Estimates on the Stokes Pressure by Partitioning the Energy of Harmonic Functions

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Abstract

We show that in a tubular domain with sufficiently small width, the normal and tangential gradients of a harmonic function have almost the same L^2 norm. This estimate yields a sharp estimate of the pressure in terms of the viscosity term in the Navier-Stokes equation with no-slip boundary condition. By consequence, one can analyze the Navier-Stokes equations simply as a perturbed vector diffusion equation instead of as a perturbed Stokes system. As an application, we describe a rather easy approach to establish a new isomorphism theorem for the non-homogeneous Stokes system.

1 Introduction

Let Ω to be a bounded, connected domain in $\mathbb{R}^N$ $(N \geq 2)$ with C^3 boundary $\Gamma = \partial\Omega$. The Navier-Stokes equations for incompressible fluid flow in Ω with no-slip boundary conditions on Γ take the form

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p = \nu\Delta\vec{u} + \vec{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\vec{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\vec{u} = 0 \quad \text{on } \Gamma. \tag{3}$$

Here $\vec{u}$ is the fluid velocity, p the pressure, and ν is the kinematic viscosity coefficient, assumed to be a fixed positive constant.

Let $\mathcal{P} : L^2(\Omega,\mathbb{R}^N) \to (\nabla H^1(\Omega))^\perp$ denote the Helmholtz-Hodge projection onto vector fields that are divergence free and have zero normal component on the boundary. One may apply $\mathcal{P}$ to both sides of (1) to obtain

$$\partial_t\vec{u} + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f} - \nu\Delta\vec{u}) = 0. \tag{4}$$

In this formulation, solutions formally satisfy $\partial_t(\nabla\cdot\vec{u}) = 0$. Consequently the zero-divergence condition (2) needs to be imposed only on initial data.

Alternatives are possible, however. Instead of (4), as in [LLP] we consider

$$\partial_t\vec{u} + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f} - \nu\Delta\vec{u}) = \nu\nabla(\nabla\cdot\vec{u}). \tag{5}$$

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There is no difference as long as $\nabla \cdot \vec{u} = 0$. However, the incompressibility constraint is enforced in a more robust way, because the divergence of velocity satisfies a weak form of the diffusion equation with no-flux (Neumann) boundary conditions — For all appropriate test functions ϕ, since $\mathcal{P}a \perp \nabla\phi$ for any $a \in L^2(\Omega, \mathbb{R}^N)$, we have

$$\int_\Omega \partial_t \vec{u} \cdot \nabla\phi = \nu \int_\Omega \nabla(\nabla \cdot \vec{u}) \cdot \nabla\phi. \tag{6}$$

Taking $\phi = \nabla \cdot \vec{u}$ we get the dissipation identity

$$\frac{d}{dt}\frac{1}{2}\int_\Omega (\nabla \cdot \vec{u})^2 + \nu \int_\Omega |\nabla(\nabla \cdot \vec{u})|^2 = 0. \tag{7}$$

Due to the Poincaré inequality and the fact that $\int_\Omega \nabla \cdot \vec{u} = 0$, the divergence of velocity is smoothed and decays exponentially in L^2 norm. And $\nabla \cdot \vec{u} = 0$ for all time if true initially, giving a solution of the standard Navier-Stokes equations (1)–(3).

A second reason to prefer (5) is much deeper. To explain, we recast (5) in the form (1). Subtracting (5) from (1), one can get

$$\nabla p = -(I - \mathcal{P})(\vec{u}\cdot\nabla\vec{u} - \vec{f}) + \nu\left((I - \mathcal{P})\Delta\vec{u} - \nabla\nabla \cdot \vec{u}\right). \tag{8}$$

To explicitly identify the separate contributions to the pressure term made by the convection and viscosity terms, we introduce the *Euler pressure* p_{E} and *Stokes pressure* p_{S} via the relations

$$\nabla p_{\mathrm{E}} = -(I - \mathcal{P})(\vec{u}\cdot\nabla\vec{u} - \vec{f}) \tag{9}$$

$$\nabla p_{\mathrm{S}} = (I - \mathcal{P})\Delta\vec{u} - \nabla(\nabla \cdot \vec{u}). \tag{10}$$

Using (9) and (10), one can put (5) into the form (1) with $p = p_{\mathrm{E}} + \nu p_{\mathrm{S}}$:

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p_{\mathrm{E}} + \nu\nabla p_{\mathrm{S}} = \nu\Delta\vec{u} + \vec{f}. \tag{11}$$

Identifying the Euler and Stokes pressure terms in this way allows one to focus separately on the difficulties peculiar to each. The Euler pressure is nonlinear, but of lower order. Since the Helmholtz projection is orthogonal, naturally the Stokes pressure satisfies

$$\int_\Omega |\nabla p_{\mathrm{S}}|^2 \le \int_\Omega |\Delta\vec{u}|^2 \qquad \text{if } \nabla \cdot \vec{u} = 0. \tag{12}$$

Actually, a better estimate is true. It is not hard to show $\nabla\nabla\cdot\vec{u} = \Delta(I - \mathcal{P})\vec{u}$ in the sense of distributions for $\vec{u} \in L^2(\Omega, \mathbb{R}^N)$ (see Lemma 1 of [LLP]), hence

$$\nabla p_{\mathrm{S}} = (I - \mathcal{P})\Delta\vec{u} - \nabla(\nabla \cdot \vec{u}) = (\Delta\mathcal{P} - \mathcal{P}\Delta)\vec{u} = [\Delta, \mathcal{P}]\vec{u}. \tag{13}$$

Then it turns out that the Stokes pressure term is actually *strictly* dominated by the viscosity term, regardless of the divergence constraint. The following theorem is proved in [LLP]:

Theorem 1 *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a connected bounded domain with C^3 boundary. Then for any $\varepsilon > 0$, there exists $C \ge 0$ such that for all vector fields $\vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$, the Stokes pressure p_S determined by (10) satisfies*

$$\int_\Omega |\nabla p_S|^2 \le \beta \int_\Omega |\Delta\vec{u}|^2 + C \int_\Omega |\nabla\vec{u}|^2, \qquad \text{where } \beta = \tfrac{1}{2} + \varepsilon. \tag{14}$$

This theorem allows one to view (5) as *fully dissipative*. Rewriting it as

$$\partial_t\vec{u} + \mathcal{P}(\vec{u}\cdot\nabla\vec{u} - \vec{f}) = \nu\Delta\vec{u} - \nu\nabla p_{\mathrm{S}}, \tag{15}$$

Theorem 1 allows us to regard the last term as a controlled perturbation and thus we can treat the Navier-Stokes equations in bounded domains simply as a perturbation of the vector diffusion equation $\partial_t\vec{u} = \nu\Delta\vec{u}$. Both the pressure and convection terms are dominated by the viscosity term. This contrasts with the standard approach that treats the Navier-Stokes equations as a perturbation of the Stokes system $\partial_t\vec{u} = \nu\Delta\vec{u} - \nabla p$, $\nabla\cdot\vec{u} = 0$.

A rather simple analytic proof of Theorem 1 has already been given in [LLP]. In this paper we will present an alternative, more geometric proof, which will be carried out in section 3. Important ingredients are: (i) an estimate near the boundary that is related to boundedness of the Neumann-to-Dirichlet map for boundary values of harmonic functions — this estimate is proved in section 2, see Theorem 2; and (ii) a representation formula for the Stokes pressure in terms of a part of velocity near and parallel to the boundary. In section 4 we deduce an apparently new result for the linear Stokes system, namely an isomorphism theorem between the solution space and a space of data for non-homogeneous side conditions in which only the average flux through the boundary vanishes.

For references to other work connected to this new formulation of Navier-Stokes equations and for further results, we refer to [LLP].

2 Integrated Neumann-to-Dirichlet estimates in tubes

2.1 Notation

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^3 boundary Γ. For any $\vec{x} \in \Omega$ we let $\Phi(\vec{x}) = \mathrm{dist}(x, \Gamma)$ denote the distance from x to Γ. For any $s > 0$ we denote the set of points in Ω within distance s from Γ by

$$\Omega_s = \{\vec{x} \in \Omega \mid \Phi(\vec{x}) < s\}, \tag{16}$$

and set $\Omega_s^c = \Omega \backslash \Omega_s$ and $\Gamma_s = \{\vec{x} \in \Omega \mid \Phi(\vec{x}) = s\}$. Since Γ is C^3 and compact, there exists $s_0 > 0$ such that Φ is C^3 in Ω_{s_0} and its gradient is a unit vector, with $|\nabla\Phi(\vec{x})| = 1$ for every $\vec{x} \in \Omega_{s_0}$. We let

$$\vec{n}(\vec{x}) = -\nabla\Phi(\vec{x}), \tag{17}$$

then $\vec{n}(\vec{x})$ is the outward unit normal to $\Gamma_s = \partial\Omega_s^c$ for $s = \Phi(\vec{x})$, and $\vec{n} \in C^2(\bar{\Omega}_{s_0}, \mathbb{R}^N)$.

We let $\langle f, g\rangle_\Omega = \int_\Omega fg$ denote the L^2 inner product of functions f and g in Ω, and let $\|\cdot\|_\Omega$ denote the corresponding norm in $L^2(\Omega)$. We drop the subscript on the inner product and norm when the domain of integration is understood in context.

2.2 Statement of results

Our strategy for proving Theorem 1 crucially involves an integrated Neumann-to-Dirichlet–type estimate for harmonic functions in the tubular domains Ω_s for small $s > 0$. Such an estimate can be obtained from a standard Neumann-to-Dirichlet estimate of the form

$$\int_{\Gamma_r} |(I - \vec{n}\vec{n}^t)\nabla p|^2 \leq \alpha_1 \int_{\Gamma_r} |\vec{n}\cdot\nabla p|^2, \tag{18}$$

by integrating over $r \in (0, s)$, provided one shows that $\alpha_1 > 0$ can be chosen independent of r for small $r > 0$. But the following theorem gives a sharper estimate on α_1 which enables us to establish the full result in Theorem 1 for any number β greater than $\frac{1}{2}$, independent of the domain. (In a half-space one has (14) with $\beta = \frac{1}{2}$ and $C = 0$ and this is sharp, see [LLP].)

Theorem 2 *Let Ω be a bounded domain with C^3 boundary. There exist positive constants s_1 and C_0 such that for any $s \le s_1$, whenever p is a harmonic function in Ω_s we have*

$$\int_{\Omega_s} |(I - \vec{n}\vec{n}^t)\nabla p|^2 \le (1 + C_0 s) \int_{\Omega_s} |\vec{n} \cdot \nabla p|^2. \tag{19}$$

Our proof here is different from the arguments in [LLP], and is motivated by the case of slab domains with periodic boundary conditions in the transverse directions. In this case the analysis reduces to estimates for Fourier series expansions in the transverse variables. For general domains, the idea is to approximate $-\Delta$ in thin tubular domains Ω_s by the Laplace-Beltrami operator on $\Gamma \times (0, s)$. This operator has a direct-sum structure, and we obtain the integrated Neumann-to-Dirichlet–type estimate by separating variables and expanding in series of eigenfunctions of the Laplace-Beltrami operator on Γ. For basic background in Riemannian geometry and the Laplace-Beltrami operator we refer to [Au] and [Ta].

2.3 Harmonic functions on $\Gamma \times (0, s)$

Geometric preliminaries. We consider the manifold $\mathcal{G} = \Gamma \times \mathcal{I}$ with $\mathcal{I} = (0, s)$ as a Riemannian submanifold of $\mathbb{R}^N \times \mathbb{R}$ with boundary $\partial\mathcal{G} = \Gamma \times \{0, s\}$. We let γ denote the metric on Γ induced from $\mathbb{R}^N$, let ι denote the standard Euclidean metric on $\mathcal{I}$, and let g denote the metric on the product space $\mathcal{G}$. Any vector $\vec{a}$ tangent to $\mathcal{G}$ at $z = (y, r)$ has components $\vec{a}_\Gamma$ tangent to Γ at y and $\vec{a}_\mathcal{I}$ tangent to $\mathcal{I}$ at r. For any two such vectors $\vec{a}$ and $\vec{b}$, we have

$$g(\vec{a}, \vec{b}) = \gamma(\vec{a}_\Gamma, \vec{b}_\Gamma) + \iota(\vec{a}_\mathcal{I}, \vec{b}_\mathcal{I}). \tag{20}$$

Given a C^1 function $z = (y, r) \mapsto f(y, r)$ on $\mathcal{G}$, its gradient $\nabla_\mathcal{G} f$ at z is a tangent vector to $\mathcal{G}$ determined from the differential via the metric, through requiring

$$g(\nabla_\mathcal{G} f, \vec{a}) = df \cdot \vec{a} \quad \text{for all } \vec{a} \in T_z\mathcal{G}. \tag{21}$$

By keeping r fixed, the function $y \mapsto f(y, r)$ determines the gradient vector $\nabla_\Gamma f$ tangent to Γ in similar fashion, and by keeping y fixed, the function $r \mapsto f(y, r)$ determines the gradient vector $\nabla_\mathcal{I} f$ tangent to $\mathcal{I}$. These gradients are also the components of $\nabla_\mathcal{G} f$:

$$(\nabla_\mathcal{G} f)_\Gamma = \nabla_\Gamma f, \qquad (\nabla_\mathcal{G} f)_\mathcal{I} = \nabla_\mathcal{I} f.$$

If $u = (u^1, \ldots, u^{N-1}) \mapsto y = (y^1, \ldots, y^N)$ is a local coordinate chart for Γ, the metric is given by $\gamma_{ij}\, du^i\, du^j$ (summation over repeated indices implied) with matrix elements

$$\gamma_{ij} = \frac{\partial y^k}{\partial u^i}\frac{\partial y^k}{\partial u^j}.$$

For $\mathcal{I} \subset \mathbb{R}$ the identity map serves as coordinate chart. In these coordinates the tangent vectors are written (in a form that aids in tracking coordinate changes) as

$$\nabla_\Gamma f = \gamma^{ij}\frac{\partial f}{\partial u^i}\frac{\partial}{\partial u^j}, \qquad \nabla_\mathcal{I} f = \frac{\partial f}{\partial r}\frac{\partial}{\partial r}. \tag{22}$$

As usual, the matrix $(\gamma^{ij}) = (\gamma_{ij})^{-1}$. Given two C^1 functions $f, \tilde{f}$ on $\mathcal{G}$,

$$\gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde{f}) = \gamma^{ij}\frac{\partial f}{\partial u^i}\frac{\partial \tilde{f}}{\partial u^j}, \qquad \iota(\nabla_\mathcal{I} f, \nabla_\mathcal{I} \tilde{f}) = \frac{\partial f}{\partial r}\frac{\partial \tilde{f}}{\partial r}. \tag{23}$$

In these coordinates, the (positive) Laplace-Beltrami operators on Γ and $\mathcal{I}$ respectively take the form

$$\Delta_\Gamma f = -\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial u_i}\left(\sqrt{\gamma}\gamma^{ij}\frac{\partial}{\partial u_j}f\right), \qquad \Delta_{\mathcal{I}} f = -\frac{\partial^2}{\partial r^2}f, \tag{24}$$

where $\sqrt{\gamma} = \sqrt{\det(\gamma_{ij})}$ is the change-of-variables factor for integration on Γ — if a function f on Γ is supported in the range of the local coordinate chart then

$$\int_\Gamma f(y)\,dS(y) = \int_{\mathbb{R}^{N-1}} f(y(u))\sqrt{\gamma}\,du. \tag{25}$$

(Since orthogonal changes of coordinates in $\mathbb{R}^N$ and $\mathbb{R}^{N-1}$ leave the integral invariant, one can understand $\sqrt{\gamma}$ as the product of the singular values of the matrix $\partial y/\partial u$.)

Whenever $f \in H^1(\Gamma)$ and $\tilde f \in H^2(\Gamma)$, one has the integration-by-parts formula

$$\int_\Gamma f\Delta_\Gamma \tilde f = \int_\Gamma \gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde f). \tag{26}$$

One may extend Δ_Γ to be a map from $H^1(\Gamma) \to H^{-1}(\Gamma)$ by using this equation as a definition of $\Delta_\Gamma \tilde f$ as a functional on $H^1(\Gamma)$. In standard fashion [Ta], one finds that $I+\Delta_\Gamma : H^1(\Gamma) \to H^{-1}(\Gamma)$ is an isomorphism, and that $(I+\Delta_\Gamma)^{-1}$ is a compact self-adjoint operator on $L^2(\Gamma)$, hence $L^2(\Gamma)$ admits an orthonormal basis of eigenfunctions of Δ_Γ. Since the coefficient functions in (24) are C^1, standard interior elliptic regularity results ([GT, Theorem 8.8], [Ta, p. 306, Proposition 1.6]) imply that the eigenfunctions belong to $H^2(\Gamma)$. We denote the eigenvalues of Δ_Γ by ν_k^2, $k = 1, 2, \ldots$, with $0 = \nu_1 \le \nu_2 \le \ldots$ where $\nu_k \to \infty$ as $k \to \infty$, and let ψ_k be corresponding eigenfunctions forming an orthonormal basis of $L^2(\Gamma)$. If $\Delta_\Gamma \psi = 0$ then ψ is constant on each component of Γ, so if m is the number of components of Γ, then $0 = \nu_m < \nu_{m+1}$.

In the coordinates $\hat u = (u, r) \mapsto z = (y, r)$ for $\mathcal{G}$, the metric g takes the form $\gamma_{ij}\,du^i\,du^j + dr^2$, and the Laplace-Beltrami operator $\Delta_{\mathcal{G}} = \Delta_\Gamma + \Delta_{\mathcal{I}}$. Similar considerations as above apply to $\Delta_{\mathcal{G}}$, except $\mathcal{G}$ has boundary. Whenever $f \in H^1_0(\mathcal{G})$ and $\tilde f \in H^2(\mathcal{G})$ we have

$$\int_{\mathcal{G}} f\Delta_{\mathcal{G}} \tilde f = \int_{\mathcal{G}} g(\nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} \tilde f). \tag{27}$$

One extends $\Delta_{\mathcal{G}}$ to map $H^1(\mathcal{G})$ to $H^{-1}(\mathcal{G})$ by using this equation as a definition of $\Delta_{\mathcal{G}} \tilde f$ as a functional on $H^1_0(\mathcal{G})$.

We introduce notation for L^2 inner products and norms on $\mathcal{G}$ as follows:

$$\langle f, \tilde f\rangle_{\mathcal{G}} = \int_{\mathcal{G}} f\tilde f \qquad\qquad \|f\|^2_{\mathcal{G}} = \int_{\mathcal{G}} |f|^2, \tag{28}$$

$$\langle \nabla_\Gamma f, \nabla_\Gamma \tilde f\rangle_{\mathcal{G}} = \int_{\mathcal{G}} \gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde f), \qquad\qquad \|\nabla_\Gamma f\|^2_{\mathcal{G}} = \int_{\mathcal{G}} \gamma(\nabla_\Gamma f, \nabla_\Gamma f), \tag{29}$$

$$\langle \nabla_{\mathcal{I}} f, \nabla_{\mathcal{I}} \tilde f\rangle_{\mathcal{G}} = \int_{\mathcal{G}} (\partial_r f)(\partial_r \tilde f), \qquad\qquad \|\nabla_{\mathcal{I}} f\|^2_{\mathcal{G}} = \int_{\mathcal{G}} (\partial_r f)^2, \tag{30}$$

$$\langle \nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} \tilde f\rangle_{\mathcal{G}} = \int_{\mathcal{G}} g(\nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} \tilde f) = \langle \nabla_\Gamma f, \nabla_\Gamma \tilde f\rangle_{\mathcal{G}} + \langle \nabla_{\mathcal{I}} f, \nabla_{\mathcal{I}} \tilde f\rangle_{\mathcal{G}}, \tag{31}$$

$$\|\nabla_{\mathcal{G}} f\|^2_{\mathcal{G}} = \int_{\mathcal{G}} g(\nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} f) = \|\nabla_\Gamma f\|^2_{\mathcal{G}} + \|\nabla_{\mathcal{I}} f\|^2_{\mathcal{G}}. \tag{32}$$

Lemma 1 *Let*

$$\gamma = \int_{-s/2}^{s/2} \sinh^2 \nu_{m+1}\tau d\tau$$

with ν_{m+1} the first non-zero eigenvalue of Δ_Γ. Suppose $f \in H^1(\mathcal{G})$ and $\Delta_\mathcal{G} f = 0$ on $\mathcal{G} = \Gamma \times (0, s)$. Then,

$$\|\nabla_\Gamma f\|_\mathcal{G}^2 \leq \left(1 + \frac{s}{\gamma}\right) \|\nabla_\mathcal{I} f\|_\mathcal{G}^2. \tag{33}$$

Proof: Suppose $\Delta_\mathcal{G} f = 0$ on $\mathcal{G}$. Since the coefficient functions in (24) are C^1, the aforementioned interior elliptic regularity results imply that that $f \in H^2_{\text{loc}}(\mathcal{G})$. For any $r \in (0, s)$, fixing r yields a trace of f in $H^1(\Gamma)$, and as a function of r, we can regard $f = f(y, r)$ as in the space $L^2([a, b], H^2(\Gamma)) \cap H^2([a, b], L^2(\Gamma))$ for any closed interval $[a, b] \subset (0, s)$. Now, for each r we have the $L^2(\Gamma)$-convergent expansion

$$f(y, r) = \sum_k \hat{f}(k, r)\psi_k(y) \tag{34}$$

where

$$\hat{f}(k, r) = \int_\Gamma f(y, r)\psi_k(y)\, dS(y). \tag{35}$$

For each $k \in \mathbb{N}$, the map $r \mapsto \hat{f}(k, r)$ is in $H^2_{\text{loc}}(0, s)$ and

$$\partial_r \hat{f}(k, r) = \int_\Gamma \partial_r f(y, r)\psi_k(y)\, dS(y). \tag{36}$$

For any smooth $\xi \in C_0^\infty(0, s)$, taking $\tilde{f}(y, r) = \psi_k(y)\xi(r)$ we compute that

$$\nabla_\Gamma \tilde{f} = \xi(r)\nabla_\Gamma \psi_k, \qquad \partial_r \tilde{f} = \psi_k \partial_r \xi, \tag{37}$$

and so by (27), (20), and (26), we have

$$\begin{aligned} 0 = \int_{\mathcal{G}_s} (\Delta_\mathcal{G} f)\tilde{f} &= \int_\mathcal{I}\int_\Gamma \Big(\gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde{f}) + (\partial_r f)(\partial_r \tilde{f})\Big) \\ &= \int_\mathcal{I} \xi(r) \int_\Gamma \gamma(\nabla_\Gamma f, \nabla_\Gamma \psi_k) + \int_\mathcal{I} (\partial_r \xi) \int_\Gamma (\partial_r f)\psi_k \\ &= \int_\mathcal{I} \xi(r) \int_\Gamma f \Delta_\Gamma \psi_k + \int_\mathcal{I} (\partial_r \xi)\partial_r \hat{f}(k, r) \\ &= \int_0^s \Big(\xi(r)\nu_k^2 \hat{f}(k, r) + (\partial_r \xi)\partial_r \hat{f}(k, r)\Big)\, dr. \end{aligned} \tag{38}$$

Therefore $\hat{f}(k, \cdot)$ is a weak solution of $\partial_r^2 \hat{f} = \nu_k^2 \hat{f}$ in $H^2_{\text{loc}}(0, s)$ and hence is C^2. It follows that whenever $\nu_k \neq 0$, there exist a_k, b_k such that

$$\hat{f}(k, r) = a_k \sinh \nu_k \tau + b_k \cosh \nu_k \tau, \qquad \tau = r - s/2. \tag{39}$$

Now

$$\|f\|_\mathcal{G}^2 = \sum_k \int_0^s |\hat{f}(k, r)|^2\, dr, \tag{40}$$

$$\|\nabla_\Gamma f\|_\mathcal{G}^2 = \sum_k \int_0^s |\nu_k \hat{f}(k, r)|^2\, dr, \tag{41}$$

$$\|\nabla_\mathcal{I} f\|_\mathcal{G}^2 = \sum_k \int_0^s |\partial_r \hat{f}(k, r)|^2\, dr. \tag{42}$$

Let $\gamma_k = \int_{-s/2}^{s/2} \sinh^2 \nu_k\tau\, d\tau$. Then $\gamma_k + s = \int_{-s/2}^{s/2} \cosh^2 \nu_k\tau\, d\tau$. When $\nu_k \neq 0$ we get

$$\int_0^s |\nu_k \hat{f}(k,r)|^2\, dr = \nu_k^2(|a_k|^2\gamma_k + |b_k|^2(\gamma_k + s)), \tag{43}$$

$$\int_0^s |\partial_r \hat{f}(k,r)|^2\, dr = \nu_k^2(|a_k|^2(\gamma_k + s) + |b_k|^2\gamma_k). \tag{44}$$

Since γ_k increases with k, $\beta_0(\gamma_k + s) \leq \gamma_k$ when $k \geq m+1$, where $\beta_0 = \gamma_{m+1}/(\gamma_{m+1} + s) = \gamma/(\gamma + s)$. It follows from (43) and (44) that

$$\beta_0 \int_0^s |\nu_k \hat{f}(k,r)|^2\, dr \leq \int_0^s |\partial_r \hat{f}(k,r)|^2\, dr. \tag{45}$$

The result follows by summing up over k. □

2.4 Global coordinates on $\Gamma \times (0, s)$

To prove Theorem 2, we need to compare the Laplacian on $\mathcal{G} = \Gamma \times (0, s)$ with the Laplacian on Ω_s. It will be important for this reason to coordinatize $\mathcal{G}$ for small $s > 0$ *globally* via the coordinate chart $\Omega_s \to \mathcal{G}$ given by

$$x \mapsto z = (y, r) = (x + \Phi(x)\vec{n}(x), \Phi(x)). \tag{46}$$

In these coordinates, the metric on $\mathcal{G}$ that is inherited from $\mathbb{R}^{N+1}$ has the representation $g_{ij}\, dx^i\, dx^j$ with matrix elements given by

$$g_{ij} = \frac{\partial z^k}{\partial x^i}\frac{\partial z^k}{\partial x^j} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j} + \frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}. \tag{47}$$

Let us write $\partial_i = \partial/\partial x^i$ and let $\nabla f = (\partial_1 f, \ldots, \partial_N f)$ denote the usual gradient vector in $\mathbb{R}^N$. The components of $\vec{n}$ are $n_i = -\partial_i\Phi$ and so $\partial_i n_j = \partial_j n_i$, meaning the matrix $\nabla\vec{n}$ is symmetric. Since $|\vec{n}|^2 = 1$ we have $n_i\partial_j n_i = 0 = n_i\partial_i n_j$. Then the $N \times N$ matrix

$$\frac{\partial y}{\partial x} = I - \vec{n}\vec{n}^t + \Phi\nabla\vec{n} = (I - \vec{n}\vec{n}^t)(I + \Phi\nabla\vec{n})(I - \vec{n}\vec{n}^t), \tag{48}$$

and the matrix

$$G = (g_{ij}) = (I - \vec{n}\vec{n}^t)(I + \Phi\nabla\vec{n})^2(I - \vec{n}\vec{n}^t) + \vec{n}\vec{n}^t = (I + \Phi\nabla\vec{n})^2. \tag{49}$$

With $\sqrt{g} = \sqrt{\det G}$, the integral of a function f on $\mathcal{G}$ in terms of these coordinates is given by

$$\int_{\mathcal{G}} f = \int_{\Omega_s} f\, \sqrt{g}\, dx. \tag{50}$$

Given two C^1 functions f, $\tilde{f}$ on $\mathcal{G}$, we claim that the following formulae are valid in the coordinates from (46):

$$g(\nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} \tilde{f}) = (\nabla f)^t G^{-1}(\nabla\tilde{f}) = g^{ij}\partial_i f \partial_j \tilde{f}, \tag{51}$$

$$\gamma(\nabla_\Gamma f, \nabla_\Gamma \tilde{f}) = (\nabla f)^t (I - \vec{n}\vec{n}^t) G^{-1} (I - \vec{n}\vec{n}^t)(\nabla\tilde{f}), \tag{52}$$

$$\iota(\nabla_{\mathcal{I}} f, \nabla_{\mathcal{I}} \tilde{f}) = (\vec{n}\cdot\nabla f)(\vec{n}\cdot\nabla\tilde{f}) = (\nabla f)^t \vec{n}\vec{n}^t (\nabla\tilde{f}). \tag{53}$$

Of course (51) simply expresses the metric in the x-coordinates from (46). To prove (53), note that along any curve $\tau \mapsto x(\tau)$ satisfying $\partial_\tau x = \vec{n}(x)$ we have

$$\partial_\tau \vec{n}(x) = n_j\partial_j n_i = n_j\partial_j\partial_i\Phi = n_j\partial_i n_j = 0.$$

So $\vec{n}(x)$ is constant and the curve is a straight line segment. Hence in the chart from (46), $\vec{n}(x) = \vec{n}(y)$ and we have $x = y - r\vec{n}(y)$. Given a C^1 function f then, we find that in these Ω_s-coordinates,

$$\partial_r f(y,r) = (\partial_r x_j)(\partial_j f) = -n_j \partial_j f = -\vec{n}\cdot\nabla f, \tag{54}$$

and (53) follows from (23). Finally, (52) follows directly from (51) and (53) using (20) — because of (49) we have $\vec{n}\vec{n}^t G = G\vec{n}\vec{n}^t = \vec{n}\vec{n}^t$, so $\vec{n}\vec{n}^t = \vec{n}\vec{n}^t G^{-1} = G^{-1}\vec{n}\vec{n}^t$ and hence

$$(I - \vec{n}\vec{n}^t)G^{-1}(I - \vec{n}\vec{n}^t) = G^{-1} - \vec{n}\vec{n}^t. \tag{55}$$

2.5 Proof of Theorem 2

There exists an $\varepsilon_1 > 0$ so that when $0 < \varepsilon \le \varepsilon_1$, $\left((1+\varepsilon)^{-6} - \varepsilon\right)^{-1} \le 1 + 20\varepsilon$. (Indeed, $\varepsilon_1 \approx 0.16$.) Assuming that the distance function Φ is C^3 in Ω_{s_0}, the s_1 in Theorem 2 will be taken as $\varepsilon_1/(\sqrt{2}C)$ with C some constant shown up later which depends only on Ω and s_0. We assume that $s_1 \le s_0$, otherwise we can make C larger and take $s_1 = s_0$.

Suppose $\Delta p = 0$ in Ω_s. We may assume $p \in H^1(\Omega_s)$ without loss of generality by establishing the result in subdomains where $\Phi(x) \in (a,b)$ with $[a,b] \subset (0,s)$ and taking $a \to 0$, $b \to s$. We write

$$p = p_1 + p_2,$$

where $p_1 \in H_0^1(\Omega_s)$ is found by solving a weak form of $\Delta_{\mathcal{G}} p_1 = \Delta_{\mathcal{G}} p$:

$$\langle \nabla_{\mathcal{G}} p_1, \nabla_{\mathcal{G}} \phi\rangle_{\mathcal{G}} = \langle \nabla_{\mathcal{G}} p, \nabla_{\mathcal{G}} \phi\rangle_{\mathcal{G}} \quad \text{for all } \phi \in H_0^1(\Omega_s). \tag{56}$$

For small $s > 0$, $G = (g_{ij}) = I + O(s)$ and $\sqrt{g} = 1 + O(s)$. Since $\langle \nabla p, \nabla p_1\rangle = 0$, taking $\phi = p_1$ we have

$$\|\nabla_{\mathcal{G}} p_1\|_{\mathcal{G}}^2 = \int_{\Omega_s} (\nabla p)^t (G^{-1}\sqrt{g} - I)\nabla p_1 \, dx \le Cs\|\nabla p\|_{\Omega_s} \|\nabla_{\mathcal{G}} p_1\|_{\mathcal{G}}, \tag{57}$$

where C is a constant independent of s.

For $0 < \varepsilon < 1$, using (52), (50) and (29) we deduce

$$\begin{aligned}
\|(I - \vec{n}\vec{n}^t)\nabla p\|_{\Omega_s}^2 &\le (1 + Cs)\|\nabla_\Gamma p\|_{\mathcal{G}}^2 \\
&\le (1 + Cs)\left((1+\varepsilon)\|\nabla_\Gamma p_2\|_{\mathcal{G}}^2 + (1+\varepsilon^{-1})\|\nabla_\Gamma p_1\|_{\mathcal{G}}^2\right) \\
&\le (1 + Cs)(1+\varepsilon)\left(\|\nabla_\Gamma p_2\|_{\mathcal{G}}^2 + \varepsilon^{-1} C^2 s^2 \|\nabla p\|_{\Omega_s}^2\right).
\end{aligned} \tag{58}$$

Now $p_2 = p - p_1$ satisfies $\Delta_{\mathcal{G}} p_2 = 0$ in Ω_s and $p_2 \in H^1(\mathcal{G})$, hence we have

$$\|\nabla_\Gamma p_2\|_{\mathcal{G}}^2 \le \left(1 + \frac{s}{\gamma}\right)\|\nabla_{\mathcal{I}} p_2\|_{\mathcal{G}}^2, \tag{59}$$

$$\begin{aligned}
\|\nabla_{\mathcal{I}} p_2\|_{\mathcal{G}}^2 &\le (1+\varepsilon)\|\nabla_{\mathcal{I}} p\|_{\mathcal{G}}^2 + (1+\varepsilon^{-1})\|\nabla_{\mathcal{I}} p_1\|_{\mathcal{G}}^2 \\
&\le (1+\varepsilon)(1+Cs)\left(\|\vec{n}\cdot\nabla p\|_{\Omega_s}^2 + \varepsilon^{-1} C^2 s^2 \|\nabla p\|_{\Omega_s}^2\right),
\end{aligned} \tag{60}$$

Without loss of generality, we can take $C > 1/\gamma$. Taking $Cs = \varepsilon/\sqrt{2}$, assembling these estimates yields

$$\begin{aligned}
\|(I - \vec{n}\vec{n}^t)\nabla p\|_{\Omega_s}^2 &\le (1+\varepsilon)^5 \left(\|\vec{n}\cdot\nabla p\|_{\Omega_s}^2 + \varepsilon\|\nabla p\|_{\Omega_s}^2\right) \\
&\le (1+\varepsilon)^6 \left(\|\vec{n}\cdot\nabla p\|_{\Omega_s}^2 + \varepsilon\|(I - \vec{n}\vec{n}^t)\nabla p\|_{\Omega_s}^2\right),
\end{aligned} \tag{61}$$

since $|\nabla p|^2 = |\vec{n} \cdot \nabla p|^2 + |(I - \vec{n}\vec{n}^t)\nabla p|^2$. From the above inequality, we can derive

$$\big((1+\varepsilon)^{-6} - \varepsilon\big)\, \|(I - \vec{n}\vec{n}^t)\nabla p\|^2_{\Omega_s} \le \|\vec{n} \cdot \nabla p\|^2_{\Omega_s}. \tag{62}$$

When $s \le s_1$, we have $\varepsilon \le \varepsilon_1$ and hence $\big((1+\varepsilon)^{-6} - \varepsilon\big)^{-1} \le 1 + 20\varepsilon$. So, we get

$$\|(I - \vec{n}\vec{n}^t)\nabla p\|^2_{\Omega_s} \le (1 + 20\varepsilon)\, \|\vec{n} \cdot \nabla p\|^2_{\Omega_s}. \tag{63}$$

Substituting $\varepsilon = \sqrt{2}Cs$ into (63) leads to (19) for any $s \le s_1$.

3 Analysis of the Stokes pressure

The main purpose of this section is to prove Theorem 1. Here we follow rather closely the arguments made in [LLP].

3.1 Identities at the boundary

A key part of the proof of Theorem 1 involves boundary values of two quantities that involve the decomposition of $\vec{u} = (I - \vec{n}\vec{n}^t)\vec{u} + \vec{n}\vec{n}^t\vec{u}$ into parts parallel and normal to the boundary, for which we have the following lemma.

Lemma 2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class C^3. Then for any $\vec{u} \in H^2(\Omega, \mathbb{R}^N)$ with $\vec{u}|_\Gamma = 0$, the following is valid on Γ:*

(i) $\nabla \cdot \big((I - \vec{n}\vec{n}^t)\vec{u}\big) = 0$ in $H^{1/2}(\Gamma)$.

(ii) $\vec{n} \cdot (\Delta - \nabla\nabla\cdot)\big(\vec{n}\vec{n}^t\vec{u}\big) = 0$ in $H^{-1/2}(\Gamma)$.

Proof: By a density argument, we only need to consider $\vec{u} \in C^2(\bar{\Omega}, \mathbb{R}^N)$. In [LLP, Lemma 3], we have proved (i) as well as

$$\nabla \vec{u}_\perp - (\nabla \vec{u}_\perp)^\tau = 0 \qquad \text{on } \Gamma \tag{64}$$

with $\vec{u}_\perp = \vec{n}\vec{n}^t\vec{u}$ in a neighborhood of Γ. Then, note that for any $\vec{v} \in H^2(\Omega, \mathbb{R}^N)$ and $\phi \in H^1(\Omega)$, by integration by parts, we have the identity

$$\int_\Gamma (\vec{n} \cdot (\Delta\vec{v} - \nabla\nabla \cdot \vec{v}))\, \phi = \int_\Omega (\Delta\vec{v} - \nabla\nabla \cdot \vec{v}) \cdot \nabla\phi = \int_\Gamma \vec{n} \cdot (\nabla\vec{v} - \nabla\vec{v}^\tau)\, \nabla\phi. \tag{65}$$

Hence (ii) follows from (64) by taking $\vec{v}$ to be equal to $\vec{u}_\perp$ in a neighborhood of Γ in (65). $\square$

3.2 Identities for the Stokes pressure

Given $\vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$, recall that $\mathcal{P}(\nabla\nabla \cdot \vec{u}) = 0$, so that the Stokes pressure defined in (10) satisfies

$$\nabla p_{\mathrm{S}} = \Delta\vec{u} - \nabla\nabla \cdot \vec{u} - \mathcal{P}\Delta\vec{u} = (I - \mathcal{P})(\Delta - \nabla\nabla\cdot)\vec{u}. \tag{66}$$

Also recall that whenever $\vec{a} \in L^2(\Omega, \mathbb{R}^N)$ and $\nabla \cdot \vec{a} \in L^2(\Omega)$, $\vec{n} \cdot \vec{a} \in H^{-1/2}(\Gamma)$ by the trace theorem for $H(\mathrm{div}; \Omega)$. If $\nabla \cdot \vec{a} = 0$ and $\vec{n} \cdot \vec{a}|_\Gamma = 0$, then we have $\langle \vec{a}, \nabla\phi \rangle = 0$ for all $\phi \in H^1(\Omega)$ and this means $(I - \mathcal{P})\vec{a} = 0$. Thus, the Stokes pressure is not affected by any part of the velocity field that contributes nothing to $\vec{n} \cdot \vec{a}|_\Gamma$ where $\vec{a} = (\Delta - \nabla\nabla\cdot)\vec{u}$. Indeed, this means that the Stokes pressure is not affected by the part of the velocity field in the interior of Ω away from the boundary, nor is it affected by the normal component of velocity near the boundary, since $\vec{n} \cdot (\Delta - \nabla\nabla\cdot)(\vec{n}\vec{n}^t\vec{u})|_\Gamma = 0$ by Lemma 2.

This suggests we focus on the part of velocity near and parallel to the boundary. We make the following decomposition. Let $\rho : [0,\infty) \to [0,1]$ be a smooth decreasing function with $\rho(t) = 1$ for $t < \frac{1}{2}$ and $\rho(t) = 0$ for $t \geq 1$. For small $s > 0$, the cutoff function given by $\xi(x) = \rho(\Phi(x)/s)$ is C^3, with $\xi = 1$ when $\Phi(x) < \frac{1}{2}s$ and $\xi = 0$ when $\Phi(x) \geq s$. Then we can write

$$\vec{u} = \vec{u}_\perp + \vec{u}_\parallel \tag{67}$$

where

$$\vec{u}_\perp = (1-\xi)\vec{u} + \xi\vec{n}\vec{n}^t\vec{u}, \qquad \vec{u}_\parallel = \xi(I - \vec{n}\vec{n}^t)\vec{u}. \tag{68}$$

Since $\vec{u}_\perp = (\vec{n}\vec{n}^t)\vec{u}$ in $\Omega_{s/2}$, with $\vec{a}_\perp = (\Delta - \nabla\nabla\cdot)\vec{u}_\perp$ we have

$$\vec{a}_\perp \in L^2(\Omega, \mathbb{R}^N), \qquad \nabla\cdot\vec{a}_\perp = 0 \quad \text{and} \quad \vec{n}\cdot\vec{a}_\perp|_\Gamma = 0 \tag{69}$$

by Lemma 2(ii). Hence $\langle \vec{a}_\perp, \nabla\phi\rangle = 0$ for all $\phi \in H^1(\Omega)$, that is,

$$(I - \mathcal{P})(\Delta - \nabla\nabla\cdot)\vec{u}_\perp = 0. \tag{70}$$

Combining this with (66) and (67) proves part (i) of the following.

Lemma 3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^3 boundary, and let $\vec{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$. Let p_S and $\vec{u}_\parallel$ be defined as in (66) and (68) respectively. Then*

(i) The Stokes pressure is determined by $\vec{u}_\parallel$ according to the formula

$$\nabla p_S = (I - \mathcal{P})(\Delta - \nabla\nabla\cdot)\vec{u}_\parallel. \tag{71}$$

(ii) For any $q \in H^1(\Omega)$ that satisfies $\Delta q = 0$ in the sense of distributions,

$$\langle \Delta\vec{u}_\parallel - \nabla p_S, \nabla q\rangle = 0. \tag{72}$$

(iii) In particular we can let $q = p_S$ in (ii), so $\langle \Delta\vec{u}_\parallel - \nabla p_S, \nabla p_S\rangle = 0$ and

$$\|\Delta\vec{u}_\parallel\|^2 = \|\Delta\vec{u}_\parallel - \nabla p_S\|^2 + \|\nabla p_S\|^2. \tag{73}$$

Proof: We already proved (i). For (ii), note by Lemma 2(i) we have

$$\nabla\cdot\vec{u}_\parallel|_\Gamma = 0, \tag{74}$$

so $\nabla\cdot\vec{u}_\parallel \in H^1_0(\Omega)$, thus $\langle \nabla\nabla\cdot\vec{u}_\parallel, \nabla q\rangle = -\langle \nabla\cdot\vec{u}_\parallel, \Delta q\rangle = 0$. Now (i) entails

$$\langle \nabla p_{\mathrm{S}}, \nabla q\rangle = \langle \Delta\vec{u}_\parallel, \nabla q\rangle. \tag{75}$$

This proves (ii), and then (iii) follows by the L^2 orthogonality. $\square$

3.3 Proof of Theorem 1

Let $\varepsilon > 0$ and $\beta = \frac{1}{2} + \varepsilon$. Fix $\beta_1 < 1$ such that $1 + \varepsilon_0 := \beta(1 + \beta_1) > 1$, and fix $s, \varepsilon_1, \varepsilon_2 > 0$ small so that $2\varepsilon_1 < \varepsilon_0$ and $1 - \varepsilon_2 - 2C_0 s > \beta_1$ with C_0 as in Theorem 2.

Let $\vec{u} \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$, define the Stokes pressure by (10), and make the decomposition $\vec{u} = \vec{u}_\perp + \vec{u}_\parallel$ as in the previous subsection. Then by Lemma 3 we have

$$\|\Delta\vec{u}\|^2 = \|\Delta\vec{u}_\perp\|^2 + 2\big\langle\Delta\vec{u}_\perp, \Delta\vec{u}_\parallel\big\rangle + \|\Delta\vec{u}_\parallel - \nabla p_{\mathrm{S}}\|^2 + \|\nabla p_{\mathrm{S}}\|^2. \tag{76}$$

We will establish the Theorem with the help of two further estimates.
Claim 1: There exists a constant $C_1 > 0$ independent of $\vec{u}$ such that

$$\big\langle\Delta\vec{u}_\perp, \Delta\vec{u}_\parallel\big\rangle \geq -\varepsilon_1\|\Delta\vec{u}\|^2 - C_1\|\nabla\vec{u}\|^2. \tag{77}$$

Claim 2: There exists a constant C_2 independent of $\vec{u}$ such that

$$\|\Delta\vec{u}_\parallel - \nabla p_{\mathrm{S}}\|^2 \geq \beta_1\|\nabla p_{\mathrm{S}}\|^2 - C_2\|\nabla\vec{u}\|^2. \tag{78}$$

Combining the two claims with (76), we get

$$(1 + 2\varepsilon_1)\|\Delta\vec{u}\|^2 \geq (1 + \beta_1)\,\|\nabla p_{\mathrm{S}}\|^2 - (C_2 + 2C_1)\|\nabla\vec{u}\|^2. \tag{79}$$

Multiplying by β and using $\beta(1 + \beta_1) = 1 + \varepsilon_0 > 1 + 2\varepsilon_1$ yields (14).
Proof of claim 1: From the definitions in (68), we have

$$\Delta\vec{u}_\perp = \xi\vec{n}\vec{n}^t\Delta\vec{u} + (1 - \xi)\Delta\vec{u} + R_1, \qquad \Delta\vec{u}_\parallel = \xi(I - \vec{n}\vec{n}^t)\Delta\vec{u} + R_2, \tag{80}$$

where $\|R_1\| + \|R_2\| \leq C\|\nabla\vec{u}\|$ with C independent of $\vec{u}$. Since $I - \vec{n}\vec{n}^t = (I - \vec{n}\vec{n}^t)^2$,

$$\big(\xi\vec{n}\vec{n}^t\Delta\vec{u} + (1 - \xi)\Delta\vec{u}\big)\cdot\big(\xi(I - \vec{n}\vec{n}^t)\Delta\vec{u}\big) = 0 + \xi(1 - \xi)|(I - \vec{n}\vec{n}^t)\Delta\vec{u}|^2 \geq 0.$$

This means the leading term of $\big\langle\Delta\vec{u}_\perp, \Delta\vec{u}_\parallel\big\rangle$ is non-negative. Using the inequality $|\big\langle a, b\big\rangle| \leq (\varepsilon_1/C)\|a\|^2 + (C/\varepsilon_1)\|b\|^2$ and the bounds on R_1 and R_2 to estimate the remaining terms, it is easy to obtain (77).
Proof of claim 2: Let $\vec{a} = \nabla p_{\mathrm{S}}$ and $\vec{b} = \Delta\vec{u}_\parallel$, and put

$$\vec{a}_\parallel = (I - \vec{n}\vec{n}^t)\vec{a}, \quad \vec{a}_\perp = (\vec{n}\vec{n}^t)\vec{a}, \quad \vec{b}_\parallel = (I - \vec{n}\vec{n}^t)\vec{b}, \quad \vec{b}_\perp = (\vec{n}\vec{n}^t)\vec{b}. \tag{81}$$

Recall $\vec{u}_\parallel$ is supported in $\Omega_s = \{x \in \Omega \mid \Phi(x) < s\}$. Due to (80), we have

$$\int_{\Omega_s}|\vec{b}_\perp|^2 = \int_{\Omega_s}|\vec{n}\cdot\Delta\vec{u}_\parallel|^2 = \int_{\Omega_s}|\vec{n}\cdot R_2|^2 \leq C\int_\Omega|\nabla\vec{u}|^2 \tag{82}$$

Since $\vec{b} = 0$ in $\Omega_s^c = \{x \in \Omega \mid \Phi(x) \geq s\}$, we have

$$\|\Delta\vec{u}_\parallel - \nabla p_{\mathrm{S}}\|^2 = \int_\Omega|\vec{a} - \vec{b}|^2 = \int_{\Omega_s^c}|\vec{a}|^2 + \int_{\Omega_s}|\vec{a}_\perp - \vec{b}_\perp|^2 + \int_{\Omega_s}|\vec{a}_\parallel - \vec{b}_\parallel|^2. \tag{83}$$

We estimate the terms in (83) as follows. First,

$$\int_{\Omega_s}|\vec{a}_\perp - \vec{b}_\perp|^2 \geq \int_{\Omega_s}(|\vec{a}_\perp|^2 - 2\vec{a}_\perp\cdot\vec{b}_\perp) \geq (1 - \varepsilon_2)\int_{\Omega_s}|\vec{a}_\perp|^2 - \frac{1}{\varepsilon_2}\int_{\Omega_s}|\vec{b}_\perp|^2. \tag{84}$$

Due to the orthogonality in Lemma 3, we have $\langle \vec{a}, \vec{a} - \vec{b} \rangle = 0$, hence

$$0 = \int_\Omega \vec{a} \cdot (\vec{a} - \vec{b}) = \int_{\Omega_s^c} |\vec{a}|^2 + \int_{\Omega_s} \vec{a}_\perp \cdot (\vec{a}_\perp - \vec{b}_\perp) + \int_{\Omega_s} \vec{a}_\| \cdot (\vec{a}_\| - \vec{b}_\|). \tag{85}$$

For a sharp estimate we need to treat $\vec{b}_\|$ carefully. Using (85) we obtain

$$\begin{aligned} \int_{\Omega_s} |\vec{a}_\| - \vec{b}_\||^2 + |\vec{a}_\||^2 &\geq -2 \int_{\Omega_s} \vec{a}_\| \cdot (\vec{a}_\| - \vec{b}_\|) \\ &= 2 \int_{\Omega_s^c} |\vec{a}|^2 + 2 \int_{\Omega_s} \vec{a}_\perp \cdot (\vec{a}_\perp - \vec{b}_\perp) \\ &\geq 2 \int_{\Omega_s^c} |\vec{a}|^2 + (2 - \varepsilon_2) \int_{\Omega_s} |\vec{a}_\perp|^2 - \frac{1}{\varepsilon_2} \int_{\Omega_s} |\vec{b}_\perp|^2, \end{aligned}$$

hence

$$\int_{\Omega_s} |\vec{a}_\| - \vec{b}_\||^2 \geq (1 - \varepsilon_2) \int_{\Omega_s} |\vec{a}_\||^2 + (2 - \varepsilon_2) \int_{\Omega_s} (|\vec{a}_\perp|^2 - |\vec{a}_\||^2) - \frac{1}{\varepsilon_2} \int_{\Omega_s} |\vec{b}_\perp|^2. \tag{86}$$

Using (84) and (86) in (83) yields

$$\int_\Omega |\vec{a} - \vec{b}|^2 \geq (1 - \varepsilon_2) \int_\Omega |\vec{a}|^2 + (2 - \varepsilon_2) \int_{\Omega_s} (|\vec{a}_\perp|^2 - |\vec{a}_\||^2) - \frac{2}{\varepsilon_2} \int_{\Omega_s} |\vec{b}_\perp|^2. \tag{87}$$

Finally, using Theorem 2 and the estimate (82) we infer

$$\int_\Omega |\nabla p_{\rm S} - \Delta \vec{u}_\||^2 \geq (1 - \varepsilon_2 - 2C_0 s) \int_\Omega |\nabla p_{\rm S}|^2 - C \int_\Omega |\nabla \vec{u}|^2. \tag{88}$$

This establishes Claim 2, and finishes the proof of Theorem 1. □

4 Isomorphism theorems for non-homogeneous Stokes systems

Consider the non-homogeneous Stokes system:

$$\begin{aligned} \partial_t \vec{u} + \nabla p - \nu \Delta \vec{u} &= \vec{f} & &(t > 0,\ x \in \Omega), & (89) \\ \nabla \cdot \vec{u} &= h & &(t \geq 0,\ x \in \Omega), & (90) \\ \vec{u} &= \vec{g} & &(t \geq 0,\ x \in \Gamma), & (91) \\ \vec{u} &= \vec{u}_{\rm in} & &(t = 0,\ x \in \Omega). & (92) \end{aligned}$$

The aim of this section is to obtain an isomorphism between the space of solutions and the space of data $\{\vec{f}, \vec{g}, h, \vec{u}_{\rm in}\}$, for the Stokes system. In examining this question we are motivated by the classic works of Lions and Magenes [LM] which provide a satisfactory description of the correspondence between solutions and data for elliptic boundary value problems. In the spirit of these results, a satisfactory theory of a given system of partial differential equations should describe exactly how, in the space of all functions involved, the manifold of solutions can be parametrized. Yet we are not aware of any such complete treatment of the non-homogeneous Stokes system. (See further remarks on this issue below.)

We will first present an unconstrained formulation of the Stokes system (89)-(92) and then study existence and uniqueness of solutions of this *new* formulation. Finally we go back to the Stokes system and establish an isomorphism theorem for it.

4.1 An unconstrained formulation

What we have done before to get an unconstrained formulation of Navier-Stokes equation with non-slip boundary condition can be viewed as replacing the divergence constraint (2) by decomposing the pressure via the formulae in (9) and (10) in such a way that the divergence constraint is enforced automatically. It turns out that in the non-homogeneous case a very similar procedure works. Although we treat linear Stokes systems here, a similar procedure works for non-homogeneous Navier-Stokes equations. One can simply use the Helmholtz decomposition to identify a Stokes pressure term *exactly as before* via the formulae (10), but in addition another term is needed in the total pressure to deal with the inhomogeneities. Dropping the nonlinear term, equation (5) is replaced by

$$\partial_t \vec{u} + \mathcal{P}(-\vec{f} - \nu\Delta\vec{u}) + \nabla p_{gh} = \nu\nabla(\nabla\cdot\vec{u}). \tag{93}$$

The equation that determines the inhomogeneous pressure p_{gh} can be found by dotting with $\nabla\phi$ for $\phi \in H^1(\Omega)$, formally integrating by parts and plugging in the side conditions: We require

$$\big\langle \nabla p_{gh}, \nabla\phi \big\rangle = -\big\langle \partial_t(\vec{n}\cdot\vec{g}), \phi \big\rangle_\Gamma + \big\langle \partial_t h, \phi \big\rangle + \big\langle \nu\nabla h, \nabla\phi \big\rangle \tag{94}$$

for all $\phi \in H^1(\Omega)$. With this definition, we see from (93) that

$$\big\langle \partial_t\vec{u}, \nabla\phi \big\rangle - \big\langle \partial_t(\vec{n}\cdot\vec{g}), \phi \big\rangle_\Gamma + \big\langle \partial_t h, \phi \big\rangle = \big\langle \nu\nabla(\nabla\cdot\vec{u} - h), \nabla\phi \big\rangle \tag{95}$$

for every $\phi \in H^1(\Omega)$. This will mean $w := \nabla\cdot\vec{u} - h$ is a weak solution of

$$\partial_t w = \nu\Delta w \ \text{ in } \Omega, \qquad \vec{n}\cdot\nabla w = 0 \ \text{ on } \Gamma, \tag{96}$$

with initial condition $w = \nabla\cdot\vec{u}_{\text{in}} - h\big|_{t=0}$. So the divergence constraint will be enforced through exponential diffusive decay as before (see (119) below).

To find the total pressure in (89), subtract (93) from (89) to get

$$\nabla p = (I - \mathcal{P})\vec{f} + \nu\nabla p_{\text{S}} + \nabla p_{gh}, \tag{97}$$

where p_{S} is defined as before via (10), and p_{gh} is determined up to a constant by the forcing functions g and h through the weak-form pressure Poisson equation (94). (See Lemma 5 below.) Our unconstrained formulation is

$$\partial_t\vec{u} + \nabla p - \nu\Delta\vec{u} = \vec{f} \qquad (t > 0,\ x \in \Omega), \tag{98}$$

$$\vec{u} = \vec{g} \qquad (t \ge 0,\ x \in \Gamma), \tag{99}$$

$$\vec{u} = \vec{u}_{\text{in}} \qquad (t = 0,\ x \in \Omega), \tag{100}$$

with p determined by (97).

Although the definition of Stokes pressure does not require a no-slip velocity field, clearly the analysis that we performed in section 2 does rely in crucial ways on no-slip boundary conditions. So in order to analyze the new unconstrained formulation, we will decompose the velocity field $\vec{u}$ in two parts. We introduce a fixed field $\tilde{u}$ in $\Omega \times [0, T]$ that satisfies $\tilde{u} = \vec{g}$ on Γ, and let

$$\vec{v} = \vec{u} - \tilde{u}. \tag{101}$$

Then $\vec{v} = 0$ on Γ. With this $\vec{v}$, similar to (10) we introduce

$$\nabla q_{\text{S}} = (I - \mathcal{P})\Delta\vec{v} - \nabla\nabla\cdot\vec{v}. \tag{102}$$

Then we can rewrite (98) as an equation for $\vec{v}$:

$$\partial_t \vec{v} + \nu \nabla q_{\mathrm{S}} = \nu \Delta \vec{v} + \tilde{f}, \tag{103}$$

where

$$\tilde{f} := -\partial_t \tilde{u} + \mathcal{P}(\nu \Delta \tilde{u}) + \nu \nabla\nabla \cdot \tilde{u} - \nabla p_{gh} + \mathcal{P}\vec{f}. \tag{104}$$

We will answer questions concerning the existence and regularity of $\tilde{u}$ and p_{gh} in the next subsection.

4.2 Regularity assumptions

Let Ω be a bounded, connected domain in $\mathbb{R}^N$ ($N \geq 2$) with boundary Γ of class C^3. Let

$$V(0,T) := L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{105}$$
$$W(0,T) := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)'). \tag{106}$$

Here $(H^1)'$ is the space dual to H^1. Note we have the embeddings ([Ta1, p. 42], [Ev, p. 288], [Te1, p. 176])

$$V(0,T) \hookrightarrow C([0,T],H^1(\Omega)), \quad W(0,T) \hookrightarrow C([0,T],L^2(\Omega)). \tag{107}$$

Our theory on strong solutions comes in two similar flavors, depending on the regularity assumed on the data. The two flavors correspond to solutions having either the regularity

$$\vec{u} \in V_{\mathrm{div}}(0,T) := V(0,T)^N \cap \{\vec{u} \mid \nabla\cdot\vec{u} \in V(0,T)\}, \tag{108}$$

or the somewhat weaker regularity

$$\vec{u} \in W_{\mathrm{div}}(0,T) := V(0,T)^N \cap \{\vec{u} \mid \nabla\cdot\vec{u} \in W(0,T)\}, \tag{109}$$

for some $T > 0$. In the first case, $\nabla\cdot\vec{u}$ is more regular ($\nabla\cdot\vec{u} = 0$ is usual), but we need to assume $\nabla\cdot\vec{u}_{\mathrm{in}} \in H^1(\Omega)$ due to the embedding (107). The condition (109) means that $\vec{u} \in V(0,T)^N$ and $\nabla\cdot\vec{u}$ has vector-valued distributional derivative $\partial_t(\nabla\cdot\vec{u})$ in $L^2(0,T;H^1(\Omega)')$, the dual of $L^2(0,T;H^1(\Omega))$.

Note the following characterization of $W_{\mathrm{div}}(0,T)$ (see [LLP, Lemma 6] for the proof):

Lemma 4 $\quad W_{\mathrm{div}}(0,T) = V(0,T)^N \cap \{\vec{u} \mid \partial_t(\vec{n}\cdot\vec{u}|_\Gamma) \in L^2(0,T;H^{-1/2}(\Gamma))\}.$

Corresponding to the regularity in (109), our precise assumptions on the data are that for some $T > 0$ we have

$$\vec{u}_{\mathrm{in}} \in H_{uin} := H^1(\Omega,\mathbb{R}^N), \tag{110}$$
$$\vec{f} \in H_f := L^2(0,T;L^2(\Omega,\mathbb{R}^N)), \tag{111}$$
$$\vec{g} \in H_g := H^{3/4}(0,T;L^2(\Gamma,\mathbb{R}^N)) \cap L^2(0,T;H^{3/2}(\Gamma,\mathbb{R}^N)) \cap \{\vec{g} \,\big|\, \partial_t(\vec{n}\cdot\vec{g}) \in L^2(0,T;H^{-1/2}(\Gamma))\}, \tag{112}$$
$$h \in H_h := W(0,T) \tag{113}$$

We also make the compatibility assumptions

$$\vec{g} = \vec{u}_{\mathrm{in}} \quad \text{when } t = 0,\ x \in \Gamma, \tag{114}$$
$$\big\langle \partial_t(\vec{n}\cdot\vec{g}), 1 \big\rangle_\Gamma = \big\langle \partial_t h, 1 \big\rangle_\Omega. \tag{115}$$

Lemma 5 *Assume (110)-(115). Then, there exists some $\tilde{u} \in V$ that satisfies*

$$\tilde{u}(0) = \vec{u}_{\rm in}, \quad \tilde{u}\big|_\Gamma = \vec{g}, \tag{116}$$

and there exists $p_{gh} \in L^2(H^1(\Omega)/\mathbb{R})$ satisfying (94). Moreover,

$$\|\tilde{u}\|_V^2 \le C\big(\|\vec{g}\|^2_{H^{3/4}(L^2(\Gamma))\cap L^2(H^{3/2}(\Gamma))} + \|\vec{u}_{\rm in}\|^2_{H^1(\Omega)}\big), \tag{117}$$

$$\|p_{gh}\|_{L^2(H^1(\Omega)/\mathbb{R})} \le C\big(\|\partial_t(\vec{n}\cdot\vec{g})\|_{L^2(H^{-1/2}(\Gamma))} + \|h\|_{W(0,T)}\big). \tag{118}$$

Proof: (i) By a trace theorem of Lions and Magenes [LM, vol II, Theorem 2.3], the fact $\vec{g} \in H^{3/4}(L^2(\Gamma)) \cap L^2(H^{3/2}(\Gamma))$ together with (110) and the compatibility condition (114) implies the existence of $\tilde{u} \in V$ satisfying (116).

(ii) One applies the Lax-Milgram lemma for a.e. t to (94) in the space of functions in $H^1(\Omega)$ with zero average. We omit the standard details. □

4.3 Existence and uniqueness for the unconstrained formulation (97)–(100)

Theorem 3 *Let Ω be a bounded, connected domain in $\mathbb{R}^N$ $(N \ge 2)$ and assume (110)-(115). Then for any $T > 0$, there is a unique strong solution of (97)-(100) exists on $[0,T]$, with*

$$\vec{u} \in V(0,T)^N, \qquad p \in L^2(0,T;H^1(\Omega)/\mathbb{R}),$$

where p_S and p_{gh} are defined in (10) and (94). Moreover, $\vec{u} \in C([0,T],H^1(\Omega,\mathbb{R}^N))$ and

$$\nabla\cdot\vec{u} - h \in W(0,T)$$

is a smooth solution of the heat equation for $t > 0$ with no-flux boundary conditions. The map $t \mapsto \|\nabla\cdot\vec{u} - h\|^2$ is smooth for $t > 0$ and we have the dissipation identity

$$\frac{d}{dt}\frac{1}{2}\|\nabla\cdot\vec{u} - h\|^2 + \nu\|\nabla(\nabla\cdot\vec{u} - h)\|^2 = 0. \tag{119}$$

If we further assume $h \in H_{h.s} := V(0,T)$ and $\nabla\cdot\vec{u}_{\rm in} \in H^1(\Omega)$, then

$$\nabla\cdot\vec{u} \in V(0,T).$$

Proof: We will consider the existence of the system (102)-(104). Consider the following spatially continuous time discretization scheme:

$$\frac{\vec{v}^{n+1} - \vec{v}^n}{\Delta t} - \nu\Delta\vec{v}^{n+1} = \tilde{f}^n - \nu\nabla q^n_{\rm S}, \tag{120}$$

$$\nabla q^n_{\rm S} = (I - \mathcal{P})\Delta\vec{v}^n - \nabla(\nabla\cdot\vec{v}^n), \tag{121}$$

$$\vec{v}^n\big|_\Gamma = 0. \tag{122}$$

We set

$$\tilde{f}^n = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \tilde{f}(t)\,dt, \tag{123}$$

and take $\vec{v}^0 = 0$ because of (116). Notice that from Lemma 5 we can conclude $\tilde{f} \in L^2(L^2(\Omega))$.

Dot (120) with $-\Delta\vec{v}^{n+1}$, we get

$$\begin{aligned}\frac{1}{2\Delta t}&\Big(\|\nabla\vec{v}^{n+1}\|^2-\|\nabla\vec{v}^{n}\|^2+\|\nabla\vec{v}^{n+1}-\nabla\vec{v}^{n}\|^2\Big)+\nu\|\Delta\vec{v}^{n+1}\|^2\\&\le\|\Delta\vec{v}^{n+1}\|\left(\|\tilde{f}^n\|+\nu\|\nabla q_{\mathrm{S}}^n\|\right)\\&\le\frac{\varepsilon_1}{2}\|\Delta\vec{v}^{n+1}\|^2+\frac{2}{\varepsilon_1}\|\tilde{f}^n\|^2+\frac{\nu}{2}\big(\|\Delta\vec{v}^{n+1}\|^2+\|\nabla q_{\mathrm{S}}^n\|^2\big)\end{aligned}\tag{124}$$

By Theorem 1, $\|\nabla q_{\mathrm{S}}^n\|^2\le\beta\|\Delta\vec{v}^n\|^2+C\|\nabla\vec{v}^n\|^2$, and we can manipulate (124) to derive

$$\begin{aligned}\frac{1}{\Delta t}&\Big(\|\nabla\vec{v}^{n+1}\|^2-\|\nabla\vec{v}^{n}\|^2\Big)+(\nu-\varepsilon_1)\big(\|\Delta\vec{v}^{n+1}\|^2-\|\Delta\vec{v}^{n}\|^2\big)+\varepsilon_2\|\Delta\vec{v}^{n}\|^2\\&\le\frac{4}{\varepsilon_1}\|\tilde{f}^n\|^2+\nu C_\beta\|\nabla\vec{v}^n\|^2,\end{aligned}\tag{125}$$

for $\varepsilon_2=\nu-\varepsilon_1-\nu\beta>0$. Using a discrete Gronwall inequality and $\vec{v}^0=0$ gives

$$\sup_{0\le k\le n}\|\nabla\vec{v}^k\|^2+\sum_{k=0}^{n}\|\Delta\vec{v}^k\|^2\Delta t+\sum_{k=0}^{n-1}\left\|\frac{\vec{v}^{k+1}-\vec{v}^k}{\Delta t}\right\|^2\Delta t\le e^{CT}\sum_{k=0}^{n}\|\tilde{f}^k\|^2\Delta t,\tag{126}$$

where we have also used (120). This implies that the function $\vec{v}_{\Delta t}(t)$ and $\vec{V}_{\Delta t}(t)$ are bounded uniformly in

$$L^2(H^2\cap H_0^1(\Omega))\cap H^1(L^2(\Omega))\quad\text{and}\quad L^2(H^2\cap H_0^1(\Omega)),\tag{127}$$

respectively, where $\vec{v}_{\Delta t}(t)$ and $\vec{V}_{\Delta t}(t)$ are defined on each subinterval $[t_n,t_n+\Delta t)$ through linear interpolation and as piecewise constant respectively:

$$\vec{v}_{\Delta t}(t_n+s)=\vec{v}^n+s\left(\frac{\vec{v}^{n+1}-\vec{v}^n}{\Delta t}\right),\qquad s\in[0,\Delta t),\tag{128}$$

$$\vec{V}_{\Delta t}(t_n+s)=\vec{v}^n,\qquad s\in[0,\Delta t).\tag{129}$$

Then (120) means that whenever $t>0$ with $t\neq t_n$,

$$\partial_t\vec{v}_{\Delta t}-\nu\mathcal{P}\Delta\vec{V}_{\Delta t}=\nu\Delta(\vec{V}_{\Delta t}(\cdot+\Delta t)-\vec{V}_{\Delta t})+\nu\nabla\nabla\cdot\vec{V}_{\Delta t}+\tilde{f}^n_{\Delta t},\tag{130}$$

where $\tilde{f}_{\Delta t}(t)=\tilde{f}^n$ for $t\in[t_n,t_n+\Delta t)$. By the boundedness in (127), we have associated weakly convergent sequences in (127) and $\vec{v}_{\Delta t}\to\vec{v}$ strongly in $L^2(L^2(\Omega))$. Then, because of (126),

$$\|\vec{v}_{\Delta t}-\vec{V}_{\Delta t}\|^2_{L^2(Q)}\le\|\vec{V}_{\Delta t}(\cdot+\Delta t)-\vec{V}_{\Delta t}\|^2_{L^2(Q)}=\sum_{k=0}^{n-1}\|\vec{v}^{n+1}-\vec{v}^n\|^2\Delta t\le C\Delta t^2.\tag{131}$$

Therefore $\vec{V}_{\Delta t},\vec{V}_{\Delta t}(\cdot+\Delta t)$ and $\vec{v}_{\Delta t}$ convergence strongly in $L^2(L^2(\Omega))$ to the same limits $\vec{v}$. And hence their weak limits in (127) are the same. Then we can pass to the limit weakly in $L^2(L^2(\Omega))$ in all terms in (130) to see $\vec{v}$ satisfies

$$\partial_t\vec{v}-\nu\mathcal{P}(\Delta\vec{v})=\nu\nabla\nabla\cdot\vec{v}+\tilde{f}.$$

That is, $\vec{v}$ is indeed a strong solution of (103) and therefore (98). At the same time, we also get the boundedness of the mapping from data to solution. For this linear equation, the uniqueness follows.

Now, we prove the regularity of $\nabla \cdot \vec{u}$. We go from (98) to (93) by using (97) and (10). Then using (94) we get (95) for any $\phi \in H^1(\Omega)$.

Recall, the operator $A := \nu\Delta$ defined on $L^2(\Omega)$ with domain

$$D(A) = \{w \in H^2(\Omega) \mid \vec{n} \cdot \nabla w = 0 \text{ on } \Gamma\} \tag{132}$$

is self-adjoint and non-positive, so generates an analytic semigroup. With $w = \nabla \cdot \vec{u} - h$, taking $\phi \in D(A)$, we have

$$\langle w, \phi \rangle = \langle \vec{n} \cdot \vec{g}, \phi \rangle_\Gamma - \langle \vec{u}, \nabla\phi \rangle - \langle h, \phi \rangle, \tag{133}$$

therefore $t \mapsto \langle w, \phi \rangle$ is absolutely continuous, and (95) yields $(d/dt)\langle w, \phi \rangle = \langle w, A\phi \rangle$ for a.e. t. By Ball's characterization of weak solutions of abstract evolution equations [Ba], $w(t) = e^{At}w(0)$ for all $t \in [0, T]$. It follows $w \in C([0, T], L^2(\Omega))$, and $w(t) \in D(A^m)$ for every $m > 0$ [Pa, theorem 6.13]. Since $A^m w(t) = e^{A(t-\tau)} A^m w(\tau)$ if $0 < \tau < t$ we infer that for $0 < t \leq T$, $w(t)$ is analytic in t with values in $D(A^m)$. Using interior estimates for elliptic equations, we find $w \in C^\infty((0, T], C^\infty(\Omega))$ as desired. The dissipation identity follows by dotting with w.

If we further assume $h \in H_{h.s}$ and $\nabla \cdot \vec{u}_{\text{in}} \in H^1(\Omega)$, then $w(0) \in H^1(\Omega)$. We claim

$$H^1(\Omega) = D((-A)^{1/2}). \tag{134}$$

Then semigroup theory yields $w \in C([0, T], D((-A)^{1/2}))$. Since

$$0 = \big\langle -\Delta w, \partial_t w - \nu\Delta w \big\rangle = \frac{d}{dt}\frac{1}{2}\|\nabla w\|^2 + \nu\|\Delta w\|^2 \tag{135}$$

for $t > 0$, we integrate (135) on $[\varepsilon, T]$ and let $\varepsilon \to 0$ to deduce $w \in L^2(0, T; H^2(\Omega))$. Then because $\partial_t w = \nu\Delta w$, we get $w \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$, and $\nabla \cdot \vec{u}$ is in the same space.

To prove (134), note $X := D((-A)^{1/2})$ is the closure of $D(A)$ from (132) in the norm given by

$$\|w\|_X^2 = \|w\|^2 + \|(-A)^{1/2}w\|^2 = \big\langle (I - \nu\Delta)w, w \big\rangle = \int_\Omega |w|^2 + \nu|\nabla w|^2.$$

Clearly $X \subset H^1(\Omega)$. For the other direction, let $w \in H^1(\Omega)$ be arbitrary. We may suppose $w \in C^\infty(\bar{\Omega})$ since this space is dense in $H^1(\Omega)$. Now we only need to construct a sequence of C^2 functions $w_n \to 0$ in H^1 norm with $\vec{n} \cdot \nabla w_n = \vec{n} \cdot \nabla w$ on Γ. This is easily accomplished using functions of the form $w_n(x) = \xi_n(\text{dist}(x, \Gamma))\vec{n} \cdot \nabla w(x)$, where $\xi_n(s) = \xi(ns)/n$ with ξ smooth and satisfying $\xi(0) = 0$, $\xi'(0) = 1$ and $\xi(s) = 0$ for $s > 1$. This proves (134).

4.4 Isomorphism theorem for the Stokes system (89)–(92)

4.4.1 From data to solutions

First, we consider the mapping from data to solutions. Given $\{\vec{f}, \vec{g}, h, \vec{u}_{\text{in}}\}$, we first solve (97)-(100). To go from the unconstrained formulation (97)-(100) to Stokes system (89)-(92), there is one more step: we need to verify the $\nabla \cdot \vec{u} = h$ is satisfied. It turns out that this is true if the following additional compatibility condition holds:

$$\nabla \cdot \vec{u}_{\text{in}} = h \qquad \text{in } \Omega \;\text{ for }\; t = 0. \tag{136}$$

From $\Pi_{F.c}$ to Π_U. Recall, from Theorem 3, when the data $\{\vec{f}, \vec{g}, h, \vec{u}_{\text{in}}\}$ lie inside the space

$$\Pi_F := H_f \times H_g \times H_h \times H_{uin} \tag{137}$$

from (110)–(113), and satisfy the compatibility conditions (114)–(115), we have a unique solution $\vec{u}$ of (97)-(100) in the space

$$H_u := W_{\rm div}(0,T). \tag{138}$$

The total pressure p lies in

$$H_p := L^2(0,T;H^1(\Omega)/\mathbb{R}), \tag{139}$$

and the pair $\{\vec{u},p\}$ satisfies (89), (91) and (92). Moreover, $w = \nabla\cdot\vec{u} - h$ satisfies a heat equation with no-flux boundary conditions and (119) is true. Equation (90) says that $w = 0$, and this will hold if and only if $w(0) = 0$, i.e., the additional compatibility condition (136) holds.

Therefore, for the non-homogeneous Stokes system (89)–(92), we should define the data and solution spaces by

$$\Pi_{F.c} := \Big\{\{\vec{f},\vec{g},h,\vec{u}_{\rm in}\} \in \Pi_F :\ (114),\ (115) \text{ and } (136) \text{ hold }\Big\}, \tag{140}$$

$$\Pi_U := H_u \times H_p. \tag{141}$$

From what we have said so far, we get a bounded map $\Pi_{F.c} \to \Pi_U$ by solving the unconstrained formulation (97)–(100), which because of (136) also solves (89)-(92).

From $\Pi_{F.c.s}$ to $\Pi_{U.s}$. Note that in Theorem 3, one has more regularity on $\nabla\cdot\vec{u}$ if one assumes more on $\nabla\cdot\vec{u}_{\rm in}$ and h. Correspondingly, like $H_{h.s}$ defined in Theorem 3, we introduce spaces of stronger regularity by

$$H_{uin.s} := H^1(\Omega,\mathbb{R}^N) \cap \{\vec{u}_{\rm in} \mid \nabla\cdot\vec{u}_{\rm in} \in H^1(\Omega)\}, \tag{142}$$

$$\Pi_{F.s} := H_f \times H_g \times H_{h.s} \times H_{uin.s}. \tag{143}$$

The solution $\vec{u}$ then lies in

$$H_{u.s} := V_{\rm div}(0,T). \tag{144}$$

So as an alternative to the spaces in (140)–(141), we also obtain an isomorphism between the data and solution spaces with stronger regularity defined by

$$\Pi_{F.c.s} := \Big\{\{\vec{f},\vec{g},h,\vec{u}_{\rm in}\} \in \Pi_{F.s} :\ (114),\ (115) \text{ and } (136) \text{ hold }\Big\}, \tag{145}$$

$$\Pi_{U.s} := H_{u.s} \times H_p. \tag{146}$$

4.4.2 From solutions to data

In the other direction, given $\{\vec{u},p\} \in \Pi_U$ (or $\Pi_{U,s}$), we simply define $\{\vec{f},\vec{g},h,\vec{u}_{\rm in}\}$ using (89)–(92) and check that this lies in $\Pi_{F.c}$ (or $\Pi_{F.c.s}$). (Note that we are not using (97)-(100) because we have trouble to determine h. See the Remark 2 at the end of this section.)

4.4.3 Isomorphism theorem

Summarizing, we have proved the following isomorphism theorem for the non-homogeneous Stokes system (89)–(92).

Theorem 4 *Let Ω be a bounded, connected domain in $\mathbb{R}^N$ with N any positive integer ≥ 2, and let $T > 0$. The map $\{\vec{f},\vec{g},h,\vec{u}_{\rm in}\} \mapsto \{\vec{u},p\}$, given by solving the unconstrained system (97)–(100), defines an isomorphism from $\Pi_{F.c}$ onto Π_U. The same solution procedure defines an isomorphism from $\Pi_{F.c.s}$ onto $\Pi_{U.s}$.*

Remark 1. For the standard Stokes system with zero-divergence constraints $\nabla \cdot \vec{u}_{\rm in} = 0$ and $h = 0$, existence and uniqueness results together with the estimates

$$\begin{aligned}\sup_{0\le t\le T} \|\vec{u}(t)\|_{H^1} + \|\vec{u}\|_{L^2(0,T;H^2)} + \|p\|_{L^2(0,T;H^1/\mathbb{R})} \\ \le C\big(\|\vec{f}\|_{L^2(0,T;L^2)} + \|\vec{u}_{\rm in}\|_{H^1} + \|\vec{g}\|_{H^{3/4}(L^2(\Gamma))} + \|\vec{g}\|_{L^2(H^{3/2}(\Gamma))}\big)\end{aligned} \tag{147}$$

were obtained in the classic work of Solonnikov [Sol, Theorem 15], where more general L^p estimates were also proved. (Also see [GS1, GS2].) However, instead of the necessary compatibility condition

$$\int_\Gamma \vec{n}\cdot\vec{g} = 0, \tag{148}$$

Solonnikov made the stronger constraining assumption that both the data $\vec{g}$ and solution $\vec{u}$ have zero normal component on Γ, and correspondingly his estimates do not contain a term $\|\partial_t(\vec{n}\cdot\vec{g})\|_{L^2(H^{-1/2}(\Gamma))}$ on the right hand side of (147). (Note that when $\nabla\cdot\vec{u}_{\rm in} = 0$ and $h = 0$, we have $\int_\Gamma \vec{n}\cdot\vec{g}|_{t=0} = \int_\Omega \nabla\cdot\vec{u}_{\rm in} = 0$ by (114), whence (148) is equivalent to (115).)

One treatment with $h = 0$ but imposing only $\int_\Gamma \vec{n}\cdot\vec{g} = 0$ is that of Fursikov et al. [Fu], who study the problem in a scale of spaces that in one case exactly corresponds to what we consider here but with zero divergence constraint. Amann recently studied very weak solutions without imposing $\vec{n}\cdot\vec{g} = 0$ on Γ, but only in spaces of very low regularity that exclude the present case [Am].

Remark 2 (Isomorphism theorem for (97)-(100)). For the unconstrained Stokes system (97)-(100) there is an extra subtlety in determining an isomorphism from data to solution. We obtain a unique solution pair $\{\vec{u}, p\} \in \Pi_U$ given any data $\{\vec{f}, \vec{g}, h, \vec{u}_{\rm in}\} \in \Pi_F$ that satisfy only the compatibility conditions (114) and (115) *without* (136). And, in the other direction, given $\{\vec{u}, p\}$, we can recover

$$\vec{f} = \partial_t\vec{u} + \nabla p - \nu\Delta\vec{u}, \qquad \vec{g} = \vec{u}\big|_\Gamma, \qquad \vec{u}_{\rm in} = \vec{u}|_{t=0}. \tag{149}$$

But how are we to recover h? We need to use the fact, that follows from the definition of p_{gh} in (94), that $\nabla\cdot\vec{u} - h$ satisfies a heat equation with no-flux boundary conditions. In fact, to be able to recover h we need to know one more item, $h_{\rm in}$, the initial value of h. We have

$$h = \nabla\cdot\vec{u} - w \tag{150}$$

where w is the solution of

$$\partial_t w = \nu\Delta w \ \text{ in } \Omega, \qquad \vec{n}\cdot\nabla w = 0 \ \text{ on } \Gamma, \qquad w(0) = \nabla\cdot\vec{u}|_{t=0} - h_{\rm in}. \tag{151}$$

This procedure indicates that we should count the triple $\{\vec{u}, p, h_{\rm in}\}$ as our solution in order to build an isomorphism with the data for system (97)-(100). Of course, the regularity of $h_{\rm in}$ must match that of h, recalling the embeddings in (107).

Consequently, we see that solving the unconstrained system (97)-(100) defines an isomorphism between the data spaces

$$\tilde\Pi_{F.c} := \Big\{\{\vec{f}, \vec{g}, h, \vec{u}_{\rm in}\} \in \Pi_F : \ (114) \text{ and } (115) \text{ hold } \Big\}, \tag{152}$$

$$\tilde\Pi_{F.c.s} := \Big\{\{\vec{f}, \vec{g}, h, \vec{u}_{\rm in}\} \in \Pi_{F.s} : \ (114) \text{ and } (115) \text{ hold } \Big\}, \tag{153}$$

and, respectively, the solution spaces for $\{\vec{u}, p, h_{\rm in}\}$ given by

$$\tilde\Pi_U = H_u \times H_p \times H_{hin}, \qquad H_{hin} = L^2(\Omega), \tag{154}$$

$$\tilde\Pi_{U.s} = H_{u.s} \times H_p \times H_{hin.s}, \qquad H_{hin.s} = H^1(\Omega). \tag{155}$$

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References

[Am] H. Amann, Nonhomogeneous Navier-Stokes equations with integrable low-regularity data, in *Nonlinear problems in mathematical physics and related topics II*, Int. Math. Ser., Kluwer/Plenum, New York (2002), 1–28.

[Au] T. Aubin, *Nonlinear analysis on manifolds, Monge-Ampère equations*. Springer-Verlag, New York, 1982.

[Ba] J. M. Ball, Strongly continuous semigroups, weak solutions, and the variation of constants formula. Proc. Amer. Math. Soc. 63 (1977) 370–373.

[Ev] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence, 1998.

[Fu] A. Fursikov, M. Gunzburger and L. Hou, Trace theorems for three-dimensional, time-dependent solenoidal vector fields and their applications, Trans. Amer. Math. Soc. 354 (2002), 1079–1116.

[GS1] G. Grubb and V. A. Solonnikov, Reduction of basic initial-boundary value problems of the Navier-Stokes equations to initial-boundary value problems for nonlinear parabolic systems of pseudodifferential equations, J. Soviet Math. 56 (1991) 2300–2308.

[GS2] G. Grubb and V. A. Solonnikov, Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudodifferential methods, Math. Scand. 69 (1991) 217–290.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 3rd ed, Springer-Verlag, Berlin, 1998.

[JL] H. Johnston and J.-G. Liu, Accurate, stable and efficient Navier-Stokes solvers based on explicit treatment of the pressure term, J. Comp. Phys. 199 (1) (2004) 221–259

[LLP] J.-G. Liu, J. Liu and R. Pego, Stability and convergence of efficient Navier-Stokes solvers via a commutator estimate, accepted by Comm. Pure Appl. Math.

[LM] J. L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, vol. I and II. Springer-Verlag, Berlin, 1972

[Pa] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[Sol] V. A. Solonnikov, Estimates of the solutions of a nonstationary linearized system of Navier-Stokes equations. Amer. Math. Soc. Transl. 75, (1968) 1–116.

[Ta1] L. Tartar, Topics in Nonlinear Analysis. Publications mathématiques d'Orsay 78.13. Université de Paris Sud, France, 1978.

[Ta] M. Taylor, *Partial Differential Equations: Basic Theory*. Springer-Verlag, New York, 1996.

[Te1] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*. AMS Chelsea, Providence, 2001.