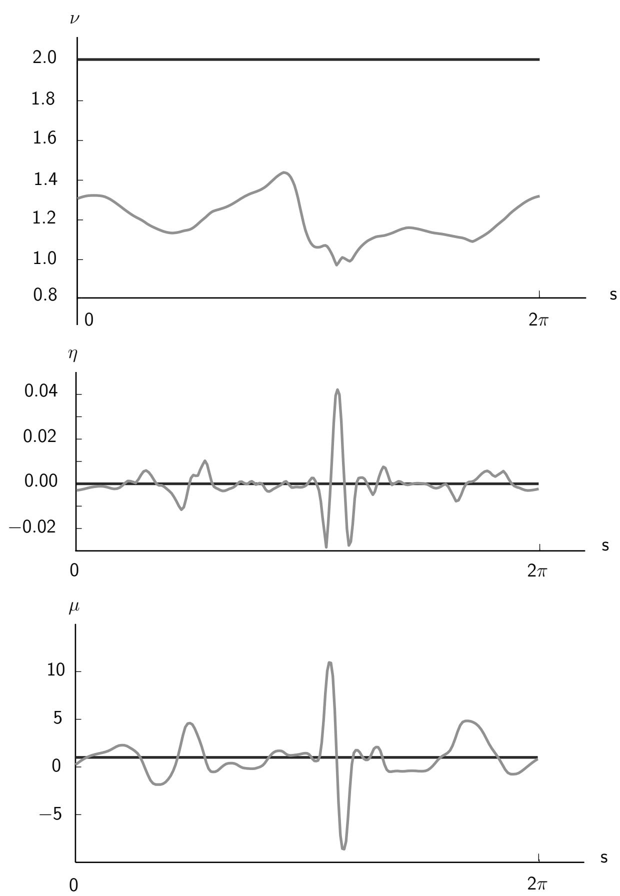



FIGURE 6. The forms of $\nu$, $\eta$, $\mu$ at a fixed time $t = 5$ as computed by the MUSCL scheme and invariant Lax-Friedrichs scheme. The graphs of the latter, which are the exact solutions, are the straight lines. Note that the scales of each of these graphs are very different.

There is an extensive literature on artificial viscosity; see [35, 43] and the references cited therein. Shokin [43] treats questions of invariance under change of coordinates, which are not the questions we have treated above. For a treatment of invariant forms of artificial viscosity for rods moving in space, see [6]. Details on the computations leading to Figures 4–6 are available upon request.

In the process of constructing invariant modifications corresponding to the Lax-Friedrichs scheme, we may introduce strain-gradient terms into the equations of motion.

These have the character of modifications of the dispersive Lax-Wendroff and Beam-Warming schemes. Our methods of constructing invariant regularizations leads to attractive new kinds of constitutive equations. Their study might throw light on open questions of shock structure, for which deep analyses have been required to study the asymptotics of systems like (12.5), (12.6) as $\alpha, \beta \to 0$ [18, 27].

## 13. Conclusion

The central issue in all of nonlinear continuum mechanics is the choice of physically and mathematically natural classes of constitutive equations. The situation for elastic materials is becoming clear, not in the sense that there is a single universal set of appropriate conditions as Truesdell once wished (cf. [46]), but that there is a range of alternative conditions, whose interrelationships and physical significance are reasonably well understood [7, 16]. In contrast, the situation for constitutive restrictions on dissipative mechanisms like viscosity and regularizing mechanisms like strain-gradient effects and capillarity is very far from being well understood. Even for ostensibly simple constitutive equations like (2.4), viscous and elastic effects are inextricably linked due to the presence of the argument $\boldsymbol{C}_t$, which involves products of $\boldsymbol{p_x}$ and $\boldsymbol{p_{xt}}$. Besides the strong viscosity we have studied above, there are many other viscous mechanisms involving memory effects. Which of these manifold mechanisms are physically appropriate for specific materials is still a largely open problem. Though there are attractive treatments of specific problems, the analysis of nonlinear problems with memory is still in its infancy [41].

In most of the dynamical problems treated above we employed a strong viscous mechanism corresponding to a uniform version of (6.12). This condition, together with a strengthening of it to cause the viscosity to become infinite at a total compression, ensures that initial-boundary-value problems for the viscoelastic rods of Section 6 do not permit shocks and have globally defined regular solutions [12]. The role of viscosity was not conspicuous in Section 7 because we were examining just ordinary differential equations. For the interesting problem of the loss of stability of these "trivial solutions" to non-radial solutions, the strong viscosity would again play a central role, as it does in various ad hoc models of parametric instability, e.g., that of [25, 37]. Viscosity, of course, does not enter in the equilibrium problem treated in Section 8. But as we mentioned in the first paragraph of Section 10, the presence of strong dissipative mechanisms is needed to make sense of the traditional doctrine of snapping, which was commented on in the last paragraph of Section 8. The treatment of Section 9 could be enriched by a discussion of how magneto-elastic controls could control viscosity [4]. The central theme of Sections 10–12 was the pervasive role of viscosity in the analysis.

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